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# An Integral Collocation Approach Based on Legendre Polynomials for Solving Riccati, Logistic and Delay Differential Equations

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## Abstract

In this paper, we propose and analyze some schemes of the integral collocation formulation based on Legendre polynomials. We implement these formulae to solve numerically Riccati, Logistic and delay differential equations with variable coefficients. The properties of the Legendre polynomials are used to reduce the proposed problems to the solution of non-linear system of algebraic equations using Newton iteration method. We give numerical results to satisfy the accuracy and the applicability of the proposed schemes.

## Keywords

## 1. Introduction

It is well known that the ordinary differential equations (ODEs) have been the focus of many studies due to their frequent appearance in various applications, such as in fluid mechanics, viscoelasticity, biology, physics and engineering applications, for more details, for example [1]-[5]. Consequently, considerable attention has been given to the efficient numerical solutions of ODEs of physical interest, because it is difficult to find exact solutions. Different numerical methods have been proposed in the literature for solving ODEs [6]-[13].

The Riccati differential equation (RDE) is named after the Italian Nobleman Count Jacopo Francesco Riccati

(1676-1754). The book of Reid [14] contains the fundamental theories of Riccati equation, with applications to random processes, optimal control, and diffusion problems. Besides important engineering science applications that today are considered classical, such as stochastic realization theory, optimal control, robust stabilization, and network synthesis, the newer applications include such areas as financial mathematics [15]. The solution of this equation can be reached using classical numerical methods such as, the forward Euler method and Runge-Kutta method. Bahnasawi *et al.* [16] presented the usage of Adomian decomposition method to solve the non-linear RDE in an analytic form. Tan and Abbasbandy [17] employed the analytic technique called homotopy analysis method to solve the quadratic RDE.

The Logistic model can be obtained by applying the derivative operator on the Logistic equation. The model is initially published by Pierre Verhulst in 1838 [18]. The continuous Logistic model is described by first order ODE. The discrete Logistic model is simple iterative equation that reveals the chaotic property in certain regions [19]. There are many variations of the population modeling [20]. The Verhulst model is the classic example to illustrate the periodic doubling and chaotic behavior in dynamical system [19]. The model describs the population growth may be limited by certain factors like population density [20].

## 2. Applications of Logistic Equation

A typical application of the Logistic equation is a common model of population growth. Let $u(t)$ represents the population size and $t$ represents the time where the constant $\rho > 0$ defines the growth rate.

Another application of Logistic curve is in medicine, where the Logistic differential equation is used to model the growth of tumors. This application can be considered an extension of the above mentioned use in the framework of ecology.

The solution of Logistic equation is explained the constant population growth rate which not includes the limitation on food supply or spread of diseases [21]. The solution curve of the model is increase exponentially from the multiplication factor up to saturation limit which is maximum carrying capacity [21],

$\dfrac{dN}{dt} = \rho N \left( 1 - \dfrac{N}{K} \right)$ where $N$ is the population with respect to time, $\rho$ is the rate of maximum population

growth and $K$ is the carrying capacity. The solution of continuous Logistic equation is in the form of constant growth rate as in formula $N(t) = N_0 e^{\rho t}$ where $N_0$ is the initial population [22].

A delay differential equation (DDE) is a differential equation in which the derivative of the function at any time depends on the solution at previous time. Introduction of delay in the model enriches its dynamics and allows a precise description of the real life phenomena. DDEs are proved useful in control systems [23], lasers, traffic models [24], metal cutting, epidemiology, neuroscience, population dynamics [25], chemical kinetics [26], etc. In DDE, one has to provide history of the system over the delay interval $[-\tau, 0]$ as the initial condition. Due to this reason delay systems are infinite dimensional in nature. Because of in infinite dimensionality the DDEs are difficult to analyze analytically and hence the numerical solutions play an important role.

In [27], Mai-Duy, *et al.* derived an integral collocation approach based on Chebyshev polynomials and used it for solving numerically the bi-harmonic equations. In [28], Bhrawy and Alofi introduced a new shifted Chebyshev operational matrix of fractional integration of arbitrary order and applied together with spectral tau method for solving linear fractional differential equations. Khader *et al.* [29] introduced a new approximate formula of the fractional derivative using Legendre series expansion and used it to solve numerically the fractional diffusion equation.

In this article, we extend the previous work and derive some schemes of the integral collocation formulation based on Legendre polynomials. We implement these formulae to solve numerically Riccati, Logistic and delay differential equations.

Our paper is organized as follows: In Section 2, we derive some integration collocation formulations using Legendre series expansion. In Section 3, we give the integral collocation approach for solving Riccati, Logistic and delay differential equations. In Section 4, the paper ends with a brief conclusion and some remarks.

## 3. Integration Collocation Formulations

The well known Legendre polynomials are defined on the interval $[-1, 1]$ and can be determined with the aid of the following recurrence formula [30]

$$L_{k+1}(z) = \frac{2k+1}{k+1}zL_k(z) - \frac{k}{k+1}L_{k-1}(z), k = 1, 2, \cdots,$$

where $L_0(z) = 1$ and $L_1(z) = z$. In order to use these polynomials on the interval $[0,1]$, we define the so called shifted Legendre polynomials by introducing the change of variable $z = 2x - 1$. Let the shifted Legendre polynomials $L_k(2x-1)$ be denoted by $L_k^*(x)$. Then $L_k^*(x)$ can be obtained as follows

$$L_{k+1}^*(x) = \frac{(2k+1)(2x-1)}{(k+1)}L_k^*(x) - \frac{k}{k+1}L_{k-1}^*(x),$$

where $L_0^*(x) = 1$ and $L_1^*(x) = 2x - 1$. The analytic form of the shifted Legendre polynomials $L_k^*(x)$ of degree $k$ is given by

$$L_k^*(x) = \sum_{i=0}^{k}(-1)^{k+i}\frac{(k+i)!}{(k-i)!(i!)^2}x^i, k = 2, 3, \cdots. \tag{1}$$

Note that $L_k^*(0) = (-1)^k$ and $L_k^*(1) = 1$. The orthogonality condition is

$$\int_0^1 L_i^*(x)L_j^*(x)\mathrm{d}x = \begin{cases} \dfrac{1}{2i+1}, & \text{for } i = j, \\ 0, & \text{for } i \neq j. \end{cases}$$

In this work, to consider the differential equation of $n$-th order, we build the integration collocation method using the truncated Legendre series of degree $m$ to represent the $n$-th derivative of the unknown function $u(x)$ in the following manner

$$\frac{\mathrm{d}^n u(x)}{\mathrm{d}x^n} \cong \sum_{k=0}^{m}a_k L_k^*(x) = \sum_{k=0}^{m}a_k I_k^{(n)}(x). \tag{2}$$

Using the integration we can obtain the lower-order derivatives and the function itself as follows

$$\frac{\mathrm{d}^{n-1}u(x)}{\mathrm{d}x^{n-1}} \cong \sum_{k=0}^{m}a_k I_k^{(n-1)}(x) + c_1, \tag{3}$$

$$\frac{\mathrm{d}^{n-2}u(x)}{\mathrm{d}x^{n-2}} \cong \sum_{k=0}^{m}a_k I_k^{(n-2)}(x) + c_1 x + c_2, \cdots \tag{4}$$

$$\frac{\mathrm{d}u(x)}{\mathrm{d}x} = \sum_{k=0}^{m}a_k I_k^{(1)}(x) + c_1\frac{x^{n-2}}{(n-2)!} + c_2\frac{x^{n-3}}{(n-3)!} + \cdots + c_{n-2}x + c_{n-1}, \tag{5}$$

$$u(x) = \sum_{k=0}^{m}a_k I_k^{(0)}(x) + c_1\frac{x^{n-1}}{(n-1)!} + c_2\frac{x^{n-2}}{(n-2)!} + \cdots + c_{n-1}x + c_n, \tag{6}$$

from (1) and (2) we have

$$I_k^{(n)}(x) = \sum_{i=0}^{k}\frac{(-1)^{k+i}(k+i)!}{(k-i)!(i!)^2}x^i,$$

$$I_k^{(n-1)}(x) = \int I_k^{(n)}(x)\mathrm{d}x = \sum_{i=0}^{k}\frac{(-1)^{k+i}(k+i)!}{(k-i)!(i!)^2(i+1)}x^{i+1},$$

$$I_k^{(n-2)}(x) = \int I_k^{(n-1)}(x)\mathrm{d}x = \sum_{i=0}^{k}\frac{(-1)^{k+i}(k+i)!}{(k-i)!(i!)^2(i+1)(i+2)}x^{i+2}, \tag{7}$$

$$\cdots$$

$$I_k^{(0)}(x) = \int I_k^{(1)}(x)\mathrm{d}x = \sum_{i=0}^{k}\frac{(-1)^{k+i}(k+i)!x^{i+n}}{(k-i)!(i!)^2(i+1)\cdots(i+n-1)(i+n)}.$$

We now collocate Equatuions (2)-(6) at $(m+1)$ points $x_p$, $p=0,1,\cdots,m$ as

$$\frac{d^n u(x_p)}{dx^n} = \Omega^{(n)} \hat{S}, \quad \frac{d^{n-1} u(x_p)}{dx^{n-1}} = \Omega^{(n-1)} \hat{S}, \cdots,$$

$$\frac{du(x_p)}{dx} = \Omega^{(1)} \hat{S}, \quad u(x_p) = \Omega^{(0)} \hat{S},$$

$$(8)$$

where $\hat{S} = [a_0, a_1, \cdots, a_m, c_1, c_2, \cdots, c_n]^T$, and $\Omega^{(n)}, \Omega^{(n-1)}, \cdots, \Omega^{(0)}$ are integrated matrices.

## 4. Integral Collocation Approach for Solving Riccati, Logistic and Delay Differential Equations

In this section, we introduce the integral collocation approach using Legendre expansion for solving the Riccati, Logistic and delay differential equations.

### 4.1. Model 1: Riccati Differential Equation

$$\frac{du(x)}{dx} + u^2(x) - 1 = 0, \qquad x > 0, \tag{9}$$

we also assume an initial condition

$$u(0) = u^0. \tag{10}$$

The exact solution to this problem at $u^0 = 0$ is

$$u(x) = \frac{e^{2x} - 1}{e^{2x} + 1}.$$

The procedure of the implementation is given by the following steps:

1) Approximate the function $u(x)$ using Formula (6) and its relevant derivatives with $m=5$, as follows

$$\frac{du(x)}{dx} \cong \sum_{k=0}^{5} a_k I_k^{(1)}(x),$$

$$u(x) \cong \sum_{k=0}^{5} a_k I_k^{(0)}(x) + c_1,$$

$$(11)$$

where $I_k^{(0)}(x)$ is defined in (7) as

$$I_k^{(0)}(x) = \sum_{i=0}^{k} (-1)^{k+i} \frac{(k+i)!}{(k-i)!(i!)^2 (i+1)} x^{i+1},$$

$$I_k^{(1)}(x) = \sum_{i=0}^{k} (-1)^{k+i} \frac{(k+i)!}{(k-i)!(i!)^2} x^{i}.$$

Then the Riccati differential Equation (9) is transformed to the following approximated form

$$\sum_{k=0}^{5} a_k L_k^*(x) + \left( \sum_{k=0}^{5} a_k I_k^{(0)}(x) + c_1 \right)^2 = 1. \tag{12}$$

We now collocate Equation (12) at $(m+1=6)$ points $x_p$, $p=0,1,2,3,4,5$ as

$$\sum_{k=0}^{5} a_k L_k^*(x_p) + \left( \sum_{k=0}^{5} a_k I_k^{(0)}(x_p) + c_1 \right)^2 = 1. \tag{13}$$

For suitable collocation points we use the roots of shifted Legendre polynomial $L_6^*(x)$.

2) Also, by substituting from the initial condition (10) in (11) we can obtain $(n=1)$ an equation which gives the value of the constant $c_1$ as follows

$$c_1 = u^0 = 0. \tag{14}$$

Equations (13) and (14) represent a system of non-linear algebraic equations which contains seven equations for the unknowns $a_k, k = 0, 1, 2, 3, 4, 5$ and $c_1$.

3) Solve the resulting system using the Newton iteration method to obtain the unknowns $a_k, k = 0, 1, 2, 3, 4, 5$ as follows

$$a_0 = 0.76160, \quad a_1 = -0.31790, \quad a_2 = -0.0506,$$
$$a_3 = 0.02896, \quad a_4 = -0.00111, \quad a_5 = -0.00112.$$

Therefore, from Formula (12) we can obtain the approximate solution in the form

$$u(x) \cong \sum_{k=0}^{5} a_k I_k^{(0)}(x) + c_1$$

$$s = 0.990x + 0.002x^2 - 0.346x^3 + 0.027x^4 + 0.125x^5 - 0.047x^6.$$

The numerical results of the proposed problem (9) are given in **Figure 1** with different values of $m(m = 5, 7)$ in the interval $[0,1]$ at $u^0 = 0$. From this figure, since the obtained numerical solutions are in excellent agreement with the exact solution, so, we can conclude that the proposed technique is well for solving such class of ODEs.
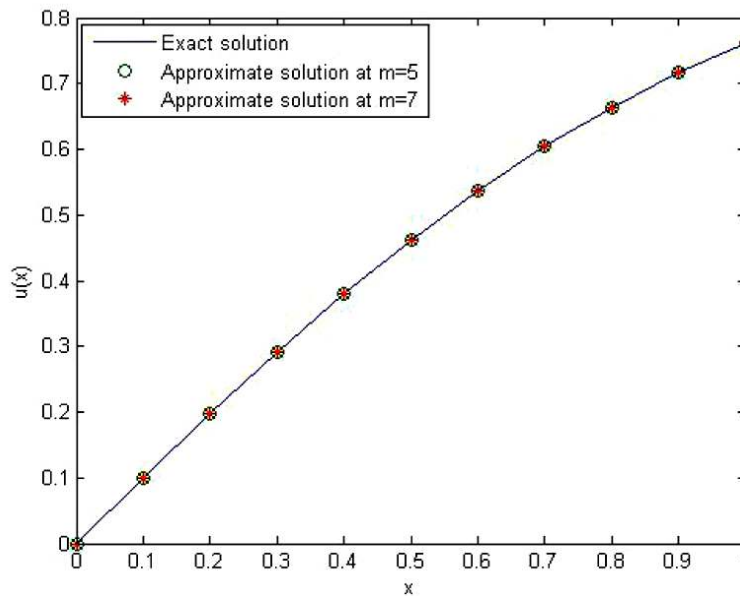
## 4.2. Model 2: Logistic Differential Equation

$$\frac{\mathrm{d}u(x)}{\mathrm{d}x} = \rho u(x)(1 - u(x)), \quad x > 0, \ \rho > 0. \tag{15}$$

We also assume an initial condition

$$u(0) = u_0, \qquad u_0 > 0. \tag{16}$$

The exact solution to this problem is given by

$$u(x) = \frac{u_0}{(1 - u_0)\mathrm{e}^{-\rho x} + u_0}.$$



**Figure 1.** A comparison between the exact solution and the approximate solution at $m = 5$ and $m = 7$.

The existence and the uniqueness of the proposed problem (15) are introduced in details in [31].

The procedure of the implementation is given by the following steps:

1) Approximate the function $u(x)$ and its relevant derivatives with $m = 5$, using Formula (11). Then the Logistic differential Equation (15) is transformed to the following approximated form

$$\sum_{k=0}^{5} a_k L_k^*(x) - \rho \left( \sum_{k=0}^{5} a_k I_k^{(0)}(x) + c_1 \right)$$
$$\left( 1 - \left( \sum_{k=0}^{5} a_k I_k^{(0)}(x) + c_1 \right) \right) = 0. \tag{17}$$

We now collocate Equation (17) at $(m+1=6)$ points $x_p$, $p = 0,1,2,3,4,5$ as

$$\sum_{k=0}^{5} a_k L_k^*(x_p) - \rho \left( \sum_{k=0}^{5} a_k I_k^{(0)}(x_p) + c_1 \right) \left( 1 - \left( \sum_{k=0}^{5} a_k I_k^{(0)}(x_p) + c_1 \right) \right) = 0. \tag{18}$$

For suitable collocation points we use roots of shifted Legendre polynomial $L_6^*(x)$ which are

$$x_0 = 0.96623, \quad x_1 = 0.03377,$$
$$x_2 = 0.38069, x_3 = 0.61930, \tag{19}$$
$$x_4 = 0.16931, \quad x_5 = 0.83060.$$

2) Also, by substituting from the initial condition (16) in (11) with $u_0 = 0.85$ we can obtain $(n = 1)$ an equation which gives the value of the constant $c_1 = 0.85$.

Equation (18) represents a system of non-linear algebraic equations which contains six equations for the unknowns $a_k, k = 0,1,2,3,4,5$.

3) Solve the resulting system using the Newton iteration method to obtain the unknowns $a_k, k = 0,1,2,3,4,5$ as follows

$$a_0 = 0.0533, \quad a_1 = -0.0101,$$
$$a_2 = 0.0004, a_3 = 0.00002, \tag{20}$$
$$a_4 = -1.642 \times 10^{-6}, \ a_5 = 5.398 \times 10^{-8}.$$

Therefore, from Formula (11) we can obtain the approximate solution in the form

$$u(x) = \sum_{k=0}^{5} a_k I_k^{(0)}(x) + c_1$$
$$= 1 + x + 0.5x^2 + 0.1667x^3 + 0.0417x^4 + 0.0083x^5 + 0.0014x^6 + 0.0002x^7 + 0.00004x^8.$$

The numerical results of the proposed problem (15) are given in **Figure 2** with different values of $m (m = 3, 5)$ in the interval $[0,1]$. From this figure, since the obtained numerical solutions are in excellent agreement with the exact solution, so, we can conclude that the proposed technique is well for solving such class of ODEs.

## 4.3. Model 3: Delay Differential Equation

Consider the linear delay differential equation of third-order

$$\frac{\mathrm{d}^3 u(x)}{\mathrm{d}x^3} = -u(x) - u(x - 0.3) + \mathrm{e}^{-x+0.3}, 0 \le x \le 1, \tag{21}$$

with the initial conditions

$$u(0) = 1, \frac{\mathrm{d}u(0)}{\mathrm{d}x} = -1, \frac{\mathrm{d}^2 u(0)}{\mathrm{d}x^2} = 1, u(x) = \mathrm{e}^{-x}. \tag{22}$$

The exact solution of this model is $u(x) = \mathrm{e}^{-x}$.

The procedure of the implementation is given by the following steps:

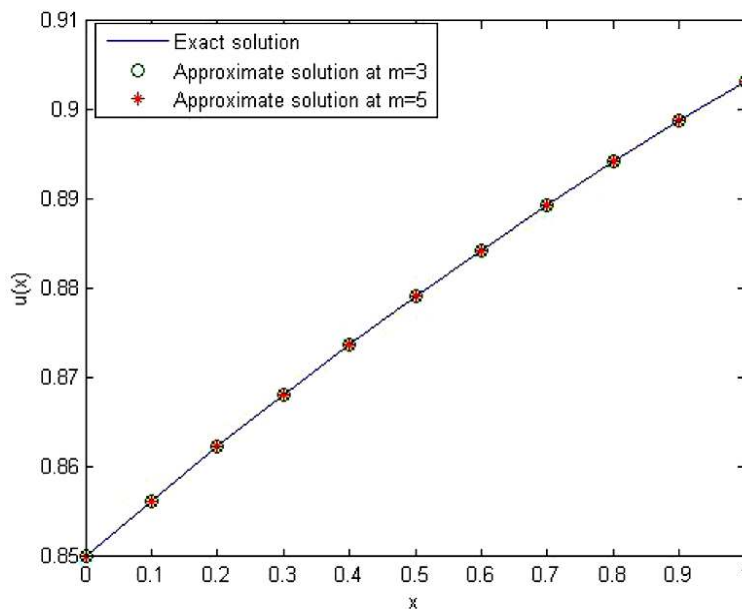1) Approximate the function $u(x)$ using Formula (6) and its relevant derivatives with $m = 5$, as follows

**Figure 2.** A comparison between the exact solution and the approximate solution at $m = 3$ and $m = 5$.

$$\frac{d^3 u(x)}{dx^3} \cong \sum_{k=0}^{5} a_k L_k^*(x) = \sum_{k=0}^{3} a_k I_k^{(3)}(x),$$

$$\frac{d^2 u(x)}{dx^2} \cong \sum_{k=0}^{5} a_k I_k^{(2)}(x) + c_1,$$

$$\frac{du(x)}{dx} \cong \sum_{k=0}^{5} a_k I_k^{(1)}(x) + c_1 x + c_2,$$

$$u(x) \cong \sum_{k=0}^{5} a_k I_k^{(0)}(x) + c_1 \frac{x^2}{2!} + c_2 \frac{x}{1!} + c_3,$$

(23)

where $I_k^{(0)}(x)$, $I_k^{(1)}(x)$ and $I_k^{(x)}(x)$ are defined in (7) as follows

$$I_k^{(0)}(x) = \sum_{i=0}^{k} \frac{(-1)^{k+i}(k+i)!}{(k-i)!(i!)^2 (i+1)(i+2)(i+3)} x^{i+3},$$

$$I_k^{(1)}(x) = \sum_{i=0}^{k} \frac{(-1)^{k+i}(k+i)!}{(k-i)!(i!)^2 (i+1)(i+2)} x^{i+2},$$

$$I_k^{(2)}(x) = \sum_{i=0}^{k} \frac{(-1)^{k+i}(k+i)!}{(k-i)!(i!)^2 (i+1)} x^{i+1}.$$

Then the delay differential Equation (21) is transformed to the following approximated form

$$\sum_{k=0}^{5} a_k L_k^*(x) + \left( \sum_{k=0}^{5} a_k I_k^{(0)}(x) + c_1 \frac{x^2}{2!} + c_2 \frac{x}{1!} + c_3 \right)$$

$$+ \left( \sum_{k=0}^{5} a_k I_k^{(0)}(x - 0.3) + c_1 \frac{(x-0.3)^2}{2!} + c_2 \frac{(x-0.3)}{1!} + c_3 \right) = e^{-x+0.3}.$$

(24)

We now collocate Equation (24) at $(m + 1 = 6)$ points $x_p$, $p = 0,1,2,3,4,5$ as

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$$\sum_{k=0}^{5} a_k L_k^* (x_p) + \left( \sum_{k=0}^{5} a_k I_k^{(0)} (x_p) + c_1 \frac{x_p^2}{2!} + c_2 \frac{x_p}{1!} + c_3 \right) + \left( \sum_{k=0}^{5} a_k I_k^{(0)} (x_p)(x_p - 0.3) + c_1 \frac{(x_p - 0.3)^2}{2!} \right.$$

$$\left. + c_2 \frac{(x_p - 0.3)}{1!} + c_3 \right) = e^{-x_p + 0.3}. \tag{25}$$

For suitable collocation points we use the roots of shifted Legendre polynomial $L_6^*(x)$.

2) Also, by substituting from the initial conditions (22) in (23) we can obtain $(n = 3)$ equations which give the values of the constants

$$c_1 = 1, \quad c_2 = -1, \quad c_3 = 1. \tag{26}$$

Equations (25) and (26) represent a system of linear algebraic equations which contains nine equations for the unknowns $a_k, k = 0,1,2,3,4,5$ and $c_1, c_2, c_3$.

3) Solve the resulting system using the conjugate gradient method to obtain the unknowns $a_k, k = 0,1,2,3,4,5$ as follows

$$a_0 = -0.6321, \quad a_1 = 0.3109, \quad a_2 = -0.0515,$$
$$a_3 = 0.0051, \quad a_4 = 0.0004, \quad a_5 = 0.00002. \tag{27}$$

Therefore, from Formula (23) we can obtain the approximate solution in the form

$$u(x) \cong \sum_{k=0}^{5} a_k I_k^{(0)} (x) + c_1 \frac{x^2}{2!} + c_2 \frac{x}{1!} + c_3$$

$$= 1 - x + 0.5x^2 - 0.1667x^3 + 0.0417x^4 - 0.0084x^5 + 0.0014x^6 - 0.0002x^7 + 0.00002x^8.$$

The numerical results of the proposed problem (21) are given in **Figure 3** with different values of $m(m = 5,7)$ in the interval $[0,1]$. From this figure, since the obtained numerical solutions are in excellent agreement with the exact solution, so, we can conclude that the proposed technique is well for solving such class of ODEs.

## 5. Conclusion and Remarks

In this article, an integral collocation approach based on Legendre polynomials is introduced for solving
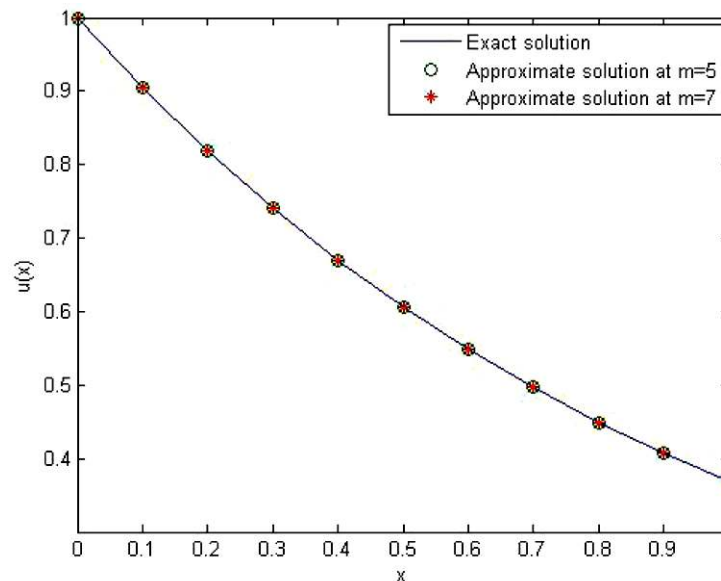


**Figure 3.** A comparison between the exact solution and the approximate solution at $m = 5$ and $m = 7$.

numerically the Riccati, Logistic and delay differential equations. The properties of the Legendre polynomials are used to reduce the proposed problems to system of algebraic equations which are solved by a suitable numerical method. From the obtained numerical results, we can conclude that this method gives results with an excellent agreement with the exact solution. All numerical results are obtained using Matlab program 8.

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