An Insurance Risk Model with Parisian Implementation Delays — **Source link** ↗

David Landriault, Jean-François Renaud, Xiaowen Zhou

Institutions: University of Waterloo, Université du Québec à Montréal, Concordia University

Related papers:

- Parisian ruin probability for spectrally negative Lévy processes

- Ruin probability with Parisian delay for a spectrally negative Lévy risk process

- Gerber–Shiu distribution at Parisian ruin for Lévy insurance risk processes

- Parisian ruin probability for spectrally negative L\'{e}vy processes

- Occupation times of intervals until first passage times for spectrally negative Lévy processes

INSURANCE RISK MODELS WITH PARISIAN IMPLEMENTATION DELAYS

DAVID LANDRIAULT, JEAN-FRANÇOIS RENAUD, AND XIAOWEN ZHOU

ABSTRACT. Inspired by Parisian barrier options in finance (see e.g. Chesney et al. (1997)), a new definition of the event *ruin* for an insurance risk model is considered. As in Dassios and Wu (2009), the surplus process is allowed to spend time under a pre-specified default level before ruin is recognized. In this paper, we capitalize on the idea of Erlangian horizons (see Asmussen et al. (2002) and Kyprianou and Pistorius (2003)) and, thus assume an implementation delay of a mixed Erlang nature. Using the modern language of scale functions, we study the Laplace transform of this Parisian time to default in an insurance risk model driven by a spectrally negative Lévy process of bounded variation. In the process, a generalization of the two-sided exit problem for this class of processes is further obtained.

1. INTRODUCTION

Historically, in actuarial risk theory, a lot of attention has been given to the analysis of events related to the time of default which is assumed to occur if and when the surplus process falls below a certain threshold level for the first time (see, e.g., Gerber and Shiu (1998), Li and Garrido (2005) and Willmot (2007)). Without loss of generality, which is due to the spatial homogeneity of most risk processes, this threshold level has commonly been assumed to be the *artificial* level 0. For solvency purposes, it is more appropriate to view this threshold level as the insurer's minimum capital requirement set by the regulatory body to ensure adequate capital levels are maintained by insurers (e.g., Solvency II, MCCSR). In this context, the existing literature in ruin theory can heavily be relied on to gather important risk management information as to the timing and the severity of a capital shortfall.

From a practical standpoint, it seems rather unlikely that the regulator and/or the insurer monitor the corresponding surplus level on a continuous basis and be immediately notified of the occurrence of a capital shortfall event. Therefore, Dassios and Wu (2009) consider the application of an implementation delay in the recognition of an insurer's capital insufficiency. More precisely, they assume that the event *ruin* occurs if the excursion below the critical threshold level is longer than a deterministic time. In the aforementioned article, the analysis of the ruin probability is done in the context of the classical compound Poisson risk model. It is worth pointing out that this new definition of *ruin* is also referred to as 'Parisian ruin' due to its ties with the concept of Parisian options (see Chesney et al. (1997)).

In the present paper, we also introduce the idea of Parisian ruin but now in the rich class of Lévy insurance risk models; see, e.g., Biffis and Kyprianou (2010) for an overview of this family of models. Furthermore, we assume that the deterministic delay is replaced by a stochastic grace

period with a pre-specified distribution. We show that the specification of this implementation delay to be of mixed Erlang nature improves the tractability of the resulting expression for the Laplace transform of the Parisian ruin time. In nature, this is similar to the use of Erlangian horizons (rather than a deterministic horizon) for the calculation of finite-time ruin probabilities in various risk models (see Asmussen et al. (2002) and Ramaswami et al. (2008)). As will be shown, all our results are expressed in terms of scale functions for which many explicit examples are known; see, e.g., Hubalek and Kyprianou (2010), Kyprianou and Rivero (2008), as well as the numerical algorithm developed by Surya (2008). Furthermore, mixed Erlang distributions are known to be a very large and flexible class of distributions for modelling purposes (see Willmot and Woo (2007)). Among others, it is well known that a sequence of Erlang distributions can be used to approximate the deterministic implementation delay strategy, as illustrated in the final section of this paper.

The rest of the paper is organized as follows. Next, we introduce Lévy insurance risk models and state some important properties of scale functions. In Section 2, a generalized version of the two-sided exit problem is studied when the first passage time below level 0 is substituted by the Parisian ruin time. These results are further particularized under the assumption that implementation delays are exponentially distributed and later, mixed Erlang distributed. Finally, in Section 3, explicit expressions for the Laplace transform of the Parisian ruin time are obtained under the same distributional assumptions for the implementation delays.

1.1. **Lévy insurance risk processes.**

A modern approach in ruin theory is to work with a spectrally negative Lévy process to describe the (free) surplus of an insurance company/portfolio. In the actuarial literature, these Lévy processes with no positive jumps are also called Lévy insurance risk processes. Such a process $X = (X_t)_{t \geq 0}$ is defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$, has independent and stationary increments, and has *càdlàg* paths (right-continuous with left limits). Its law when $X_0 = x$ is denoted by P_x and the expectation by E_x. To avoid trivialities, it is implicitly assumed that X does not have monotone sample paths, that is X is not a negative subordinator, as for example a compound Poisson process with a negative drift, or just a deterministic drift.

It is well known that the Laplace transform of X is given by

$$E_0\left[e^{\theta X_t}\right] = e^{t\psi(\theta)},$$

for $\theta \geq 0$ and $t \geq 0$, where

$$\psi(\theta) = \gamma\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{-\infty}^0 \left(e^{\theta z} - 1 - \theta z \mathbb{I}_{(-1,0)}(z)\right)\Pi(\mathrm{d}z),$$

for $\gamma \in \mathbb{R}$ and $\sigma \geq 0$. Also, Π is a measure on $(-\infty, 0)$ such that

$$\int_{-\infty}^0 (1 \wedge z^2)\Pi(\mathrm{d}z) < \infty.$$

When X has paths of bounded variation, we may write

(1)
$$\psi(\theta) = c\theta - \int_{-\infty}^0 \left(1 - e^{\theta z}\right)\Pi(\mathrm{d}z),$$

where $c = \gamma - \int_{-1}^{0} z\Pi(\mathrm{d}z)$ is strictly positive. Therefore, $X_t = X_0 + ct - S_t$, where $c > 0$ denotes the constant premium intensity, and S is a pure-jump subordinator representing the aggregate claims. Note that the compound Poisson risk process corresponds to the case

$$S_t = \sum_{i=1}^{N_t} C_i,$$

where $N = (N_t)_{t \geq 0}$ is a Poisson process and the claim amounts $(C_i)_{i \geq 1}$ form a sequence of positive independent and identically distributed (iid) random variables. Equivalently, this corresponds to $\Pi(\mathrm{d}z) = \lambda F(-\mathrm{d}z)$, where λ is the jump intensity of N and F is the distribution of the C_i's.

Finally, note that the net profit condition for a general Lévy insurance risk process is given by $E_0[X_1] = \psi'(0+) > 0$, which agrees with the classical formulation. In the sequel, we assume that this condition is satisfied.

1.2. **Scale functions.** As the Laplace exponent ψ is strictly convex and $\lim_{\theta \to \infty} \psi(\theta) = \infty$, there exists a function $\Phi \colon [0, \infty) \to [0, \infty)$ defined by $\Phi(\theta) = \sup\{\xi \geq 0 \mid \psi(\xi) = \theta\}$ (its right-inverse) and such that $\psi(\Phi(\theta)) = \theta$, for $\theta \geq 0$.

We now define the scale functions associated with the process X. For $q \geq 0$, the q-scale function $W^{(q)}$ is defined as the unique continuous function with Laplace transform

$$(2) \qquad \int_0^{\infty} e^{-\theta z} W^{(q)}(z)\mathrm{d}z = \frac{1}{\psi(\theta) - q}, \quad \text{for } \theta > \Phi(q).$$

We shall mention that $W^{(q)}$ is positive and strictly increasing. If X has bounded variation, the initial value of $W^{(q)}$ is given by $W^{(q)}(0+) = 1/c$. In general, although we primarily regard $W^{(q)}$ as a function on $(0, \infty)$, if needed, we extend it to the entire real line by setting $W^{(q)}(0) = W^{(q)}(0+)$ and $W^{(q)}(x) = 0$ for $x < 0$. Also, we define

$$Z^{(q)}(x) = 1 + q\overline{W}^{(q)}(x),$$

where

$$\overline{W}^{(q)}(x) = \int_0^x W^{(q)}(z)\mathrm{d}z.$$

Finally, it is known that $W^{(q)}(x) = e^{-\Phi(q)x} W_{\Phi(q)}(x)$, where W_ζ is the 0-scale function of X under P^ζ given by

$$(3) \qquad \left.\frac{\mathrm{d}P^\zeta}{\mathrm{d}P}\right|_{\mathcal{F}_t} = e^{\zeta X_t - \psi(\zeta)t},$$

for $\zeta \geq 0$.

1.3. **Standard fluctuation identities.** If we denote the *standard* time of default/ruin, i.e., absorption in $(-\infty, 0)$, by

$$(4) \qquad \tau_0^- = \inf\{t > 0 \colon X_t < 0\},$$

with the convention $\inf \emptyset = \infty$, then, for $x \geq 0$,

$$(5) \qquad E_x\left[e^{-q\tau_0^-}; \tau_0^- < \infty\right] = Z^{(q)}(x) - \frac{q}{\Phi(q)} W^{(q)}(x).$$

More generally, it is also known that

$$(6) \quad E_x \left[e^{-q\tau_0^- + rX_{\tau_0^-}}; \tau_0^- < \infty \right] = e^{rx} + (q - \psi(r))e^{rx} \int_0^x e^{-rz} W^{(q)}(z) \mathrm{d}z - \frac{q - \psi(r)}{\Phi(q) - r} W^{(q)}(x).$$

Now, define the first passage above a given level b by

$$\tau_b^+ = \inf\{t > 0 \colon X_t > b\}.$$

It is known that, for $0 \le x \le b$,

$$(7) \qquad E_x \left[e^{-q\tau_b^+}; \tau_b^+ < \tau_0^- \right] = \frac{W^{(q)}(x)}{W^{(q)}(b)}.$$

As $\mathbb{I}_{\{\tau_0^- < \tau_b^+\}} = \mathbb{I}_{\{\tau_0^- < \infty\}} - \mathbb{I}_{\{\tau_b^+ < \tau_0^-\}}$, the strong Markov property of the process X together with (7) and (6) yield

$$(8) \quad E_x \left[e^{-q\tau_0^- + rX_{\tau_0^-}}; \tau_0^- < \tau_b^+ \right]$$

$$= e^{rx} + (q - \psi(r))e^{rx} \int_0^x e^{-rz} W^{(q)}(z) \mathrm{d}z$$

$$- \frac{W^{(q)}(x)}{W^{(q)}(b)} \left\{ e^{rb} + (q - \psi(r))e^{rb} \int_0^b e^{-rz} W^{(q)}(z) \mathrm{d}z \right\}.$$

In particular,

$$(9) \qquad E_x \left[e^{-q\tau_0^-}; \tau_0^- < \tau_b^+ \right] = Z^{(q)}(x) - \frac{W^{(q)}(x)}{W^{(q)}(b)} Z^{(q)}(b).$$

For more details on Lévy insurance risk processes and their scale functions, we refer the reader to the monograph of Kyprianou (2006).

In the sequel, the following representation of the bivariate Laplace transform of $(\tau_0^-, X_{\tau_0^-})$ in terms of the Dickson-Hipp operator of the scale function $W^{(q)}$ will be particularly useful.

Remark 1.1. *Using* (2), *simple manipulations of* (6) *result in*

$$E_x \left[e^{-q\tau_0^- + rX_{\tau_0^-}}; \tau_0^- < \infty \right] = e^{rx} + (q - \psi(r)) e^{rx} \left(\int_0^\infty e^{-rz} W^{(q)}(z) \, dz - \int_x^\infty e^{-rz} W^{(q)}(z) \, dz \right)$$

$$- \frac{q - \psi(r)}{\Phi(q) - r} W^{(q)}(x)$$

$$= (\psi(r) - q) e^{rx} \int_x^\infty e^{-rz} W^{(q)}(z) \, dz - \frac{q - \psi(r)}{\Phi(q) - r} W^{(q)}(x)$$

$$(10) \qquad = (\psi(r) - q) \left(\mathcal{T}_r W^{(q)}(x) + \frac{1}{\Phi(q) - r} W^{(q)}(x) \right),$$

for $r > \Phi(q)$, *where* $\mathcal{T}_r$ *is the well-known* Dickson-Hipp operator *defined as*

$$\mathcal{T}_r f(x) = \int_0^\infty e^{-ry} f(y + x) \, dy,$$

for any r such that the integral converges. Using (10), *it is clear that*

$$E_x\left[e^{-q\tau_0^- + rX_{\tau_0^-}}; \tau_0^- < \tau_b^+\right] = E_x\left[e^{-q\tau_0^- + rX_{\tau_0^-}}; \tau_0^- < \infty\right] - \frac{W^{(q)}(x)}{W^{(q)}(b)} E_b\left[e^{-q\tau_0^- + rX_{\tau_0^-}}; \tau_0^- < \infty\right]$$

$$(11) \qquad\qquad = (\psi(r) - q)\left(\mathcal{T}_r W^{(q)}(x) - \frac{W^{(q)}(x)}{W^{(q)}(b)}\mathcal{T}_r W^{(q)}(b)\right),$$

for $r > \Phi(q)$.

2. Risk models with Parisian implementation delays

In this paper, we assume that the (underlying) Lévy insurance risk process X has bounded variation and satisfies the net profit condition. We first give a descriptive definition of the time to ruin τ_d. We assume that each excursion below the critical level 0 is accompanied by an iid copy of an independent (of X) and positive random variable e_d. We will refer to it as the implementation clock. If the duration of a given excursion below 0 is less than its associated implementation clock, then the 'short' excursion below 0 process is neglected as far as ruin is concerned. More precisely, we assume that ruin occurs at the first time that an implementation clock rings before the end of its corresponding excursion below 0. It is worth pointing out that the time to ruin τ_d is properly defined when there are countably many drops below 0 which explains our restriction to Lévy insurance risk processes of bounded variation in this paper.

Let $(e_d^k)_{k\geq 1}$ be a sequence of iid copies of e_d. Recall that the convention $\inf \emptyset = \infty$ is used. Also, let

$$\tau_{0,1}^- = \tau_0^- \equiv \inf\{t > 0\colon X_t < 0\},$$

be the first time that the process X enters $(-\infty, 0)$, and, correspondingly,

$$\tau_{0,1}^+ = \inf\left\{t > \tau_{0,1}^-\colon X_t > 0\right\},$$

be the first time (after $\tau_{0,1}^-$) that the process X enters $(0, \infty)$. Recursively, we define two sequences of stopping times $(\tau_{0,k}^-)_{k\geq 1}$ and $(\tau_{0,k}^+)_{k\geq 1}$ as follows: for $k \geq 2$, let

$$\tau_{0,k}^- = \inf\left\{t > \tau_{0,k-1}^+\colon X_t < 0\right\},$$

and

$$\tau_{0,k}^+ = \inf\left\{t > \tau_{0,k}^-\colon X_t > 0\right\}.$$

Thus, the Parisian ruin time τ_d is defined as follows:

$$\tau_d = \tau_{0,k_d}^- + e_d^{k_d},$$

where

$$k_d = \inf\left\{k \geq 1\colon \tau_{0,k}^- + e_d^k < \tau_{0,k}^+\right\}.$$

In this section, we propose a generalization of the two-sided exit problem in Kyprianou (2006) (see Eqs. (8.8) and (8.9)) when the first passage time below 0, namely τ_0^-, is substituted by the Parisian ruin time τ_d.

2.1. **General implementation delays.** Let us consider implementation delays with an arbitrary distribution to begin with. A general structure for the Laplace transform of the two exit times will be identified in the following two lemmas. For an initial surplus $x < 0$, we silently assume that the distribution of the first implementation delay has an identical distribution as the others.

Lemma 2.1. *In the context of a Lévy insurance risk model with paths of bounded variation,*

$$(12) \qquad E_x \left[e^{-q\tau_b^+}; \tau_b^+ < \tau_d \right] = \frac{H_d^{(q)}(x)}{H_d^{(q)}(b)}, \qquad x \leq b,$$

where

$$(13) \qquad H_d^{(q)}(x) = \begin{cases} \frac{W^{(q)}(x)}{W^{(q)}(0)} \left(1 - E_0 \left[e^{-q\tau_0^-} E_{X_{\tau_0^-}} \left[e^{-q\tau_0^+}; \tau_0^+ < e_d \right]; \tau_0^- < \tau_x^+ \right] \right), & x \geq 0, \\ E_x \left[e^{-q\tau_0^+}; \tau_0^+ < e_d \right], & x < 0. \end{cases}$$

Proof: For $0 \leq x \leq b$, we condition on whether the process reaches level b or level 0 first (as well as the relevant characteristics associated to this first passage). Capitalizing on the strong Markov property of the underlying Lévy process, we have
$$(14)$$
$$E_x \left[e^{-q\tau_b^+}; \tau_b^+ < \tau_d \right] = \frac{W^{(q)}(x)}{W^{(q)}(b)} + E_x \left[e^{-q\tau_0^-} E_{X_{\tau_0^-}} \left[e^{-q\tau_0^+}; \tau_0^+ < e_d \right]; \tau_0^- < \tau_b^+ \right] E_0 \left[e^{-q\tau_b^+}; \tau_b^+ < \tau_d \right],$$

for $0 \leq x \leq b$. In particular, for $x = 0$, Eq. (14) together with (13) yields

$$(15) \qquad E_0 \left[e^{-q\tau_b^+}; \tau_b^+ < \tau_d \right] = \frac{1}{H_d^{(q)}(b)}.$$

Substituting (15) into (14) leads to

$$E_x \left[e^{-q\tau_b^+}; \tau_b^+ < \tau_d \right] = \frac{W^{(q)}(x)}{W^{(q)}(b)} + \frac{E_x \left[e^{-q\tau_0^-} E_{X_{\tau_0^-}} \left[e^{-q\tau_0^+}; \tau_0^+ < e_d \right]; \tau_0^- < \tau_b^+ \right]}{H_d^{(q)}(b)}.$$

Utilizing once again the strong Markov property of the underlying Lévy insurance risk process,

$$E_x \left[e^{-q\tau_0^-} E_{X_{\tau_0^-}} \left[e^{-q\tau_0^+} ; \tau_0^+ < e_d \right] ; \tau_0^- < \tau_b^+ \right]$$

$$= \frac{W^{(q)}(x)}{W^{(q)}(0)} E_0 \left[e^{-q\tau_0^-} E_{X_{\tau_0^-}} \left[e^{-q\tau_0^+} ; \tau_0^+ < e_d \right] ; \tau_0^- < \tau_b^+ \right]$$

$$- \frac{W^{(q)}(x)}{W^{(q)}(0)} E_0 \left[e^{-q\tau_0^-} E_{X_{\tau_0^-}} \left[e^{-q\tau_0^+} ; \tau_0^+ < e_d \right] ; \tau_0^- < \tau_x^+ \right]$$

$$= \frac{W^{(q)}(x)}{W^{(q)}(0)} \left(1 - E_0 \left[e^{-q\tau_0^-} E_{X_{\tau_0^-}} \left[e^{-q\tau_0^+} ; \tau_0^+ < e_d \right] ; \tau_0^- < \tau_x^+ \right] \right)$$

$$- \frac{W^{(q)}(x)}{W^{(q)}(0)} \left(1 - E_0 \left[e^{-q\tau_0^-} E_{X_{\tau_0^-}} \left[e^{-q\tau_0^+} ; \tau_0^+ < e_d \right] ; \tau_0^- < \tau_b^+ \right] \right)$$

$$(16) \qquad = H_d^{(q)}(x) - \frac{W^{(q)}(x)}{W^{(q)}(b)} H_d^{(q)}(b),$$

which implies that

$$E_x \left[e^{-q\tau_b^+} ; \tau_b^+ < \tau_d \right] = \frac{W^{(q)}(x)}{W^{(q)}(b)} + \frac{1}{H_d^{(q)}(b)} \left(H_d^{(q)}(x) - \frac{W^{(q)}(x)}{W^{(q)}(b)} H_d^{(q)}(b) \right)$$

$$= \frac{H_d^{(q)}(x)}{H_d^{(q)}(b)},$$

for $0 \leq x \leq b$.

For $x < 0$, it is immediate that

$$(17) \qquad E_x \left[e^{-q\tau_b^+} ; \tau_b^+ < \tau_d \right] = E_x \left[e^{-q\tau_0^+} ; \tau_0^+ < e_d \right] E_0 \left[e^{-q\tau_b^+} ; \tau_b^+ < \tau_d \right].$$

Combining (17) and (15) completes the proof of (12). □

Remark 2.1. *When e_d is a random variable with a degenerate distribution at 0, i.e., when $e_d = 0$, it is immediate that $H_d^{(q)}(x) = \frac{W^{(q)}(x)}{W^{(q)}(0)}$ for $x \geq 0$ which yields*

$$(18) \qquad E_x \left[e^{-q\tau_b^+} ; \tau_b^+ < \tau_d \right] = \frac{W^{(q)}(x)}{W^{(q)}(b)}, \qquad 0 \leq x \leq b,$$

which is Eq. (8.8) of Kyprianou (2006).

Remark 2.2. *Given that the spectrally negative Lévy insurance risk model is skip-free upwards and possesses the strong Markov property, a passage from level x to level b shall occur with (at least) one visit to an intermediate level $y \in (x, b)$ in the interim. As a result, it can be argued probabilistically that $L_q(x; b)$ shall be of the form (12) (see Gerber et al. (2006)).*

We now consider the Laplace transform of the time of a Parisian exit below 0 before reaching level b.

Lemma 2.2. *For the Lévy insurance risk model with paths of bounded variation,*

$$(19) \qquad E_x\left[e^{-q\tau_d}; \tau_d < \tau_b^+\right] = P_d^{(q)}(x) - \frac{H_d^{(q)}(x)}{H_d^{(q)}(b)} P_d^{(q)}(b), \qquad x \le b,$$

where

$$(20) \qquad P_d^{(q)}(x) = \begin{cases} -\frac{W^{(q)}(x)}{W^{(q)}(0)} E_0\left[e^{-q\tau_0^-} E_{X_{\tau_0^-}}\left[e^{-qe_d}; e_d < \tau_0^+\right]; \tau_0^- < \tau_x^+\right], & x \ge 0, \\ E_x\left[e^{-qe_d}; e_d < \tau_0^+\right], & x < 0. \end{cases}$$

    Proof: For $0 \le x \le b$, one capitalizes on the strong Markov property of the Lévy insurance risk model at the time of the first passage to level b or 0 to obtain

$$E_x\left[e^{-q\tau_d}; \tau_d < \tau_b^+\right] = E_x\left[e^{-q\tau_0^-} E_{X_{\tau_0^-}}\left[e^{-qe_d}; e_d < \tau_0^+\right]; \tau_0^- < \tau_b^+\right]$$

$$(21) \qquad\qquad + E_x\left[e^{-q\tau_0^-} E_{X_{\tau_0^-}}\left[e^{-q\tau_0^+}; \tau_0^+ < e_d\right]; \tau_0^- < \tau_b^+\right] E_0\left[e^{-q\tau_d}; \tau_d < \tau_b^+\right].$$

In particular, for $x = 0$, we have

$$E_0\left[e^{-q\tau_d}; \tau_d < \tau_b^+\right] = \frac{E_0\left[e^{-q\tau_0^-} E_{X_{\tau_0^-}}\left[e^{-qe_d}; e_d < \tau_0^+\right]; \tau_0^- < \tau_b^+\right]}{1 - E_0\left[e^{-q\tau_0^-} E_{X_{\tau_0^-}}\left[e^{-q\tau_0^+}; \tau_0^+ < e_d\right]; \tau_0^- < \tau_b^+\right]}$$

$$= \frac{\frac{W^{(q)}(b)}{W^{(q)}(0)} E_0\left[e^{-q\tau_0^-} E_{X_{\tau_0^-}}\left[e^{-qe_d}; e_d < \tau_0^+\right]; \tau_0^- < \tau_b^+\right]}{\frac{W^{(q)}(b)}{W^{(q)}(0)}\left(1 - E_0\left[e^{-q\tau_0^-} E_{X_{\tau_0^-}}\left[e^{-q\tau_0^+}; \tau_0^+ < e_d\right]; \tau_0^- < \tau_b^+\right]\right)}.$$

Using (13) and (20), one concludes that

$$(22) \qquad\qquad E_0\left[e^{-q\tau_d}; \tau_d < \tau_b^+\right] = -\frac{P_d^{(q)}(b)}{H_d^{(q)}(b)}.$$

Substituting (22) into (21) yields

$$E_x\left[e^{-q\tau_d}; \tau_d < \tau_b^+\right] = E_x\left[e^{-q\tau_0^-} E_{X_{\tau_0^-}}\left[e^{-qe_d}; e_d < \tau_0^+\right]; \tau_0^- < \tau_b^+\right]$$

$$(23) \qquad\qquad - E_x\left[e^{-q\tau_0^-} E_{X_{\tau_0^-}}\left[e^{-q\tau_0^+}; \tau_0^+ < e_d\right]; \tau_0^- < \tau_b^+\right] \frac{P_d^{(q)}(b)}{H_d^{(q)}(b)}.$$

Using sample path arguments, it can be shown that

$$
E_x \left[e^{-q\tau_0^-} E_{X_{\tau_0^-}} \left[e^{-qe_d}; e_d < \tau_0^+ \right]; \tau_0^- < \tau_b^+ \right] = \frac{W^{(q)}(x)}{W^{(q)}(0)} E_0 \left[e^{-q\tau_0^-} E_{X_{\tau_0^-}} \left[e^{-qe_d}; e_d < \tau_0^+ \right]; \tau_0^- < \tau_b^+ \right]
$$

$$
- \frac{W^{(q)}(x)}{W^{(q)}(0)} E_0 \left[e^{-q\tau_0^-} E_{X_{\tau_0^-}} \left[e^{-qe_d}; e_d < \tau_0^+ \right]; \tau_0^- < \tau_x^+ \right]
$$

$$
(24) \qquad = P_d^{(q)}(x) - \frac{W^{(q)}(x)}{W^{(q)}(b)} P_d^{(q)}(b).
$$

Using (16) and (24), (23) becomes

$$
E_x \left[e^{-q\tau_d}; \tau_d < \tau_b^+ \right] = P_d^{(q)}(x) - \frac{W^{(q)}(x)}{W^{(q)}(b)} P_d^{(q)}(b) - \left(\frac{H_d^{(q)}(x)}{H_d^{(q)}(b)} - \frac{W^{(q)}(x)}{W^{(q)}(b)} \right) P_d^{(q)}(b)
$$

$$
(25) \qquad = P_d^{(q)}(x) - \frac{H_d^{(q)}(x)}{H_d^{(q)}(b)} P_d^{(q)}(b).
$$

For $x < 0$, we condition on whether the implementation clock or the first passage to level 0 will occur first. It follows that

$$
(26) \qquad E_x \left[e^{-q\tau_d}; \tau_d < \tau_b^+ \right] = E_x \left[e^{-qe_d}; e_d < \tau_0^+ \right] + E_x \left[e^{-q\tau_0^+}; \tau_0^+ < e_d \right] E_0 \left[e^{-q\tau_d}; \tau_d < \tau_b^+ \right].
$$

Using (22) and (13), (26) becomes

$$
E_x \left[e^{-q\tau_d}; \tau_d < \tau_b^+ \right] = E_x \left[e^{-qe_d}; e_d < \tau_0^+ \right] - H_d^{(q)}(x) \frac{P_d^{(q)}(b)}{H_d^{(q)}(b)}.
$$

From the definition of $P_d^{(q)}(x)$ for $x < 0$, the proof is now complete. $\square$

Remark 2.3. *Assuming that e_d is a random variable with a degenerate distribution at 0, we have*

$$
(27) \qquad P_d^{(q)}(x) = -\frac{W^{(q)}(x)}{W^{(q)}(0)} E_0 \left[e^{-q\tau_0^-}; \tau_0^- < \tau_x^+ \right], \qquad x > 0.
$$

With the help of (9), (27) can be rewritten as

$$
(28) \qquad P_d^{(q)}(x) = Z^{(q)}(x) - \frac{W^{(q)}(x)}{W^{(q)}(0)}.
$$

Substituting (28) and (18) into (19)

$$
E_x \left[e^{-q\tau_d}; \tau_d < \tau_b^+ \right] = Z^{(q)}(x) - \frac{W^{(q)}(x)}{W^{(q)}(0)} - \frac{W^{(q)}(x)}{W^{(q)}(b)} \left(Z^{(q)}(b) - \frac{W^{(q)}(b)}{W^{(q)}(0)} \right)
$$

$$
= Z^{(q)}(x) - \frac{W^{(q)}(x)}{W^{(q)}(b)} Z^{(q)}(b),
$$

for $x \geq 0$, which is Eq. (8.9) of Kyprianou (2006).

In what follows, we characterize $H_d^{(q)}$ and $P_d^{(q)}$ in the representation of the two-sided exit problem when implementation delays are exponentially distributed and mixed Erlang distributed respectively.

2.2. Exponentially distributed implementation delays.

Let e_d be an exponentially distributed random variable with mean $1/\beta$. From Theorem 3.12 of Kyprianou (2006), it is clear that, for $x < 0$,

$$(29) \qquad E_x\left[e^{-q\tau_0^+} ; \tau_0^+ < e_d\right] = e^{\Phi(q+\beta)x}.$$

Substituting (29) into (13) yields

$$(30) \qquad H_d^{(q)}(x) = \begin{cases} \frac{W^{(q)}(x)}{W^{(q)}(0)}\left(1 - E_0\left[e^{-q\tau_0^- + \Phi(q+\beta)X_{\tau_0^-}} ; \tau_0^- < \tau_x^+\right]\right), & x \geq 0, \\ e^{\Phi(q+\beta)x}, & x < 0. \end{cases}$$

Using (11) and (2), it is well known that

$$E_0\left[e^{-q\tau_0^- + \Phi(q+\beta)X_{\tau_0^-}} ; \tau_0^- < \tau_x^+\right] = \beta\left\{\mathcal{T}_{\Phi(q+\beta)}W^{(q)}(0) - \frac{W^{(q)}(0)}{W^{(q)}(x)}\mathcal{T}_{\Phi(q+\beta)}W^{(q)}(x)\right\}$$

$$(31) \qquad\qquad = 1 - \beta\frac{W^{(q)}(0)}{W^{(q)}(x)}\mathcal{T}_{\Phi(q+\beta)}W^{(q)}(x),$$

for $x \geq 0$. Combining (30) and (31) results in

$$(32) \qquad H_d^{(q)}(x) = \begin{cases} \beta\mathcal{T}_{\Phi(q+\beta)}W^{(q)}(x), & x \geq 0, \\ e^{\Phi(q+\beta)x}, & x < 0. \end{cases}$$

As for $P_d^{(q)}$, it is easy to show that, for $x > 0$,

$$E_x\left[e^{-qe_d} ; e_d < \tau_0^+\right] = E\left[e^{-qe_d}\right] - E_x\left[e^{-q\tau_0^+} ; \tau_0^+ < e_d\right]E\left[e^{-qe_d}\right]$$

$$(33) \qquad\qquad = \frac{\beta}{\beta+q}\left(1 - e^{\Phi(q+\beta)x}\right).$$

Substituting (33) into (20) yields

$$(34) \qquad P_d^{(q)}(x) = \begin{cases} -\frac{\beta}{\beta+q}\frac{W^{(q)}(x)}{W^{(q)}(0)}E_0\left[e^{-q\tau_0^-}\left(1 - e^{\Phi(q+\beta)X_{\tau_0^-}}\right) ; \tau_0^- < \tau_x^+\right], & x \geq 0, \\ \frac{\beta}{\beta+q}\left(1 - e^{\Phi(q+\beta)x}\right), & x < 0. \end{cases}$$

With the help of (9) and (31),

$$E_0\left[e^{-q\tau_0^-}\left(1 - e^{\Phi(q+\beta)X_{\tau_0^-}}\right) ; \tau_0^- < \tau_x^+\right] = \left(1 - \frac{W^{(q)}(0)}{W^{(q)}(x)}Z^{(q)}(x)\right) - \left(1 - \beta\frac{W^{(q)}(0)}{W^{(q)}(x)}\mathcal{T}_{\Phi(q+\beta)}W^{(q)}(x)\right)$$

$$= \frac{W^{(q)}(0)}{W^{(q)}(x)}\left(\beta\mathcal{T}_{\Phi(q+\beta)}W^{(q)}(x) - Z^{(q)}(x)\right),$$

which implies that (34) becomes

$$(35) \qquad P_d^{(q)}(x) = \begin{cases} \frac{\beta}{\beta+q}\left(Z^{(q)}(x) - \beta\mathcal{T}_{\Phi(q+\beta)}W^{(q)}(x)\right), & x \geq 0, \\ \frac{\beta}{\beta+q}\left(1 - e^{\Phi(q+\beta)x}\right), & x < 0. \end{cases}$$

By extending the domain of definition of $Z^{(q)}$ such that $Z^{(q)}(x) = 1$ for $x < 0$, one concludes, by comparing (32) and (35), that

$$P_d^{(q)}(x) = \frac{\beta}{\beta+q}\left(Z^{(q)}(x) - H_d^{(q)}(x)\right),$$

which, in turn, yields

$$E_x\left[e^{-q\tau_d}; \tau_d < \tau_b^+\right] = \frac{\beta}{\beta+q}\left(Z^{(q)}(x) - \frac{H_d^{(q)}(x)}{H_d^{(q)}(b)}Z^{(q)}(b)\right).$$

2.3. **Mixed Erlang implementation delays.** We now generalize the results of Section 2.2 by assuming that e_d is mixed Erlang distributed with Laplace transform

$$(36) \qquad \widetilde{f}(s) = C\left(\frac{\beta}{\beta+s}\right),$$

where

$$C(z) = \sum_{n=1}^{r} c_n z^n,$$

with $c_n \geq 0$ for $n = 1, ..., r$ and $\sum_{n=1}^{r} c_n = 1$. Its survival function is given by

$$(37) \qquad \overline{F}(x) = \sum_{n=0}^{r-1} \overline{C}_n \frac{(\beta x)^n}{n!} e^{-\beta x}, \qquad x > 0,$$

where $\overline{C}_n = \sum_{j=n+1}^{r} c_j$. We point out that the integer-valued parameter r can either be finite or infinite. The reader is referred to Tijms (1994) for a proof that any positive and continuous random variable can be approximated arbitrary accurately by a mixed Erlang density and to Willmot and Woo (2007) for an extensive treatment of this class of distributions.

The following lemma will be helpful to characterize both $H_d^{(q)}$ and $P_d^{(q)}$ in our generalization of the two-sided exit problem. The proof is provided in the Appendix. In order to state these results, we first present a combinatorial identity known as *di Bruno's formula* (see e.g. Riordan (1980)).

Proposition 2.1. *(di Bruno's formula) For two functions f and g sufficiently differentiable,*

$$(38) \qquad \frac{d^n}{d\theta^n}f(g(\theta)) = \sum_{i=0}^{n} f^{(i)}(g(\theta))B_{i,n}\left(g^{(1)}(\theta), g^{(2)}(\theta), ..., g^{(n-i+1)}(\theta)\right),$$

where $B_{i,n}(x_1, ..., x_{n-i+1})$ is the Bell polynomial

$$B_{i,n}(x_1, ..., x_{n-i+1}) = \sum \frac{n!}{k_1!\, k_2!\, ...\, k_{n-i+1}!}\left(\frac{x_1}{1!}\right)^{k_1}\left(\frac{x_2}{2!}\right)^{k_2}...\left(\frac{x_{n-i+1}}{(n-i+1)!}\right)^{k_{n-i+1}},$$

with the sum extends over all sequences $k_1, k_2, ..., k_{n-i+1}$ of non-negative integers such that $k_1 + k_2 + ... + k_{n-i+1} = i$ and $k_1 + 2k_2 + ... + (n - i + 1) k_{n-i+1} = n$. By definition, let $B_{0,0}(x_1) = 1$ for all x_1.

Lemma 2.3. *For e_d a mixed Erlang random variable with Laplace transform (36),*

$$(39) \qquad E_x\left[e^{-q\tau_0^+}; \tau_0^+ < e_d\right] = \sum_{i=0}^{r-1} \zeta_i \left\{x^i e^{\Phi(q+\beta)x}\right\}, \qquad x < 0,$$

and

$$(40) \qquad E_x\left[e^{-qe_d}; e_d < \tau_0^+\right] = C\left(\frac{\beta}{\beta+q}\right) - \sum_{i=0}^{r-1} \chi_i \left\{x^i e^{\Phi(q+\beta)x}\right\}, \qquad x < 0,$$

where

$$\zeta_i = \sum_{n=i}^{r-1} \overline{C}_n \vartheta_{i,n},$$

$$\chi_i = \sum_{j=i}^{r-1} \vartheta_{i,j} \sum_{n=j+1}^{r} c_n \left(\frac{\beta}{\beta+q}\right)^{n-j},$$

and

$$\vartheta_{i,n} = \frac{\beta^n}{n!}(-1)^n B_{i,n}\left(\Phi^{(1)}(q+\beta), ..., \Phi^{(n-i+1)}(q+\beta)\right).$$

Let

$$b_{r,l,i} = q\mathbf{1}_{\{l=i\}} - \binom{i}{l}\psi^{(i-l)}(r).$$

An explicit expression for $H_d^{(q)}$ and $P_d^{(q)}$ is presented in the following proposition. The reader is referred to the Appendix for the proof of this result.

Proposition 2.2. *When e_d has Laplace transform (36),*

(a)

$$(41) \qquad H_d^{(q)}(x) = \begin{cases} \sum_{l=0}^{r-1} \nu_l \left\{\mathcal{T}_{\Phi(q+\beta)}^{l+1} W^{(q)}(x)\right\}, & x \geq 0, \\ \sum_{i=0}^{r-1} \zeta_i \left\{x^i e^{\Phi(q+\beta)x}\right\}, & x < 0, \end{cases}$$

where

$$(42) \qquad \nu_l = (-1)^{l+1}(l!) \sum_{i=l}^{r-1} \zeta_i b_{\Phi(q+\beta),l,i}.$$

(b)

$$P_d^{(q)}(x) = \begin{cases} \frac{\chi_0 - C\left(\frac{\beta}{\beta+q}\right)}{W^{(q)}(0)} W^{(q)}(x) + C\left(\frac{\beta}{\beta+q}\right) Z^{(q)}(x) + \sum_{l=0}^{r-1} \varsigma_l \left\{\mathcal{T}_{\Phi(q+\beta)}^{l+1} W^{(q)}(x)\right\}, & x \geq 0, \\ C\left(\frac{\beta}{\beta+q}\right) - \sum_{i=0}^{r-1} \chi_i \left\{x^i e^{\Phi(q+\beta)x}\right\}, & x < 0, \end{cases}$$

(43)

where

$$\varsigma_l = (-1)^l \, (l!) \sum_{i=l}^{r-1} \chi_i b_{\Phi(q+\beta),l,i}.$$

3. LAPLACE TRANSFORM OF THE TIME TO RUIN

In this section, we consider the analysis of the Laplace transform of the time to ruin, namely

$$\phi_q(x) = E_x \left[e^{-q\tau_d}; \tau_d < \infty \right],$$

in the Lévy insurance risk model with paths of bounded variation. We rely heavily on the two-sided exit problem studied in Section 3 given that

$$\phi_q(x) = \lim_{b \to \infty} E_x \left[e^{-q\tau_d}; \tau_d < \tau_b^+ \right].$$

Corollary 3.1. *The Laplace transform of the Parisian ruin time can be expressed as*

$$(44) \qquad \phi_q(x) = P_d^{(q)}(x) - \sigma_q H_d^{(q)}(x),$$

where

$$(45) \qquad \sigma_q = \lim_{x \to \infty} \frac{P_d^{(q)}(x)}{H_d^{(q)}(x)}.$$

From their definitions, we easily see that $P_d^{(q)}(0) = 0$ and $H_d^{(q)}(0) = 1$. As a consequence, we have

$$\phi_q(0) = -\sigma_q$$

and then the probability of Parisian ruin when starting from zero is given by

$$P_0 \{ \tau_d < \infty \} = -\sigma_0.$$

Remark 3.1. *For e_d a degenerate random variable at 0, we recall that $H_d^{(q)}(x) = W^{(q)}(x) / W^{(q)}(0)$ and $P_d^{(q)}(x) = Z^{(q)}(x) - W^{(q)}(x) / W^{(q)}(0)$. From Exercise 8.5 in Kyprianou (2006), it is known that*

$$(46) \qquad \lim_{x \to \infty} \frac{Z^{(q)}(x)}{W^{(q)}(x)} = \frac{q}{\Phi(q)},$$

which implies that

$$\sigma_q = W^{(q)}(0) \frac{q}{\Phi(q)} - 1.$$

One concludes that

$$\phi_q(x) = Z^{(q)}(x) - \frac{q}{\Phi(q)} W^{(q)}(x),$$

for $x \geq 0$, therefore recovering Equation (5).

We revisit the two examples of Section 3 to identify σ_q under those distributional assumptions. But first, an identity of particular interest in the sequel is proved. Indeed, it is known (see Kyprianou (2006)) that the scale function $W^{(q)}$ satisfies $W^{(q)}(x) = e^{\Phi(q)x} W(x)$ where $W(x)$ is

a non-decreasing and bounded function, the latter being provided by the net profit condition. As a result,

$$\frac{\mathcal{T}_r^l W^{(q)}(x)}{W^{(q)}(x)} = \int_0^\infty \frac{y^{l-1}e^{-ry}}{(l-1)!} \frac{W^{(q)}(x+y)}{W^{(q)}(x)} dy$$

$$= \int_0^\infty \frac{y^{l-1}e^{-(r-\Phi(q))y}}{(l-1)!} \frac{W(x+y)}{W(x)} dy.$$

for $l = 1, 2, ...$ and $r > \Phi(q)$. Letting $x \to \infty$ and using the dominated convergence theorem, one concludes that

$$\lim_{x\to\infty} \frac{\mathcal{T}_r^l W^{(q)}(x)}{W^{(q)}(x)} = \int_0^\infty \frac{y^{l-1}e^{-(r-\Phi(q))y}}{(l-1)!} dy$$

$$\tag{47} = \left(\frac{1}{r - \Phi(q)}\right)^l,$$

for $l = 1, 2, ...$ and $r > \Phi(q)$.

3.1. **Exponentially distributed implementation delays.** When e_d is exponentially distributed with mean $1/\beta$, it is known from (32) and (35) that

$$\tag{48} \sigma_q = \lim_{x\to\infty} \frac{\frac{\beta}{\beta+q}\left(Z^{(q)}(x) - \beta\mathcal{T}_{\Phi(q+\beta)}W^{(q)}(x)\right)}{\beta\mathcal{T}_{\Phi(q+\beta)}W^{(q)}(x)}.$$

From (46) and (47), (48) becomes

$$\sigma_q = \frac{\frac{1}{\beta+q}\left(\frac{q}{\Phi(q)} - \frac{\beta}{\Phi(q+\beta)-\Phi(q)}\right)}{\frac{1}{\Phi(q+\beta)-\Phi(q)}}$$

$$= \frac{q\Phi(q+\beta) - (\beta+q)\Phi(q)}{(\beta+q)\Phi(q)}.$$

Therefore, we get the following expression for the probability of Parisian ruin when $X_0 = 0$:

Corollary 3.2. *If the net profit condition is satisfied, then*

$$\phi_q(0) = 1 - \frac{q}{\beta+q}\left(\frac{\Phi(\beta+q)}{\Phi(q)}\right)$$

and

$$P_0\{\tau_d < \infty\} = 1 - \psi'(0+)\frac{\Phi(\beta)}{\beta}.$$

Proof. The first result follows from the previous discussion and calculations. The second result is a consequence of the first one and the following fact:

$$\lim_{q\to 0} \frac{q}{\Phi(q)} = \psi'(0+).$$

$\square$

3.2. **Mixed Erlang implementation delays.** Similarly, when e_d is a mixed Erlang random variable with Laplace transform (36), it is known from Proposition 2.2 that

$$(49) \qquad \sigma_q = \lim_{x \to \infty} \frac{\frac{\chi_0 - C\left(\frac{\beta}{\beta+q}\right)}{W^{(q)}(0)} W^{(q)}(x) + C\left(\frac{\beta}{\beta+q}\right) Z^{(q)}(x) + \sum_{l=0}^{r-1} \varsigma_l \left\{ \mathcal{T}_{\Phi(q+\beta)}^{l+1} W^{(q)}(x) \right\}}{\sum_{l=0}^{r-1} \nu_l \left\{ \mathcal{T}_{\Phi(q+\beta)}^{l+1} W^{(q)}(x) \right\}}.$$

From (46) and (47), (49) becomes

$$\sigma_q = \frac{\frac{\chi_0 - C\left(\frac{\beta}{\beta+q}\right)}{W^{(q)}(0)} + C\left(\frac{\beta}{\beta+q}\right) \frac{q}{\Phi(q)} + \sum_{l=0}^{r-1} \varsigma_l \left(\frac{1}{\Phi(q+\beta) - \Phi(q)} \right)^{l+1}}{\sum_{l=0}^{r-1} \nu_l \left(\frac{1}{\Phi(q+\beta) - \Phi(q)} \right)^{l+1}}.$$

4. Numerical example

In this section, we focus on the calculation of the probability of Parisian ruin within an infinite-time horizon (namely ϕ_0). We consider a deterministic implementation delay (say T) for the recognition of ruin, and approximate this deterministic time T by a sequence of Erlang distributed implementation delays with mean T and variance T^2/n (for n a positive integer). We show numerically the convergence of these Parisian ruin probabilities (to the Parisian ruin probability with a deterministic implementation delay) as n goes to infinity.

For illustrative purposes, we consider the classical compound Poisson risk model. We assume that claims arrive at rate $1/3$, and claim sizes are exponentially distributed with mean 9. Incoming premiums are collected at a rate of 4 per unit time. In the following two tables, we display the values of Parisian ruin probability for an initial surplus of $X_0 = 0$ and $X_0 = 50$ respectively. For comparative measures, we indicate that the traditional ruin probability (i.e. Parisian ruin probability with $T = 0$) is 0.7500 and 0.1870 respectively for these two initial surplus values.

n	Parisian Ruin Probability			
	T=0	T=1	T=2	T=5
1	0.6886	0.6478	0.5676	0.4867
5	0.6767	0.6195	0.5020	0.3879
10	0.6748	0.6144	0.4910	0.3737
15	0.6741	0.6126	0.4873	0.3690
20	0.6737	0.6117	0.4854	0.3667
25	0.6735	0.6112	0.4842	0.3653
30	0.6733	0.6108	0.4835	0.3644
35	0.6732	0.6105	0.4829	0.3637
40	0.6732	0.6103	0.4825	0.3633
45	0.6731	0.6102	0.4822	0.3629
50	0.6731	0.6100	0.4820	0.3626
55	0.6730	0.6099	0.4817	0.3623

n	Parisian Ruin Probability			
	T=0	T=1	T=2	T=5
1	0.1717	0.1615	0.1415	0.1213
5	0.1687	0.1545	0.1252	0.0967
10	0.1683	0.1532	0.1224	0.0932
15	0.1681	0.1528	0.1215	0.0920
20	0.1680	0.1525	0.1210	0.0914
25	0.1679	0.1524	0.1207	0.0911
30	0.1679	0.1523	0.1206	0.0909
35	0.1679	0.1522	0.1204	0.0907
40	0.1679	0.1522	0.1203	0.0906
45	0.1679	0.1521	0.1202	0.0905
50	0.1678	0.1521	0.1202	0.0904
55	0.1678	0.1521	0.1201	0.0903

REFERENCES

S. Asmussen, F. Avram, and M. Usabel. Erlangian approximations for finite-horizon ruin probabilities. *Astin Bull.*, 32(2):267–281, 2002.

E. Biffis and A. E. Kyprianou. A note on scale functions and the time value of ruin for Lévy insurance risk processes. *Insurance Math. Econom.*, 46(1):85–91, 2010.

M. Chesney, M. Jeanblanc-Picqué, and M. Yor. Brownian excursions and Parisian barrier options. *Adv. in Appl. Probab.*, 29(1):165–184, 1997.

A. Dassios and S. Wu. Parisian ruin with exponential claims. *submitted*, 2009.

H. U. Gerber and E. S. W. Shiu. On the time value of ruin. *N. Am. Actuar. J.*, 2(1):48–78, 1998.

H. U. Gerber, X. S. Lin, and H. Yang. A note on the dividends-penalty identity and the optimal dividend barrier. *Astin Bull.*, 36(2):489–503, 2006.

F. Hubalek and A. E. Kyprianou. Old and new examples of scale functions for spectrally negative Lévy processes. *Sixth Seminar on Stochastic Analysis, Random Fields and Applications, eds R. Dalang, M. Dozzi, F. Russo. Progress in Probability, Birkhäuser*, 2010.

A. E. Kyprianou. *Introductory lectures on fluctuations of Lévy processes with applications*. Universitext. Springer-Verlag, 2006.

A. E. Kyprianou and M. R. Pistorius. Perpetual options and Canadization through fluctuation theory. *Ann. Appl. Probab.*, 13(3):1077–1098, 2003.

A. E. Kyprianou and V. Rivero. Special, conjugate and complete scale functions for spectrally negative Lévy processes. *Electron. J. Probab.*, 13:1672–1701, 2008.

S. Li and J. Garrido. On ruin for the Erlang(n) risk process. *Insurance Math. Econom.*, 34(3): 391–408, 2004.

S. Li and J. Garrido. On a general class of renewal risk process: analysis of the Gerber-Shiu function. *Adv. in Appl. Probab.*, 37(3):836–856, 2005.

V. Ramaswami, D. G. Woolford, and D. A. Stanford. The Erlangization method for Markovian fluid flows. *Ann. Oper. Res.*, 160:215–225, 2008.

J. Riordan. *An introduction to combinatorial analysis*. Princeton University Press, 1980.

B. A. Surya. Evaluating scale functions of spectrally negative Lévy processes. *J. Appl. Probab.*, 45(1):135–149, 2008.

H. C. Tijms. *Stochastic models.* John Wiley & Sons Ltd., 1994.

G. E. Willmot. On the discounted penalty function in the renewal risk model with general interclaim times. *Insurance Math. Econom.*, 41(1):17–31, 2007.

G. E. Willmot and J.-K. Woo. On the class of Erlang mixtures with risk theoretic applications. *N. Am. Actuar. J.*, 11(2):99–115, 2007.

5. Appendix

5.1. **Proof of (39) in Lemma 2.3.** Using the mixed Erlang survival function (37), one readily finds

$$
E_x\left[e^{-q\tau_0^+};\tau_0^+ < e_d\right] = E_x\left[e^{-q\tau_0^+}\left(\sum_{n=0}^{r-1}\overline{C}_n\frac{(\beta\tau_0^+)^n}{n!}e^{-\beta\tau_0^+}\right);\tau_0^+ < \infty\right]
$$

$$
= \sum_{n=0}^{r-1}\overline{C}_n\frac{\beta^n}{n!}E_x\left[\left(\tau_0^+\right)^n e^{-(q+\beta)\tau_0^+};\tau_0^+ < \infty\right]
$$

$$
= \sum_{n=0}^{r-1}\overline{C}_n\frac{\beta^n}{n!}(-1)^n E_x\left[\frac{d^n}{d\xi^n}e^{-\xi\tau_0^+};\tau_0^+ < \infty\right]\Bigg|_{\xi=q+\beta},
$$

for $x \geq 0$. Interchanging the order of the expectation sign and the n-th derivative, one finds that

$$
(50) \qquad E_x\left[e^{-q\tau_0^+};\tau_0^+ < e_d\right] = \sum_{n=0}^{r-1}\overline{C}_n\frac{\beta^n}{n!}(-1)^n \frac{d^n}{d\xi^n}e^{\Phi(\xi)x}\Bigg|_{\xi=q+\beta}
$$

Using *di Bruno's formula*, (50) becomes
(51)
$$
E_x\left[e^{-q\tau_0^+};\tau_0^+ < e_d\right] = e^{\Phi(q+\beta)x}\sum_{n=0}^{r-1}\overline{C}_n\frac{\beta^n}{n!}(-1)^n\sum_{i=0}^{n}x^i B_{i,n}\left(\Phi^{(1)}(q+\beta),...,\Phi^{(n-i+1)}(q+\beta)\right),
$$

Interchanging the order of summation in (51) leads to (39).

5.2. **Proof of (40) in Lemma 2.3.** Using the mixed Erlang density followed by a series of simple manipulations, one arrives at

$$
E_x\left[e^{-qe_d};e_d < \tau_0^+\right] = \sum_{n=1}^{r}c_n E_x\left[e^{-qe_{n,\beta}};e_{n,\beta} < \tau_0^+\right]
$$

$$
= \sum_{n=1}^{r}c_n\left(E\left[e^{-qe_{n,\beta}}\right] - E_x\left[e^{-qe_{n,\beta}};\tau_0^+ < e_{n,\beta}\right]\right)
$$

$$
(52) \qquad = \sum_{n=1}^{r}c_n\left(E\left[e^{-qe_{n,\beta}}\right] - \sum_{j=0}^{n-1}E_x\left[e^{-qe_{n,\beta}};e_{j,\beta} < \tau_0^+ < e_{j+1,\beta}\right]\right),
$$

where $e_{j,\beta}$ is an Erlang-j random variable with density

$$f(y) = \frac{\beta^j y^{j-1} e^{-\beta y}}{(j-1)!}, \qquad y > 0,$$

(with $e_{0,\beta}$ a degenerate random variable at 0). Given that the Lévy insurance risk model is skip-free upwards and due to the memoryless property of the exponential distribution, we have

$$(53) \qquad E_x\left[e^{-qe_{n,\beta}}; e_{j,\beta} < \tau_0^+ < e_{j+1,\beta}\right] = E_x\left[e^{-q\tau_0^+} E\left[e^{-qe_{n-j,\beta}}\right]; e_{j,\beta} < \tau_0^+ < e_{j+1,\beta}\right],$$

where

$$(54) \qquad\qquad E\left[e^{-qe_{j,\mu}}\right] = \left(\frac{\beta}{\beta+q}\right)^j, \qquad j = 0, 1, 2, ...$$

Substituting (53) and (54) into (52) yields

$$E_x\left[e^{-qe_d}; e_d < \tau_0^+\right] = C\left(\frac{\beta}{\beta+q}\right)$$

$$- \sum_{n=1}^r c_n \sum_{j=0}^{n-1} \left(\frac{\beta}{\beta+q}\right)^{n-j} E_x\left[e^{-q\tau_0^+}; e_{j,\beta} < \tau_0^+ < e_{j+1,\beta}\right].$$

(55)

Using (39) with

$$\overline{C}_n = \begin{cases} 1, & n = 0, 1, ..., j, \\ 0, & n = j+1, j+2, ..., \end{cases}$$

and

$$\overline{C}_n = \begin{cases} 1, & n = 0, 1, ..., j-1, \\ 0, & n = j, j+1, ..., \end{cases}$$

respectively, one deduces that

$$E_x\left[e^{-q\tau_0^+}; e_{j,\beta} < \tau_0^+ < e_{j+1,\beta}\right] = E_x\left[e^{-q\tau_0^+}; \tau_0^+ < e_{j+1,\beta}\right] - E_x\left[e^{-q\tau_0^+}; \tau_0^+ < e_{j,\beta}\right]$$

$$(56) \qquad\qquad\qquad = \sum_{i=0}^j \vartheta_{i,j} \left\{x^i e^{\Phi(q+\beta)x}\right\}.$$

Substituting (56) into (55) (followed by some simple manipulations) yields (40).

5.3. **Proof of Proposition 2.2.** For $x < 0$, (41) and (43) follows immediately from Lemma 2.3 together with (13) and (20) respectively. For $x \geq 0$, the results of Lemma 2.3 allows to rewrite (13) and (20) respectively as

$$(57) \qquad H_d^{(q)}(x) = \frac{W^{(q)}(x)}{W^{(q)}(0)}\left(1 - \sum_{i=0}^{r-1} \zeta_i\, E_0\left[\left(X_{\tau_0^-}\right)^i e^{-q\tau_0^- + \Phi(q+\beta)X_{\tau_0^-}}; \tau_0^- < \tau_x^+\right]\right),$$

and

$$P_d^{(q)}(x) = \frac{W^{(q)}(x)}{W^{(q)}(0)} \sum_{i=0}^{r-1} \chi_i E_0\left[\left(X_{\tau_0^-}\right)^i \left\{e^{-q\tau_0^- + \Phi(\beta+q)X_{\tau_0^-}}\right\}; \tau_0^- < \tau_x^+\right]$$

$$(58) \qquad - \frac{W^{(q)}(x)}{W^{(q)}(0)} C\left(\frac{\beta}{\beta+q}\right) E_0\left[e^{-q\tau_0^-}; \tau_0^- < \tau_x^+\right].$$

An expression for $E_0\left[\left(X_{\tau_0^-}\right)^i e^{-q\tau_0^- + rX_{\tau_0^-}}; \tau_0^- < \tau_x^+\right]$ for $r > \Phi(q)$ is needed to further explicit (57) and (58).

For $i = 0$, (11) immediately yields

$$(59) \qquad E_0\left[e^{-q\tau_0^- + rX_{\tau_0^-}}; \tau_0^- < \tau_x^+\right] = 1 - W^{(q)}(0)(\psi(r) - q)\frac{\mathcal{T}_r W^{(q)}(x)}{W^{(q)}(x)}.$$

Otherwise, for $i = 1, 2, \ldots$,

$$E_0\left[\left(X_{\tau_0^-}\right)^i e^{-q\tau_0^- + rX_{\tau_0^-}}; \tau_0^- < \tau_x^+\right] = \frac{d^i}{d\xi^i} E_0\left[e^{-q\tau_0^- + \xi X_{\tau_0^-}}; \tau_0^- < \tau_x^+\right]\Bigg|_{\xi=r}$$

$$(60) \qquad\qquad\qquad = W^{(q)}(0)\frac{d^i}{d\xi^i}\left((q - \psi(\xi))\frac{\mathcal{T}_\xi W^{(q)}(x)}{W^{(q)}(x)}\right)\Bigg|_{\xi=r}.$$

From Property 5 of the Dickson-Hipp operator on page 393 of Li and Garrido (2004), namely

$$\frac{d^l}{d\xi^l}\mathcal{T}_\xi W^{(q)}(x) = (-1)^l (l!)\,\mathcal{T}_\xi^{l+1} W^{(q)}(x), \qquad l = 0, 1, \ldots,$$

(60) becomes

$$(61) \qquad E_0\left[\left(X_{\tau_0^-}\right)^i e^{-q\tau_0^- + rX_{\tau_0^-}}; \tau_0^- < \tau_x^+\right] = W^{(q)}(0)\sum_{l=0}^{i} b_{r,l,i}\frac{(-1)^l (l!)\,\mathcal{T}_r^{l+1} W^{(q)}(x)}{W^{(q)}(x)},$$

for $r > \Phi(q)$ where

$$b_{r,l,i} = q\mathbf{1}_{\{l=i\}} - \binom{i}{l}\psi^{(i-l)}(r).$$

Then, substituting (59) and (61) into (57) and using the fact that $\zeta_0 = 1$, one concludes that

$$H_d^{(q)}(x) = \frac{W^{(q)}(x)}{W^{(q)}(0)}\left(1 - \zeta_0\left(1 - \beta W^{(q)}(0)\frac{\mathcal{T}_{\Phi(q+\beta)} W^{(q)}(x)}{W^{(q)}(x)}\right)\right)$$

$$- \sum_{i=1}^{r-1} \zeta_i \sum_{l=0}^{i} b_{\Phi(q+\beta),l,i}\left\{(-1)^l (l!)\,\mathcal{T}_{\Phi(q+\beta)}^{l+1} W^{(q)}(x)\right\}$$

$$= \beta\mathcal{T}_{\Phi(q+\beta)} W^{(q)}(x) + \sum_{i=1}^{r-1} \zeta_i \sum_{l=0}^{i} b_{\Phi(q+\beta),l,i}\left\{(-1)^{l+1} (l!)\,\mathcal{T}_{\Phi(q+\beta)}^{l+1} W^{(q)}(x)\right\}$$

$$(62) \qquad = \sum_{i=0}^{r-1} \zeta_i \sum_{l=0}^{i} b_{\Phi(q+\beta),l,i}\left\{(-1)^{l+1} (l!)\,\mathcal{T}_{\Phi(q+\beta)}^{l+1} W^{(q)}(x)\right\}.$$

Interchanging the order of summation, (62) has the equivalent representation given in (41). Using a similar line of logic, (43) can be obtained from (58) through (59), (61) and (9). The proof is therefore omitted.

5.4. **Derivatives of $\Phi(\theta)$.** In this section, we propose a recursive formula for the calculation of the derivatives of the inverse of the Laplace exponent, namely $\Phi(\theta)$. Under the positive security loading condition, we have

$$(63) \qquad \theta = \psi(\Phi(\theta)).$$

Using the chain rule, the differentiation of (63) w.r.t. θ yields

$$1 = \psi'(\Phi(\theta))\Phi'(\theta),$$

or equivalently

$$\Phi'(\theta) = \frac{1}{\psi'(\Phi(\theta))}.$$

In general, the same procedure can be repeated to obtain the higher-order derivatives of $\Phi(\theta)$. Indeed, using di Bruno's formula (see Eq. (38)), we have

$$(64) \qquad \frac{d^n}{d\theta^n}\psi(\Phi(\theta)) = \sum_{i=1}^{n}\psi^{(i)}(\Phi(\theta))B_{i,n}\left(\Phi^{(1)}(\theta),\Phi^{(2)}(\theta),...,\Phi^{(n-i+1)}(\theta)\right),$$

where $B_{i,n}$ is the Bell polynomial. It is easy to show that $B_{1,n}(x_1,...,x_n) = x_n$ which permits to re-write (64) as
(65)
$$\frac{d^n}{d\theta^n}\psi(\Phi(\theta)) = \psi^{(1)}(\Phi(\theta))\Phi^{(n)}(\theta) + \sum_{i=2}^{n}\psi^{(i)}(\Phi(\theta))B_{i,n}\left(\Phi^{(1)}(\theta),\Phi^{(2)}(\theta),...,\Phi^{(n-i+1)}(\theta)\right),$$

where it is worth pointing out that the second term on the right-hand side of (65) is a function of the first $(n-1)$ derivatives of $\Phi(\theta)$. For $n \geq 2$, an application of the n-th derivative operator on each side of (63) results in

$$0 = \psi^{(1)}(\Phi(\theta))\Phi^{(n)}(\theta) + \sum_{i=2}^{n}\psi^{(i)}(\Phi(\theta))B_{i,n}\left(\Phi^{(1)}(\theta),\Phi^{(2)}(\theta),...,\Phi^{(n-i+1)}(\theta)\right),$$

or equivalently

$$\Phi^{(n)}(\theta) = \frac{-1}{\psi^{(1)}(\Phi(\theta))}\left(\sum_{i=2}^{n}\psi^{(i)}(\Phi(\theta))B_{i,n}\left(\Phi^{(1)}(\theta),\Phi^{(2)}(\theta),...,\Phi^{(n-i+1)}(\theta)\right)\right)$$

$$(66) \qquad = \left(-\Phi'(\theta)\right)\sum_{i=2}^{n}\psi^{(i)}(\Phi(\theta))B_{i,n}\left(\Phi^{(1)}(\theta),\Phi^{(2)}(\theta),...,\Phi^{(n-i+1)}(\theta)\right).$$

Department of Statistics and Actuarial Science, University of Waterloo, 200 University Avenue West, Waterloo (Ontario) N2L 3G1, Canada
E-mail address: `dlandria@math.uwaterloo.ca`

Département de mathématiques, UQAM, 201 av. Président-Kennedy, Montréal (Québec) H2X 3Y7, Canada
E-mail address: `renaud.jf@uqam.ca`

Department of Mathematics and Statistics, Concordia University, 1455 de Maisonneuve Blvd W., Montréal (Québec) H3G 1M8, Canada
E-mail address: `xzhou@mathstat.concordia.ca`