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An integrable decomposition of the Manakov equation

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Abstract. An integrable decomposition of the Manakov equation is presented. A pair of new finite-dimensional integrable Hamiltonian systems which constitute the integrable decomposition of the Manakov equation are obtained.

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Key words: binary nonlinearization of the spectral problem, the 4-component AKNS equation, the Manakov equation, integrable decomposition.

1 Introduction

The Manakov equation

$$\begin{cases} q_{1t} = \dfrac{i}{3}\big(q_{1xx} + 2(|q_1|^2 + |q_2|^2)q_1\big), \\[2ex] q_{2t} = \dfrac{i}{3}\big(q_{2xx} + 2(|q_1|^2 + |q_2|^2)q_2\big), \end{cases} \tag{1}$$

where q_1, q_2 are potentials, is nothing but the 2-component vector nonlinear Schrödinger equation and sometimes is referred to as the coupled nonlinear Schrödinger equation. Manakov first examined equation (1) as an asymptotic model for the propagation of the electric field in a waveguide [1]. Subsequently, the system (1) was derived as a key model for lightwave propagation in optical

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fibers [2]. The system admits vector-soliton solutions, the solution collision is elastic and the dynamics of soliton interactions can be explicitly computed [3].

The nonlinearization of spectral problem (NSP), which was put forward by Cao [4, 5], is a powerful tool to study integrable systems in (1+1)-dimensions. With it we can generate new finite-dimensional integrable Hamiltonian systems, decompose (1+1) dimensional integrable systems into a pair of compatible finite-dimensional integrable Hamiltonian systems, and thus construct explicit or numerical solutions of the (1+1) dimensional integrable systems. During the past two decades, many powerful techniques of generalizations of the NSP have been obtained. For example, the binary NSP, which was presented for the first time by Ma and Strampp [6], has been studied [7, 8]. After that the higher-order symmetry constraint method [9] and binary nonlinearization of spectral problems under higher-order symmetry constraints [10] were discussed. The nonlinearization and binary nonlinearization of the discrete eigenvalue problem were introduced [11, 12], respectively. And then Ma and Zhou proposed the adjoint symmetry constraint method [13, 14]. The adjoint symmetry constraint can be used to solve the multicomponent AKNS equations associated with degenerate spectral problems. Because the spectral matrix of the Manakov equation is degenerate, general methods of the NSPs are impossible for the success of making integrable decompositions. Therefore, the introduction of the adjoint symmetry is very crucial for the integrable decomposition of the Manakov equation.

In this paper, we will present an integrable decomposition of the Manakov equation. As is well known, equation (1) can be reduced from the 4-component AKNS equation by imposing the reality condition $r = -q^{\dagger}$. Therefore, we can make the integrable decomposition of equation (1) through the following procedure: couple the spectral problems of equation (1) with their complex conjugates and then reduce these problems to that of the 4-component AKNS equation by introducing new variables. The paper is organized as follows. Firstly, we recall the integrable decomposition of the 4-component AKNS equation with the help of the adjoint symmetry constraints and the binary NSP. Secondly, on the basis of section 2, we study the integrable decomposition of the Manakov equation and finally obtain a pair of finite-dimensional completely integrable Hamiltonian systems.

2 An integrable decomposition of the 4-component AKNS equation

In Ref. [15] the multi-wave interaction equations associated with the 3×3 matrix AKNS spectral problem were decomposed into finite-dimensional Liouville integrable Hamiltonian systems by carrying out binary Bargmann symmetry constraints. As a special case, in this section, following [13, 15] we review the integrable decomposition of the 4-component AKNS equation.

2.1 The 4-component AKNS hierarchy of equations

It is well known that the 4-component AKNS hierarchy of equations is associated with the following spectral problem

$$\Phi_x = U(u,\lambda)\Phi, \qquad U(u,\lambda) = \begin{pmatrix} -2i\lambda & q_1 & q_2 \\ r_1 & i\lambda & 0 \\ r_2 & 0 & i\lambda \end{pmatrix}, \tag{2}$$

where λ is a spectral parameter and q_1, q_2, r_1, r_2 are four potentials,

$$\Phi = (\phi_1, \phi_2, \phi_3)^T, \quad u = (q_1, q_2, r_1, r_2)^T, \quad q = (q_1, q_2), \quad r = (r_1, r_2)^T.$$

From the adjoint representation equation

$$V_x = [U, V] = UV - VU, \tag{3}$$

with

$$V = \sum_{k=0}^{\infty} \begin{pmatrix} a^{(k)} & b^{(k)} \\ c^{(k)} & d^{(k)} \end{pmatrix} \lambda^{-k},$$

$$b^{(k)} = \left(b_1^{(k)}, b_2^{(k)}\right),\ c^{(k)} = \left(c_1^{(k)}, c_2^{(k)}\right)^T,\ d^{(k)} = \left(d_{ij}^{(k)}\right)_{2\times 2},$$

and then through a tedious and straightforward calculation, we work out

$$b_i^{(1)} = q_i, \quad c_i^{(1)} = r_i, \quad a^{(1)} = 0, \quad d_{ij}^{(1)} = 0,$$

$$b_i^{(2)} = -\frac{1}{3i} q_{i,x}, \quad c_i^{(2)} = \frac{1}{3i} r_{i,x},$$

$$a^{(2)} = \frac{1}{3i}(q_1 r_1 + q_2 r_2), \quad d_{ij}^{(2)} = -\frac{1}{3i} r_i q_j,$$

$$b_i^{(3)} = -\frac{1}{9}\big[q_{i,xx} - 2(q_1r_1 + q_2r_2)q_i\big],$$

$$c_i^{(3)} = -\frac{1}{9}\big[r_{i,xx} - 2(q_1r_1 + q_2r_2)r_i\big],$$

where $1 \le i, j \le 2$.

Consider the following auxiliary problem

$$\Phi_{t_n} = V^{(n)}\Phi, \quad V^{(n)} = (\lambda^n V)_+, \quad n \ge 0. \tag{4}$$

From the compatibility condition of (2) and (4)

$$U_t - V_x^{(n)} + [U, V^{(n)}] = 0,$$

we obtain the 4-component AKNS hierarchy of equations

$$u_{t_n} = \begin{pmatrix} q^T \\ r \end{pmatrix}_{t_n} = \begin{pmatrix} -3ib^{(n+1)T} \\ 3ic^{(n+1)} \end{pmatrix}. \tag{5}$$

The first nontrivial equation or the 4-component AKNS equation is

$$\begin{cases} q_{j,t_2} = -\dfrac{1}{3i}\big[q_{j,xx} - 2(q_1r_1 + q_2r_2)q_j\big], \\ r_{j,t_2} = \dfrac{1}{3i}\big[r_{j,xx} - 2(q_1r_1 + q_2r_2)r_j\big], \end{cases} \quad 1 \le j \le 2, \tag{6}$$

which admits spectral problem (2) and

$$\Phi_{t_2} = V^{(2)}(q, r, \lambda)\Phi, \tag{7}$$

where

$$\begin{aligned} V^{(2)}(u, \lambda) \; = \; & \begin{pmatrix} -2i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix}\lambda^2 + \begin{pmatrix} 0 & q_1 & q_2 \\ r_1 & 0 & 0 \\ r_2 & 0 & 0 \end{pmatrix}\lambda \\ & + \frac{i}{3}\begin{pmatrix} -(q_1r_1 + q_2r_2) & q_{1x} & q_{2x} \\ -r_{1x} & r_1q_1 & r_1q_2 \\ -r_{2x} & r_2q_1 & r_2q_2 \end{pmatrix}. \end{aligned} \tag{8}$$

2.2 Adjoint symmetry constraint

In order to preform the binary NSPs of the 4-component AKNS hierarchy, we take N distinct eigenvalue parameters λ_j, $1 \leq j \leq N$, and N copies of the 3×3 matrix spectral problem for AKNS hierarchy (2) and their adjoint spectral problems as follows

$$\begin{cases} \phi_{1j,x} = -2i\lambda_j\phi_{1j} + q_1\phi_{2j} + q_2\phi_{3j}, \\ \phi_{2j,x} = i\lambda_j\phi_{2j} + r_1\phi_{1j}, \\ \phi_{3j,x} = i\lambda_j\phi_{3j} + r_2\phi_{1j}, \\ \psi_{1j,x} = 2i\lambda_j\psi_{1j} - r_1\psi_{2j} - r_2\psi_{3j}, \\ \psi_{2j,x} = -i\lambda_j\psi_{2j} - q_1\psi_{1j}, \\ \psi_{3j,x} = -i\lambda_j\psi_{3j} - q_2\psi_{1j}, \end{cases} \quad 1 \leq j \leq N, \tag{9}$$

which can be written as the following compact form

$$\begin{cases} \phi_{1x} = -2iA\phi_1 + q_1\phi_2 + q_2\phi_3, \\ \phi_{2x} = iA\phi_2 + r_1\phi_1, \\ \phi_{3x} = iA\phi_3 + r_2\phi_1, \\ \psi_{1x} = 2iA\psi_1 - r_1\psi_2 - r_2\psi_3, \\ \psi_{2x} = -iA\psi_2 - q_1\psi_1, \\ \psi_{3x} = -iA\psi_3 - q_2\psi_1, \end{cases} \tag{10}$$

where

$$\phi_j = (\phi_{j1}, \ldots, \phi_{jN})^T, \ \psi_j = (\psi_{j1}, \ldots, \psi_{jN})^T, \ A = \text{diag}(\lambda_1, \ldots, \lambda_N).$$

Following [13], we consider the following Bargmann adjoint symmetry constraint

$$q_j = \frac{3i}{\gamma_{j+1} - \gamma_1}\langle\phi_1, \psi_{j+1}\rangle, \quad 1 \leq j \leq 2, \tag{11}$$

$$r_j = \frac{3i}{\gamma_{j+1} - \gamma_1}\langle\phi_{j+1}, \psi_1\rangle, \quad 1 \leq j \leq 2, \tag{12}$$

where $< \cdot, \cdot >$ refers to the standard inner product of the Euclidian space and $\gamma_1, \gamma_2, \gamma_3$ are distinct parameters.

2.3 Completely integrable Hamiltonian system

Substituting (11) and (12) into (10), then we get the finite-dimensional Hamiltonian system

$$\begin{cases} \phi_{1x} = -2iA\phi_1 + \dfrac{3i}{\gamma_2-\gamma_1}\langle\phi_1,\psi_2\rangle\phi_2 \\ \qquad + \dfrac{3i}{\gamma_3-\gamma_1}\langle\phi_1,\psi_3\rangle\phi_3 = -\dfrac{\partial H^x}{\partial\psi_1}, \\ \phi_{2x} = iA\phi_2 + \dfrac{3i}{\gamma_2-\gamma_1}\langle\phi_2,\psi_1\rangle\phi_1 = -\dfrac{\partial H^x}{\partial\psi_2}, \\ \phi_{3x} = iA\phi_3 + \dfrac{3i}{\gamma_3-\gamma_1}\langle\phi_3,\psi_1\rangle\phi_1 = -\dfrac{\partial H^x}{\partial\psi_3}, \\ \psi_{1x} = -2iA\psi_1 - \dfrac{3i}{\gamma_2-\gamma_1}\langle\phi_2,\psi_1\rangle\psi_2 \\ \qquad - \dfrac{3i}{\gamma_3-\gamma_1}\langle\phi_3,\psi_1\rangle\psi_3 = \dfrac{\partial H^x}{\partial\phi_1}, \\ \psi_{2x} = -iA\psi_2 - \dfrac{3i}{\gamma_2-\gamma_1}\langle\phi_1,\psi_2\rangle\psi_1 = \dfrac{\partial H^x}{\partial\phi_2}, \\ \psi_{3x} = -iA\psi_3 - \dfrac{3i}{\gamma_3-\gamma_1}\langle\phi_1,\psi_3\rangle\psi_1 = \dfrac{\partial H^x}{\partial\phi_3}, \end{cases} \tag{13}$$

where

$$\begin{aligned} H^x &= 2i\langle A\phi_1,\psi_1\rangle - \sum_{j=1}^{2}\frac{3i}{\gamma_{j+1}-\gamma_1}\langle\phi_1,\psi_{j+1}\rangle\langle\phi_{j+1},\psi_1\rangle \\ &\quad -i\sum_{j=1}^{2}\langle A\phi_{j+1},\psi_{j+1}\rangle. \end{aligned}$$

Under the control of (11), (12) and (13), we can get the following Hamiltonian system

$$\begin{aligned} &\phi_{1,t_2} = -\frac{\partial H^{t_2}}{\partial\psi_1}, \quad \phi_{2,t_2} = -\frac{\partial H^{t_2}}{\partial\psi_2}, \quad \phi_{3,t_2} = -\frac{\partial H^{t_2}}{\partial\psi_3}, \\ &\psi_{1,t_2} = \frac{\partial H^{t_2}}{\partial\phi_1}, \qquad \psi_{2,t_2} = \frac{\partial H^{t_2}}{\partial\phi_2}, \qquad \psi_{3,t_2} = \frac{\partial H^{t_2}}{\partial\phi_3}, \end{aligned} \tag{14}$$

where

$$\begin{aligned} H^{t_2} &= 2i\langle A^2\phi_1,\psi_1\rangle - i\langle A^2\phi_2,\psi_2\rangle - i\langle A^2\phi_3,\psi_3\rangle \\ &- \frac{3i}{\gamma_2-\gamma_1}\langle\phi_1,\psi_2\rangle\langle A\phi_2,\psi_1\rangle - \frac{3i}{\gamma_3-\gamma_1}\langle\phi_1,\psi_2\rangle\langle A\phi_3,\psi_1\rangle \\ &- \frac{3i}{\gamma_2-\gamma_1}\langle\phi_2,\psi_1\rangle\langle A\phi_1,\psi_2\rangle - \frac{3i}{\gamma_3-\gamma_1}\langle\phi_3,\psi_1\rangle\langle A\phi_1,\psi_3\rangle \\ &- \frac{3i}{(\gamma_2-\gamma_1)^2}\langle\phi_1,\psi_2\rangle\langle\phi_2,\psi_1\rangle\langle\phi_1,\psi_1\rangle \\ &- \frac{3i}{(\gamma_3-\gamma_1)^2}\langle\phi_1,\psi_3\rangle\langle\phi_3,\psi_1\rangle\langle\phi_1,\psi_1\rangle \\ &+ \frac{3i}{(\gamma_2-\gamma_1)^2}\langle\phi_1,\psi_2\rangle\langle\phi_2,\psi_1\rangle\langle\phi_2,\psi_2\rangle \\ &+ \frac{3i}{(\gamma_2-\gamma_1)(\gamma_3-\gamma_1)}\langle\phi_1,\psi_2\rangle\langle\phi_3,\psi_1\rangle\langle\phi_2,\psi_3\rangle \\ &+ \frac{3i}{(\gamma_2-\gamma_1)(\gamma_3-\gamma_1)}\langle\phi_1,\psi_3\rangle\langle\phi_2,\psi_1\rangle\langle\phi_3,\psi_2\rangle \\ &+ \frac{3i}{(\gamma_3-\gamma_1)^2}\langle\phi_1,\psi_3\rangle\langle\phi_3,\psi_1\rangle\langle\phi_3,\psi_3\rangle. \end{aligned}$$

Hamiltonian systems (13) and (14) allow Lax representations

$$\frac{d}{dx}L(\lambda) = \big[U(\widetilde{u},\lambda), L(\lambda)\big], \quad \frac{d}{dt_2}L(\lambda) = \big[\widetilde{V}^{(2)}(\widetilde{u},\lambda), L(\lambda)\big], \tag{15}$$

respectively, where

$$L(\lambda) = \begin{pmatrix} \gamma_1 & 0 & 0 \\ 0 & \gamma_2 & 0 \\ 0 & 0 & \gamma_3 \end{pmatrix} + \sum_{j=1}^{N} \frac{1}{\lambda-\lambda_j} \begin{pmatrix} \phi_{1j}\psi_{1j} & \phi_{1j}\psi_{2j} & \phi_{1j}\psi_{3j} \\ \phi_{2j}\psi_{1j} & \phi_{2j}\psi_{2j} & \phi_{2j}\psi_{3j} \\ \phi_{3j}\psi_{1j} & \phi_{3j}\psi_{2j} & \phi_{3j}\psi_{3j} \end{pmatrix}, \tag{16}$$

$\lambda_1, \lambda_2, \cdots, \lambda_N$ are N distinct parameters, and $U(\widetilde{u},\lambda)$, $\widetilde{V}^{(2)}(\widetilde{u},\lambda)$ are the constraint spectral matrices generated from U, $V^{(2)}$ under constraint (11) and (12).

To analyze the integrability of (13) and (14), we define a symplectic structure

$$\omega^2 = \sum_{i=1}^{3} d\phi_i \wedge d\psi_i = \sum_{i=1}^{3}\sum_{s=1}^{N} d\phi_{is} \wedge d\psi_{is}. \tag{17}$$

The corresponding Poisson bracket is given by

$$\{f, g\} = \sum_{i=1}^{3}\sum_{s=1}^{N}\left(\frac{\partial f}{\partial \psi_{is}}\frac{\partial g}{\partial \phi_{is}} - \frac{\partial f}{\partial \phi_{is}}\frac{\partial g}{\partial \psi_{is}}\right), \quad f, g \in C^{\infty}(\mathbb{C}^{6N}). \tag{18}$$

Furthermore, it has been shown that Lax matrix (16) satisfies an r-matrix relation [13]. Therefore, we can get 3N conserved integrals

$$F_{1k} = \langle A^{k-1}\phi_1, \psi_1\rangle + \langle A^{k-1}\phi_2, \psi_2\rangle + \langle A^{k-1}\phi_3, \psi_3\rangle, \tag{19}$$

$$\begin{aligned} F_{2k} = {} & (\gamma_2+\gamma_3)\langle A^{k-1}\phi_1, \psi_1\rangle + (\gamma_1+\gamma_3)\langle A^{k-1}\phi_2, \psi_2\rangle \\ & + (\gamma_1+\gamma_2)\langle A^{k-1}\phi_3, \psi_3\rangle + \sum_{\substack{i+l=k-2\\ i,l\geq 0}}\left[\begin{vmatrix}\langle A^i\phi_1, \psi_1\rangle & \langle A^i\phi_1, \psi_2\rangle\\ \langle A^l\phi_2, \psi_1\rangle & \langle A^l\phi_2, \psi_2\rangle\end{vmatrix}\right. \\ & \left. + \begin{vmatrix}\langle A^i\phi_1, \psi_1\rangle & \langle A^i\phi_1, \psi_3\rangle\\ \langle A^l\phi_3, \psi_1\rangle & \langle A^l\phi_3, \psi_3\rangle\end{vmatrix} + \begin{vmatrix}\langle A^i\phi_2, \psi_2\rangle & \langle A^i\phi_2, \psi_3\rangle\\ \langle A^l\phi_3, \psi_2\rangle & \langle A^l\phi_3, \psi_3\rangle\end{vmatrix}\right], \end{aligned} \tag{20}$$

$$\begin{aligned} F_{3k} = {} & \gamma_2\gamma_3\langle A^{k-1}\phi_1, \psi_1\rangle + \gamma_1\gamma_3\langle A^{k-1}\phi_2, \psi_2\rangle + \gamma_1\gamma_2\langle A^{k-1}\phi_3, \psi_3\rangle \\ & + \sum_{\substack{i+l=k-2\\ i,l\geq 0}}\left[\gamma_1\begin{vmatrix}\langle A^i\phi_2, \psi_2\rangle & \langle A^i\phi_2, \psi_3\rangle\\ \langle A^l\phi_3, \psi_2\rangle & \langle A^l\phi_3, \psi_3\rangle\end{vmatrix}\right. \\ & \left. + \gamma_2\begin{vmatrix}\langle A^i\phi_1, \psi_1\rangle & \langle A^i\phi_1, \psi_3\rangle\\ \langle A^l\phi_3, \psi_1\rangle & \langle A^l\phi_3, \psi_3\rangle\end{vmatrix} + \gamma_3\begin{vmatrix}\langle A^i\phi_1, \psi_1\rangle & \langle A^i\phi_1, \psi_2\rangle\\ \langle A^l\phi_2, \psi_1\rangle & \langle A^l\phi_2, \psi_2\rangle\end{vmatrix}\right] \\ & + \sum_{\substack{i+l+n=k-3\\ i,l,n\geq 0}}\begin{vmatrix}\langle A^i\phi_1, \psi_1\rangle & \langle A^i\phi_1, \psi_2\rangle & \langle A^i\phi_1, \psi_3\rangle\\ \langle A^l\phi_2, \psi_1\rangle & \langle A^l\phi_2, \psi_2\rangle & \langle A^l\phi_2, \psi_3\rangle\\ \langle A^n\phi_3, \psi_1\rangle & \langle A^n\phi_3, \psi_2\rangle & \langle A^n\phi_3, \psi_3\rangle\end{vmatrix}. \end{aligned} \tag{21}$$

A direct check shows that

$$F_{ik}, \ 1 \leq i \leq 3, \ 1 \leq k \leq N,$$

are in involution and functionally independent. Thus Hamiltonian systems (13) and (14) are completely integrable in the sense of Liouville. Bringing these together, we arrive at the following proposition.

Proposition 1. *If* $\phi_1, \phi_2, \phi_3, \psi_1, \psi_2, \psi_3$ *satisfy both* (13) *and* (14)*, then*

$$q_j = \frac{3i}{\gamma_{j+1}-\gamma_1}\langle \phi_1, \psi_{j+1}\rangle, \quad r_j = \frac{3i}{\gamma_{j+1}-\gamma_1}\langle \phi_{j+1}, \psi_1\rangle, \quad 1 \le j \le 2,$$

solve the 4-component AKNS equation (6).

3 An integrable decomposition of the Manakov equation

This section is devoted to an integrable decomposition of the Manakov equation. It is known that equation (1) may be reduced from the 4-component AKNS equation by imposing the reality condition $r = -q^\dagger$. Moreover, the Manakov equation associates with the following spectral problems

$$\Phi_x = U(q, r, \lambda)\Phi, \quad r = -q^\dagger, \tag{22}$$

$$\Phi_{t_2} = V^{(2)}(q, r, \lambda)\Phi, \quad r = -q^\dagger, \tag{23}$$

and the adjoint spectral problems

$$\Psi_x = -U^T(q, r, \lambda)\Psi, \quad r = -q^\dagger, \tag{24}$$

$$\Psi_{t_2} = -V^{(2)T}(q, r, \lambda)\Psi, \quad r = -q^\dagger, \tag{25}$$

where $U(q, r, \lambda)$, $V^{(2)}(q, r, \lambda)$ are defined by (2), (8).

On the basis of the reality condition $r = -q^\dagger$ we have the following lemma.

Lemma 1. *Let* $r = -q^\dagger$, $(\phi_1, \phi_2, \phi_3)^T$ *solves* (22), (23)*, and* $(\psi_1, \psi_2, \psi_3)^T$ *solves* (24), (25)*,*

(i) *If parameter* λ *is nonreal complex, then* $(\psi_1^*, \psi_2^*, \psi_3^*)^T$ *solves* (22) *and* (23) *with parameter* λ^**. In the same time,* $(\phi_1^*, \phi_2^*, \phi_3^*)^T$ *solves* (24) *and* (25) *with parameter* λ^**.*

(ii) *If parameter* λ *is real, then* $(\phi_1^*, \phi_2^*, \phi_3^*)^T$ *solves the adjoint spectral problems* (24) *and* (25) *with parameter* λ*.*

Now let us carry out the binary NSPs of the Manakov equation. We take N parameters: $\lambda_1, \cdots, \lambda_r, \lambda_{r+1}, \cdots, \lambda_N$, where $\lambda_1, \cdots, \lambda_r$ are distinct nonreal complex numbers, $\lambda_i \ne \lambda_j^*$, $1 \le i, j \le r$, and $\lambda_{r+1}, \cdots, \lambda_N$ are $N - r$ distinct real numbers.

Guided by Lemma 1, we introduce new variables and notation

$$\begin{aligned}
\widehat{\phi}_1 &= \left(\phi_1^{(r)}, \psi_1^{*(r)}, \phi_1^{(N-r)}\right)^T, & \widehat{\phi}_2 &= \left(\phi_2^{(r)}, \psi_2^{*(r)}, \phi_2^{(N-r)}\right)^T, \\
\widehat{\phi}_3 &= \left(\phi_3^{(r)}, \psi_3^{*(r)}, \phi_3^{(N-r)}\right)^T, \\
\widehat{\psi}_1 &= \left(\psi_1^{(r)}, \phi_1^{*(r)}, \phi_1^{*(N-r)}\right)^T, & \widehat{\psi}_2 &= \left(\psi_2^{(r)}, \phi_2^{*(r)}, \phi_2^{*(N-r)}\right)^T, \\
\widehat{\psi}_3 &= \left(\psi_3^{(r)}, \phi_3^{*(r)}, \phi_3^{*(N-r)}\right)^T, & \widehat{A} &= \mathrm{diag}\left(\Lambda^{(r)}, \overline{\Lambda^{(r)}}, \Lambda^{(N-r)}\right),
\end{aligned} \tag{26}$$

where

$$\begin{aligned}
\Lambda^{(r)} &= \mathrm{diag}(\lambda_1, \cdots, \lambda_r), & \Lambda^{(N-r)} &= \mathrm{diag}(\lambda_{r+1}, \cdots, \lambda_N), \\
\phi_k^{(r)} &= (\phi_{k1}, \cdots, \phi_{kr})^T, & \phi_k^{(N-r)} &= (\phi_{k,r+1}, \cdots, \phi_{k,N})^T, \\
\psi_k^{(r)} &= (\psi_{k1}, \cdots, \psi_{kr})^T, & \psi_k^{(N-r)} &= (\psi_{k,r+1}, \cdots, \psi_{k,N})^T,
\end{aligned} \tag{27}$$

and then we can get the following vector form of spectral problem (22) and (24) coupling with their complex conjugates

$$\begin{cases}
\widehat{\phi}_{1x} = -2i\widehat{A}\widehat{\phi}_1 + q_1\widehat{\phi}_2 + q_2\widehat{\phi}_3, \\
\widehat{\phi}_{2x} = i\widehat{A}\widehat{\phi}_2 + r_1\widehat{\phi}_1, \\
\widehat{\phi}_{3x} = i\widehat{A}\widehat{\phi}_3 + r_2\widehat{\phi}_1, \qquad\qquad r = -q^\dagger, \\
\widehat{\psi}_{1x} = 2i\widehat{A}\widehat{\psi}_1 - r_1\widehat{\psi}_2 - r_2\widehat{\psi}_3, \\
\widehat{\psi}_{2x} = -i\widehat{A}\widehat{\psi}_2 - q_1\widehat{\psi}_1, \\
\widehat{\psi}_{3x} = -i\widehat{A}\widehat{\psi}_3 - q_2\widehat{\psi}_1,
\end{cases} \tag{28}$$

which is just a compact form of special 3×3 matrix spectral problems for AKNS hierarchy and their adjoint spectral problems. Therefore, we only need to modify the procedure of the binary NSPs of the 4-component AKNS hierarchy by replacing variables $\phi_1, \phi_2, \phi_3, \psi_1, \psi_2, \psi_3$ with $\widehat{\phi}_1, \widehat{\phi}_2, \widehat{\phi}_3, \widehat{\psi}_1, \widehat{\psi}_2, \widehat{\psi}_3$ such that the reality condition $r = -q^\dagger$ holds.

Now we consider the constraint of the Manakov equation

$$\begin{aligned}
q_j &= \frac{3i}{\gamma_{j+1} - \gamma_1}\langle \widehat{\phi}_1, \widehat{\psi}_{j+1}\rangle = \frac{3i}{\gamma_{j+1} - \gamma_1}(\langle \phi_1^{(r)}, \psi_{j+1}^{(r)}\rangle \\
&\quad + \langle \phi_1^{(N-r)}, \phi_{j+1}^{*(N-r)}\rangle + \langle \psi_1^{*(r)}, \phi_{j+1}^{*(r)}\rangle), \\
r_j &= \frac{3i}{\gamma_{j+1} - \gamma_1}\langle \widehat{\phi}_{j+1}, \widehat{\psi}_1\rangle = \frac{3i}{\gamma_{j+1} - \gamma_1}(\langle \phi_{j+1}^{(r)}, \psi_1^{(r)}\rangle \\
&\quad + \langle \phi_{j+1}^{(N-r)}, \phi_1^{*(N-r)}\rangle + \langle \psi_{j+1}^{*(r)}, \phi_1^{*(r)}\rangle),
\end{aligned} \tag{29}$$

which solves

$$r = -q^{\dagger}, 1 \leq j \leq 2.$$

Substituting (29) into (28), then we get the finite-dimensional Hamiltonian system

$$\begin{array}{lll}
\phi_{1x}^{(r)} = -\dfrac{\partial \widehat{H}^x}{\partial \psi_1^{(r)}}, & \psi_{1x}^{(r)} = \dfrac{\partial \widehat{H}^x}{\partial \phi_1^{(r)}}, & \phi_{1x}^{*(r)} = \dfrac{\partial \widehat{H}^x}{\partial \psi_1^{*(r)}}, \\
\phi_{2x}^{(r)} = -\dfrac{\partial \widehat{H}^x}{\partial \psi_2^{(r)}}, & \psi_{2x}^{(r)} = \dfrac{\partial \widehat{H}^x}{\partial \phi_2^{(r)}}, & \phi_{2x}^{*(r)} = \dfrac{\partial \widehat{H}^x}{\partial \psi_2^{*(r)}}, \\
\phi_{3x}^{(r)} = -\dfrac{\partial \widehat{H}^x}{\partial \psi_3^{(r)}}, & \psi_{3x}^{(r)} = \dfrac{\partial \widehat{H}^x}{\partial \phi_3^{(r)}}, & \phi_{3x}^{*(r)} = \dfrac{\partial \widehat{H}^x}{\partial \psi_3^{*(r)}}, \\
\psi_{1x}^{*(r)} = -\dfrac{\partial \widehat{H}^x}{\partial \phi_1^{*(r)}}, & \phi_{1x}^{(N-r)} = -\dfrac{\partial \widehat{H}^x}{\partial \phi_1^{*(N-r)}}, & \phi_{1x}^{*(N-r)} = \dfrac{\partial \widehat{H}^x}{\partial \phi_1^{(N-r)}}, \\
\psi_{2x}^{*(r)} = -\dfrac{\partial \widehat{H}^x}{\partial \phi_2^{*(r)}}, & \phi_{2x}^{(N-r)} = -\dfrac{\partial \widehat{H}^x}{\partial \phi_2^{*(N-r)}}, & \phi_{2x}^{*(N-r)} = \dfrac{\partial \widehat{H}^x}{\partial \phi_2^{(N-r)}}, \\
\psi_{3x}^{*(r)} = -\dfrac{\partial \widehat{H}^x}{\partial \phi_3^{*(r)}}, & \phi_{3x}^{(N-r)} = -\dfrac{\partial \widehat{H}^x}{\partial \phi_3^{*(N-r)}}, & \phi_{3x}^{*(N-r)} = \dfrac{\partial \widehat{H}^x}{\partial \phi_3^{(N-r)}},
\end{array} \tag{30}$$

where

$$\begin{aligned}
\widehat{H}^x \;=\; & 2i\big(\langle \Lambda^{(r)}\phi_1^{(r)}, \psi_1^{(r)}\rangle + \langle \overline{\Lambda^{(r)}}\psi_1^{*(r)}, \phi_1^{*(r)}\rangle + \langle \Lambda^{(N-r)}\phi_1^{(N-r)}, \phi_1^{*(N-r)}\rangle\big) \\
& - i\big(\langle \Lambda^{(r)}\phi_2^{(r)}, \psi_2^{(r)}\rangle + \langle \overline{\Lambda^{(r)}}\psi_2^{*(r)}, \phi_2^{*(r)}\rangle + \langle \Lambda^{(N-r)}\phi_2^{(N-r)}, \psi_2^{*(N-r)}\rangle\big) \\
& - i\big(\langle \Lambda^{(r)}\phi_3^{(r)}, \psi_3^{(r)}\rangle + \langle \overline{\Lambda^{(r)}}\psi_3^{*(r)}, \phi_3^{*(r)}\rangle + \langle \Lambda^{(N-r)}\phi_3^{(N-r)}, \psi_3^{*(N-r)}\rangle\big) \\
& - \frac{3i}{\gamma_2 - \gamma_1}\big(\langle \phi_1^{(r)}, \psi_2^{(r)}\rangle + \langle \psi_1^{*(r)}, \phi_2^{*(r)}\rangle + \langle \phi_1^{(N-r)}, \phi_2^{*(N-r)}\rangle\big) \\
\times \; & \big(\langle \phi_2^{(r)}, \psi_1^{(r)}\rangle + \langle \psi_2^{*(r)}, \phi_1^{*(r)}\rangle + \langle \phi_2^{(N-r)}, \phi_1^{*(N-r)}\rangle\big) \\
& - \frac{3i}{\gamma_3 - \gamma_1}\big(\langle \phi_1^{(r)}, \psi_3^{(r)}\rangle + \langle \psi_1^{*(r)}, \phi_3^{*(r)}\rangle + \langle \phi_1^{(N-r)}, \phi_3^{*(N-r)}\rangle\big) \\
\times \; & \big(\langle \phi_3^{(r)}, \psi_1^{(r)}\rangle + \langle \psi_3^{*(r)}, \phi_1^{*(r)}\rangle + \langle \phi_3^{(N-r)}, \phi_1^{*(N-r)}\rangle\big).
\end{aligned}$$

Similarly, we can get the temporal Hamiltonian system

$$
\begin{aligned}
&\phi^{(r)}_{1t_2} = -\frac{\partial \widehat{H}^{t_2}}{\partial \psi^{(r)}_1}, && \psi^{(r)}_{1t_2} = \frac{\partial \widehat{H}^{t_2}}{\partial \phi^{(r)}_1}, && \phi^{*(r)}_{1t_2} = \frac{\partial \widehat{H}^{t_2}}{\partial \psi^{*(r)}_1},\\
&\phi^{(r)}_{2t_2} = -\frac{\partial \widehat{H}^{t_2}}{\partial \psi^{(r)}_2}, && \psi^{(r)}_{2t_2} = \frac{\partial \widehat{H}^{t_2}}{\partial \phi^{(r)}_2}, && \phi^{*(r)}_{2t_2} = \frac{\partial \widehat{H}^{t_2}}{\partial \psi^{*(r)}_2},\\
&\phi^{(r)}_{3t_2} = -\frac{\partial \widehat{H}^{t_2}}{\partial \psi^{(r)}_3}, && \psi^{(r)}_{3t_2} = \frac{\partial \widehat{H}^{t_2}}{\partial \phi^{(r)}_3}, && \phi^{*(r)}_{3t_2} = \frac{\partial \widehat{H}^{t_2}}{\partial \psi^{*(r)}_3},\\
&\psi^{*(r)}_{1t_2} = -\frac{\partial \widehat{H}^{t_2}}{\partial \phi^{*(r)}_1}, && \phi^{(N-r)}_{1t_2} = -\frac{\partial \widehat{H}^{t_2}}{\partial \phi^{*(N-r)}_1}, && \phi^{*(N-r)}_{1t_2} = \frac{\partial \widehat{H}^{t_2}}{\partial \phi^{(N-r)}_1},\\
&\psi^{*(r)}_{2t_2} = -\frac{\partial \widehat{H}^{t_2}}{\partial \phi^{*(r)}_2}, && \phi^{(N-r)}_{2t_2} = -\frac{\partial \widehat{H}^{t_2}}{\partial \phi^{*(N-r)}_2}, && \phi^{*(N-r)}_{2t_2} = \frac{\partial \widehat{H}^{t_2}}{\partial \phi^{(N-r)}_2},\\
&\psi^{*(r)}_{3t_2} = -\frac{\partial \widehat{H}^{t_2}}{\partial \phi^{*(r)}_3}, && \phi^{(N-r)}_{3t_2} = -\frac{\partial \widehat{H}^{t_2}}{\partial \phi^{*(N-r)}_3}, && \phi^{*(N-r)}_{3t_2} = \frac{\partial \widehat{H}^{t_2}}{\partial \phi^{(N-r)}_3},
\end{aligned}
\tag{31}
$$

where

$$
\begin{aligned}
\widehat{H}^{t_2} = {} & 2i\big(\langle \Lambda^{(r)^2}\phi^{(r)}_1, \psi^{(r)}_1\rangle + \langle \overline{\Lambda^{(r)}}^2 \psi^{*(r)}_1, \phi^{*(r)}_1\rangle + \langle \Lambda^{(N-r)^2}\phi^{(N-r)}_1, \phi^{*(N-r)}_1\rangle\big)\\
& - i\big(\langle \Lambda^{(r)^2}\phi^{(r)}_2, \psi^{(r)}_2\rangle + \langle \overline{\Lambda^{(r)}}^2 \psi^{*(r)}_2, \phi^{*(r)}_2\rangle + \langle \Lambda^{(N-r)^2}\phi^{(N-r)}_2, \psi^{*(N-r)}_2\rangle\big)\\
& - i\big(\langle \Lambda^{(r)^2}\phi^{(r)}_3, \psi^{(r)}_3\rangle + \langle \overline{\Lambda^{(r)}}^2 \psi^{*(r)}_3, \phi^{*(r)}_3\rangle + \langle \Lambda^{(N-r)^2}\phi^{(N-r)}_3, \psi^{*(N-r)}_3\rangle\big)\\
& - \frac{3i}{\gamma_2-\gamma_1}\big(\langle \phi^{(r)}_1, \psi^{(r)}_2\rangle + \langle \psi^{*(r)}_1, \phi^{*(r)}_2\rangle + \langle \phi^{(N-r)}_1, \phi^{*(N-r)}_2\rangle\big)\\
\times & \big(\langle \Lambda^{(r)}\phi^{(r)}_2, \psi^{(r)}_1\rangle + \langle \overline{\Lambda^{(r)}} \psi^{*(r)}_2, \phi^{*(r)}_1\rangle + \langle \Lambda^{(N-r)}\phi^{(N-r)}_2, \phi^{*(N-r)}_1\rangle\big)\\
& - \frac{3i}{\gamma_3-\gamma_1}\big(\langle \phi^{(r)}_1, \psi^{(r)}_3\rangle + \langle \psi^{*(r)}_1, \phi^{*(r)}_3\rangle + \langle \phi^{(N-r)}_1, \phi^{*(N-r)}_3\rangle\big)\\
\times & \big(\langle \Lambda^{(r)}\phi^{(r)}_3, \psi^{(r)}_1\rangle + \langle \overline{\Lambda^{(r)}} \psi^{*(r)}_3, \phi^{*(r)}_1\rangle + \langle \Lambda^{(N-r)}\phi^{(N-r)}_3, \phi^{*(N-r)}_1\rangle\big)\\
& - \frac{3i}{\gamma_2-\gamma_1}\big(\langle \phi^{(r)}_2, \psi^{(r)}_1\rangle + \langle \psi^{*(r)}_2, \phi^{*(r)}_1\rangle + \langle \phi^{(N-r)}_2, \phi^{*(N-r)}_1\rangle\big)\\
\times & \big(\langle \Lambda^{(r)}\phi^{(r)}_1, \psi^{(r)}_2\rangle + \langle \overline{\Lambda^{(r)}} \psi^{*(r)}_1, \phi^{*(r)}_2\rangle + \langle \Lambda^{(N-r)}\phi^{(N-r)}_1, \phi^{*(N-r)}_2\rangle\big)\\
& - \frac{3i}{\gamma_3-\gamma_1}\big(\langle \phi^{(r)}_3, \psi^{(r)}_1\rangle + \langle \psi^{*(r)}_3, \phi^{*(r)}_1\rangle + \langle \phi^{(N-r)}_3, \phi^{*(N-r)}_1\rangle\big)
\end{aligned}
$$

$$\times \left(\langle\Lambda^{(r)}\phi_1^{(r)}, \psi_3^{(r)}\rangle + \langle\overline{\Lambda^{(r)}}\psi_1^{*(r)}, \phi_3^{*(r)}\rangle + \langle\Lambda^{(N-r)}\phi_1^{(N-r)}, \phi_3^{*(N-r)}\rangle\right)$$

$$- \frac{3i}{(\gamma_2-\gamma_1)^2}\left(\langle\phi_1^{(r)}, \psi_2^{(r)}\rangle + \langle\psi_1^{*(r)}, \phi_2^{*(r)}\rangle + \langle\phi_1^{(N-r)}, \phi_2^{*(N-r)}\rangle\right)$$

$$\times \left(\langle\phi_2^{(r)}, \psi_1^{(r)}\rangle + \langle\psi_2^{*(r)}, \phi_1^{*(r)}\rangle + \langle\phi_2^{(N-r)}, \phi_1^{*(N-r)}\rangle\right)$$

$$\times \left(\langle\phi_1^{(r)}, \psi_1^{(r)}\rangle + \langle\psi_1^{*(r)}, \phi_1^{*(r)}\rangle + \langle\phi_1^{(N-r)}, \phi_1^{*(N-r)}\rangle\right)$$

$$- \frac{3i}{(\gamma_3-\gamma_1)^2}\left(\langle\phi_1^{(r)}, \psi_3^{(r)}\rangle + \langle\psi_1^{*(r)}, \phi_3^{*(r)}\rangle + \langle\phi_1^{(N-r)}, \phi_3^{*(N-r)}\rangle\right)$$

$$\times \left(\langle\phi_3^{(r)}, \psi_1^{(r)}\rangle + \langle\psi_3^{*(r)}, \phi_1^{*(r)}\rangle + \langle\phi_3^{(N-r)}, \phi_1^{*(N-r)}\rangle\right)$$

$$\times \left(\langle\phi_1^{(r)}, \psi_1^{(r)}\rangle + \langle\psi_1^{*(r)}, \phi_1^{*(r)}\rangle + \langle\phi_1^{(N-r)}, \phi_1^{*(N-r)}\rangle\right)$$

$$+ \frac{3i}{(\gamma_2-\gamma_1)^2}\left(\langle\phi_1^{(r)}, \psi_2^{(r)}\rangle + \langle\psi_1^{*(r)}, \phi_2^{*(r)}\rangle + \langle\phi_1^{(N-r)}, \phi_2^{*(N-r)}\rangle\right).$$

Hamiltonian systems (30) and (31) are completely integrable systems. In fact, first of all, it is not difficult to check that (30) and (31) allow the following Lax matrix

$$L(\lambda) = \begin{pmatrix} \gamma_1 & 0 & 0 \\ 0 & \gamma_2 & 0 \\ 0 & 0 & \gamma_3 \end{pmatrix} + \sum_{j=1}^{r} \frac{1}{\lambda-\lambda_j} \begin{pmatrix} \phi_{1j}\psi_{1j} & \phi_{1j}\psi_{2j} & \phi_{1j}\psi_{3j} \\ \phi_{2j}\psi_{1j} & \phi_{2j}\psi_{2j} & \phi_{2j}\psi_{3j} \\ \phi_{3j}\psi_{1j} & \phi_{3j}\psi_{2j} & \phi_{3j}\psi_{3j} \end{pmatrix}$$
$$+ \sum_{j=1}^{r} \frac{1}{\lambda-\lambda_j^*} \begin{pmatrix} \psi_{1j}^*\phi_{1j}^* & \psi_{1j}^*\phi_{2j}^* & \psi_{1j}^*\phi_{3j}^* \\ \psi_{2j}^*\phi_{1j}^* & \psi_{2j}^*\phi_{2j}^* & \psi_{2j}^*\phi_{3j}^* \\ \psi_{3j}^*\phi_{1j}^* & \psi_{3j}^*\phi_{2j}^* & \psi_{3j}^*\phi_{3j}^* \end{pmatrix} \tag{32}$$
$$+ \sum_{j=r+1}^{N} \frac{1}{\lambda-\lambda_j} \begin{pmatrix} |\phi_{1j}|^2 & \phi_{1j}\phi_{2j}^* & \phi_{1j}\phi_{3j}^* \\ \phi_{2j}\phi_{1j}^* & |\phi_{2j}|^2 & \phi_{2j}\phi_{3j}^* \\ \phi_{3j}\phi_{1j}^* & \phi_{3j}\phi_{2j}^* & |\phi_{3j}|^2 \end{pmatrix}.$$

Introducing symplectic structure

$$\widehat{\omega}^2 = \sum_{i=1}^{3}\sum_{s=1}^{r}\left(d\phi_{is}\wedge d\psi_{is} + d\psi_{is}^*\wedge d\phi_{is}^*\right) + \sum_{i=1}^{3}\sum_{j=1}^{N-r} d\phi_{i,r+j}\wedge d\phi_{i,r+j}^*. \tag{33}$$

The corresponding Poisson bracket is given by

$$\{f, g\} = \sum_{i=1}^{3}\sum_{s=1}^{r}\left(\frac{\partial f}{\partial \psi_{is}}\frac{\partial g}{\partial \phi_{is}} - \frac{\partial f}{\partial \phi_{is}}\frac{\partial g}{\partial \psi_{is}} + \frac{\partial f}{\partial \phi^*_{is}}\frac{\partial g}{\partial \psi^*_{is}} - \frac{\partial f}{\partial \psi^*_{is}}\frac{\partial g}{\partial \phi^*_{is}}\right) + \sum_{i=1}^{3}\sum_{j=1}^{N-r}\left(\frac{\partial f}{\partial \phi^*_{i,r+j}}\frac{\partial g}{\partial \phi_{i,r+j}} - \frac{\partial f}{\partial \phi_{i,r+j}}\frac{\partial g}{\partial \phi^*_{i,r+j}}\right).$$

By a direct check, we come to a conclusion that $L(\lambda)$ satisfies the r-matrix relation

$$\{L_1(\lambda), L_2(\mu)\} = [r_{12}(\lambda, \mu), L_1(\lambda)] + [r_{21}(\lambda, \mu), L_2(\mu)],$$

where $L_1(\lambda) = L(\lambda) \otimes I,\ L_2(\mu) = I \otimes L(\mu),\ I$ is the 3×3 unit matrix,

$$r_{12}(\lambda, \mu) = -\sum_{1\le i,j\le 3}\frac{e_{ij}\otimes e_{ji}}{\mu-\lambda}, \quad r_{21}(\lambda, \mu) = -\sum_{1\le i,j\le 3}\frac{e_{ij}\otimes e_{ji}}{\mu-\lambda}, \tag{34}$$

and e_{ij} is the matrix with the element 1 at the (i, j) position and zeros elsewhere.

We can expand (19), (20), (21) as follows

$$\begin{aligned}\widehat{F}_{1k} = &\left(\langle \Lambda^{(r)^{k-1}}\phi_1^{(r)}, \psi_1^{(r)}\rangle + \langle \overline{\Lambda^{(r)}}^{k-1}\psi_1^{*(r)}, \phi_1^{*(r)}\rangle\right.\\ &\left.+ \langle \Lambda^{(N-r)^{k-1}}\phi_1^{(N-r)}, \phi_1^{*(N-r)}\rangle\right) + \left(\langle \Lambda^{(r)^{k-1}}\phi_2^{(r)}, \psi_2^{(r)}\rangle\right.\\ &\left.+ \langle \overline{\Lambda^{(r)}}^{k-1}\psi_2^{*(r)}, \phi_2^{*(r)}\rangle + \langle \Lambda^{(N-r)^{k-1}}\phi_2^{(N-r)}, \phi_2^{*(N-r)}\rangle\right)\\ &+ \left(\langle \Lambda^{(r)^{k-1}}\phi_3^{(r)}, \psi_3^{(r)}\rangle + \langle \overline{\Lambda^{(r)}}^{k-1}\psi_3^{*(r)}, \phi_3^{*(r)}\rangle\right.\\ &\left.+ \langle \Lambda^{(N-r)^{k-1}}\phi_3^{(N-r)}, \phi_3^{*(N-r)}\rangle\right),\end{aligned} \tag{35}$$

$$\begin{aligned}\widehat{F}_{2k} = &(\gamma_2+\gamma_3)\left(\langle \Lambda^{(r)^{k-1}}\phi_1^{(r)}, \psi_1^{(r)}\rangle + \langle \overline{\Lambda^{(r)}}^{k-1}\psi_1^{*(r)}, \phi_1^{*(r)}\rangle\right.\\ &\left.+ \langle \Lambda^{(N-r)^{k-1}}\phi_1^{(N-r)}, \phi_1^{*(N-r)}\rangle\right) + (\gamma_1+\gamma_3)\left(\langle \Lambda^{(r)^{k-1}}\phi_2^{(r)}, \psi_2^{(r)}\rangle\right.\\ &\left.+ \langle \overline{\Lambda^{(r)}}^{k-1}\psi_2^{*(r)}, \phi_2^{*(r)}\rangle + \langle \Lambda^{(N-r)^{k-1}}\phi_2^{(N-r)}, \phi_2^{*(N-r)}\rangle\right)\\ &+ (\gamma_1+\gamma_2)\left(\langle \Lambda^{(r)^{k-1}}\phi_3^{(r)}, \psi_3^{(r)}\rangle + \langle \overline{\Lambda^{(r)}}^{k-1}\psi_3^{*(r)}, \phi_3^{*(r)}\rangle\right.\\ &\left.+ \langle \Lambda^{(N-r)^{k-1}}\phi_3^{(N-r)}, \phi_3^{*(N-r)}\rangle\right)\\ &+ \sum_{\substack{i+l=k-2\\ i,l\ge 0}}\left[\begin{vmatrix}\Delta_{11}^i & \Delta_{12}^i\\ \Delta_{21}^l & \Delta_{22}^l\end{vmatrix} + \begin{vmatrix}\Delta_{11}^i & \Delta_{13}^i\\ \Delta_{31}^l & \Delta_{33}^l\end{vmatrix} + \begin{vmatrix}\Delta_{22}^i & \Delta_{23}^i\\ \Delta_{32}^l & \Delta_{33}^l\end{vmatrix}\right],\end{aligned} \tag{36}$$

$$
\begin{aligned}
\widehat{F}_{3k} \;=\;& \gamma_2\gamma_3\big(\langle \Lambda^{(r)^{k-1}}\phi_1^{(r)}, \psi_1^{(r)}\rangle + \langle \overline{\Lambda^{(r)}}^{k-1}\psi_1^{*(r)}, \phi_1^{*(r)}\rangle \\
&+ \langle \Lambda^{(N-r)^{k-1}}\phi_1^{(N-r)}, \phi_1^{*(N-r)}\rangle\big) + \gamma_1\gamma_3\big(\langle \Lambda^{(r)^{k-1}}\phi_2^{(r)}, \psi_2^{(r)}\rangle \\
&+ \langle \overline{\Lambda^{(r)}}^{k-1}\psi_2^{*(r)}, \phi_2^{*(r)}\rangle + \langle \Lambda^{(N-r)^{k-1}}\phi_2^{(N-r)}, \phi_2^{*(N-r)}\rangle\big) \\
&+ \gamma_1\gamma_2\big(\langle \Lambda^{(r)^{k-1}}\phi_3^{(r)}, \psi_3^{(r)}\rangle + \langle \overline{\Lambda^{(r)}}^{k-1}\psi_3^{*(r)}, \phi_3^{*(r)}\rangle \\
&+ \langle \Lambda^{(N-r)^{k-1}}\phi_3^{(N-r)}, \phi_3^{*(N-r)}\rangle\big) \\
&+ \sum_{\substack{i+l=k-2\\ i,l\geq 0}} \left[\gamma_3 \begin{vmatrix} \Delta_{11}^i & \Delta_{12}^i \\ \Delta_{21}^l & \Delta_{22}^l \end{vmatrix} + \gamma_2 \begin{vmatrix} \Delta_{11}^i & \Delta_{13}^i \\ \Delta_{31}^l & \Delta_{33}^l \end{vmatrix} + \gamma_1 \begin{vmatrix} \Delta_{22}^i & \Delta_{23}^i \\ \Delta_{32}^l & \Delta_{33}^l \end{vmatrix}\right] \\
&+ \sum_{\substack{i+l+n=k-3\\ i,l,n\geq 0}} \begin{vmatrix} \Delta_{11}^i & \Delta_{12}^i & \Delta_{13}^i \\ \Delta_{21}^l & \Delta_{22}^l & \Delta_{23}^l \\ \Delta_{31}^n & \Delta_{32}^n & \Delta_{33}^n \end{vmatrix},
\end{aligned}
\tag{37}
$$

where $1 \leq k \leq N + r$,

$$
\Delta_{ab}^s = \langle \Lambda^{(r)^s}\phi_a^{(r)}, \psi_b^{(r)}\rangle + \langle \overline{\Lambda^{(r)}}^s\psi_a^{*(r)}, \phi_b^{*(r)}\rangle + \langle \Lambda^{(N-r)^s}\phi_a^{(N-r)}, \phi_b^{*(N-r)}\rangle.
$$

It can be checked that $\widehat{F}_{1k}, \widehat{F}_{2k}, \widehat{F}_{3k}$, $1 \leq k \leq N + r$, are $3(N + r)$ integrals in involution. To establish the completely integrability of Hamiltonian systems (30) and (31), it is essential to prove the functional independence of the above $3(N + r)$ integrals by making use of a small epsilon technique [16].

In fact, let P_0 be a point of $\mathbb{C}^{3(N+r)}$ satisfying $\widehat{\phi}_{is} = \varepsilon$, $1 \leq i \leq 3$, $1 \leq s \leq N + r$, where ε is a small constant, and $\widehat{\phi}_{is}, \widehat{\psi}_{is}$, $1 \leq i \leq 3$, $1 \leq s \leq N + r$, are defined by (26). At this point P_0, by a direct calculation, we have

$$
\begin{aligned}
J \;=\;& \frac{\partial\big(F_{11}, \ldots, F_{1,N+r}, F_{21}, \ldots, F_{2,N+r}, F_{31}, \ldots, F_{3,N+r}\big)}{\partial\big(\widehat{\psi}_{11}, \ldots, \widehat{\psi}_{1,N+r}, \widehat{\psi}_{21}, \ldots, \widehat{\psi}_{2,N+r}, \widehat{\psi}_{31}, \ldots, \widehat{\psi}_{3,N+r}\big)} \\
=\;& \varepsilon^{3(N+r)} \begin{vmatrix} \theta_{N+r} & \theta_{N+r} & \theta_{N+r} \\ (\gamma_2+\gamma_3)\theta_{N+r} & (\gamma_1+\gamma_3)\theta_{N+r} & (\gamma_1+\gamma_2)\theta_{N+r} \\ \gamma_2\gamma_3\theta_{N+r} & \gamma_1\gamma_3\theta_{N+r} & \gamma_1\gamma_2\theta_{N+r} \end{vmatrix} + O(\varepsilon^{3(N+r)+1}) \\
=\;& \varepsilon^{3(N+r)} \det(\Omega_3 \otimes \theta_{N+r}) + O(\varepsilon^{3(N+r)+1}) \\
=\;& \varepsilon^{3(N+r)} \prod_{1\leq i<j\leq 3} (\gamma_i - \gamma_j)^N \prod_{1\leq i<j\leq N+r} (\lambda_j - \lambda_i)^3 + O(\varepsilon^{3(N+r)+1}),
\end{aligned}
$$

where

$$\Omega_3 = \begin{pmatrix} 1 & 1 & 1 \\ \gamma_2+\gamma_3 & \gamma_1+\gamma_3 & \gamma_1+\gamma_2 \\ \gamma_2\gamma_3 & \gamma_1\gamma_3 & \gamma_1\gamma_2 \end{pmatrix},$$

$$\theta_N = (\theta_{ij}^{(N)})_{N\times N}, \quad \lambda_{N+i} = \lambda_i^*, \quad 1 \le i \le r.$$

Since the Jacobian is a polynomial function of ϕ_{is}, ψ_{is}, ϕ_{is}^*, ψ_{is}^*, $1 \le i \le 3$, $1 \le s \le N+r$, it is not zero over a dense open subset of $\mathbb{C}^{3(N+r)}$. Therefore, the functions $\widehat{F}_{ik}$, $1 \le i \le 3$, $1 \le k \le N+r$, are functionally independent over that dense open subset of $\mathbb{C}^{3(N+r)}$.

Moreover, a direct computation shows that the following proposition holds.

Theorem 1. *If* $\phi_j^{(r)}$, $\psi_j^{(r)}$, $\phi_j^{*(r)}$, $\psi_j^{*(r)}$, $\phi_j^{(N-r)}$, $\phi_j^{*(N-r)}$, $1 \le j \le 3$, *satisfy both* (30) *and* (31), *then*

$$q_j = \frac{3i}{\gamma_{j+1}-\gamma_1}\big(\langle\phi_1^{(r)}, \psi_{j+1}^{(r)}\rangle + \langle\psi_1^{*(r)}, \phi_{j+1}^{*(r)}\rangle + \langle\phi_1^{(N-r)}, \phi_{j+1}^{*(N-r)}\rangle\big), \quad 1 \le j \le 2,$$

solve the Manakov equation (1). *This means that* (30) *and* (31) *constitute an integrable decomposition of the Manakov equation.*

4 Concluding remarks

In this paper, we have presented the integrable decomposition of the Manakov equation. The main idea of this method is to couple the spectral problems of the Manakov equation with their complex conjugates and then reduce these problems to that of 4-component AKNS equation by introducing new variables. Finally, the integrable decompositions of the Manakov equation are obtained by applying the corresponding results of the 4-component AKNS equation.

As indicated in [17], usually it is very complicated to deal with the soliton equation with reduction conditions or reality conditions. In [18, 19], one of the author (Zhou) proposed an approach to construct integrable decompositions of soliton equations with reality conditions such as the nonlinear Schrödinger equation, real-valued mKdV equation and the nonlinear derivative Schrödinger equation. But the approach can only be used to the cases that all the eigenvalue parameters $\lambda_1, \ldots, \lambda_N$ are nonreal complex. In the present paper, such restrictions are released by using binary nonlinearizations of spectral problem, instead of mono-nonlinearization of spectral problem. Viewing from (33), we can easily find that whether the eigenvalue parameters are real or not leads to completely different symplectic structures.

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