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AN INFINITELY DIVISIBLE DISTRIBUTION INVOLVING MODIFIED BESSEL FUNCTIONS

MOURAD E. H. ISMAIL[1] AND KENNETH S. MILLER

ABSTRACT. We prove that the function

$$\left(\frac{b}{a}\right)^{\mu-\nu}\frac{K_\mu(bx^{1/2})K_\nu(ax^{1/2})}{K_\mu(ax^{1/2})K_\nu(bx^{1/2})}$$

is the Laplace transform of an infinitely divisible probability distribution when $\nu > \mu \geq 0$ and $b > a > 0$. This implies the complete monotonic ity of the function. We also establish a representation as a Stieltjes transform, which implies in particular that the function has positive real part when x lies in the right half-plane. We conjecture that

$$\left(\frac{b}{a}\right)^{\mu-\nu}\frac{I_\mu(ax^{1/2})I_\nu(bx^{1/2})}{I_\mu(bx^{1/2})I_\nu(ax^{1/2})}$$

also is the Laplace transform of an infinitely divisible probability distribution. It is known that in the limit as $\nu \to \infty$, the infinite divisibility property holds for both functions.

1. Introduction. A probability distribution on $[0,\infty)$ is said to be infinitely divisible if and only if for every natural number n, the nth root of its Laplace transform is a Laplace transform of a probability distribution. A function f defined on $(0,\infty)$ and of class C^∞ is said to be completely monotonic if $(-1)^n f^{(n)}(x) \geq 0$ on $(0,\infty)$ for all n. The connection between infinitely divisible distributions and completely monotonic functions is expressed in the following theorem (see [**5**, p. 450]):

THEOREM 1. *The function w is the Laplace transform of an infinitely divisible probability distribution on $[0,\infty)$ if and only if $w = e^{-h}$ where $h(0+) = 0$ and h' is completely monotonic.*

In §2 we shall prove the following result:

THEOREM 2. *Let K_λ be the modified Bessel function of the second kind. Then*

$$F(x;\mu,\nu) = \left(\frac{b}{a}\right)^{\mu-\nu}\frac{K_\mu(bx^{1/2})K_\nu(ax^{1/2})}{K_\mu(ax^{1/2})K_\nu(bx^{1/2})} \tag{1.1}$$

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is the Laplace transform of an infinitely divisible probability distribution when $\nu > \mu \geq 0$ *and* $b > a > 0$.

As ν increases without limit,

$$K_\nu(x) \sim (2\nu)^{\nu-1/2}e^{-\nu}x^{-\nu}\pi^{1/2}$$

(see, for example [8]). Hence

$$F(x;\mu,\infty) = \left(\frac{b}{a}\right)^\mu \frac{K_\mu(bx^{1/2})}{K_\mu(ax^{1/2})}.$$

The limiting case ($\nu \to \infty$) of Theorem 2 was proved by Ismail and Kelker [**9**] using complex variable and special functions methods. Kent [**11**] and Wendel [**14**] independently encountered the same distribution ($\nu = \infty$) in their studies of Bessel processes and Brownian motion, and established its infinite divisibility property by probabilistic methods. Ismail and May [**10**] provided an integral representation of the probability density function as the Laplace transform of a transcendental function. Miller [**12**, p. 82] proved that $F(x;\mu,\nu)$ is monotone decreasing, a result needed in certain hypothesis testing problems.

In §3 we shall establish an integral representation of $F(x;\mu,\nu)$ as a Stieltjes transform. This implies, for example, that $F(z;\mu,\nu)$ has positive real part when z lies in the right half-plane. The Stieltjes transform seems to play an important role in infinite divisibility problems, and in the theory of orthogonal polynomials (see [**2**, **3**, **9**, **10**]).

We conjecture that if $\nu > \mu \geq 0$ and $b > a > 0$, then the function

$$G(x;\mu,\nu) = \left(\frac{b}{a}\right)^{\mu-\nu} \frac{I_\mu(ax^{1/2})I_\nu(bx^{1/2})}{I_\mu(bx^{1/2})I_\nu(ax^{1/2})} \tag{1.2}$$

is also the Laplace transform of an infinitely divisible probability distribution; here I_λ is the modified Bessel function of the first kind.

As λ increases without limit

$$I_\lambda(x) \sim (\tfrac{1}{2}x)^\lambda \Gamma^{-1}(\lambda+1)$$

(see, for example [**4**, pp. 22 and 23]). Hence

$$G(x;\mu,\infty) = \left(\frac{b}{a}\right)^\mu \frac{I_\mu(ax^{1/2})}{I_\mu(bx^{1/2})}$$

and the conjecture reduces to a known result, see [**9**, **10**, **14**]. In §4 we shall discuss this conjecture and formulate an equivalent conjecture involving functions that resemble theta functions.

2. Proof of Theorem 2. Our argument will be to show that

$$h(x) = -\ln F(x;\mu,\nu) \tag{2.1}$$

has a completely monotonic derivative and that $h(0+) = 0$. Then we may invoke Theorem 1 to complete the proof.

Clearly h is well defined since $F(x;\mu,\nu)$ is positive on $[0,\infty)$. Now (see [**1**, p. 375]) as $x \to 0+$,

$$K_\nu(x) \sim \tfrac{1}{2}\Gamma(\nu)(\tfrac{1}{2}x)^{-\nu}, \quad \operatorname{Re}\nu > 0,$$
$$K_0(x) \sim -\ln x.$$

This implies that $h(0+) = 0$ since $F(0+;\mu,\nu) = 1$. Using the recursion relation [**13**, p.79]

$$K'_\nu(x) = -K_{\nu-1}(x) - \nu K_\nu(x)/x \tag{2.2}$$

we obtain

$$\begin{aligned} -2x^{1/2}h'(x) = a\frac{K_{\mu-1}(ax^{1/2})}{K_\mu(ax^{1/2})} - b\frac{K_{\mu-1}(bx^{1/2})}{K_\mu(bx^{1/2})} \\ - a\frac{K_{\nu-1}(ax^{1/2})}{K_\nu(ax^{1/2})} + b\frac{K_{\nu-1}(bx^{1/2})}{K_\nu(bx^{1/2})}. \end{aligned} \tag{2.3}$$

The integral representation [**7**]

$$\frac{K_{\nu-1}(x^{1/2})}{K_\nu(x^{1/2})} = \frac{2}{\pi^2}\int_0^\infty \frac{x^{1/2}t^{-1}}{x+t}[J_\nu^2(t^{1/2}) + Y_\nu^2(t^{1/2})]^{-1}\,dt, \quad \nu \geq 0, \tag{2.4}$$

where J_ν and Y_ν are the Bessel functions of the first and second kind, respectively, enables us to express h' in the form

$$h'(x) = \frac{b^2 - a^2}{\pi^2}\int_0^\infty (xa^2 + t)^{-1}(xb^2 + t)^{-1}\phi(t)\,dt \tag{2.5}$$

where

$$\phi(t^2) = [J_\mu^2(t) + Y_\mu^2(t)]^{-1} - [J_\nu^2(t) + Y_\nu^2(t)]^{-1}.$$

Now, Nicholson's formula [**13**, p. 444]

$$J_\lambda^2(x) + Y_\lambda^2(x) = \frac{8}{\pi^2}\int_0^\infty K_0(2x \sinh t)\cosh 2\lambda t\,dt \tag{2.6}$$

shows that $J_\lambda^2(x) + Y_\lambda^2(x)$ is a strictly increasing function of λ for $\lambda \geq 0$ and x a fixed positive number. Therefore $\phi(t)$ is strictly positive for $t > 0$; and the integrand in equation (2.5) is a positive multiple of the completely monotonic function $(t + a^2x)^{-1}(t + b^2x)^{-1}$ (as a function of x). This then implies the complete monotonicity of h'. Thus the function $F(x;\mu,\nu) = e^{-h(x)}$ is also completely monotonic (see Criterion 2 in [**5**, p. 441]). By Bernstein's theorem [**5**, p. 439] we see that $F(x;\mu,\nu)$ must be the Laplace transform of a probability distribution since $F(0;\mu,\nu) = 1$.

Note that in the process of proving Theorem 2 we also have shown that:

COROLLARY. *The function $F(x;\mu,\nu)$, defined in (1.1), is a completely monotonic function of x for $\nu \geq \mu \geq 0$ and $b > a > 0$.*

3. An integral representation. The purpose of this section is to establish an integral representation for the function F of Theorem 2.

THEOREM 3. *If K_λ is the modified Bessel function of the second kind and of order λ, then*

$$\begin{aligned} &\frac{K_\mu(bz^{1/2})K_\nu(az^{1/2})}{K_\mu(az^{1/2})K_\nu(bz^{1/2})} - 1 \\ &= \frac{1}{\pi}\int_0^\infty \frac{M(t,a,b,\mu,\nu)\,dt}{(z+t)[J_\mu^2(at^{1/2}) + Y_\mu^2(at^{1/2})][J_\nu^2(bt^{1/2}) + Y_\nu^2(bt^{1/2})]}, \\ &\qquad |\arg z| < \pi, \end{aligned} \tag{3.1}$$

for $\mu > 0$, $\nu > 0$, $a > 0$, $b > 0$ where

$$\begin{aligned} M(t^2, a, b, \mu, \nu) = {} & [J_\mu(at)J_\nu(at) + Y_\mu(at)Y_\nu(at)] \\ & \times [J_\nu(bt)Y_\mu(bt) - J_\mu(bt)Y_\nu(bt)] \\ & - [J_\mu(bt)J_\nu(bt) + Y_\mu(bt)Y_\nu(bt)] \\ & \times [J_\nu(at)Y_\mu(at) - J_\mu(at)Y_\nu(at)] \end{aligned} \tag{3.2}$$

and J_μ, Y_ν are the Bessel functions of the first and second kind respectively.

The proof utilizes the representation and inversion theorems for the Stieltjes transform, namely,

LEMMA (REPRESENTATION THEOREM). *If*
(i) *$H(z)$ is analytic for $|\arg z| < \pi/\alpha$ for some α, $0 < \alpha < 1$, and*
(ii) *$H(z) = o(1)$ as $|z| \to \infty$ and $H(z) = o(|z|^{-1})$ as $|z| \to 0$ uniformly in every sector $|\arg z| < \pi/\alpha'$ with $\alpha' > \alpha$, then*

$$H(z) = \frac{1}{\pi}\int_0^\infty \frac{dt}{x+t}\frac{1}{2\pi i}\int_C \frac{\varsigma H(te^\varsigma)e^{\varsigma/2}}{\varsigma^2+\pi^2}\,d\varsigma \tag{3.3}$$

where C is a rectifiable closed curve going around $[-i\pi, i\pi]$ in the positive direction and lying in the strip $|\mathrm{Im}\,\varsigma| < \pi/\alpha$.

For a proof of this lemma see [**6**, pp. 210 and 235]. We are now in a position to prove our theorem.

PROOF OF THEOREM 3. Denote the left-hand side of (3.1) by $H(z)$. Then $H(z)$ satisfies condition (i) of the Lemma, for some $\alpha > 2/3$, because $K_\lambda(z)$ has finitely many zeros and none in the half-plane $\mathrm{Re}\, z \geq 0$. For $\lambda \geq 0$ we have $K_\lambda(z) \sim (\pi/2z)^{1/2}e^{-z}$ uniformly as $|z| \to \infty$ in the sector $|\arg z| \leq 3/2 - \delta$ for any $\delta \in (0, 3\pi/2)$, (see [**13**, p. 203]). As $|z| \to 0$ we have $K_\lambda(z) \sim \frac{1}{2}\Gamma(\lambda)(\frac{1}{2}z)^{-\lambda}$ for $\lambda > 0$ and $K_0(z) \sim -\ln z$, (see [**1**, p. 375]). Hence $H(z)$ also satisfies condition (ii) of the Lemma.

The contour integral in (3.3) is now readily seen to be

$$\tfrac{i}{2}[H(te^{i\pi}) - H(te^{-i\pi})]. \tag{3.4}$$

Equation (3.1) now follows, for $z > 0$, from (3.4) and

$$K_\nu(ze^{i\pi/2}) = -\tfrac{1}{2}i\pi e^{-i\pi\nu/2}[J_\nu(z) - iY_\nu(z)]$$

and

$$K_\nu(ze^{-i\pi/2}) = \tfrac{1}{2}i\pi e^{i\pi\nu/2}[J_\nu(z) + iY_\nu(z)],$$

(see, for example, [**4**, pp. 4 and 6]). Finally, (3.1) follows for $|\arg z| < \pi$ from $z > 0$ by analytic continuation.

4. Discussion of conjecture. We shall show that the conjecture made in §1 is equivalent to the statement that the function

$$\Theta_\nu(x; a, b) = \sum_{n=1}^\infty \left(e^{-j^2_{\nu,n}ax} - e^{-j^2_{\nu,n}bx}\right), \quad 0 < a < b, \quad \nu \geq 0, \tag{4.1}$$

is a decreasing function of ν, where $j_{\nu,1} < j_{\nu,2} < \cdots < j_{\nu,n} < \cdots$ are the successive positive zeros of the Bessel function of the first kind and order ν.

From (1.2)

$$-2x^{1/2}\frac{d}{dx}\ln G(x;\mu,\nu) = b\frac{I_{\mu+1}(bx^{1/2})}{I_\mu(bx^{1/2})} - a\frac{I_{\mu+1}(ax^{1/2})}{I_\mu(ax^{1/2})} - b\frac{I_{\nu+1}(bx^{1/2})}{I_\nu(bx^{1/2})} + a\frac{I_{\nu+1}(ax^{1/2})}{I_\nu(ax^{1/2})} \tag{4.2}$$

where we have used [**13**, p. 79]

$$I'_\lambda(x) = I_{\lambda+1}(x) + \lambda I_\lambda(x)/x.$$

An analog of (2.4) is the Mittag-Leffler expansion [**4**, p. 61]

$$\frac{I_{\lambda+1}(z)}{I_\lambda(z)} = 2z\sum_{n=1}^{\infty}(z^2 + j^2_{\lambda,n})^{-1}, \quad \lambda > -1. \tag{4.3}$$

Combining (4.1), (4.2) and (4.3) we get

$$\begin{aligned}-\frac{d}{dx}G(x;\mu,\nu) &= \sum_{n=1}^{\infty}[(x + j^2_{\mu,n}b^{-2})^{-1} - (x + j^2_{\mu,n}a^{-2})^{-1}] \\ &\quad - \sum_{n=1}^{\infty}[(x + j^2_{\nu,n}b^{-2})^{-1} - (x + j^2_{\nu,n}a^{-2})^{-1}] \\ &= \int_0^\infty e^{-xt}[\Theta_\mu(t;b^{-2},a^{-2}) - \Theta_\nu(t;b^{-2},a^{-2})]\,dt\end{aligned}$$

which shows that the conjecture is equivalent to the above-mentioned property of $\Theta_\nu(x;a,b)$.

The sums appearing in (4.1) resemble theta functions, and, in fact, when $\nu = \pm\frac{1}{2}$ the function Θ_ν reduces to the difference of two theta functions,

$$\Theta_{1/2}(x;a,b) = \tfrac{1}{2}[\vartheta_3(0,q) - \vartheta_3(0,q')],$$
$$\Theta_{-1/2}(x;a,b) = \tfrac{1}{2}[\vartheta_2(0,q) - \vartheta_2(0,q')],$$

where $q = e^{-\pi^2 ax}$ and $q' = e^{-\pi^2 bx}$.

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Department of Mathematics, Arizona State University, Tempe, Arizona 85287

Riverside Research Institute, 1701 North Fort Myer Drive, Arlington, Virginia 22209