Advances in Differential Equations Volume 6, Number 12, December 2001, Pages 1463–1492

AN INITIAL–BOUNDARY VALUE PROBLEM FOR THE KORTEWEG–DE VRIES EQUATION POSED ON A FINITE INTERVAL

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Abstract. The Korteweg–de Vries equation occurs as a model for unidirectional propagation of small amplitude long waves in numerous physical systems. The aim of this work is to propose a well-posed mixed initial–boundary value problem when the spacial domain is of finite extent. More precisely, we establish local existence of solutions for arbitrary initial data in the Sobolev space H^1 and global existence for small initial data in this space. In a second step we show global strong regularizing effects.

1. Setting of the problem and main results

In 1895 Korteweg and de Vries [18] introduced the equation that bears their names in the context of unidirectional water waves propagating in an infinite channel. These authors have considered a layer of incompressible fluid over a flat bottom and assumed that the flow was irrotational (see Colin–Dias and Ghidaglia [9] and [10] for rotational flows). Under the influence of gravity, the motion of surface waves leads to a system of p.d.e.'s in the variables $\eta(x,t)$ and $v(x,t)$ where η is the wave height measured from an undisturbed water level and $v(x,t)$ is the spatial derivative of the restriction to the surface of the velocity potential. Considering waves with wavelength l and amplitude a which satisfy $a = O(\varepsilon)$ and $l = O(\varepsilon^{-1/2})$ (where ε denotes a small parameter) and in the hypothesis of unidirectional waves, one gets,

Accepted for publication: December 2000.
AMS Subject Classifications: 35A07, 35Q53, 76B15.

at the leading order in ε and after scaling transforms, the following equation for η :

$$\eta_t + h\eta_x + \frac{\varepsilon h^3}{12}\eta_{xxx} + \frac{3}{2}\eta\eta_x = 0. \tag{1.1}$$

Here h denotes the (finite) depth of the fluid. In the case of an infinite channel (i.e., $x \in \mathbb{R}$) it is possible to perform further scaling and change in the independent variables x and t in order to obtain the classical form the KdV equation, namely,

$$u_t + uu_x + u_{xxx} = 0. \tag{1.2}$$

However, in case of a domain which is not invariant by translation with respect to x the term in η_x cannot be removed and the relevant equation reads now as (still after scaling)

$$u_t + u_x + uu_x + u_{xxx} = 0. \tag{1.3}$$

In [3] and [4], Bona and Winther have considered the initial–boundary value problem for (1.3) in the quarter plane $\{(x,t), x > 0, t > 0\}$. As boundary condition, they have prescribed the value of u at the boundary $x = 0$, namely,

$$u(0,t) = g(t),\ t \geqslant 0. \tag{1.4}$$

As x goes to $+\infty$, Bona and Winther have imposed a kind of decay by looking for solutions which are continuous (with respect to time) functions from $[0,+\infty)$ into $H^m(0,\infty)$, the Sobolev space of order $m \geqslant 0$ constructed on $L^2(0,\infty)$.

In this paper we are interested in the case where (1.3) is posed on the strip $(0,+\infty) \times (0,L)$ where L is a finite positive number. Namely, we would like to model the case where a wave-maker is putting energy in a finite-length channel from the left ($x = 0$) while the right end ($x = L$) of the channel is free.

As it is well-known, (1.3), when it is posed on the whole line, has a Hamiltonian structure associated to the Hamiltonian

$$H(u) = \frac{1}{2}\int_{-\infty}^{+\infty} (u_x^2 - u^2 - u^3/3)\ dx \tag{1.5}$$

(in fact the structure of (1.3) is even much richer, and we refer to Olver [19] for further elements pertaining to the Hamiltonian structure; see also Ablowitz and Segur [1] concerning exact integrability of (1.3)).

On the other hand, $\int_{-\infty}^{+\infty} u^2 dx$ is also conserved and the conservation of $H(u)$ and $\int_{-\infty}^{+\infty} u^2 dx$ are indeed sufficient for the proof of a priori estimates on the H^1 norm of $u(\cdot,t)$ which lead to existence results (see Temam [20] and Bona and Smith [2]) in H^m-spaces for $m \geqslant 1$. The pure initial value problem for (1.3) has been very much investigated and sharp (and quite surprising) results have been obtained; e.g., Kenig, Ponce and Vega [17] have proved local-in-time existence results for initial data in $H^{-3/4+\varepsilon}(\mathbb{R})$, a space of distributions. Bourgain [5] has proven global existence results for small initial data in $L^2(\mathbb{R})$.

All these results, for data with very weak regularity, are based on local regularizing effects of the Airy equation and are possible because the problem is posed on $\mathbb{R}$.

Concerning the problem on a finite interval, we first try to construct solutions by the classical energy method. It is therefore necessary to consider the evolution of the two integrals $\int_0^L u^2 dx$ and $\int_0^L (u_x^2 - u^2 - u^3/3)\, dx$. This leads us to propose two boundary conditions at $x = L$ that will indeed allow us to obtain suitable a priori estimates that can be viewed as perturbations of $\frac{d}{dt}H(u) = 0$ and $\frac{d}{dt}\int_{-\infty}^{+\infty} u^2\, dx = 0$; see (2.10), (2.11), (3.10) and (3.11).

These boundary conditions read as

$$\frac{\partial u}{\partial x}(L,t) = h(t),\ \frac{\partial^2 u}{\partial x^2}(L,t) = k(t),\ t \geqslant 0. \tag{1.6}$$

In a second step we address the question of smoothing effects. These are not possible on a finite interval with periodic boundary condition since the problem is then time-reversible. In our case we do not have time reversibility since changing t into $-t$ amounts to exchanging the left and right boundaries. Actually we are able to show global smoothing effects and therefore construct solutions for L^2 initial data.

The results in this paper concern the following IBVP for the KdV equation. Given g, h, and k from $[0,T)$ into $\mathbb{R}$ and u_0 from $[0,L]$ into $\mathbb{R}$, find u from $[0,L] \times [0,T)$ into $\mathbb{R}$ such that

$$u_t + u_x + uu_x + u_{3x} = 0, \quad \text{for } (x,t) \in (0,L) \times (0,T), \tag{1.7}$$

$$u(0,t) = g(t), \quad \text{for } t \in [0,T), \tag{1.8}$$

$$\frac{\partial u}{\partial x}(L,t) = h(t), \quad \text{for } t \in [0,T), \tag{1.9}$$

$$\frac{\partial^2 u}{\partial x^2}(L,t) = k(t), \quad \text{for } t \in [0,T), \tag{1.10}$$

$$u(x,0) = u_0(x), \quad \text{for } x \in [0,L]. \tag{1.11}$$

Here $L > 0$ and $T \in (0, +\infty]$.

We are going to establish the following results.

Theorem 1.1. (*Local existence*) *Let* $u_0 \in H^1(0,L)$ *and* $g,h,k \in \mathcal{C}^1([0,+\infty))$ *satisfy the compatibility condition* $u_0(0) = g(0)$. *Then there exists* $T > 0$ *and a function* $u \in L^\infty(0,T;H^1(0,L)) \cap \mathcal{C}([0,T],L^2(0,L))$ *with traces at* $x = L$: $u_x(L,\cdot) \in H^{-1}(0,T), u_{xx}(L,\cdot) \in H^{-2}(0,T)$ *which solves* (1.7) *to* (1.11) *in the distribution sense and* (1.11) *for almost every* $x \in [0,L]$.

By imposing conditions on the size of the data u_0, g, h and k it is actually possible to show a global existence result. We refer to Section 3 for the precise meaning of the condition on the data (see (3.12) and (3.15)) and state:

Theorem 1.2. (*Global existence*) *Let* u_0, g, h, *and* k *be as in Theorem* 1 *and satisfy the smallness assumptions* (3.12) *and* (3.15). *Then we can take* $T = +\infty$ *in the statement of Theorem* 1.1.

On the other hand, we show that the Airy equation,

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} = 0, \;\; 0 < x < L, \;\; t \geqslant 0,$$
$$u(0,t) = \frac{\partial u}{\partial x}(L,t) = \frac{\partial^2 u}{\partial x^2}(L,t) = 0, \;\; t \geqslant 0$$
$$u(x,0) = u_0(x), \;\; 0 < x < L,$$

has some parabolic-type smoothing effects:

$$\int_0^L xu^2(x,t)\,dt + 3\int_0^t \int_0^L \left(\frac{\partial u}{\partial x}\right)^2 (x,s)\,dx\;ds = \int_0^L x|u_0(x)|^2\,dx.$$

Using this property, we prove

Theorem 1.3. (*Smoothing effects*) *Let* $u_0 \in L^2(0,L)$, *there exists a unique maximal weak solution* $u \in \mathcal{C}([0,T\max);L^2) \cap L^2_{loc}([0,T\max);H^1)$ *to* (1.7)-(1.11) *with* $g = h = k \equiv 0$.

See Definition 4.3 for the precise meaning of a weak solution to (1.7).

The paper is organized as follows. In the next section we address what we call the homogeneous case, i.e., the case where $h \equiv 0$ and $k \equiv 0$ (but still $g \not\equiv 0$). This allows us to prove Theorems 1.1 and 1.2 in detail without too many technicalities. The method of the proof is based on the approach of Bona and Smith [2], namely, the introduction of a regularized equation, which has its own interest (see (2.41)). Then Section 3 is devoted to the proofs of Theorems 1.1 and 1.2 in full generality. In Section 4 we prove global smoothing effects of the Airy equation and give the proof of Theorem 1.3. Finally, in Section 5 we list open questions and directions of further investigations.

We have announced in [11] and [12] the results of this work. Some technical proofs have been omitted but can be found in [13].

Acknowledgment. We would like to thank Jerry Bona, Jean-Claude Saut and Roger Temam for very stimulating discussions pertaining to this work.

2. The homogeneous case

In this section we consider the case where $k \equiv 0$ and $h \equiv 0$; namely, we address the following IBVP. Find $u = u(x,t),\ x \in [0, L]$ and $t \geqslant 0$ a solution to

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0,\ 0 < x < L,\ t > 0, \tag{2.1}$$

$$u(0,t) = g(t),\ t \geqslant 0, \tag{2.2}$$

$$\frac{\partial u}{\partial x}(L,t) = 0,\ \frac{\partial^2 u}{\partial x^2}(L,t) = 0,\ t \geqslant 0, \tag{2.3}$$

$$u(x,0) = u_0(x),\ 0 < x < L. \tag{2.4}$$

First, as usual, we will establish (in Section 2.1) *a priori* estimates on the solutions to (2.1)–(2.4). Then we will construct suitable approximations u^ε to u for $\varepsilon > 0$, ε small (Section 2.2) via a regularized equation. Finally, Section 2.3 will be devoted to the construction of a solution to (2.1)–(2.4) by letting ε go to zero.

2.1. Two a priori estimates.

In this section, we do not care about regularity and perform formal estimates on the solutions to (2.1)–(2.4). More precisely, we assume that u (and therefore u_0 and g) are so smooth that all the computations made hereafter are justified.

Let v be defined by

$$v(x,t) := u(x,t) - g(t). \tag{2.5}$$

Then (2.1)–(2.4) reads as

$$\frac{\partial v}{\partial t} + (1+g)\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3} = -\frac{dg}{dt}, \tag{2.6}$$

$$v(0,t) = 0,\ t \geqslant 0, \tag{2.7}$$

$$\frac{\partial v}{\partial x}(L,t) = 0,\ \ \frac{\partial^2 v}{\partial x^2}(L,t) = 0,\ \ t \geqslant 0, \tag{2.8}$$

$$v(x,0) = v_0(x) \equiv u_0(x) - g(0),\ 0 < x < L. \tag{2.9}$$

We multiply (2.6) by v and integrate on $[0, L]$ the resulting identity, and, after some integrations by parts, it follows that

$$\frac{d}{dt}\Big(\int_0^L v^2(x,t)\,dx\Big) + (1+g(t))v^2(L,t) + \frac{2v^3(L,t)}{3} + v_x^2(0,t)$$
$$= -2g_t(t)\int_0^L v(x,t)\,dx, \tag{2.10}$$

where subscripts denote partial differentiation. Since the left-hand side of (2.6) can be written as $\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}\left\{(1+g)v + \frac{v^2}{2} + \frac{\partial^2 v}{\partial x^2}\right\}$ we deduce after multiplication by $(1+g)v + v^2/2 + v_{xx}$ that

$$\frac{d}{dt}\Big(\int_0^L (v_x^2 - (1+g)v^2 - \frac{v^3}{3})\,dx\Big) + v_{xx}^2(0,t) = \tag{2.11}$$
$$\Big((1+g)v(L,t) + \frac{v^2}{2}(L,t)\Big)^2 - 2g_t(t)v_x(0,t) + 2g_t(t)(1+g(t))\int_0^L v(x,t)\,dx.$$

2.2. **A local-in-time estimate.** Our first goal is to deduce from (2.10) and (2.11) an estimate on the H^1 norm of v for arbitrarily large data L, g and u_0. To that aim we introduce the time-dependent functions

$$X(t) = \int_0^t |v(\cdot,s)|_\infty^4\,ds, \tag{2.12}$$
$$Y(t) = \int_0^L \{v_x^2(x,t) - (1+g(t))v^2(x,t) - \frac{v^3(x,t)}{3}\}\,dx, \tag{2.13}$$

where $|\ |_\infty$ denotes the L^∞ norm and more generally $|\ |_p$ the L^p norm for $p \in (1,\infty)$.

All the calculations rely on the classical Young inequality and on the interpolation inequality,

$$w^4(x) \leqslant 4\int_0^L w^2(y)\,dy \ \int_0^L w_x^2(y)\ dy,\ \forall x, \tag{2.14}$$

which holds for functions vanishing at $x = 0$.

Time integration of (2.10) leads then to

$$|v|_2^2(t) + \int_0^t v_x^2(0,s)\,ds \leqslant \gamma_1(t)X^{1/2} + \frac{2t^{1/4}}{3}X^{3/4} + 2L\gamma_2(t)X^{1/4} + |v_0|_2^2, \tag{2.15}$$

where the γ_i depend on g :

$$\gamma_1(t) = \Big(\int_0^t \{(1+g)_-(s)\}^2\, ds\Big)^{1/2},\ \gamma_2(t) = \Big(\int_0^t |g_t(s)|^{4/3} ds\Big)^{3/4},$$
$$\gamma_3(t) = \Big(\int_0^t g_t^2(s) ds\Big)^{1/2} \tag{2.16}$$

and $x_- = \text{ Min } (x, 0),\ x \in \mathbb{R}$.

On the other hand, time integration of (2.11) leads in its turn to

$$Y(t) \leqslant \Big(\gamma_4(t) X^{1/4} + \frac{X^{1/2}}{2}\Big)^2 + \int_0^t v_x^2(0,s)\, ds + \gamma_3^2(t) + 2L\gamma_5(t) X^{1/4} + Y(0) \tag{2.17}$$

where

$$\gamma_4(t) = \Big(\int_0^t (1+g(s))^4 ds\Big)^{\frac{1}{4}} \text{ and } \gamma_5(t) = \Big(\int_0^t |g_t(s)|^{\frac{4}{3}} |(1+g(s))|^{\frac{4}{3}} ds\Big)^{\frac{3}{4}}. \tag{2.18}$$

Next we observe that, by definition, $\frac{dX}{dt} = |v|_\infty^4$, and therefore by (2.14)

$$\frac{dX}{dt} \leqslant 4|v|_2^2 |v_x|_2^2. \tag{2.19}$$

Clearly, $|v(t)|_2^2$ is estimated thanks to (2.15). In order to estimate $|v_x(\cdot,t)|_2^2$ we are going to make use of $Y(t)$ as follows now. We write (at time t)

$$|v_x|_2^2 = Y + (1+g)|v|_2^2 + \frac{1}{3}\int_0^L v^3 dx \tag{2.20}$$

and, making use of (2.14),$|\int_0^L v^3 dx| \leqslant 2^{1/2} |v|_2^{5/2} |v_x|_2^{1/2}$. Hence, it follows from (2.14), (2.20) and Young's inequality that

$$|v_x|_2^2 \leqslant \frac{4Y}{3} + \frac{4}{3}(1+g)_+ |v|_2^2 + \big(\frac{2^{1/2}}{3}\big)^{4/3} |v|_2^{10/3}. \tag{2.21}$$

Combining (2.15) and (2.17) we have

$$\begin{aligned} Y &\leqslant Y(0) + |v_0|_2^2 + \gamma_3^2 + 2L(\gamma_2 + \gamma_5) X^{1/4} \\ &\quad + (\gamma_1 + \gamma_4^2) X^{1/2} + \big(\gamma_4 + \frac{2t^{1/4}}{3}\big) X^{3/4} + X/4. \end{aligned} \tag{2.22}$$

Clearly, (2.21) and (2.22) give with (2.15) an explicit estimate of $|v_x|_2^2$ in terms of X^σ, $\sigma > 0$ and of the data L, g and u_0.

Finally, replacing this last estimate on $|v_x|_2^2$ in (2.19), we obtain

$$\frac{1}{4}\frac{dX}{dt} \leqslant \Big\{|v_0|_2^2 + 2L\gamma_2 X^{1/4} + \gamma_1 X^{1/2} + \frac{2t^{1/4}}{3}X^{3/4}\Big\}$$
$$\Big\{\frac{4}{3}\Big(\,(\gamma_1+\gamma_4^2)\,X^{1/2} + \Big(\gamma_4 + \frac{2t}{3}\Big)^{1/4}X^{3/4} + \frac{1}{4}X + \gamma_3^2 + 2L\gamma_5 X^{1/4}$$
$$+\,Y(0) + |v_0|_2^2\Big) + \frac{4}{3}(1+g)_+\Big(|v_0|_2^2 + 2L\gamma_2 X^{1/4} + \gamma_1 X^{1/2} + \frac{2t^{1/4}}{3}X^{3/4}\Big)$$
$$+\,\frac{4^{1/3}}{3^{4/3}}\Big(|v_0|_2^2 + 2L\gamma_2 X^{1/4} + \gamma_1 X^{1/2} + \frac{2t^{1/4}}{3}X^{3/4}\Big)^{5/3}\Big\}. \tag{2.23}$$

By several applications of Young's inequality, it follows from (2.23) that

$$\frac{dX}{dt} \leqslant \alpha(t) + (1+t^{2/3})X^2, \tag{2.24}$$

where α depends only on the data through the γ_i's, L, $|v_0|_2$ and $|v_{0x}|_2$.

Let $T \in (0,\infty)$ be given; we have

$$X(t) \leqslant X^\star(t), \;\; 0 \leqslant t < T^\star, \tag{2.25}$$

where $X^\star$ is the solution to

$$\begin{cases} \frac{dX^\star}{dt} = \alpha_T + (1+T^{2/3})(X^\star)^2, \\ X^\star(0) = 0, \end{cases} \tag{2.26}$$

here $\alpha_T = \sup_{0\leqslant t\leqslant T}\alpha(t)$ and $T^\star$ denotes the life span of the solution to (2.26), $T^\star = \frac{\pi}{2\sqrt{\alpha_T(1+T^{2/3})}}$. Hence, for $t \leqslant T^\star/2 = \frac{\pi}{4\sqrt{\alpha_T(1+T^{2/3})}}$, we deduce a bound on $X(t)$:

$$X(t) \leqslant (\alpha_T/(1+T^{2/3}))^{1/2}, \;\; 0 \leqslant t \leqslant \frac{\pi}{4\sqrt{\alpha_T(1+T^{2/3})}}. \tag{2.27}$$

Then (2.15), (2.17) and (2.21) allow us to bound the H^1 norm of $v : \int_0^L (v^2 + v_x^2)(x,t)\,dx$ for t in the previous range.

2.2.1. **A global-in-time estimate.** Here we want to show that the previous estimates, which are local in time for arbitrarily large data, can be turned into global ones for small data. Our first assumption is that

$$\inf_{t\geqslant 0}(1+g(t)) \equiv \alpha_0 > 0. \tag{2.28}$$

As will be seen in the sequel, the following constants appear naturally in the estimates :

$$\alpha_1 = \int_0^\infty |g_t|(t)\,dt, \;\; \alpha_2 = \sup_{t\geqslant 0}(1+g(t)), \;\; \alpha_3 = \int_0^\infty |g_t|^2(t)\,dt. \tag{2.29}$$

Let us then denote by M_0 and M_1 the function of the data:

$$M_0 = |v_0|_2^2 + 2\alpha_1\sqrt{L}(\sqrt{L}\alpha_1 + |v_0|_2), \tag{2.30}$$

$$M_1 = |v_{0x}|_2^2 + \alpha_3 + 2\alpha_1\alpha_2\sqrt{L}(\sqrt{L}\alpha_1 + |v_0|_2) + M_0(1 + \frac{17}{32}\alpha_0 + \frac{7}{4}\alpha_2 + \frac{2\alpha_2^2}{\alpha_0}). \tag{2.31}$$

We observe that $|v_0|_2^2 \leqslant M_0$ and $|v_{0x}|_2^2 \leqslant M_1$ so that if we impose on the initial data

$$M_0 M_1 \leqslant 2^{-6}\alpha_0^4, \tag{2.32}$$

then by (2.14) we have

$$|v_0|_\infty^4 \leqslant 4M_0M_1 \leqslant 2^{-4}\alpha_0^4. \tag{2.33}$$

It follows by continuation that

$$|v(\cdot,t)|_\infty \leqslant 3\alpha_0/4 \text{ for } 0 \leqslant t \leqslant T, \tag{2.34}$$

and we cannot take $T = +\infty$ only if $|v(\cdot,t)|_\infty$ indeed reaches the value $\frac{3\alpha_0}{4}$. Actually, we are going to show that, thanks to (2.34), we have

$$|v(\cdot,t)|_2^2 \leqslant M_0 \text{ and } |v_x(\cdot,t)|_2^2 \leqslant M_1,\ \ 0 \leqslant t \leqslant T. \tag{2.35}$$

Hence, by (2.14), $|v(\cdot,t)|_\infty \leqslant \alpha_0/2$, which shows that $T = +\infty$, and therefore

$$|v(\cdot,t)|_\infty \leqslant \alpha_0/2,\ \ |v(\cdot,t)|_2^2 \leqslant M_0,\ |v_x(\cdot,t)|_2^2 \leqslant M_1,\ \forall t \geqslant 0. \tag{2.36}$$

It remains to show that as long as (2.34) holds true, the estimates (2.35) are satisfied. The first step relies on the observation that (2.34) implies that $(1+g)v^2(L,t) + \frac{2v^3(L,t)}{3} \geqslant \alpha_0 v^2(L,t)/2$ so that the following inequality transpires from (2.10):

$$\frac{d}{dt}|v|_2^2 + \frac{\alpha_0}{2}v^2(L,t) + v_x^2(0,t) \leqslant 2\sqrt{L}|v|_2|g_t|. \tag{2.37}$$

Then we deduce readily that

$$|v(\cdot,t)|_2 \leqslant |v_0|_2 + \sqrt{L}\alpha_1 \tag{2.38}$$

and

$$|v(\cdot,t)|_2^2 + \frac{\alpha_0}{2}\int_0^t v^2(L,s)\,ds + \int_0^t v_x^2(0,s)\,ds \leqslant M_0. \tag{2.39}$$

Hence, the first part of (2.35) is established. The second step makes use of (2.11). Let us first observe that

$$(1+g)v_0^2(x,t) - v_0^3(x,t)/3 \geqslant (\alpha_0 - \frac{3\alpha_0}{4}\cdot\frac{1}{3})v_0^2(x,t) \geqslant 0.$$

Integrating (2.11) with respect to time leads then to (after utilization of (2.34) and (2.38))

$$\int_0^L (v_x^2 - (1+g)v^2 - \frac{v^3}{3})(x,t)\,dx \leqslant |v_{0x}|_2^2 + \alpha_2^2 \int_0^t v^2(L,s)\,ds \tag{2.40}$$
$$+ \alpha_2 \frac{3\alpha_0}{4} \int_0^t v^2(L,s)\,ds + \left(\frac{3}{4}\right)^2 (\alpha_0^2/4) \int_0^t v^2(L,s)\,ds$$
$$+ \int_0^t |g_t(s)|^2\,ds + \int_0^t v_x^2(0,s)\,ds + 2\int_0^t |g_t(1+g)|(s)\,ds\sqrt{L}\big(|v_0|_2 + 2\sqrt{L}\alpha_1\big).$$

Now, since

$$\int_0^L ((1+g)v^2 + \frac{v^3}{3})\,dx \leqslant |v|_2^2\big(\alpha_2 + \frac{1}{3}\frac{3\alpha_0}{4}\big),$$

we deduce from (2.40) that $|v_x(\cdot,t)|_2^2 \leqslant M_1$, and (2.35) is proved. Once (2.28) is satisfied, (2.32) is indeed achieved by making the data v_0 and g small.

2.3. **A regularized problem.** Let $\varepsilon > 0$ be given. Our aim in this section is to construct a solution to the following problem. Find $u^\varepsilon = u^\varepsilon(x,t),\ x \in [0,L]$ and $t \geqslant 0$ a solution to

$$\frac{\partial u^\varepsilon}{\partial t} + \frac{\partial u^\varepsilon}{\partial x} + u^\varepsilon \frac{\partial u^\varepsilon}{\partial x} + \frac{\partial^3 u^\varepsilon}{\partial x^3} - \varepsilon \frac{\partial^3 u^\varepsilon}{\partial x^2 \partial t} = 0,\ \ 0 < x < L,\ \ t > 0 \tag{2.41}$$

together with the boundary and initial conditions (2.2), (2.3) and (2.4).

Let us make the following change of independent variables by setting (the dependence of w below with respect to ε is dropped for simplification of notation):

$$w(x,t) = \varepsilon u^\varepsilon(\varepsilon^{1/2}(x-t),\, \varepsilon^{3/2}t) \tag{2.42}$$

so that (2.41), (2.2), (2.3), (2.4) is transformed into

$$\frac{\partial w}{\partial t} + (1+\varepsilon)\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial x} - \frac{\partial^3 w}{\partial x^2 \partial t} = 0,\ t < x < t + \varepsilon^{-1/2}L,\ t > 0, \tag{2.43}$$

$$w(t,t) = \varepsilon g(\varepsilon^{3/2}t), t \geqslant 0 \tag{2.44}$$

$$\frac{\partial w}{\partial x}(t + \varepsilon^{-1/2}L, t) = 0,\ \frac{\partial^2 w}{\partial x^2}(L + \varepsilon^{-1/2}L, t) = 0,\ t \geqslant 0 \tag{2.45}$$

$$w(x,0) = w_0(x) \equiv \varepsilon u_0(\varepsilon^{1/2}x),\ 0 < x < \varepsilon^{-1/2}L. \tag{2.46}$$

In order to solve (2.43)–(2.46), which is posed on a tilted domain, we notice that (2.43) can also be written as

$$\big(1-\frac{\partial^2}{\partial x^2}\big)\frac{\partial w}{\partial t}=-\frac{\partial}{\partial x}\big\{(1+\varepsilon)w+\frac{w^2}{2}\big\}. \tag{2.47}$$

Inverting formally the operator $1-\frac{\partial^2}{\partial x^2}$, this equation can be seen as an o.d.e. since $\big(1-\frac{\partial^2}{\partial x^2}\big)^{-1}\frac{\partial}{\partial x}$ is a bounded operator. This o.d.e. is then solved via a fixed-point procedure. This is the subject of the next section.

2.3.1. **An integral formulation of the regularized problem.** Since L and ε are fixed, we change a little bit the notations and study

$$w_t+cw_x+ww_x-w_{xxt}=0,\ t<x<t+\lambda,\, t>0, \tag{2.48}$$

$$w(t,t)=\gamma(t),\ t\geqslant 0, \tag{2.49}$$

$$w_x(t+\lambda,t)=0,\ w_{xx}(t+\lambda,t)=0,\ t\geqslant 0, \tag{2.50}$$

$$w(x,0)=\varphi(x),\ 0<x<\lambda. \tag{2.51}$$

Next we set $v(x,t)=w(x,t)-\gamma(t)$ and obtain

$$v_t-v_{xxt}=-\gamma_t-V_x,\ t<x<t+\lambda,\ t>0, \tag{2.52}$$

$$v(t,t)=0,\ t\geqslant 0, \tag{2.53}$$

$$v_x(t+\lambda,t)=0,\ v_{xx}(t+\lambda,t)=0,\ t\geqslant 0, \tag{2.54}$$

$$v(x,0)=\varphi(x)-\gamma(0),\ 0<x<L \tag{2.55}$$

where

$$V(x,t)\equiv (C+\gamma(t))\,v(x,t)+\frac{v^2(x,t)}{2}. \tag{2.56}$$

As previously said, one of the key points in the resolution of (2.52)–(2.55) is the study of the inverse of the operator $1-\frac{\partial^2}{\partial x^2}$.

Of course we can only retain two boundary conditions on $(t,t+\lambda)$, and we will take the homogeneous Dirichlet boundary condition at $x=t$ and the homogeneous Neumann condition at $x=t+\lambda$. It is then straightforward to check the following proposition.

Proposition 2.1. *Let $t\geqslant 0$ and $\lambda>0$ be given. For every $h\in L^2(t,t+\lambda)$, there exists a unique solution $\psi\in H^2(t,t+\lambda)$ to the following boundary* *value problem*:

$$\big(1 - \frac{\partial^2}{\partial x^2}\big)\psi(x) = h(x),\ t < x < t+\lambda, \tag{2.57}$$

$$\psi(t) = 0,\ \frac{\partial \psi}{\partial x}(t+\lambda) = 0. \tag{2.58}$$

Moreover, this function is given by the formula

$$\psi(x) = \int_t^x h(z)\sinh(z-x)\,dz + \frac{\sinh(x-t)}{\cosh\lambda}\int_t^{t+\lambda} h(z)\cosh(t+\lambda-z)\,dz. \tag{2.59}$$

Let us now transform (2.52)–(2.55) into an integral equation thanks to this proposition. First we integrate (2.52) with respect to time. Two cases are to be considered: (i) $x \leqslant \lambda$ and (ii) $x \geqslant \lambda$.

Case (i), $x \leqslant \lambda$. We obtain, integrating between 0 and t,

$$\big(1 - \frac{\partial^2}{\partial x^2}\big)v(x,t) = \big(1 - \frac{\partial^2}{\partial x^2}\big)(\varphi(x) - \gamma(0)) - \int_0^t [\gamma_t(s) + V_x(x,s)]\,ds.$$

Case (ii), $x \geqslant \lambda$. We obtain, integrating between $x-\lambda$ and t,

$$\begin{aligned}\big(1 - \frac{\partial^2}{\partial x^2}\big)v(x,t) &= \big(1 - \frac{\partial^2}{\partial x^2}\big)v(x,x-\lambda) - \int_{x-\lambda}^t [\gamma_t(s) + V_x(x,s)]\,ds \\ &= v(x,x-\lambda) - \int_{x-\lambda}^t [\gamma_t(s) + V_x(x,s)]\,ds\end{aligned}$$

by using the second condition in (2.50).

It follows that if we set

$$W(x,t) = \begin{cases} \big(1 - \frac{\partial^2}{\partial x^2}\big)\varphi(x) - \gamma(t) - \int_0^t V_x(x,s)\,ds,\ x < \lambda, \\ v(x,x-\lambda) - \gamma(t) + \gamma(x-\lambda) - \int_{x-\lambda}^t V_x(x,s)\,ds,\ x \geqslant \lambda, \end{cases} \tag{2.60}$$

we have obtained that

$$\big(1 - \frac{\partial^2}{\partial x^2}\big)v(x,t) = W(x,t),\ t < x < t+\lambda. \tag{2.61}$$

The next step is to use Proposition 2.1 in order to obtain v explicitly as a function of W. This is done by setting (for a fixed t) $h(x) = W(x,t)$ in the formula (2.59). After a straightforward but long and technical computation (see the appendix of [13]) one finds

$$v(x,t) = (\mathcal{T}v)(x,t),\ t < x < t+\lambda,\ 0 \leqslant t \leqslant \lambda, \tag{2.62}$$

where $\mathcal{T}$ is an integral operator defined hereinafter. Let us denote by S_λ^∞ the tilted strip $S_\lambda^\infty = \{(x,t) \in \mathbb{R}^2 : t \geqslant 0 \text{ and } t \leqslant x \leqslant t+\lambda\}$ and also for $T > 0$

$$S_\lambda^T = \left\{(x,t) \in \mathbb{R}^2 : 0 \leqslant t \leqslant T \text{ and } t \leqslant x \leqslant t+\lambda\right\}.$$

Given a function $v : S_\lambda^T$ into $\mathbb{R}$, we first construct $V : S_\lambda^T$ into $\mathbb{R}$ by setting

$$V(x,t) = (c+\gamma(t))v(x,t) + \frac{1}{2}v^2(x,t). \tag{2.63}$$

Then we set

$$\begin{aligned}
&(\mathcal{T}v)(x,t) = \frac{\sinh(x-t)}{\cosh\lambda}\Big[\varphi(t)\sinh\lambda - \varphi(\lambda)\sinh t - \gamma(t)\sinh\lambda \\
&+ \int_\lambda^{t+\lambda} (v(z,z-\lambda)+\gamma(z-\lambda))\cosh(t+\lambda-z)\,dz \\
&- \int_0^t V(s+\lambda,s)\cosh(t-s)\,ds - \int_0^t\int_t^{s+\lambda} V(z,s)\sinh(t+\lambda-z)\,dz\,ds\Big] \\
&- \varphi(t)\cosh(t-x) + \gamma(t)\left[\cosh(t-x)-1\right] \\
&+ \chi_\lambda(x)\left(\varphi(x) + \int_t^x\int_0^t V(z,s)\cosh(z-x)\,ds\,dz\right) \\
&+ (1-\chi_\lambda(x))\Big(\varphi(\lambda)\cosh(\lambda-x) + \int_\lambda^x\int_{\lambda z-\lambda}^{+} V(z,s)\cosh(z-x)\,ds\,dz \\
&+ \int_t^\lambda\int_0^t V(z,s)\cosh(z-x)\,ds\,dz - \int_0^{x-\lambda} V(s+\lambda,s)\sinh(s+\lambda-x)\,ds \\
&+ \int_\lambda^x \{V(z,z-\lambda)+\gamma(z-\lambda)\}\sinh(z-x)\,dz\Big),
\end{aligned} \tag{2.64}$$

where χ_λ is the characteristic function of the set $\{x \in \mathbb{R} : x \leqslant \lambda\} : \chi_\lambda(x) = 1$ for $x \leqslant \lambda$ and $\chi_\lambda(x) = 0$ for $x > \lambda$.

Remark 2.2. The discontinuity of χ_λ at $x = \lambda$ does not introduce a discontinuity of $\mathcal{T}$ at this point since the coefficients of χ_λ and $1-\chi_\lambda$ are equal at $x = \lambda$.

As a result, we have *formally* transformed the original p.d.e. problem (2.48)–(2.51) into a fixed-point problem (2.62). Our strategy now is very classical: (i) to prove that this fixed-point problem has indeed a solution on a suitable space of functions; (ii) to establish to what extent a solution to (2.62) solves (2.48)–(2.51).

2.3.2. **Existence of solutions to the integral equation.** Let us assume that $\gamma \in \mathcal{C}^0(\mathbb{R}_+)$, the space of continuous function on $\mathbb{R}_+$, and that $\varphi \in H^1(0,\lambda)$. Since $H^1(0,\lambda) \subset \mathcal{C}^0([0,\lambda]), \varphi(0)$ makes sense, the only compatibility condition which is needed in order to solve (2.62) will be

$$\gamma(0) = \varphi(0). \tag{2.65}$$

Let us then denote by $\mathcal{E}_T$ the Banach space of those (classes of) functions which are defined on S_T and such that for almost every $t \in (0,T), w(\cdot,t)$ belongs to $H^1(t;t+\lambda)$ and satisfies

$$||w||_{\mathcal{E}_t} = \operatorname*{ess\,sup}_{0<t<T} ||w(\cdot,t)||_{H^1(t;t+\lambda)} < \infty. \tag{2.66}$$

It is easy to check that for every $0 < T < \infty$, $\mathcal{T}$ maps $\mathcal{E}_T$ into itself. This follows mainly from the fact that the mapping $v \mapsto V$ (see (2.63)) is locally Lipschitzian on H^1. Hence, the existence of a solution to the fixed-point problem

$$w = \mathcal{T}w, \ w \in \mathcal{E}_T \tag{2.67}$$

will follow if we can exhibit R such that

$$||w||_{\mathcal{E}_T} \leqslant R \Longrightarrow ||\mathcal{T}w||_{\mathcal{E}_T} \leqslant R, \tag{2.68}$$

$$||\mathcal{T}w_1 - \mathcal{T}w_2||_{\mathcal{E}_T} \leqslant ||w_1 - w_2||_{\mathcal{E}_T}/2, \ \forall w_i \quad \text{such that } ||w_i||_{\mathcal{E}_T} \leqslant R. \tag{2.69}$$

We are now in a position to state and prove the following result.

Proposition 2.3. *Let $\gamma \in \mathcal{C}^0(\mathbb{R}_+)$ and $\varphi \in H^1(0,\lambda)$ be given. There exists $T > 0$ such that* (2.67) *has a unique solution $w \in \mathcal{E}_T$. Moreover, the function w is continuous on S_λ^T and satisfies* (*provided the compatibility condition $\gamma(0) = \varphi(0)$ holds true*)

$$w(x,0) = \varphi(x) - \gamma(0), \ w(t,t) = 0. \tag{2.70}$$

Proof. Since it is only a local-in-time result (see however below), it is sufficient to assume a priori that $T \leqslant \lambda$. We write

$$\mathcal{T}w = \mathcal{T}0 + \mathcal{T}w - \mathcal{T}0$$

and observe that there exists $R_0 > 0$ independent of T such that $||\mathcal{T}0||_{\mathcal{E}_T} \leqslant R_0$. Moreover, since H^1 is an algebra we have $||\mathcal{T}w - \mathcal{T}_0||_{\mathcal{E}_T} \leqslant T^\alpha C_1(C_2 + ||w||_{\varepsilon_T})^2$ for some $\alpha > 0$, where C_1 and C_2 are constants which are independent of T, $0 < T \leqslant \lambda$. Hence

$$||\mathcal{T}w||_{\mathcal{E}_T} \leqslant R_0 + T^\alpha C_1(C_2 + ||w||_{\mathcal{E}_T})^2,$$

and choosing $R = 2R_0$, we know that (2.68) will hold true for sufficiently small T. Then in order to obtain (2.69) we observe that for $||w||_{\mathcal{E}_T} \leqslant R =$

$2R_0$, there exist $\beta > 0$ and C_3, independent of t, such that (C_3 depends on R)

$$||\mathcal{T}w_1 - \mathcal{T}w_2||_{\mathcal{E}_T} \leqslant C_3 T^\beta ||w_1 - w_2||_{\mathcal{E}_T},$$

and by making T even smaller, we can ensure $C_3 T^\beta \leqslant 1/2$, and (2.69) is proved.

Now by the contraction principle we have shown that for T sufficiently small, (2.67) has a unique solution $w \in \mathcal{E}_T$. By Remark 2.1, $\mathcal{T}w$ is a continuous function of $(x,t) \in S_\lambda^T$, and therefore $w = \mathcal{T}w$ is continuous on S_λ^T. Then $w(t,t) = (\mathcal{T}w)(t,t) =$ (by (2.64))$= 0$, and $w(x,0) = (\mathcal{T}w)(x,0) =$ (by (2.64)) $= \varphi(x) - \varphi(0) + \frac{\cosh(x-\lambda)}{\cosh\lambda}(\gamma(0) - \varphi(0))$, which proves (2.70).

Remark 2.4. Classically, (2.67) satisfies a continuation principle; however, the situation is slightly different from the usual Cauchy–Lipschitz setting for o.d.e's for two reasons. The first one is that the problem is not posed on a cylindrical domain but on a tilted strip, but this does not introduce major changes. The second reason is related to the time delay λ. This also does not introduce a difficulty. By the continuation principle, we have the following alternative. Either $T = \lambda$ or $T < \lambda$, and in this last case, either the solution can be continued on $[T, T+\tau]$, $\tau > 0$, $T + \tau \leqslant \lambda$ or $\lim_{t\to T} ||w||_{\mathcal{E}_T} = +\infty$. This allows us to construct a maximal solution on $[0, T_{\max}[$ with $T_{\max} \leqslant \lambda$. At this point we see that the restriction $T \leqslant \lambda$ which appears in the course of the proof of Proposition 2.3 can be removed because if $T_{\max} = \lambda$, we can start again at time λ and Proposition 2.2 allows us to continue the solution beyond this point provided we ensure the ad hoc compatibility condition at the point $x = t = \lambda$. It is indeed the case since γ and the solution $w(\cdot,\cdot)$ are continuous.

Proposition 2.3 and the previous remark allow us to state the following result.

Theorem 2.5. *Let $\gamma \in \mathcal{C}^0(\mathbb{R}_+)$ and $\varphi \in H^1(0,\lambda)$ be such that $\gamma(0) = \varphi(0)$. There exists $T \in (0,+\infty]$ such that the integral equation* (2.67) *has a unique solution $w \in \mathcal{E}_T$ and either $T = +\infty$ or $0 < T < \infty$, and then $\lim\limits_{t\to T} ||w||_{\mathcal{E}_t} = +\infty$.*

2.4. **About smoothness of the solution to the integral equation.** Since we want to satisfy (2.54), it is clear that we need more smoothness on the solution to (2.67). It is straightforward to show the following result.

Theorem 2.6. *Let $\gamma \in \mathcal{C}^0(\mathbb{R}_+)$ and $\varphi \in H^1(0,\lambda)$ be such that $\gamma(0) = \varphi(0)$, and let w denote the maximal solution $w \in \varepsilon_T$ obtained in Theorem* 2.1.

(i) *If* $\varphi \in H^2(0,\lambda)$ *satisfies* $\varphi_x(\lambda) = 0$, *then*

$$\sup_{0\leqslant t\leqslant\tau} ||w(\cdot,t)||_{H^2(t,t+\lambda)} < \infty, \ \forall\tau < T. \tag{2.71}$$

Moreover, $w_x \in \mathcal{C}^0(S_\lambda^T)$ *and*

$$w_x(t+\lambda,t) = 0. \tag{2.72}$$

(ii) *If* $\varphi \in H^3(0,\lambda)$ *satisfies* $\varphi_x(\lambda) = 0$ *and* $\varphi_{xx}(\lambda) = 0$, *then*

$$\sup_{0\leqslant t\leqslant\tau} ||w(\cdot,t)||_{H^3(t,t+\lambda)} < \infty, \ \ \forall\tau < T. \tag{2.73}$$

Moreover, $w_{xx} \in \mathcal{C}^0(S_\lambda^T)$ *and*

$$w_{xx}(t+\lambda,t) = 0. \tag{2.74}$$

(iii) *If* $\varphi \in H^3(0,\lambda)$ *satisfies* $\varphi_x(\lambda) = 0$, $\varphi_{xx}(\lambda) = 0$ *and* $\gamma \in \mathcal{C}^1(\mathbb{R}_+)$, *then*

$$\sup_{0\leqslant t\leqslant\tau} ||\frac{\partial w}{\partial t}||_{H^4(t,t+\lambda)} < \infty. \tag{2.75}$$

The points (i) and (ii) are obvious consequences of the fact that $w = \mathcal{T}w$, while (iii) follows from the expression of $\frac{\partial}{\partial t}(\mathcal{T}w)$.

Corollary 2.7. *Under the hypotheses of Theorem* 2.6 *point* (iii), w *solves* (2.52)–(2.55) *in the classical sense.*

2.4.1. **Estimates on the solutions to the regularized problem.** From Theorem 2.6 (Corollary 2.7) we have constructed solutions to (2.52)–(2.55) which satisfy (2.72) and (2.74) on a time interval $0 < t < T_\varepsilon$ with a continuation principle in $\mathcal{E}_t$ (Theorem 2.5). Performing the inverse change of variables in (2.42), i.e., setting

$$u^\varepsilon(x,t) = \varepsilon^{-1}w(\varepsilon^{-1/2}x + \varepsilon^{-3/2}t, \varepsilon^{-3/2}t), \tag{2.76}$$

we have shown the existence of classical solutions to (2.41) satisfying the boundary and initial conditions (2.2), (2.3) and (2.4). At this point, we use (2.5) and then obtain for v^ε

$$v^\varepsilon(x,t) \equiv u^\varepsilon(x,t) - g(t), \tag{2.77}$$

and for $0 \leqslant t < T_\varepsilon$

$$\frac{\partial v^\varepsilon}{\partial t} + (1+g)\frac{\partial v^\varepsilon}{\partial x} + v^\varepsilon \frac{\partial v^\varepsilon}{\partial x} + \frac{\partial^3 v^\varepsilon}{\partial x^3} - \varepsilon \frac{\partial^3 v^\varepsilon}{\partial x^2 \partial t} = -\frac{dg}{dt}, \tag{2.78}$$

$$v^\varepsilon(0,t) = 0, \tag{2.79}$$

$$\frac{\partial v^\varepsilon}{\partial x}(L,t) = 0, \ \frac{\partial^2 v^\varepsilon}{\partial x^2}(L,t) = 0. \tag{2.80}$$

Proposition 2.8. *Let* $v^\varepsilon \in \mathcal{C}([0,T_\varepsilon); H^3(0,L))$ *satisfy*

$$\frac{\partial v^\varepsilon}{\partial t} \in \mathcal{C}([0,T_\varepsilon); H^4(0,L))$$

and (2.78)–(2.80); *then one has the two following "energy" equations*:

$$\frac{d}{dt}\int_0^L ((v^\varepsilon)^2 + \varepsilon(v_x^\varepsilon)^2)\,dx + (1+g)(v^\varepsilon)^2(L) + \frac{2(v^\varepsilon)^3(L)}{3} + (v_x^\varepsilon)^2(0)$$
$$= -2g_t \int_0^L v^\varepsilon ds; \tag{2.81}$$

$$\frac{d}{dt}\int_0^L \left((v_x^\varepsilon)^2 - ((1+g)v^\varepsilon)^2 - \frac{(v^\varepsilon)^3}{3}\right) dx + \varepsilon (v_t^\varepsilon)^2(L) \tag{2.82}$$
$$+ (v_{xx}^\varepsilon - \varepsilon v_{xt})^2(0) = g_t \int_0^L (v^\varepsilon)^2 dx + \left((1+g)v^\varepsilon(L) + \frac{(v^\varepsilon)^2}{2}(L)\right)^2$$
$$+ 2g_t(1+g)\int_0^L v^\varepsilon dx - 2g_t v(0) - 2\varepsilon v_t^\varepsilon(L) g_t.$$

Proof. Due to the smoothness properties of v^ε, it is allowed to take the scalar product in $L^2(0,L)$ of (2.78) with v^ε and with $(1+g)v^\varepsilon + \frac{(v^\varepsilon)^2}{2} + \frac{\partial v^\varepsilon}{\partial x^2} - \varepsilon \frac{\partial v^\varepsilon}{\partial x \partial t}$. Then (2.81) and (2.82) follow after several integrations by parts and use of (2.79) and (2.80). □

We observe now that we can mimic what we did in Section 2.11; i.e., we denote by $X(\cdot)$ and $Y(\cdot)$ the functions defined in (2.12)–(2.13) with this time v^ε instead of v and observe that (2.15) and (2.22) are again satisfied. Hence, combining these estimates with Proposition 2.8, we obtain the following result.

Proposition 2.9. *Let* $u_0 \in H^3([0,L])$ *and* $g \in \mathcal{C}^1(\mathbb{R}^+)$ *be such that* $u_0(0) = g(0)$, $\frac{\partial}{\partial x}u_0(L) = \frac{\partial^2}{\partial x^2}u_0(L) = 0$. *Then there exists* $T(u_0, g)$ *depending only on* $|u_0|_{H^1}$ *and* $|g|_{\mathcal{C}^1}$ *such that there exists a unique solution* u^ε *to* (2.41) *satisfying* $u^\varepsilon(x,0) = u_0(x)$ *and the boundary conditions* (2.2)–(2.3). *Moreover,* $u^\varepsilon \in$

$\mathcal{C}([0,T(u_0,g)];H^3)\cap\mathcal{C}^1([0,T(u_0,g)];H^4)$ *and* $|u^\varepsilon|_{L^\infty(0,T(u_0,g);H^1)}$ *is bounded independently of* ε, *and this bound depends only on* $|u_0|_{H^1([0,L])}$ *and* $|g|_{\mathcal{C}^1(\mathbb{R}^+)}$.

Proposition 2.10. *If moreover* $g\in L^\infty(\mathbb{R}^+)$, *suppose that* $\inf_{t\geqslant 0}(1+g(t))>0$. *There exists* $\delta>0,\delta$ *depending only on* $|g|_{L^\infty(\mathbb{R}^+)}$, *such that if* $||u_0(\cdot)-g(0)||_{H^1([0;L])}\leqslant\delta$ $|g_t|_{L^1(\mathbb{R}^+)}\leqslant\delta$ *and* $|g_t|_{L^2(\mathbb{R}^+)}\leqslant\delta$, *then one can take* $T(u_0,g)=+\infty$ *in Proposition* 2.9.

2.5. **Proof of Theorems 1.1 and 1.2 in the homogeneous case.** Let $u_0\in H^1([0,L])$ and $g\in\mathcal{C}^1(\mathbb{R}^+)$ be such that $u_0(0)=g(0)$. Take a sequence $u_0^\varepsilon\in H^3([0,L])$ such that $u_0^\varepsilon(0)=g(0)$, $\frac{\partial}{\partial x}u_0^\varepsilon(L)=\frac{\partial^2}{\partial x^2}u_0^\varepsilon(L)=0$ and $u_0^\varepsilon\to u_0$ in $H^1([0,L])$. We still call u^ε the solution to (2.41) given by Proposition 2.9. Since $u_0^\varepsilon\to u_0$ in $H^1([0,L])$ and since $T(u_0^\varepsilon)$ depends only on $|u_0^\varepsilon|_{H^1}$, there exists $T_0>0$ such that $T(u_0^\varepsilon)>T_0$ for ε sufficiently small.

Moreover, the sequence $u^\varepsilon(x,t)$ is bounded in $L^\infty(0,T_0;H^1([0,L]))$. Now, since u^ε satisfies (2.41), $\partial_t u^\varepsilon$ is bounded in $L^\infty(0,T_0;H^{-2}([0,L])$, and therefore up to a subsequence, $u^\varepsilon\to u$ in $\mathcal{C}([0,T_0];L^2([0,L]))$ strongly and $u^\varepsilon\rightharpoonup u$ in $L^\infty(0,T_0;H^1([0,L]))$ weakly. It is then easy to see that u satisfies (2.1), (2.2) and (2.4). Moreover, if (2.28) is satisfied and if $g\in L^\infty(\mathbb{R}^+),\partial_t g\in L^1\cap L^2(\mathbb{R}^+)$ and if $v_0(x)\equiv u_0(x)-g(0)$ and the constants $\alpha_1,\alpha_2, and \alpha_3$ in (2.29) are small enough (in the sense of (2.32), then $T_\varepsilon=+\infty$ $\forall\varepsilon>0$ and $u\in\mathcal{C}(\mathbb{R}^+;L^2([0,L]))\cap L^\infty(\mathbb{R}^+;H^1([0,L]))$, and u satisfies (2.1), (2.2) and (2.4).

We still have to show that (2.3) is satisfied in a suitable sense. To this aim, we prove the following result.

Proposition 2.11. *Let* $T>0$ *and* A *be the following space*:

$$A=\Big\{(v,f)\in\mathcal{C}([0,T];H^3)\times\mathcal{C}([0,T];H^1)\quad\textit{such that}\quad v_t+v_x+v_{xxx}=f_x,\,0<x<L,\,0<t<T\Big\};$$

also, let $\mathcal{T}_1,\mathcal{T}_2$ *be the following linear maps defined on* A *by*

$$\mathcal{T}_1(v,f)=v_x(L,t),\quad \mathcal{T}_2(v,f)=v_{xx}(L,t).$$

One has

$$|\mathcal{T}_1(v,f)|_{H^{-1}([0,T])}\leqslant C\left(|v|_{L^\infty(0,T;L^2)}+|f|_{L^\infty(0,T;L^2)}\right)\tag{2.83}$$

and

$$|\mathcal{T}_2(v,f)|_{H^{-1}([0,T])}\leqslant C\Big(|v|^{\frac12}_{L^\infty(0,T,L^2)}|v|^{\frac12}_{L^\infty(0,T;H^1)}+|f|^{\frac12}_{L^\infty(0,T;L^2)}|f|^{\frac12}_{L^\infty(0,T,H^1)}\Big).\tag{2.84}$$

Proof. In order to establish (2.83), one multiplies (2.83) by a function $\varphi(x) \in \mathcal{C}^3([0,L])$ that will be chosen later on and one integrates over $x \in [0,L]$:

$$\frac{d}{dt}\int_0^L v\varphi + \int_0^L v_x\varphi + \int_0^L v_{xxx}\varphi = \int_0^L f_x\varphi.$$

Integration by parts leads to

$$\begin{aligned}&\frac{d}{dt}\int_0^L v\varphi - \int_0^L v\varphi_x + [v\varphi]_0^L - \int_0^L v\varphi_{xxx} \\ &+ [v_{xx}\varphi]_0^L - [v_x\varphi_x]_0^L + [v\varphi_{xx}]_0^L = -\int_0^L f\varphi_x + [f\varphi]_0^L.\end{aligned} \tag{2.85}$$

Choose now φ such that $\varphi(0) = \varphi(L) = \varphi_x(0) = \varphi_{xx}(0) = \varphi_{xx}(L) = 0$ and $\varphi_x(L) = 1$. Equation (2.85) becomes

$$\frac{d}{dt}\int_0^L v\varphi - \int_0^L v\varphi_{xxx} - \int_0^L v\varphi_x + v_{xx}(L,t) = -\int_0^L f\varphi_x,$$

which implies (2.83).

For (2.84), one takes φ satisfying $\varphi(0) = \varphi_x(0) = \varphi_x(L) = 0$ and $\varphi(L) = 1$, and (2.85) gives

$$\begin{aligned}&\frac{d}{dt}\int_0^L v\varphi - \int_0^L v\varphi_x + v(L,t) - \int_0^L v\varphi_{xxx} + v_{xx}(L,t) \\ &+ v(L,t)\varphi_{xx}(L) - v(0,t)\varphi_{xx}(0) = -\int_0^L f\varphi_x + f(L,t).\end{aligned} \tag{2.86}$$

Using the fact that $\forall\psi \in H^1([0,L])$, $|\psi|_{L^\infty} \leqslant C|\psi|_{L^2}^{1/2}|\psi|_{H^1}^{1/2}$ in (2.86), one gets (2.84).

Applying this proposition to equation (2.1), i.e., with $f = \frac{u^2}{2}$, one gets that for any $u \in L^\infty(0,T;H^1([0.L]))$ a solution to (2.1), $u_x(L,t)$ and $u_{xx}(L,t)$ exist in $H^{-1}([0,T])$ and that the application $u \mapsto (u_x(L,t), u_{xx}(L,t))$ is continuous on bounded sets of $L^\infty(0,T;H^1([0,L]))$ for the norm of $L^\infty(0,T;L^2([0,T]))$ with values in $(H^{-1}([0,T]))^2$. □

For the regularized problem (2.41) one has:

Proposition 2.12. *Let $T > 0$ and B^ε be the following space*:

$$B^\varepsilon = \Big\{(v^\varepsilon, f^\varepsilon) \in \mathcal{C}([0,T];3) \times \mathcal{C}([0,T];H^1) \quad \text{such that} \\ v_t^\varepsilon + v_x^\varepsilon + v_{xxx}^\varepsilon - \varepsilon v_{xxt}^\varepsilon = f_x^\varepsilon, \quad 0 < x < L,\ 0 < t < T\Big\}.$$

We consider the maps $\mathcal{T}_1, \mathcal{T}_2$ *as in Proposition* 2.9. *One has*

$$|\mathcal{T}_1(v^\varepsilon, f^\varepsilon)|_{H^{-1}([0,T])} \leqslant C\left(|v^\varepsilon|_{L^\infty(0,T;L^2)} + |f|_{L^\infty(0,T;L^2)} + \varepsilon|v^\varepsilon|_{L^\infty(0,T;H^1)}\right) \tag{2.87}$$

and

$$\begin{aligned}|\mathcal{T}_2(v^\varepsilon, f^\varepsilon)|_{H^{-2}([0,T])} \leqslant C\Big(&|v^\varepsilon|^{1/2}_{L^\infty(0,T;L^2)}|v^\varepsilon|^{1/2}_{L^\infty(0,T;H^1)} \\ &+|f^\varepsilon|^{1/2}_{L^\infty(0,T;L^2)}|f^\varepsilon|^{1/2}_{L^\infty(0,T;H^1)} + \varepsilon|v^\varepsilon|_{L^\infty(0,T;H^1)}\Big).\end{aligned} \tag{2.88}$$

Proof. We still have to deal with the term $-\varepsilon v^\varepsilon_{xxt}$. For (2.87), one obtains

$$-\varepsilon\int_0^L v^\varepsilon_{xxt}\varphi(x) = \varepsilon\int_0^L v^\varepsilon_{xt}\varphi - \varepsilon[v_{xt}\varphi]_0^L = \varepsilon\frac{d}{dt}\int_0^L v^\varepsilon_x\varphi - 0,$$

thereby proving (2.87). For (2.88), one gets

$$-\varepsilon\int_0^L v^\varepsilon_{xxt}\varphi(x) = \varepsilon\int_0^L v^\varepsilon_{xt}\varphi - \varepsilon[v_{xt}\varphi]_0^L = \varepsilon\frac{d}{dt}\int_0^L v^\varepsilon_x\varphi - \varepsilon\frac{d}{dt}v_x(L).$$

Using (2.87), we get (2.88). □

Remark 2.13. If $v^\varepsilon_x(L) = 0$, then (2.88) holds with $|\mathcal{T}_2(v^\varepsilon, f^\varepsilon)|_{H^{-1}([0,T])}$.

Applying these results to $v^\varepsilon(x,t) = u^\varepsilon(x,t)$, the solution to the regularized problem, and $f^\varepsilon = \frac{(u^\varepsilon)^2}{2}$, finishes the proofs of Theorems 1.1 and 1.2.

3. General case

In this section, we address the nonhomogeneous case. Namely, find $u = u(x,t)$, $x \in [0,L]$ and $t \geqslant 0$, a solution to

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} + u\frac{\partial^3 u}{\partial x^3} = 0,\ 0 < x < L,\ t > 0, \tag{3.1}$$

$$u(0,t) = g(t),\ \ t \geqslant 0, \tag{3.2}$$

$$\frac{\partial u}{\partial x}(L,t) = h(t),\ \frac{\partial^2 u}{\partial x^2}(L,t) = k(t),\ t \geqslant 0, \tag{3.3}$$

$$u(x,0) = u_0(x),\ 0 < x < L. \tag{3.4}$$

We will extend the proofs of the previous section.

3.1. **Extension of the a priori estimates.** As usual, we perform formal computations on the solutions to (3.1)–(3.4). Let v be defined by

$$v(x,t) = u(x,t) - [g(t) + (h(t) - k(t)L)x + \frac{k(t)}{2}x^2] \equiv u(x,t) - f(x,t). \quad (3.5)$$

Equations (3.1)–(3.4) read

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}((1+f)v) + v\frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3} = -\frac{\partial f}{\partial t} - f\frac{\partial f}{\partial x}, \quad (3.6)$$

$$v(0,t) = 0, \ t \geqslant 0, \quad (3.7)$$

$$\frac{\partial v}{\partial x}(L,t) = 0, \ \frac{\partial^2 v}{\partial x^2}(L,t) = 0, \ t \geqslant 0, \quad (3.8)$$

$$v(x,0) = v_0(x) \equiv u_0(x) - f(x,0). \quad (3.9)$$

Multiplying (3.6) by v leads to

$$\frac{d}{dt}\int_0^L v^2(x,t)\,dx + \int_0^L v^2(x,t)\frac{\partial f}{\partial x}(x,t)\,dx + (1+f(L,t))v^2(L,t) \quad (3.10)$$
$$+ \frac{2v^3}{3}(L,t) + v_x^2(0,t) = -2\int_0^L \left(\frac{\partial f}{\partial t}(x,t) + \frac{\partial}{\partial x}\frac{f^2}{2}(x,t)\right) v(x,t)\,dx.$$

Since the left-hand side of (3.6) can be written $\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}\big((1+f)v + \frac{v^2}{2} + \frac{\partial^2 v}{\partial x^2}\big)$, we deduce, after multiplication by $(1+f)v + \frac{v^2}{2} + v_{xx}$, that

$$\frac{d}{dt}\left(\int_0^L v_x^2 - (1+f)v^2 - \frac{v^3}{3}\,dx\right) + v_{xx}^2(0,t) - \int_0^L \frac{\partial f^2}{\partial x^2}v^2\,dx \quad (3.11)$$
$$= \left((1+f(L,t))\,v(L,t) + \frac{v^2}{2}(L,t)\right)^2 - 2\left(\frac{\partial f}{\partial t} + \frac{\partial f^2}{\partial x}\right)(0,t)v_x(0,t)$$
$$+ 2\int_0^L \left(\frac{\partial f}{\partial t} + \frac{\partial}{\partial x}\frac{f^2}{2}\right)(1+f)v\,dx + 2\int_0^L \left(\frac{\partial^3}{\partial t\partial x^2}f + \frac{\partial^3}{\partial x^3}\frac{f^2}{2}\right)v\,dx.$$

It is clear that the same kind of proof as that of Section 2.1.1. yields a local-in-time estimate. The new terms do not imply a serious problem. For the global-in-time estimate, the situation is more complicated. The two basic assumptions are

$$\inf_{t\geqslant 0}(1+f(L,t)) = \alpha_0 > 0 \quad (3.12)$$

and

$$\frac{\partial f}{\partial x}(x,t) \geqslant 0, \ 0 \leqslant x \leqslant L, \ t \geqslant 0, \quad (3.13)$$

which is equivalent to

$$h(t) - k(t)(L - x) \geqslant 0, \quad 0 \leqslant x \leqslant L,\ t \geqslant 0. \tag{3.14}$$

Remark 3.1. The condition (3.13) can be replaced by a weaker condition. Let m denote the function defined by $m(t) = h(t) - L(k(t) + |k(t)|)/2$. It is easy to check that $\forall t \geqslant 0$, $\forall x \in [0, L]$, $\frac{\partial f}{\partial x}(x,t) \geqslant m(t) = \min_{0 \leqslant y \leqslant L} \frac{\partial f}{\partial x}(y,t)$. Hence, (3.13) reads $m \geqslant 0$. Actually all the estimates can be done assuming that

$$\sup_{t \geqslant 0}(\ \exp\ -\int_0^t m(s)\, ds) < \infty \tag{3.15}$$

(although they become much more technical).

Now the proof follows along the same lines as in Section 2.1.2, assuming that the following quantities are small enough:

$$\mu_1 = \int_0^{+\infty} \Big|\frac{\partial f}{\partial t} + \frac{\partial}{\partial x}\frac{f^2}{2}\Big|_2 (s)\, ds,\ \mu_2 = \sup_{t \geqslant 0}(1 + f(L,t)), \tag{3.16}$$
$$\mu_3 = \int_0^{+\infty} \Big|\frac{\partial f}{\partial t} + \frac{\partial}{\partial x}\frac{f^2}{2}\Big|^2 (0,s)\, ds,\ \mu_4 = \int_0^{t} \Big|\frac{\partial^3}{\partial t \partial x^2} f + \frac{\partial^3}{\partial x^3}\frac{f^2}{2}\Big|_2 (s)\, ds,$$
$$\mu_5 = \int_0^{+\infty} \Big|\frac{\partial}{\partial x}\frac{f^2}{2}\Big|_\infty (s)\, ds,$$

and we refer to [13] for details.

3.1.1. **Solutions to a regularized problem.** Let $\varepsilon > 0$ be given. We want to construct a solution to the following problem. Find $u^\varepsilon = u^\varepsilon(x,t)$, $x \in [0, L]$ and $t \geqslant 0$, a solution to

$$\frac{\partial u^\varepsilon}{\partial t} + \frac{\partial u^\varepsilon}{\partial x} + u^\varepsilon \frac{\partial u^\varepsilon}{\partial x} + \frac{\partial^3 u^\varepsilon}{\partial x^3} - \varepsilon \frac{\partial^3 u^\varepsilon}{\partial x^2 \partial t} = 0, \quad 0 < x < L,\ \ t \geqslant 0, \tag{3.17}$$

together with the boundary and initial conditions (3.2), (3.3) and (3.4). The same change of variables as in Section 2.2 (formula (2.42)) $w(x,t) = \varepsilon u^\varepsilon(\varepsilon^{1/2}(x - t),\ \varepsilon^{3/2} t)$ yields

$$\frac{\partial w}{\partial t} + (1+\varepsilon)\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial x} - \frac{\partial^3 w}{\partial x^2 \partial t} = 0,\ t < x < t + \varepsilon^{-1/2} L,\ t > 0, \tag{3.18}$$
$$w(t,t) = \varepsilon g(\varepsilon^{3/2} t), \quad t \geqslant 0, \tag{3.19}$$
$$\partial_x w(t + \varepsilon^{-1/2} L, t) = \varepsilon^{3/2} h(\varepsilon^{3/2} t), \tag{3.20}$$
$$\partial^2_{xx} w(t + \varepsilon^{-1/2} L, t) = \varepsilon^2 k(\varepsilon^{3/2} t), \quad t \geqslant 0.$$

Since L and ε are fixed, as in Section 2.2.1, we have to study

$$w_t + cw_x + ww_x - w_{xxt} = 0, \quad t < x < t+\lambda,\ t > 0, \tag{3.21}$$

$$w(t,t) = \gamma(t),\ t \geqslant 0, \tag{3.22}$$

$$w_x(t+\lambda,t) = \eta(t),\ w_{xx}(t+\lambda,t) = K(t), \quad t \geqslant 0, \tag{3.23}$$

$$w(x,0) = \varphi(x), \quad 0 < x < \lambda. \tag{3.24}$$

We introduce $f_\lambda(x,t) = \gamma(t) + (\eta(t) - K(t)\lambda)(x-t) + \frac{K(t)^2}{2}(x-t)^2$, and $v(x,t) = w(x,t) - f_\lambda(x,t)$, and obtain

$$\frac{\partial v}{\partial t} - v_{xxt} = -f_{\lambda t} - v_x, \quad t < x < t+\lambda,\ t > 0, \tag{3.25}$$

$$v(t,t) = 0, \quad t \geqslant 0, \tag{3.26}$$

$$v_x(t+\lambda,\lambda,t) = 0, \quad v_{xx}(t+\lambda,t) = 0, \quad t \geqslant 0, \tag{3.27}$$

$$v(x,0) = \varphi(x) - (\gamma(0) + (\eta(0) - K(0)\lambda)x + \frac{K(0)^2}{2}x^2), \tag{3.28}$$

where

$$V(x,t) = (1+f_\lambda)v + \frac{v^2}{2} + \frac{f_\lambda^2}{2}. \tag{3.29}$$

It is clear that one can solve (3.25)–(3.29) as in Section 2.2 using an integral formulation.

One then obtains the same kind of local existence result as Theorems 2.5 and 2.6 for the integral equation; we omit the details. Concerning the regularized equation (3.17), one has the analogues of Proposition (2.9) and (2.10):

Proposition 3.2. *Let $u_0 \in H^3([0,L]), g, h, k, \in \mathcal{C}^1(\mathbb{R}^+)$ satisfying $u_0(0) = g(0)$, $u_x(L) = h(0)$, and $u_{xx}(L) = k(0)$. Then there exists T_0 depending only on $|u_0|_{H^1}, |g|_{\mathcal{C}^1}, |h|_{\mathcal{C}^1}$ and $|k|_{\mathcal{C}^1}$ such that there exists a unique solution u^ε to* (3.17) *satisfying $u^\varepsilon(x,0) = u_0(x)$ and the boundary conditions* (3.2)–(3.3). *Moreover, $u^\varepsilon \in \mathcal{C}([0,T_0]; H^3) \cap \mathcal{C}^1([0,T_0]; H^4)$ and $|u^\varepsilon|_{L^\infty(0,T_0;H^1)}$ is bounded independently of ε, and this bound depends only on $|u_0|_{H^1([0,L])}$, $|g|_{\mathcal{C}^1}$, $|h|_{\mathcal{C}^1}$, $|k|_{\mathcal{C}^1}$.*

Proposition 3.3. *Let $f(x,t) = g(t) + (h(t) - k(t)L)x + \frac{k^2(t)}{2}x^2$. Suppose that $f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial t} \in L^\infty([0,L] \times \mathbb{R}^+)$ and that*

$$\inf(1 + f(L,t)) = \alpha_0 > 0 \quad \textit{and} \quad h(t) - k(t)(L,x) \geqslant 0,\ 0 \leqslant x \leqslant L,\ t \geqslant 0.$$

Then there exists $\delta > 0$, such that if $|u_0(\cdot) - f(\cdot,0)|_{H^1} \leqslant \delta$ and if the quantities $(\mu_i)_{i=1}^5$ defined in (3.16) *satisfy $\mu_i \leqslant \delta$, $i = 1$ to 5, the one can take $T_0 = +\infty$ in Proposition* 3.1.

The only thing that one has to prove is that the "energy" estimates proved in Section 3.1 are still valid for the regularized problem (3.17). It is the same as in Section 2.2. □

3.2. **Proof of Theorem 1.1 and Theorem 1.2 in the general case.** Let $u_0 \in H^1([0,L]), (g,h,k) \in (\mathcal{C}^1(\mathbb{R}^+))^3$ be such that $u_0(0) = g(0)$. Take a sequence $u_0^\varepsilon \in H^3([0,L])$ such that $u_0^\varepsilon(0) = g(0)$, $\frac{\partial u_0^\varepsilon}{\partial x}(L) = h(0)$, $\frac{\partial^2 u_0^\varepsilon}{\partial x^2}(L) = k(0)$, and $u_0^\varepsilon \to u_0$ in $H^1([0,L])$. We still call u^ε the solution to (3.17) satisfying $u^\varepsilon(x,0) = u_0^\varepsilon$ and the boundary conditions (3.2) and (3.3), given by Proposition 3.1.

Since $u_0^\varepsilon \to u_0$ in $H^1([0,L])$ and since T_0 depends only on $|u_0^\varepsilon|_{H^1}$, one can take T_0 depending only on $|u_0|_{H^1}$. One concludes as in Section 2.3.

Of course the boundary conditions $u_x(L,t) = h(t)$ and $u_{xx}(L,t) = k(t)$ are satisfied respectively in $H^{-1}([0,T])$ and $H^{-2}([0,T])$ in the sense of Proposition 2.11. □

4. Local weak solutions in $L^2([0,L])$

Let us consider the homogeneous problem

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \quad 0 < x < L, \quad t > 0 \tag{4.1}$$

$$u(0,t) = \frac{\partial u}{\partial x}(L,t) = \frac{\partial^2 u}{\partial x^2}(L,t) = 0, \quad t \geqslant 0 \tag{4.2}$$

$$u(0,t) = u_0(x), \quad 0 < x < L. \tag{4.3}$$

The aim of this section is to construct local weak solutions to (4.1)–(4.3).

4.1. **Construction of the linear semigroup.** We first construct a solution u to

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} = 0, \quad 0 < x < L, \quad t > 0, \tag{4.4}$$

satisfying (4.2) and (4.3). One has

Theorem 4.1.

i) *Let* $u_0 \in H^3(0,L)$ *such that* $u_0(0) = \frac{\partial u_0}{\partial x}(L) = \frac{\partial^2 u_0}{\partial x^2}(L) = 0$. *There exists a unique solution denoted by* $S(\cdot)u_0$ *to* (4.4), (4.2), (4.3) *satisfying* $S(\cdot)u_0 \in L^\infty(\mathbb{R}^+; H^3(0;L))$.

ii) *There exists a constant* $C(L)$ *such that*

$$|S(\cdot)u_0|_{L^\infty(\mathbb{R}^+;L^2)} + \left|\frac{\partial}{\partial x}S(\cdot)u_0\right|_{L^2(\mathbb{R}^+;L^2)} + \left|\sqrt{t}\frac{\partial^2}{\partial x^2}S(\cdot)u_0\right|_{L^2_{loc}(\mathbb{R}^+;L^2)} \leqslant C(L)|u_0|_{L^2}.$$

Let

$$E = \{V \in \mathcal{C} \cap L^\infty(\mathbb{R}^+; L^2) \cap L^2(\mathbb{R}^+; H^1)$$

such that $\sqrt{t}\frac{\partial^2 V}{\partial x^2} \in L^2_{loc}(\mathbb{R}^+; L^2)\}$. We therefore obtain

Corollary 4.2. *The semigroup $(S(t))_{t\geqslant 0}$ extends continuously from $L^2(0,L)$ into E.*

Proof of Theorem 4.1. In order to prove i), one first constructs regularized solutions as in Section 2. Let $u^\varepsilon(x,t)$ be the solution to

$$\frac{\partial u^\varepsilon}{\partial t} + \frac{\partial^3 u^\varepsilon}{\partial x^3} - \varepsilon \frac{\partial^3 u^\varepsilon}{\partial x^2 \partial t} = 0, \quad 0 < x < L,\ t \geqslant 0 \tag{4.5}$$

and suppose u^ε satisfies (4.2) and (4.3).

Local-in-time solutions to this problem can be obtained as in Section 2. In order to show that the solutions are global, it is enough to obtain global bounds in $H^3(0,L)$.

Multiplying (4.5) by u^ε leads to

$$\frac{d}{dt}\int_0^L (u^\varepsilon)^2 + \varepsilon\left(\frac{\partial u^\varepsilon}{\partial x}\right)^2 dx + \left(\frac{\partial u^\varepsilon}{\partial x}\right)^2(0,t) = 0. \tag{4.6}$$

Multiplying $\frac{\partial}{\partial t}$ (4.5) by $\frac{\partial u^\varepsilon}{\partial t}$ leads to

$$\frac{d}{dt}\int_0^L \left(\frac{\partial u^\varepsilon}{\partial t}\right)^2 + \varepsilon\left(\frac{\partial^2}{\partial x \partial t}u^\varepsilon\right)^2 dx + \left(\frac{\partial^2 u^\varepsilon}{\partial x \partial t}\right)^2(0,t) = 0. \tag{4.7}$$

Using (4.6) and (4.7), one gets that $u^\varepsilon \rightharpoonup u$ in $L^\infty(\mathbb{R}^+, H^3)$ $w\star$ and $\frac{\partial u^\varepsilon}{\partial t} \rightharpoonup \frac{\partial u}{\partial t}$ in $L^\infty(\mathbb{R}^+; L^2)$ $w\star$ with $\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} = 0$, thereby proving that $u \in L^\infty(\mathbb{R}^+; H^3)$. Hence i) of Theorem 4.2 follows (of course $u \in \mathcal{C}(\mathbb{R}^+; H^2)$).

In order to prove ii), first multiply (4.4) successively by $u(x,t)$ and $xu(x,t)$; one obtains

$$\frac{d}{dt}\int_0^L u^2 dx + \left(\frac{\partial u}{\partial x}\right)^2(0,t) = 0, \tag{4.8}$$

$$\frac{d}{dt}\int_0^L xu^2 dx + 3\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx = 0. \tag{4.9}$$

Now multiplying (4.4) by $\frac{\partial^2 u}{\partial x^2}$ and $x\frac{\partial^2 u}{\partial x^2}$ yields

$$\frac{d}{dt}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial^2 u}{\partial x^2}\right)^2 (0,t) = 0, \tag{4.10}$$

$$\frac{d}{dt}\int_0^L x\left(\frac{\partial u}{\partial x}\right)^2 + 3\int_0^L \left(\frac{\partial^2 u}{\partial x^2}\right)^2 + 2\frac{\partial^2 u}{\partial x^2}(0,t)\frac{\partial u}{\partial x}(0,t) = 0. \tag{4.11}$$

Equations (4.8) and (4.9) give readily that

$$|S(t)u_0|_{L^\infty(\mathbb{R}^+;L^2)} + \Big|\frac{\partial}{\partial x}S(t)u_0\Big|_{L^2(\mathbb{R}^+;L^2)} \leqslant C(L)|u_0|_{L^2}.$$

Note that (4.8) implies

$$\Big|\frac{\partial u}{\partial x}(0,t)\Big|_{L^2(\mathbb{R}^+)} \leqslant C(L)|u_0|_{L^2}. \tag{4.12}$$

Let us multiply (4.10) by t and integrate the result with respect to t; one gets

$$t\int_0^L \Big(\frac{\partial u}{\partial x}\Big)^2(x,t)\,dx + \int_0^t s\Big(\frac{\partial^2 u}{\partial x^2}\Big)^2(0,s)\,ds = \int_0^t\int_0^L \Big(\frac{\partial u}{\partial x}\Big)^2 dx\,ds. \tag{4.13}$$

Multiplying (4.11) by t and integrating in time leads to

$$-\int_0^t\int_0^L x\left(\frac{\partial u}{\partial x}\right)^2 dx\,ds + t\int_0^L x\left(\frac{\partial u}{\partial x}\right)^2 dx$$
$$+\,3\int_0^t s\int_0^L \left(\frac{\partial^2 u}{\partial x^2}\right)^2 dx\,ds + 2\int_0^t s u_{xx}(0,s)u_x(0,s)\,ds = 0,$$

so that thanks to (4.13)

$$\Big|\sqrt{t}\frac{\partial^2 u}{\partial x^2}\Big|_{L^2(0,T;L^2)} \leqslant C(L)\Big[\Big|\frac{\partial u}{\partial x}\Big|_{L^2(\mathbb{R}^+;L^2)} + \Big|\frac{\partial u}{\partial x}\Big|^{\frac{1}{2}}_{L^2(\mathbb{R}^+;L^2)} 2\Big(\int_0^T s u_x^2(0,s)\,ds\Big)^{\frac{1}{2}}\Big]$$
$$\leqslant C(L)[1+T^{1/2}]|u_0|_{L^2},$$

thereby proving ii) of Theorem 4.1. □

Definition 4.3. A weak solution to (4.1)–(4.3) is a function $u \in \mathcal{C}([0,T];L^2)$ $\cap L^2_{loc}([0,T);H^1)$ satisfying

$$u = S(t)u_0 - \int_0^t S(t-s)\Big[\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x}\Big](s)\,ds.$$

4.2. **Existence and uniqueness of weak solutions.** We will prove

Theorem 4.4. *Let $u_0 \in L^2(0,L)$. There exists a unique maximal weak solution to* (4.1)–(4.3) $u \in \mathcal{C}([0,T_{max}),L^2) \cap L^2_{loc}([0,T_{max}),H^1(0,L))$. *Moreover,* $\sqrt{t}\frac{\partial^2 u}{\partial x^2} \in L^2_{loc}([0,T_{max}),L^2)$, *and if* $T_{max} < +\infty$, *then*

$$\lim_{t\to T_{max}} |u(t)|_{L^2} = +\infty.$$

Proof. We first introduce for any $f \in L^1(0,T;L^2)$

$$\Lambda f(t) = \int_0^t S(t-s)f(s)\,ds.$$

One has the following dual estimates on Λ :

Proposition 4.5. *There exists $C > 0$ such that for any $f \in L^1(0,T;L^2)$*

$$|\Lambda f(t)|_{L^\infty(0,T;L^2)} \leqslant C|f|_{L^1(0,T;L^2)}, \tag{4.14}$$

$$\Big|\frac{\partial}{\partial x}\Lambda f(t)\Big|_{L^2(0,T;L^2)} \leqslant C|f|_{L^1(0,T;L^2)}. \tag{4.15}$$

Proof of the proposition. One has $|\Lambda f(t)|_{L^2(0,L)} \leqslant \int_0^t |f|_{L^2}(s)\,ds$ and (4.14) follows. On the other hand $\big|\frac{\partial}{\partial x}\Lambda f(t)\big|_{L^2} \leqslant \int_0^t \big|\frac{\partial}{\partial x}S(t-s)f(s)\big|_{L^2}\,ds$, so that for all $\varphi \geqslant 0$

$$\begin{aligned}\int_0^T \Big|\frac{\partial\Lambda}{\partial x}f(t)\Big|\varphi(t)\,dt &\leqslant \int_0^T\int_0^t \Big|\frac{\partial}{\partial x}S(t-s)f(s)\Big|_{L^2}\,ds\varphi(t)\,dt\\ &= \int_0^T\int_s^T \Big|\frac{\partial}{\partial x}S(t-s)f(s)\Big|_{L^2}\varphi(t)\,dt\,ds\\ &\leqslant \int_0^T\int_s^T \Big|\frac{\partial}{\partial x}S(t-s)f(s)\Big|_{L^2}^2\,dt\Big)^{1/2}\Big(\int_s^T \varphi(t)^2 dt\Big)^{1/2}\,ds.\end{aligned}$$

Now, thanks to Theorem 4.1,

$$\Big(\int_0^T \Big|\frac{\partial}{\partial x}S(t-s)f(s)\Big|_{L^2}^2\,dt\Big)^{1/2} \leqslant C|f(s)|_{L^2(0,L)}$$

so that

$$\int_0^T \Big|\frac{\partial}{\partial x}\Lambda f(t)\Big|\varphi(t)\,dt \leqslant C\int_0^T |f(s)|_{L^2(0,L)}\,ds\Big(\int_0^T \varphi(t)^2\,dt\Big)^{1/2}$$

from which (4.15) follows. $\square$

In order to prove Theorem 4.4, we introduce the following functional $\mathcal{T}$, defined on $L^\infty(0,T;L^2)\cap L^2(0,T;H^1)$ by

$$\mathcal{T}(u) = S(t)u_0 - \int_0^t S(t-s)[\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x}](s)\,ds.$$

One has

$$\Big|\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x}\Big|_{L^1(0,T;L^2)} \leqslant |u|_{L^1(0,T;H^1)} + \Big|u\frac{\partial u}{\partial x}\Big|_{L^1(0,T;L^2)}.$$

Now

$$\Big|u\frac{\partial u}{\partial x}\Big|_{L^2(0,L)} \leqslant |u|_{L^\infty}\Big|\frac{\partial u}{\partial x}\Big|_{L^2} \leqslant C|u|_{L^2}^{1/2}\Big|\frac{\partial u}{\partial x}\Big|_{L^2}^{3/2},$$

so that

$$\Big|u\frac{\partial u}{\partial x}\Big|_{L^1(0,T;L^2)} \leqslant C|u|_{L^\infty(0,T;L^2)}^{1/2}\Big|\frac{\partial u}{\partial x}\Big|_{L^{3/2}(0,T;L^2)}^{3/2}.$$

It follows that

$$\Big|\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x}\Big|_{L^1(0,T;L^2)} \leqslant T^{1/2}|u|_{L^2(0,T;H^1)} + C|u|_{L^\infty(0,T;L^2)}^{1/2}T^{1/4}\Big|\frac{\partial u}{\partial x}\Big|_{L^2(0,T,H^1)}^{3/2}.$$

Applying Proposition 4.5 and Theorem 4.1, one gets

$$\begin{aligned}&|\mathcal{T}(u)|_{L^\infty(0,T;L^2)\cap L^2(0,T;H^1)} \leqslant C_1|u_0|_{L^2}\\ &+C_2\left(T^{1/2}|u|_{L^2(0,T;H^1)} + T^{1/4}|u|_{L^\infty(0,T;L^2)}^{1/2}\left|\tfrac{\partial u}{\partial x}\right|_{L^2(0,T;H^1)}^{3/2}\right).\end{aligned} \tag{4.16}$$

Let $R = 2C_1|u_0|_{L^2}$ and B_R denote the ball of radius R in $L^\infty(0,T;L^2)\cap L^2(0,T;H^1)$; we have proved thanks to (4.16) the following.

Lemma 4.6. *If T is sufficiently small, $\mathcal{T}$ maps B_R into itself.*

In the same way, one can show

Lemma 4.7. *If T is small enough, $\mathcal{T}$ is a contraction on B_R.*

Applying Banach's fixed-point theorem for $\mathcal{T}$ on B_R (which is a complete metric space) yields Theorem 4.4.

5. Open questions and further investigations

5.1. In view of Theorems 1.1 and 1.2 a natural question arises: is it possible to prove global existence of solutions for (1.7)–(1.11) for e.g. smooth solutions (as is the case for both the quarter plane and the whole line cases)? For these problems, uniqueness relies on a priori estimates in H^2 that we are not able to extend here and therefore establish the existence of more regular solutions.

R. Temam [21] (see also Bubnov [6]) has proposed a different set of boundary conditions at $x = L$ which leads to global estimates with respect to time in $H^1(0, L)$. These conditions are nonlinear and read

$$u_x(L,t) = 0, \;\; u_{xx}(L,t) + \frac{u^2}{2}(L,t) = 0. \tag{5.1}$$

A natural question which arises then is the study of the long-time behavior of solutions to this problem. One could also consider the dissipative case (KdV–Burgers)

$$u_t + u_x + uu_x + u_{xxx} - \gamma u_{xx} = 0 \tag{5.2}$$

with time-periodic forcing of the form $u(0,t) = g(t)$ and (4.1). In this case it would be interesting to obtain the existence of global attractors as in the space-periodic case (see also Ghidaglia [15] and [16]).

5.2. Another question is the study of the solutions constructed in Theorem 1.1 and 1.2 when $L \to \infty$; see [14].

5.3. We have not addressed the study of the mapping $(g, h, k) \to u(L, \cdot)$. This question is inspired from scattering theory.

5.4. Since the linear equation

$$u_t + u_{xxx} = 0,$$

$$u(0,t) = 0, \; u_x(L,t) = 0, \; u_{xx}(L,t) = 0$$

has regularizing effects, one can expect, as in the case of the whole line, existence results to the IBVP (1.7)–(1.11) for initial values in $H^s(0, L)$, $s < 0$. Let us also mention the paper of Cattabriga [8] for a complete study of Airy's equation $u_t + u_{xxx} = 0$ on a finite interval.

5.5. Finally we refer to Bubnov [6] and [7] for various a priori estimates on a similar equation.

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