The Annals of Probability
1973, Vol. 1, No. 2, 240–253

AN INFINITE PARTICLE SYSTEM WITH ZERO RANGE INTERACTIONS[1]

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Existence and uniqueness is proved for Spitzer's zero range interaction process. It is then shown that a certain class of measures on the configuration space is invariant for this process.

1. Introduction. One of the simplest models of an infinite particle system with interactions introduced by Spitzer in [7] is the one he calls "zero range interaction." It describes the random motion of infinitely many indistinguishable particles on a countable set S, which have the property that at any given time, each particle interacts only with the other particles which occupy its site at that time. Furthermore, the interaction that does occur is of the speed change type. The motion is described in terms of (a) a probability transition function $p(x, y)$ on S, and (b) a nonnegative speed change function $c(k)$ which is defined for $k \in Z_+$, the nonnegative integers. At any time $t \geqq 0$, there are to be finitely many particles at each point of S. A particle which is at $x \in S$ at time t will make a transition to $y \in S$ during the period $(t, t + \Delta t)$ with probability

$$c(k)p(x, y)\Delta t + o(\Delta t),$$

where k is the number of particles at x at time t. In particular, if $c(k)$ were constant, all the particles would move according to independent Markov chains on S with transition function $p(x, y)$ and exponential holding times with constant parameter. On the other hand, if $c(k) = k^{-1}$, the interpretation would be that each site has an exponential clock with parameter one, and that when the clock at site x rings, a particle in chosen at random from those at x and it is moved to y with probability $p(x, y)$. The interaction among particles, then is one which changes the speed with which a given particle undergoes its motion according to the total number of particles which occupy its site.

The general problem of obtaining sufficient conditions for the existence of processes of the type described above has been treated by several authors (see [1], [2], and [4]). These results, when applied to the zero range interaction

Received March 7, 1972; revised July 7, 1972.

[1] The author wishes to express his appreciation to the University of Paris and the Wiezmann Institute of Science for their hospitality during the Fall of 1971, when some of these results were obtained. The preparation of this paper was sponsored in part by NSF GP-28258 and the Air Force Office of Scientific Research, Air Force Systems Command, USAF, under Grant No. AFOSR 69-1781. The United States Government is authorized to reproduce and distribute reprints for Governmental purposes notwithstanding any copyright notation hereon.

AMS 1970 *subject classifications.* Primary 60K35; Secondary 60J35.

Key words and phrases. Infinite particle systems, interacting particle systems, semigroups of linear transformations.

model, always require that $c(k)$ satisfy the condition

$$(1.1) \qquad \sup_k kc(k) < \infty .$$

An interpretation of this is that there are strong attractive forces between particles at the same site. One reason this condition is unsatisfactory is that, in particular, it does not include the case of independent motions (i.e., $c(k)$ constant). In this paper, we will obtain an existence theorem for the zero range interaction process under the assumption

$$(1.2) \qquad L = \sup_k |(k+1)c(k+1) - kc(k)| < \infty .$$

Following this, we will prove a conjecture of Spitzer [7] concerning the set of invariant measures for this process under the same assumption.

The reason that condition (1.1) arises in earlier work on this subject is that it is, in fact, needed if one is to prove the existence of the process beginning with an arbitrary configuration of particles. This can be seen most clearly by considering the case $c(k) \equiv 1$ with an initial configuration which puts $\eta(x)$ particles at x where $\sum_x \eta(x)p(x, y) = \infty$ for some y. In this case, one would have infinitely many particles at y at all times $t > 0$. Hence one of the principal problems involved in extending the existence results beyond condition (1.1), is to determine a class of configurations which is sufficiently large to be interesting, but also has the property that the process remains in this set for all finite times with probability one if it begins in it. A similar problem arose in a deterministic context in [3].

Motivated by the example given above, we introduce a positive function $\alpha(x)$ defined on the countable set S which satisfies $\alpha(x) \to 0$ as $x \to \infty$. The space of configurations of particles which will be considered is then

$$\mathscr{X} = \{\eta(\cdot) \mid \eta(x) \in Z_+ \quad \text{for} \quad x \in S \quad \text{and} \quad \textstyle\sum_x \eta(x)\alpha(x) < \infty\} .$$

For $\eta \in \mathscr{X}$, put $||\eta|| = \sum_x \eta(x)\alpha(x)$. The σ-algebra of subsets of $\mathscr{X}$ which we will use is the smallest one with respect to which $\eta(x)$ is measurable for each $x \in S$.

Let $c_0(\alpha)$ be the space of all functions $\beta(x)$ on S such that $\lim_{x\to\infty} [\beta(x)/\alpha(x)] = 0$, with norm given by $||\beta|| = \sup_x |\beta(x)/\alpha(x)|$. Throughout this work, we will make the following assumption relating $\alpha(\cdot)$ and $p(\cdot, \cdot)$:

$$(1.3) \quad (P\beta)(x) = \textstyle\sum_y p(x, y)\beta(y) \quad \text{defines a bounded operator on} \quad c_0(\alpha) .$$

A discussion of this assumption will appear at the end of Section 3. In particular, it will be shown there that for a given p, an α exists which satisfies our requirements if and only if $\lim_{x\to\infty} p(x, y) = 0$ for each $y \in S$. A simple example of a permissible α in a special case is the following. Suppose $S = Z$, and p satisfies $p(x, y) = p(0, y - x)$ and $\sum_x p(0, x)e^{|x|} < \infty$. Then $\alpha(x) = e^{-|x|}$ satisfies assumption (1.3), and thus any configuration of particles for which $\sum_x \eta(x)e^{-|x|} < \infty$ is in $\mathscr{X}$, and therefore can be used for an initial configuration for the process. A somewhat more general situation which illustrates the breadth of the allowed initial configurations is that in which $p(x, y)$ is doubly stochastic. It will be seen in Section 3 that in this case, there exists an α which satisfies $\sum_x \alpha(x) < \infty$ in addition to (1.3). Then if $\{\eta(x)\}$ is any collection of nonnegative integer-valued random variables for which $\sup_x E(\eta(x)) < \infty$, then $\{\eta(x)\} \in \mathscr{A}$ with probability one, and therefore they can serve as an initial distribution for the process.

In order to state our main existence theorem, we need the following notation. If $\eta \in \mathscr{A}$ and $x, y \in S$, then η_x and $\eta_{xy} \in \mathscr{A}$ are defined by

$$\begin{aligned} \eta_x(u) &= \eta(u) + 1 \qquad \text{if} \quad u = x \\ &= \eta(u) \qquad \text{otherwise}, \end{aligned}$$

$$\begin{aligned} \eta_{xy}(u) &= \eta(x) - 1 \qquad \text{if} \quad u = x \quad \text{and} \quad \eta(x) \geqq 1 \\ &= \eta(y) + 1 \qquad \text{if} \quad u = y \quad \text{and} \quad \eta(x) \geqq 1 \\ &= \eta(u) \qquad \text{otherwise}, \end{aligned}$$

if $x \neq y$, and $\eta_{xx} = \eta$. So, η_{xy} is the state reached if a transition occurs from x to y.

THEOREM (1.4). *Assume that* $c(\cdot)$ *satisfies* (1.2), *and* $p(\cdot, \cdot)$ *and* $\alpha(\cdot)$ *satisfy* (1.3). *Then there exists a Markov process* $\{\eta_t\}$ *with state space* $\mathscr{A}$ *such that* $E^\eta(\|\eta_t\|) \leqq e^{\omega t}(\|\eta\| + 1)$ *for some* $\omega > 0$, *and for each* $x \in S$ *and* $k \in Z_+$, *the function* $h(t, \eta) = P^\eta(\eta_t(x) = k)$ *satisfies:*

$$\sup_{t \leqq t_0} \sup_\eta |h(t, \eta_y) - h(t, \eta)| \in c_0(\alpha) \quad \textit{for each} \quad t_0 > 0,$$

$h(t, \eta)$ *is continuously differentiable in* t *for each* $\eta \in \mathscr{A}$, *and*

$$\frac{d}{dt} h(t, \eta) = \sum_{x,y \in S} \eta(x) c(\eta(x)) p(x, y)[h(t, \eta_{xy}) - h(t, \eta)] \tag{1.5}$$

for each $\eta \in \mathscr{A}$ *and* $t \geqq 0$. *Conversely, if* $\{\zeta_t\}$ *is a Markov process on* $\mathscr{A}$ *so that for each* $x \in S$ *and* $k \in Z_+$, *the function* $h(t, \eta) = P^\eta(\zeta_t(x) = k)$ *satisfies:*

$h(t, \eta)$ *is weak * continuous on bounded sets of* $\mathscr{A}$ *for* $t \geqq 0$,

and

$h(t, \eta)$ *is continuously differentiable in* t *and satisfies* (1.5) *for each* η *such that* $\sum_x \eta(x) < \infty$,

then ζ_t *is a version of* η_t.

We also prove the following conjecture of Spitzer [7], which was proved by Holley [2] under assumption (1.1) and additional assumptions on $p(\cdot, \cdot)$. For $0 < \rho < \liminf_{k \to \infty} kc(k)$, define a measure ν on Z_+ by

$$\nu(k) = \gamma \frac{\rho^k}{k!\, c(1) \cdots c(k)}$$

where γ is chosen so that $\sum_{k=0}^\infty \nu(k) = 1$. Let $(\zeta(x), x \in S)$ be independent and identically distributed random variables with distribution ν.

THEOREM (1.6). *In addition to the assumptions of Theorem* (1.4), *take* $p(\cdot, \cdot)$ *to* *be doubly stochastic and $\alpha(\cdot)$ to satisfy $\sum_x \alpha(x) < \infty$. Then $(\zeta(x), x \in S)$ lies in $\mathscr{A}$ with probability one, and the induced measure on $\mathscr{A}$ is an invariant measure for the process η_t.*

REMARK. It will be shown in Section 3 that if $p(\cdot, \cdot)$ is doubly stochastic, then there exists an $\alpha(\cdot)$ which satisfies (1.3) and $\sum_x \alpha(x) < \infty$.

The remainder of the paper is devoted to the proofs of Theorems (1.4) and (1.6). In Section 2, the relevant semigroup of operators is constructed on a space of continuous functions on $\mathscr{A}$. One interesting technical point that arises here is that the semigroup turns out not to be strongly continuous unless the space is appropriately renormed. In Section 3, the proof of Theorem (1.4) will be completed, and assumption (1.3) will be discussed. Theorem (1.6) will be proved in Section 4.

2. Construction of the semigroup. Throughout this section, we assume that $c(\cdot)$, $p(\cdot, \cdot)$ and $\alpha(\cdot)$ satisfy (1.2) and (1.3). As defined in Section 1, $\mathscr{A}$ is a closed subset of $l_1(\alpha)$, which is the dual of $c_0(\alpha)$. So $B(\mathscr{A})$ can be defined to be the space of all real-valued functions on $\mathscr{A}$ which are weak * continuous on (norm) bounded subsets of $\mathscr{A}$ and which satisfy the growth condition

$$(2.1) \qquad \lim_{||\eta||\to\infty} \frac{|f(\eta)|}{||\eta||} = 0 .$$

For $f \in B(\mathscr{A})$, put $||f|| = \sup_\eta |f(\eta)|/(1 + ||\eta||)$. Let

$$C(\mathscr{A}) = \{f \in B(\mathscr{A}\} \mid |||f||| \equiv \sup_\eta |f(\eta)| < \infty\} .$$

As will be seen later, the required semigroup of operators will be strongly continuous on $B(\mathscr{A})$ but not on $C(\mathscr{A})$. The idea of the construction is, then, to apply the Hille-Yosida theorem to an appropriate generator on $B(\mathscr{A})$, and then to show that when the semigroup so generated is restricted to $C(\mathscr{A})$, one obtains a semigroup of contractions.

In order to define the generator of the semigroup, put

$$G_1 = \{f \in C(\mathscr{A}) \,|\sup_\eta | f(\eta_x) - f(\eta)| \in c_0(\alpha)\} .$$

Since $\sup_k c(k) < \infty$ (by (1.2)) and

$$\sup_\eta |f(\eta_{xy}) - f(\eta)| \leqq \sup_\eta |f(\eta_x) - f(\eta)| + \sup_\eta |f(\eta_y) - f(\eta)| ,$$

the sum

$$(2.2) \qquad \Omega_1 f(\eta) = \sum_x \eta(x)c(\eta(x)) \sum_y p(x, y)[f(\eta_{xy}) - f(\eta)]$$

converges absolutely and uniformly on bounded sets of $\mathscr{A}$ for $f \in G_1$. Put

$$G = \{f \in G_1 \,|\, \Omega_1 f \in B(\mathscr{A})\} .$$

LEMMA (2.3). *G is dense in $B(\mathscr{A})$.*

PROOF. First we show that G_1 is dense in $B(\mathscr{A})$. Let S_n be finite subsets of S

so that $S_n \uparrow S$, and define

$$(2.4) \qquad \begin{aligned} T_n\eta(x) &= \eta(x) \qquad x \in S_n \\ &= 0 \qquad x \notin S_n . \end{aligned}$$

For $f \in B(\mathscr{A})$, put

$$\begin{aligned} f_n(\eta) &= f(T_n\eta) \qquad \text{if } |f(T_n\eta)| \leqq n \\ &= 0 \qquad \text{otherwise} . \end{aligned}$$

Then $f_n \in G_1$, since $\sup_\eta |f(\eta_x) - f(\eta)| = 0$ for all but finitely many x. We must show that $||f_n - f|| \to 0$. Take $\eta_n \in \mathscr{A}$ and consider two cases. If $||\eta_n|| \to \infty$, then $|f(\eta_n) - f_n(\eta_n)| \leqq |f(\eta_n)| + |f(T_n\eta_n)|$, and hence

$$|f(\eta_n) - f_n(\eta_n)|[1 + ||\eta_n||]^{-1} \to 0$$

since $||T_n\eta_n|| \leqq ||\eta_n||$ and f satisfies (2.1). If $\eta_n \to_{w^*} \eta \in \mathscr{A}$, then $T_n\eta_n \to_{w^*} \eta$ and therefore $f_n(\eta_n) \to f(\eta)$ and $f(\eta_n) \to f(\eta)$. Since bounded sets in $\mathscr{A}$ have weak * compact closures, it follows that $||f_n - f|| \to 0$.

Now, to show that G is dense in G_1, take $f \in G_1$ and take $\beta \in c_0(\alpha)$ so that $\beta(x) > 0$ and $\sup_\eta |f(\eta_x) - f(\eta)| \leqq \beta(x)$. For $\lambda > 0$, define

$$g_\lambda(\eta) = f(\eta)\exp\{-\lambda \textstyle\sum_x \eta(x)\beta(x)\} .$$

Since $||g_\lambda - f|| \leqq \lambda(|||f|||)(||\beta||)$, $g_\lambda \to f$ as $\lambda \to 0$. $g_\lambda \in G_1$, so it remains to be shown that $\Omega_1 g_\lambda \in B(\mathscr{A})$. $\Omega_1 g_\lambda$ is weak * continuous on bounded sets since the convergence of (2.2) is uniform on bounded sets. Now

$$|g_\lambda(\eta_x) - g_\lambda(\eta)| \leqq [1 + \lambda\, |||f|||]\beta(x)\exp\{-\lambda \textstyle\sum_u \eta(u)\beta(u)\}$$

so that

$$|\Omega_1 g_\lambda(\eta)| \leqq [1 + \lambda\, |||f|||] \sup_k c(k) \textstyle\sum_x \eta(x)[\beta(x) + P\beta(x)] \times \exp\{-\lambda \textstyle\sum_u \eta(u)\beta(u)\} .$$

Hence, $\Omega_1 g_\lambda$ satisfies (2.1).

Before proceeding, we need the following simple result.

LEMMA (2.5). *For $f \in B(\mathscr{A})$, put*

$$M = \sup_\eta \frac{f(\eta)}{1 + ||\eta||} \quad \text{and} \quad m = \inf_\eta \frac{f(\eta)}{1 + ||\eta||} .$$

Then if $M > 0$, there is an $\eta_M \in \mathscr{A}$ so that $M = f(\eta_M)/(1 + ||\eta_M||)$, and if $m < 0$, there is an $\eta_m \in \mathscr{A}$ so that $m = f(\eta_m)/(1 + ||\eta_m||)$.

PROOF. It suffices to prove the first statement. Take $\eta_n \in \mathscr{A}$ so that $f(\eta_n)/(1 + ||\eta_n||) \to M > 0$. By (2.1), $||\eta_n||$ is bounded in n, so by extracting a subsequence we may assume that $\eta_n \to_{w^*} \eta$. Then $f(\eta_n) \to f(\eta)$ and

$$||\eta|| \leqq \liminf_{n\to\infty} ||\eta_n|| .$$

So

$$M \geqq \frac{f(\eta)}{1 + ||\eta||} \geqq \lim_{n\to\infty} \frac{f(\eta_n)}{1 + ||\eta_n||} = M ,$$

and $M = f(\eta)/(1 + ||\eta||)$.

If Γ is a linear operator on a Banach space X, $\mathscr{D}(\Gamma)$ and $\mathscr{R}(\Gamma)$ will denote its domain and range respectively. Γ is called dissipative if $f - \lambda\Gamma f = g$ implies that $||f|| \leqq ||g||$ whenever $f \in \mathscr{D}(\Gamma)$ and $\lambda > 0$. Let Ω_2 be the restriction of Ω_1 to G.

LEMMA (2.6). *Put* $\omega = (||P|| + 1)\sup_k c(k)$ *where* P *is the operator on* $c_0(\alpha)$ *defined by* (1.3). *Then* $\Omega_2 - \omega I$ *is dissipative in* $B(\mathscr{S})$.

PROOF. Take $f \in G$ and $\lambda > 0$, and define $g = f - \lambda(\Omega_2 - \omega I)f$. In order to show that $||f|| \leqq ||g||$, assume that $||f|| = \sup_\eta f(\eta)/(1 + ||\eta||)$. The other case is treated similarly. By Lemma (2.5), there is a ζ so that $f(\zeta)/(1 + ||\zeta||) = ||f||$. Since $||\eta_{xy}|| = ||\eta|| + \alpha(y) - \alpha(x)$ whenever $\eta(x) \neq 0$,

$$\frac{f(\eta)}{1 + ||\eta||}(1 + \lambda\omega) - \lambda \sum_{x,y} \eta(x)c(\eta(x))p(x, y)\left[\frac{f(\eta_{xy})}{1 + ||\eta_{xy}||} - \frac{f(\eta)}{1 + ||\eta||}\right]$$

$$= \frac{g(\eta)}{1 + ||\eta||} + \lambda \sum_{x,y} \eta(x)c(\eta(x))p(x, y)\frac{f(\eta_{xy})}{1 + ||\eta_{xy}||}\,\frac{\alpha(y) - \alpha(x)}{1 + ||\eta||}.$$

Evaluate this expression at ζ and use the nonnegativity of $\zeta(x)$, $c(\zeta(x))$ and $p(x, y)$ to obtain

$$||f||(1 + \lambda\omega) \leqq ||g|| + \lambda||f|| \sum_{x,y} \zeta(x)c(\zeta(x))p(x, y)\frac{\alpha(y) + \alpha(x)}{1 + ||\zeta||}$$

$$\leqq ||g|| + \lambda||f|| \sup_k c(k)[||P|| + 1],$$

which completes the proof. The last inequality follows from $\sum_y p(x, y)\alpha(y) \leqq ||P||\,\alpha(x)$.

Now, by Lemmas (2.3) and (2.6), $\Omega_2 - \omega I$ is a dissipative operator on $B(\mathscr{S})$ with dense domain. Therefore Ω_2 has a minimal closed extension which will be called Ω (see Lemma 3.3 of [5], for example). We will need the following technical lemma in Section 3.

LEMMA (2.7). $G = G_1 \cap \mathscr{D}(\Omega)$.

PROOF. $G \subset G_1$ by definition of G, and $G \subset \mathscr{D}(\Omega)$ since Ω is an extension of Ω_2, which has domain G. To show that $G_1 \cap \mathscr{D}(\Omega) \subset G$, take $f \in G_1 \cap \mathscr{D}(\Omega)$. Since $f \in \mathscr{D}(\Omega)$, there is a sequence $f_n \in G$ so that $f_n \to f$ and $\Omega f_n \to \Omega f$. Since $f \in G_1$, $\Omega_1 f$ is well-defined and is continuous on bounded sets of $\mathscr{S}$. Since $f_n \to f$, $\Omega f_n(\eta) \to \Omega_1 f(\eta)$ for all η such that $\sum_x \eta(x) < \infty$. Therefore $\Omega f(\eta) = \Omega_1 f(\eta)$ for all such η, and $\Omega f = \Omega_1 f$ follows by continuity since $\{\eta \in \mathscr{S} \mid \sum_x \eta(x) < \infty\}$ is dense. But $\Omega f \in B(\mathscr{S})$, so $\Omega_1 f \in B(\mathscr{S})$ and hence $f \in G$.

The next result shows that Ω satisfies one of the important properties of the generator of a semigroup of operators.

LEMMA (2.8). *Take* $0 < \lambda < [L(1 + ||P||)]^{-1}$. *Then* $\mathscr{R}(I - \lambda\Omega) = B(\mathscr{S})$. *Furthermore,*

(a) *if* $f - \lambda\Omega f = g$ *and* $\sup_\eta |g(\eta_x) - g(\eta)| \leqq \beta(x) \in c_0(\alpha)$, *then* $\sup_\eta |f(\eta_x) - f(\eta)| \leqq \tilde\beta(x)$ *where* $\tilde\beta \in c_0(\alpha)$ *is the unique solution of* $(1 - \lambda L)\tilde\beta(u) = \beta(u) + \lambda L \sum_y p(u, y)\tilde\beta(y)$, *and*

(b) *if* $f - \lambda\Omega f = g$ *and* $g \geqq 0$, *then* $f \geqq 0$.

PROOF. Let $\mathscr{A}_n = \{\eta \in \mathscr{A} \mid \sum_x \eta(x) \leqq n\}$, and B_n be the set of real-valued functions f on $\mathscr{A}_n$ such that $f(0) = 0$ and $\sup_\eta |f(\eta_x) - f(\eta)| \in c_0(\alpha)$. Then B_n is a Banach space with norm

$$N(f) = \sup_{x,\eta} \frac{|f(\eta_x) - f(\eta)|}{\alpha(x)} .$$

Since for each $x, y \in S$, $\eta \in \mathscr{A}_n$ if and only if $\eta_{xy} \in \mathscr{A}_n$, the expression on the right side of (2.2) defines a bounded operator Ω_n on B_n with $||\Omega_n|| \leqq L[1 + ||P||] + 2n \sup_k c(k)$. Suppose $f, g \in B_n$, $0 < \lambda L(1 + ||P||) < 1$, and

$$f - \lambda\Omega_n f = g . \tag{2.9}$$

Let $\gamma(x) = \sup_\eta |g(\eta_x) - g(\eta)|$ and $\tilde{\gamma}(x) = \sup_\eta |f(\eta_x) - f(\eta)|$. Evaluate (2.9) at η and at η_u, and substract to obtain

$$|f(\eta_u) - f(\eta)| [1 + \lambda \textstyle\sum_x \eta(x)c(\eta(x))] \leqq |g(\eta_u) - g(\eta)|$$
$$+ \lambda\tilde{\gamma}(u) \textstyle\sum_x \eta(x)c(\eta(x)) + \lambda L \sum_y p(u, y)[\tilde{\gamma}(u) + \tilde{\gamma}(y)] .$$

Since $\sum_x \eta(x)c(\eta(x))$ is bounded on $\mathscr{A}_n$, it follows that $(1 - \lambda L)\tilde{\gamma}(u) \leqq \gamma(u) + \lambda L \sum_y p(u, y)\tilde{\gamma}(y)$. Since γ and $\tilde{\gamma}$ are in $c_0(\alpha)$, and $\lambda L(1 + ||P||) < 1$, this may be iterated to obtain $\tilde{\gamma} \leqq [(1 - \lambda L)I - \lambda LP]^{-1}\gamma$. Therefore

$$||\tilde{\gamma}|| \leqq (1 - \lambda L - \lambda L\, ||P||)^{-1}\, ||\gamma|| .$$

Since $||\tilde{\gamma}|| = N(f)$ and $||\gamma|| = N(g)$, this says that $\Omega_n - L(1 + ||P||)I$ is dissipative. Since Ω_n is also bounded, it follows that $\mathscr{R}(I - \lambda\Omega_n) = B_n$. Now take $g \in G_1$ and $\beta \in c_0(\alpha)$ so that $g(0) = 0$ and $\sup_\eta |g(\eta_x) - g(\eta)| \leqq \beta(x)$. Then g restricted to $\mathscr{A}_n$ is in B_n, so for each n there is a unique $f_n \in B_n$ for which

$$f_n(\eta) - \lambda \textstyle\sum_x \eta(x)c(\eta(x)) \sum_y p(x, y)[f_n(\eta_{xy}) - f_n(\eta)] = g(\eta)$$

for $\eta \in \mathscr{A}_n$. By the uniqueness statement, $f_{n+1}(\eta) = f_n(\eta)$ for $\eta \in \mathscr{A}_n$, so there is a function f defined on $A = \bigcup_{n=1}^\infty \mathscr{A}_n$ for which

$$f(\eta) - \lambda \textstyle\sum_x \eta(x)c(\eta(x)) \sum_y p(x, y)[f(\eta_{xy}) - f(\eta)] = g(\eta)$$

for all $\eta \in A$. Note that A is dense in $\mathscr{A}$. In order to extend f to all of $\mathscr{A}$, define a metric on $\mathscr{A}$ by

$$d(\eta, \zeta) = \textstyle\sum_x |\eta(x) - \zeta(x)|\, \tilde{\beta}(x) ,$$

where $\tilde{\beta}$ is defined in (a) above. This gives the weak * topology on bounded sets of $\mathscr{A}$, since $\tilde{\beta} \in c_0(\alpha)$, and a bounded sequence $\eta_n \in \mathscr{A}$ converges to η in the weak * topology if and only if $\eta_n(x) \to \eta(x)$ for each $x \in S$. Since $|f(\eta_x) - f(\eta)| \leqq \tilde{\beta}(x)$ for $\eta \in A$, $|f(\eta) - f(\zeta)| \leqq d(\eta, \zeta)$ on A and hence f can be extended uniquely to all of $\mathscr{A}$ as a continuous function on bounded sets. It is easy to check that $|||f||| \leqq |||g|||$, and that if $g \geqq 0$, then $f \geqq 0$. So $f \in G_1$ and $f - \lambda\Omega_1 f = g$. Therefore $\Omega_1 f \in G_1 \subset B(\mathscr{A})$, so $f \in G \subset \mathscr{D}(\Omega)$ and $\Omega_1 f = \Omega f$. We have now shown that $\mathscr{R}(I - \lambda\Omega) \supset \{g \in G_1 \mid g(0) = 0\}$. Since $1 \in \mathscr{D}(\Omega)$ and $\Omega 1 = 0$, $\mathscr{R}(I - \lambda\Omega) \supset G_1$. But $\mathscr{R}(I - \lambda\Omega)$ is closed, since Ω is a closed operator and $\Omega - \omega I$ is dissipative. The proof is then completed by using Lemma (2.3).

Let $p_t(x, y) = e^{-t} \sum_{n=0}^{\infty} (t^n/n!)p^{(n)}(x, y)$, where $p^{(n)}(x, y)$ is the n-step transition probability for the Markov chain which has 1-step transition probabilities $p(x, y)$.

THEOREM (2.10). *Ω generates a strongly continuous semigroup $S(t)$ of operators on $B(\mathscr{X})$ such that $\|S(t)f\| \leqq e^{\omega t} \|f\|$ for all $f \in B(\mathscr{X})$. Furthermore,*

(a) *$S(t)1 = 1$,*
(b) *if $f \in B(\mathscr{X})$ and $f \geqq 0$, then $S(t)f \geqq 0$, and*
(c) *if $f \in G_1$, then $S(t)f \in G_1$ and*

$$\sup_\eta |S(t)f(\eta_x) - S(t)f(\eta)| \leqq e^{2tL} \sum_y p_{tL}(x, y) \sup_\eta |f(\eta_y) - f(\eta)| .$$

PROOF. The main statement is an immediate consequence of the Hille-Yosida theorem since $\mathscr{D}(\Omega)$ is dense in $B(\mathscr{X})$, $\Omega - \omega I$ is dissipative, and $\mathscr{R}(I - \lambda\Omega) = B(\mathscr{X})$ for all sufficiently small $\lambda > 0$. Another consequence is that $S(t)f = \lim_{n\to\infty} (I - (t/n)\Omega)^{-n}f$. To prove (a), it suffices to show that $1 \in \mathscr{D}(\Omega)$ and $\Omega 1 = 0$, which is true because $1 \in G_1$ and $\Omega_1 1 = 0$. Finally, (b) and (c) follow from (b) and (a) of Lemma (2.8) respectively. In order to obtain (c), note that if $f = (I - \lambda\Omega)^{-n}g$ and $\sup_\eta |g(\eta_x) - g(\eta)| \leqq \beta(x) \in c_0(\alpha)$, then

$$|f(\eta_x) - f(\eta)| \leqq [I - \lambda L(I + P)]^{-n}\beta(x) .$$

The proof is concluded by substituting t/n for λ and taking the limit as $n \to \infty$.

THEOREM (2.11). *If $f \in C(\mathscr{X})$, then $S(t)f \in C(\mathscr{X})$ and $|||S(t)f||| \leqq |||f|||$. Therefore, when restricted to $C(\mathscr{X})$, $S(t)$ is a (not necessarily strongly continuous) semigroup of contractions on $C(\mathscr{X})$.*

PROOF. If $m \leqq f \leqq M$, then $m \leqq S(t)f \leqq M$ by (a) and (b) of the previous theorem.

REMARK. To see that $S(t)$ is virtually never strongly continuous on $C(\mathscr{X})$, it suffices to consider the case of independent motions, $c(k) \equiv 1$. Fix x and y and take $f \in C(\mathscr{X})$ and $\eta_n \in \mathscr{X}$ to be given by

$$f(\eta) = 1 \quad \text{if} \quad \eta(y) = 0 \quad \text{and} \quad \eta_n(u) = n \quad \text{if} \quad u = x$$
$$= 0 \quad \text{if} \quad \eta(y) \geqq 1 \qquad\qquad = 0 \quad \text{if} \quad u \neq x .$$

Then $S(t)f(\eta_n) = [1 - p_t(x, y)]^n$, so $S(t)f$ does not converge to f uniformly on $\mathscr{X}$.

We conclude this section with some simple results which will be needed for the construction of the Markov process which corresponds to $S(t)$.

LEMMA (2.12). (a) *If $f \in C(\mathscr{X})$ and $|f(\eta)| \leqq \|\eta\|$ for each $\eta \in \mathscr{X}$, then*

$$|S(t)f(\eta)| \leqq e^{\omega t}[\|\eta\| + 1] .$$

(b) *If $f_n \in C(\mathscr{X})$, $\sup_n |||f_n||| < \infty$, and $f_n \to 0$ uniformly on bounded sets of $\mathscr{X}$, then $S(t)f_n \to 0$ uniformly on bounded sets.*

PROOF. (a) By assumption, $f \in B(\mathscr{X})$ and $\|f\| \leqq 1$. So $\|S(t)f\| \leqq e^{\omega t}$ by Theorem (2.10), and the conclusion follows from the definition of the norm in $B(\mathscr{X})$.

(b) The assumption implies that $\|f_n\| \to 0$. Therefore $\|S(t)f_n\| \to 0$ and the conclusion follows.

3. Construction of the process. Since $\mathscr{X}$ is not a locally compact Hausdorff space, the standard theorems which permit the construction of a Markov process from its semigroup of operators do not apply directly. Therefore, we devote the first part of this section to the consideration of this construction problem in our context.

We begin with a (not necessarily strongly continuous) semigroup of contractions on $C(\mathscr{X})$ which satisfies

(3.1) $\sup\{|S(t)f(\eta)| \mid f \in C(\mathscr{X}) \text{ and } |f(\zeta)| \leqq \|\zeta\| \text{ for all } \zeta \in \mathscr{X}\} < \infty$ for each $\eta \in \mathscr{X}$, and

(3.2) if $f_n \in C(\mathscr{X})$, $\sup_n |||f_n||| < \infty$, and $f_n \to 0$ uniformly on bounded sets, then $S(t)f_n(\eta) \to 0$ for each $\eta \in \mathscr{X}$.

Note that the semigroup constructed in Section 2 has these properties by Lemma (2.12).

THEOREM (3.3). *Under these assumptions, if $t > 0$ and $\eta \in \mathscr{X}$, there exists a probability measure $\pi_t(\eta, d\xi)$ on $\mathscr{X}$ so that for each $f \in C(\mathscr{X})$*

$$S(t)f(\eta) = \int f(\zeta)\pi_t(\eta, d\zeta) . \tag{3.4}$$

PROOF. If $T = \{x_1, \cdots, x_n\}$ is a finite subset of S, define

$$\pi_T(k_1, \cdots, k_n) = S(t)1_{\{\zeta(x_1)=k_1,\cdots,\zeta(x_n)=k_n\}}(\eta)$$

for $k_i \in Z_+$. By (3.2),

$$\sum_{k_1,\cdots,k_n} \pi_T(k_1, \cdots, k_n) = 1 ,$$

so $\{\pi_T\}$ forms a consistent set of probability measures. Therefore, there is a measure μ on Z_+^S so that

$$\mu\{\zeta \in Z_+^S \mid \zeta(x_1) = k_1, \cdots, \zeta(x_n) = k_n\} = \pi_T(k_1, \cdots, k_n) .$$

So, if f is any bounded function on Z_+^S which depends on only finitely many coordinates, $\int f(\zeta)\mu(d\zeta) = S(t)f(\eta)$. Let S_n be finite subsets of S so that $S_n \uparrow S$. Then if $f_n(\zeta) = \min\{n, \sum_{x \in S_n} \zeta(x)\alpha(x)\}$, (3.1) gives

$$\sup_n |\textstyle\int f_n(\zeta)\mu(d\zeta)| = \sup_n |S(t)f_n(\eta)| < \infty .$$

By the monotone convergence theorem, $\int \|\zeta\| \, \mu(d\zeta) < \infty$, so $\mu(\mathscr{X}) = 1$. Let $\pi_t(\eta, d\zeta)$ be μ restricted to $\mathscr{X}$. Then (3.4) holds for those $f \in C(\mathscr{X})$ which depend on finitely many coordinates. If $f \in C(\mathscr{X})$ is arbitrary, define $f_n \in C(\mathscr{X})$ by $f_n(\zeta) = f(T_n\zeta)$, where T_n is defined in (2.4). Then $f_n \to f$ uniformly on bounded subsets of $\mathscr{X}$, and (3.4) follows from (3.2).

LEMMA (3.5). *If F is a measurable set in $\mathscr{X}$, then $\pi_t(\eta, F)$ is a measurable function of η.*

PROOF. If F is a finite dimensional set, $1_F \in C(\mathscr{X})$, so $\pi_t(\eta, F) = S(t)1_F(\eta)$ which is continuous in η on bounded sets, and hence measurable. The conclusion for general measurable F follows now from the monotone class theorem.

THEOREM (3.6). *There exists a Markov process η_t with state space $\mathscr{X}$ so that*

$$P^{\eta}(\eta_t \in A) = \pi_t(\eta, A) .$$

PROOF. By Lemma (3.5), the finite dimensional distributions of the process can be written down in the standard way. The existence of the process then follows from a result in [6] (the corollary on page 83) since $\mathscr{X}$, with the topology of weak * convergence on bounded sets is a complete separable metric space. It is not difficult to see that the σ-algebra of Borel sets relative to this topology is the same as the one generated by the finite dimensional sets.

The following two results will complete respectively the existence and uniqueness parts of Theorem (1.4). Let $S(t)$ be the semigroup constructed in Section 2, and η_t be the Markov process constructed from it in Theorems (3.3) and (3.6).

THEOREM (3.7). *If $u \in S$ and $k \in Z_+$, $h(t, \eta) = P^{\eta}(\eta_t(u) = k)$ satisfies*

(3.8) $$\sup_{t \leq t_0} \sup_{\eta} |h(t, \eta_x) - h(t, \eta)| \in c_0(\alpha) \text{ for all } t_0,$$

(3.9) *$h(t, \eta)$ is continuously differentiable in t for each $\eta \in \mathscr{X}$,*

and

(3.10) $$\frac{d}{dt} h(t, \eta) = \sum_{x,y \in S} \eta(x)c(\eta(x))p(x, y)[h(t, \eta_{xy}) - h(t, \eta)] .$$

PROOF. Put

$$h(\eta) = h(0, \eta) = 1 \quad \text{if} \quad \eta(u) = k$$
$$= 0 \quad \text{if} \quad \eta(u) \neq k .$$

Then $h \notin \mathscr{D}(\Omega)$, in general. However, as shown in the proof of Lemma (2.3), if $\beta \in c_0(\alpha)$ with $\beta(x) > 0$ for all x and $h_\lambda(\eta) = h(\eta) \exp\{-\lambda \sum_x \eta(x)\beta(x)\}$, then

(a) $h_\lambda \in G \subset \mathscr{D}(\Omega)$,
(b) $||h_\lambda - h|| \to 0$ as $\lambda \to 0$, and
(c) $|h_\lambda(\eta_x) - h_\lambda(\eta)| \leqq \lambda\beta(x)$ for $x \neq u$.

By the Hille-Yosida theorem, $S(t)h_\lambda \in \mathscr{D}(\Omega)$, $S(t)h_\lambda$ is continuously differentiable in t, and

$$\frac{d}{dt} S(t)h_\lambda(\eta) = \Omega S(t)h_\lambda(\eta) .$$

By part (c) of Theorem (2.10), $S(t)h_\lambda \in G_1$, so $S(t)h_\lambda \in G$ by Lemma (2.7). So,

(3.11) $$\frac{d}{dt} S(t)h_\lambda(\eta) = \sum_{x,y} \eta(x)c(\eta(x))p(x, y)[S(t)h_\lambda(\eta_{xy}) - S(t)h_\lambda(\eta)] .$$

Since $h(t, \eta) = S(t)h(\eta)$, it suffices to take the limit in (3.11) as $\lambda \to 0$. This can be done because the sum on the right-hand side converges uniformly in λ for each $\eta \in \mathscr{A}$ by (c) above, and $S(t)h_\lambda(\eta) \to S(t)h(\eta)$ for each $\eta \in \mathscr{A}$ by (b) and the fact that $S(t)$ is a bounded operator on $B(\mathscr{A})$.

THEOREM (3.12). *Suppose ζ_t is a Markov process on $\mathscr{A}$ such that for each $u \in S$ and $k \in Z_+$, the function $h(t, \eta) = P^\eta(\zeta_t(u) = k)$ satisfies:*

(a) *$h(t, \eta)$ is weak * continuous on bounded sets for $t \geqq 0$, and*

(b) *$h(t, \eta)$ is continuously differentiable in t and satisfies* (3.10) *for all $\eta \in \mathscr{A}$ such that $\sum_x \eta(x) < \infty$.*

Then ζ_t is a version of the process η_t constructed in Theorems (3.3) *and* (3.6).

PROOF. When restricted to $\mathscr{A}_n = \{\eta \mid \sum_x \eta(x) \leqq n\}$, Equation (3.10) takes the form $dw/dt = \Gamma w(t)$ where Γ is a bounded operator. Using standard uniqueness theorems for ordinary differential equations in Banach spaces, we conclude that $h(t, \eta) = P^\eta(\eta_t(u) = k)$ for $\eta \in \mathscr{A}_n$. Since $\bigcup_n \mathscr{A}_n$ is dense in $\mathscr{A}$ and both sides of this identity are continuous in η, it holds for all $\eta \in \mathscr{A}$. So, the processes η_t and ζ_t have the same finite dimensional distributions, and are therefore versions of one another.

We conclude this section with some brief remarks concerning assumption (1.3). First note that this assumption is equivalent to

$$\text{(3.13)} \qquad \lim_{x\to\infty} \frac{p(x, y)}{\alpha(x)} = 0 \quad \text{for each} \quad y \in S\,,$$

and

$$\text{(3.14)} \qquad \sum_y p(x, y)\alpha(y) \leqq M\alpha(x) \quad \text{for some constant} \quad M\,.$$

We will only verify that these two conditions imply (1.3), since the proof of the converse is similar. If $\beta \in c_0(\alpha)$, then for $\varepsilon > 0$ there is a finite set $T \subset S$ such that $|\beta(x)| \leqq \varepsilon\alpha(x)$ for $x \notin T$. By (3.14),

$$|P\beta(x)| \leqq \sum_{y \in T} p(x, y)\, |\beta(y)| + \varepsilon M\alpha(x)\,.$$

Therefore, since ε is arbitrary, $P\beta \in c_0(\alpha)$ by (3.13). Finally,

$$|P\beta(x)| \leqq ||\beta|| \sum_y p(x, y)\alpha(y) \leqq M\, ||\beta||\, \alpha(x)\,,$$

so $||P||$ is the smallest M for which (3.14) holds.

Given a transition function $p(x, y)$, we wish to be able to construct an α which satisfies (3.13) and (3.14). This is done in two parts. First it is shown that if α satisfies the second requirement, then it can be modified slightly so as to make it satisfy both. Then it is noted that there are many choices available which satisfy (3.14).

THEOREM (3.15). *If α satisfies* (3.14) *and φ is a concave, non-decreasing function on $[0, \infty)$ such that $\varphi(0) = 0$ and $\varphi'(0) = \infty$, then $\tilde{\alpha}(x) = \varphi(\alpha(x))$ satisfies* *both* (3.13) *and* (3.14). *Furthermore, if* $\sum_x \alpha(x) < \infty$, *then* φ *can be chosen so that* $\sum_x \tilde{\alpha}(x) < \infty$.

PROOF. From the assumptions, it follows that $\varphi(Mt)/\varphi(t) \leqq \max(1, M) = \tilde{M}$ for all $t > 0$. Apply φ to (3.14) to obtain

$$\sum_y p(x, y)\tilde{\alpha}(y) \leqq \varphi(\sum_y p(x, y)\alpha(y)) \leqq \varphi(M\alpha(x)) \leqq \tilde{M}\tilde{\alpha}(x) .$$

From (3.14), $\sup_x p(x, y)/\alpha(x) < \infty$ for each $y \in S$. So (3.13) follows from $\lim_{x\to\infty} \alpha(x)/\tilde{\alpha}(x) = 0$. For the second part of the proof, assume in addition that $\sum_x \alpha(x) < \infty$. Let α_n be the enumeration of the values of α, ordered so that $\alpha_n \downarrow 0$. φ will be constructed by defining $\varphi(\alpha_n)$ appropriately and then interpolating linearly. Since $\sum_{i=1}^\infty i(\alpha_i - \alpha_{i+1}) < \infty$, we may put

$$\Delta_m = [\sum_{i=m}^\infty i(\alpha_i - \alpha_{i+1})]^{-\gamma} \quad \text{where} \quad 0 < \gamma < 1 .$$

Then $0 \leqq \Delta_1 \leqq \Delta_2 \leqq \cdots \leqq \Delta_m \uparrow \infty$ and $\sum_{m=1}^\infty m(\alpha_m - \alpha_{m+1})\Delta_m < \infty$. Define $\varphi(\alpha_m) = \sum_{i=m}^\infty (\alpha_i - \alpha_{i+1})\Delta_i$. Since

$$\Delta_m = \frac{\varphi(\alpha_m) - \varphi(\alpha_{m+1})}{\alpha_m - \alpha_{m+1}} \quad \text{if} \quad \alpha_{m+1} \neq \alpha_m ,$$

$\varphi(\alpha_m)$ satisfies $\varphi(\alpha_m) \downarrow 0$, $[\varphi(\alpha_m) - \varphi(\alpha_{m+1})]/(\alpha_m - \alpha_{m+1}) \uparrow \infty$, and $\sum_{m=1}^\infty \varphi(\alpha_m) < \infty$. Therefore the function φ defined by linearly interpolating these values has the properties required in the statement of the theorem.

THEOREM (3.16). *There exists a positive function* α *on* S *such that* $\lim_{x\to\infty} \alpha(x) = 0$ *which satisfies* (1.3) *if and only if* $\lim_{x\to\infty} p(x, y) = 0$ *for each* $y \in S$. *Furthermore, if* p *satisfies*

$$K \equiv \sup_y \sum_x p(x, y) < \infty , \tag{3.17}$$

then α *can be chosen so that* $\sum_x \alpha(x) < \infty$.

PROOF. To prove the sufficiency of the condition, take $0 < \lambda < 1$ and a positive function γ on S such that $\sum_x \gamma(x) < \infty$. Define

$$\alpha(x) = \sum_y \sum_{n=0}^\infty \lambda^n p^{(n)}(x, y)\gamma(y) .$$

Then α satisfies (3.14) with $M = \lambda^{-1}$, and $\alpha(x) > 0$ for all $x \in S$. To show that $\lim_{x\to\infty} \alpha(x) = 0$, it suffices to that

$$\lim_{x\to\infty} p^{(n)}(x, y) = 0 \tag{3.18}$$

for $n \geqq 1$ and $y \in S$. The proof is by induction on n. For $n = 1$, it is true by assumption. Assume then that it is true for $n - 1$, and write

$$p^{(n)}(x, y) = \sum_z p(x, z)p^{(n-1)}(z, y) .$$

If $\varepsilon > 0$, then by the induction hypothesis there is a finite set $T \subset S$ such that $p^{(n-1)}(z, y) < \varepsilon$ for $z \notin T$. Therefore,

$$p^{(n)}(x, y) \leqq \sum_{z\in T} p(x, z) + \varepsilon ,$$

and (3.18) follows immediately. Finally, α can be modified slightly in such a way that it satisfies (3.13) by Theorem (3.15). If (3.17) holds, then

$$\sup_y \sum_x p^{(n)}(x, y) \leqq K^n ,$$

so $\sum_x \alpha(x) < \infty$ if $\lambda K < 1$. For the proof of necessity, it suffices to observe that

$$p(x, z)\alpha(z) \leqq \sum_y p(x, y)\alpha(y) \leqq M\alpha(x)$$

for $z \in S$ by (3.14).

REMARK. It is interesting to note that (3.17) arose as an assumption in the existence theorem for the exclusion model in [4]. This is consistent with the fact that if $\sum_x \alpha(x) < \infty$, then any configuration of particles for which $\sup_x \eta(x) < \infty$ can be used as an initial configuration.

4. Invariant measures. In this section, we present a proof of Theorem (1.6). In addition to (1.2) and (1.3), it will be assumed throughout that $p(x, y)$ is doubly stochastic and $\sum_x \alpha(x) < \infty$. The following lemma will be needed.

LEMMA (4.1). *For each finite subset $T \subset S$, there exists a function $p_T(x, y)$ on $S \times S$ so that*

$$p_T(x, y) \geqq 0,\ p_T(x, y) = 0 \quad \text{if} \quad x \notin T \quad \text{or} \quad y \notin T ,$$

$\sum_x p_T(x, y) = 1$ *for* $y \in T$, $\sum_y p_T(x, y) = 1$ *for* $x \in T$, *and*

$$|p_T(x, y) - p(x, y)| \leqq \min\{\sum_{z \notin T} p(x, z), \sum_{z \notin T} p(z, y)\} \tag{4.2}$$

for $x, y \in T$. *Furthermore,*

$$p_T(x, y) \to p(x, y) , \tag{4.3}$$

$$\sum_z |p_T(x, z) - p(x, z)| \to 0 \quad \text{and} \quad \sum_z |p_T(z, y) - p(z, y)| \to 0 \tag{4.4}$$

as $T \uparrow S$ *for each* $x, y \in S$.

PROOF. Since $p(x, y)$ is doubly stochastic,

$$Q \equiv \sum_{y \in T} \sum_{x \notin T} p(x, y) = \sum_{x \in T} \sum_{y \notin T} p(x, y) .$$

Define

$$p_T(x, y) = p(x, y) + Q^{-1} [\sum_{z \notin T} p(x, z)][\sum_{z \notin T} p(z, y)]$$

for $x, y \in T$, and $p_T(x, y) = 0$ otherwise. Then $p_T(x, y)$ has the properties required in the first statement of the lemma. But (4.3) follows from (4.2), and (4.4) follows from (4.3) and an application of Scheffé's theorem.

Now, let $(\zeta(x), x \in S)$ be the random variables defined at the end of Section 1. Since $E(\zeta(x)) < \infty$ and is independent of x, $E[\sum_x \zeta(x)\alpha(x)] < \infty$. Therefore $(\zeta(x))$ lines in $\mathscr{X}$ with probability one, so μ can be defined as the induced measure on $\mathscr{X}$. Then $\int ||\eta|| \mu(d\eta) < \infty$.

LEMMA (4.5). *If $f \in G_1$, then $\int |\Omega_1 f(\eta)| \mu(d\eta) < \infty$ and $\int \Omega_1 f(\eta)\mu(d\eta) = 0$.*

PROOF. The integrability of $\Omega_1 f$ is immediate because $\Omega_1 f(\eta)$ is bounded by a constant multiple of $||\eta||$. For finite subsets T of S, let $p_T(x, y)$ be the functions obtained in Lemma (4.1), and put

$$\Omega_T f(\eta) = \sum_x \eta(x)c(\eta(x)) \sum_y p_T(x, y)[f(\eta_{xy}) - f(\eta)] .$$

Then Ω_T is the generator for the zero range interaction process on the finite set T. Spitzer ([7], page 255) has shown that $\int \Omega_T f(\eta)\mu(d\eta) = 0$. To carry out the limiting argument, let $\beta(x) = \sup_\eta |f(\eta_x) - f(\eta)|$, and note that

$$|\Omega_T f(\eta) - \Omega_1 f(\eta)| \leqq \sup_k c(k) \sum_{x,y} \eta(x)[\beta(x) + \beta(y)]\,|p_T(x, y) - p(x, y)| .$$

Since $\int \eta(x)\mu(d\eta) < \infty$ and is independent of x, this implies that $\int |\Omega_T f(\eta) - \Omega_1 f(\eta)|\,\mu(d\eta)$ is bounded by a constant multiple of

$$\sum_{x,y} [\beta(x) + \beta(y)]\,|p_T(x, y) - p(x, y)| .$$

This tends to zero as $T \uparrow S$ by Lemma (4.1) and the fact that $\sum_x \beta(x) < \infty$, which is a consequence of $\beta \in c_0(\alpha)$ and $\sum_x \alpha(x) < \infty$. Therefore, $\int \Omega_1 f(\eta)\mu(d\eta) = 0$.

PROOF OF THEOREM (1.6). By Lemma (4.5),

$$\int \Omega f(\eta)\mu(d\eta) = 0 \tag{4.6}$$

for all $f \in G$. Since $g_n \to g$ in $B(\mathscr{X})$ implies that $\int g_n(\eta)\mu(d\eta) \to \int g(\eta)\mu(d\eta)$, it follows that (4.6) holds for all $f \in \mathscr{D}(\Omega)$. So,

$$\int S(t)f(\eta)\mu(d\eta) = \int f(\eta)\mu(d\eta)$$

for all $f \in B(\mathscr{X})$, and therefore for all $f \in C(\mathscr{X})$. By (3.4), $\int \mu(d\eta)\pi_t(\eta, F) = \mu(F)$ for each finite dimensional set F in $\mathscr{X}$, and hence for each measurable set.

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