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AN INTEGRAL EQUATION MODELING OF ELECTROMAGNETIC SCATTERING FROM THE SURFACES OF ARBITRARY RESISTANCE DISTRIBUTION

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Abstract—In this paper the problem of electromagnetic scattering from the resistive surfaces is carefully surveyed. We model this problem by the integral equations of the second kind. A new set of orthogonal basis functions is used to solve these integral equations via collocation method. Numerical solutions of these equations are given for some cases of resistance distributions. Presented method in this paper can be easily generalized to apply to other cases.

1. INTRODUCTION

The development of numerical methods for solving integral equations in Electromagnetics has attracted intensive researches for more than four decades [1, 2]. The use of high-speed computers allows one to make more computations than ever before. During these years, careful analysis has paved the way for the development of efficient and effective numerical methods and, of equal importance, has provided a solid foundation for a through understanding of the techniques.

Over several decades, electromagnetic scattering problems have been the subject of extensive researches (see [3–27]). Scattering from arbitrary surfaces such as square, cylindrical, circular, spherical [3–9] are commonly used as test cases in computational Electromagnetics, because analytical solutions for scattered fields can be derived for these geometries [3].

The problem of electromagnetic scattering from the resistive surfaces leads to solve the integral equations of the second kind with complex kernels. For solving integral equations of the second kind, several numerical approaches have been proposed. These numerical methods often use the basis functions and transform the integral equation to a linear system that can be solved by direct or iterative methods [28]. It is important in these methods to select an appropriate set of basis functions so that the approximate solution of integral equation has a good accuracy.

In this paper we use a new set of orthogonal basis functions called triangular functions (TFs) and apply them to the collocation method. Using this method, the second kind integral equation reduces to a well-condition linear system of algebraic equations. Solving this system gives a stable approximate solution with good accuracy for these problems.

First of all, the problem of electromagnetic scattering from resistive surfaces is treated in detail. Then, an effective implementation of collocation method based on using TFs is presented. Finally, the problem of electromagnetic scattering from these resistive surfaces is solved via presented approach and the numerical results are obtained for some cases of resistance distributions.

2. ELECTROMAGNETIC SCATTERING FROM RESISTIVE SURFACES

In this section, we survey the problem of electromagnetic scattering from resistive surfaces and model this problem by an integral equation of the second kind. In Fig. 1, there is a resistive strip that is very long in the $\pm z$ direction. This strip is encountered by an incoming plane wave that has a polarization with its electric field parallel to the z-axis. The magnetic field of this wave is entirely in the x-y plane, and is therefore transverse to the z-axis. It is called transverse magnetic (TM) polarized wave. This polarization therefore produces a current on the strip that flows along the z-axis.

The magnetic vector potential of the current flowing along the strip is given by [29]:

$$A_z = \frac{\mu_0}{4j} \int_{-a/2}^{a/2} I_z(x') H_0^{(2)}(k|x - x'|) dx' \tag{1}$$

where:

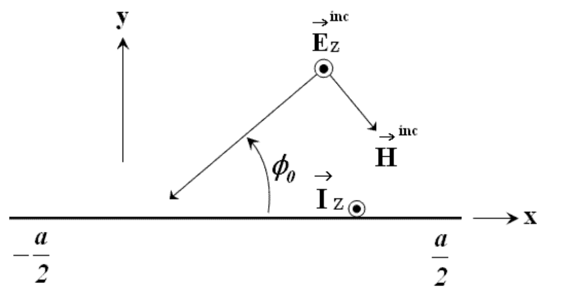

Figure 1. A resistive strip of width a is encountered by an incoming TM-polarized plane wave.

$k = \frac{2\pi}{\lambda}$, free space wave number.
λ is the wave length.
$\mu_0 = 4\pi \times 10^{-7}$ H/m, free space permeability.
$G(x, x') = \frac{1}{4j} H_0^{(2)}(k|x - x'|)$, 2D free space Green's function.
$H_0^{(2)}(x)$ is a Hankel function of the second kind 0th order.
So, the electric field is given by:

$$E_z(x) = j\omega A_z(x) \tag{2}$$

or

$$E(x) = \frac{\omega\mu_0}{4} \int_{-a/2}^{a/2} I_z(x') H_0^{(2)}(k|x - x'|) dx' \tag{3}$$

Assume that $R_s(x)$ is the surface resistance of the strip and note that the units of surface resistance are in Ω/m^2. The boundary condition at the surface of a thin resistive strip is given by the following equation [29]:

$$-E^{inc} = E^{scat} + R_s(x) J(x) \tag{4}$$

where:
$J(x)$ is the surface current of the strip.
E^{scat} is the scattered electric field produced by the surface current.
Assuming $E^{inc} = e^{jkx\cos\phi_0}$, from Eq. (3) and Eq. (4) it follows:

$$R_s(x) I(x) + \frac{\omega\mu_0}{4} \int_{-a/2}^{a/2} I(x') H_0^{(2)}(k|x - x'|) dx' = -e^{jkx\cos\phi_0} \tag{5}$$

where $I(x)$ is the current of the strip.

Equation (5) can be converted to the following equation:

$$h(x) + \int_a^b G(x,x')h(x')dx' = g(x) \tag{6}$$

where:
$h(x) = I(x)$
$G(x,x') = \frac{\omega\mu_0}{4}\frac{1}{R_s(x)}H_0^{(2)}(k|x-x'|)$
$g(x) = -\frac{1}{R_s(x)}e^{jkx\cos\phi_0}$

It is a Fredholm integral equation of the second kind and must be solved by an appropriate method. We will present a new approach to solve this type of integral equation in Section 3. However, from Eq. (5) $I(x)$ can be obtained and then the radar cross section (RCS) of the strip can be computed easily.

RCS in two dimensions is defined mathematically as [29]:

$$\sigma(\phi) = \lim_{r\to\infty} 2\pi r \frac{|\mathbf{E}^{scat}|^2}{|\mathbf{E}^{inc}|^2} \tag{7}$$

In two dimensions, the free space Green's function is:

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{4j}H_0^{(2)}(k|\mathbf{r}-\mathbf{r}'|) \tag{8}$$

The magnetic vector potential in two-dimensional space is:

$$\mathbf{A}(\mathbf{r}) = \mu\int\int \mathbf{J}(\mathbf{r}')G(\mathbf{r},\mathbf{r}')ds' \tag{9}$$

The electric field is given by:

$$\mathbf{E} = j\omega\mathbf{A} \tag{10}$$

Combining (8), (9), and (10) we obtain:

$$\mathbf{E}(\mathbf{r}) = \frac{\omega\mu}{4}\int\int \mathbf{J}(\mathbf{r}')H_0^{(2)}(k|\mathbf{r}-\mathbf{r}'|)ds' \tag{11}$$

In the TM situation, the incident electric field along the strip is 1 V/m ($|\mathbf{E}^{inc}|^2 = 1$). So, the denominator of Eq. (7) is unity. This allows us to turn our attention to the numerator. To evaluate (11), we note that as $r \longrightarrow \infty$, we can use the large argument approximation for the Hankel function [29]:

$$H_0^{(2)}(r) \approx \sqrt{\frac{2}{\pi r}}e^{-j(r-\frac{\pi}{4})} \tag{12}$$

Substituting this into (11) and implementing Eq. (7) for the TM case, we obtain:

$$\sigma(\phi) = \frac{k\eta^2}{4}\left|\int_{strip} I(x', y')e^{jk(x'\cos\phi + y'\sin\phi)}dl'\right|^2 \tag{13}$$

where $\eta = 376.73\,\Omega$.

In the presented case, the strip is restricted to the x-axis, which simplifies Eq. (13):

$$\sigma(\phi) = \frac{k\eta^2}{4}\left|\int_{-a/2}^{a/2} I(x')e^{jkx'\cos\phi}dx'\right|^2 \tag{14}$$

Also, it is possible to define a logarithmic quantity with respect to the RCS, so that:

$$\sigma_{\mathrm{dBlm}} = 10\log_{10}\sigma \tag{15}$$

3. SOLVING THE CURRENT INTEGRAL EQUATION OF THE SECOND KIND VIA COLLOCATION METHOD

3.1. Basis Functions

One very important step in any numerical solution is the choice of basis functions. In general, one chooses as basis functions the set that has the ability to accurately represent and resemble the anticipated unknown function, while minimizing the computational effort required to employ it.

Triangular functions have been introduced by Deb et al. [30] as a new set of orthogonal functions.

Two m-sets of triangular functions (TFs) are defined over the interval $[0, T)$ as:

$$\begin{aligned} T1_i(t) &= \begin{cases} 1 - \dfrac{t - ih}{h} & ih \leq t < (i+1)h, \\ 0 & \text{otherwise} \end{cases} \\ T2_i(t) &= \begin{cases} \dfrac{t - ih}{h} & ih \leq t < (i+1)h, \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{16}$$

where $i = 0, 1, \dots, m-1$, with a positive integer value for m. Also, consider $h = T/m$, and $T1_i$ as the ith left-handed triangular function and $T2_i$ as the ith right-handed triangular function.

These functions are orthogonal [30], so:

$$\begin{aligned}
&\int_0^1 T1_i(t)T1_j(t)dt = \begin{cases} \dfrac{h}{3} & i=j, \\ 0 & i \neq j \end{cases} \\
&\int_0^1 T2_i(t)T2_j(t)dt = \begin{cases} \dfrac{h}{3} & i=j, \\ 0 & i \neq j \end{cases}
\end{aligned} \tag{17}$$

Now, consider the first m terms of left-handed triangular functions and the first m terms of right-handed triangular functions and write them concisely as m-vectors:

$$\begin{aligned}
\mathbf{T1}(t) &= [T1_0(t), T1_1(t), \ldots, T1_{m-1}(t)]^t \\
\mathbf{T2}(t) &= [T2_0(t), T2_1(t), \ldots, T2_{m-1}(t)]^t
\end{aligned} \tag{18}$$

where, $\mathbf{T1}(t)$ and $\mathbf{T2}(t)$ are called left-handed triangular functions (LHTF) vector and right-handed triangular functions (RHTF) vector respectively.

The expansion of a function $f(t)$ with respect to TFs, may be compactly written as:

$$f(t) \simeq \sum_{i=0}^{m-1} c_i T1_i(t) + \sum_{i=0}^{m-1} d_i T2_i(t) = \mathbf{c}^T\mathbf{T1}(t) + \mathbf{d}^T\mathbf{T2}(t) \tag{19}$$

where c_i and d_i are constant coefficients with respect to $T1_i$ and $T2_i$ for $i = 0, 1, \ldots, m-1$, respectively.

Above coefficients can be determined by sampling $f(t)$ such that:

$$\begin{aligned}
&c_i = f(ih), \\
&d_i = f((i+1)h), \qquad \text{for } \ i = 0, 1, \ldots, m-1
\end{aligned} \tag{20}$$

But the optimal representation of $f(t)$ can be obtained if the coefficients c_i and d_i are determined from the following two equations [30]:

$$\begin{aligned}
\int_{ih}^{(i+1)h} f(t)T1_i(t)dt &= c_i \int_{ih}^{(i+1)h} [T1_i(t)]^2 dt + d_i \int_{ih}^{(i+1)h} [T1_i(t)T2_i(t)]dt \\
\int_{ih}^{(i+1)h} f(t)T2_i(t)dt &= c_i \int_{ih}^{(i+1)h} [T1_i(t)T2_i(t)]dt + d_i \int_{ih}^{(i+1)h} [T2_i(t)]^2 dt
\end{aligned} \tag{21}$$

Note that:

$$\int_{ih}^{(i+1)h} [T1_i(t)T2_i(t)]dt = \frac{h}{6} \tag{22}$$

From Eqs. (21) and Eq. (22) coefficients c_i and d_i for $i = 0, 1, \dots, m-1$ can be easily computed.

It is clear that for piecewise linear functions, optimal and non-optimal representations are identical.

3.2. Collocation Method Using TFs

Here, the definition of triangular functions is extended over any interval $[a, b)$. Then, these functions as the basis functions are applied to solve the integral equations of the second kind by collocation method.

Consider the following Fredholm integral equation of the second kind:

$$x(s) + \int_a^b k(s,t)x(t)dt = y(s) \tag{23}$$

where $k(s,t)$ and $y(s)$ are known functions but $x(t)$ is not. Moreover, $k(s,t) \in \mathcal{L}^2([a,b) \times [a,b))$ and $y(s) \in \mathcal{L}^2([a,b))$. Approximating the function $x(s)$ with respect to triangular functions by (4) gives:

$$x(s) \simeq \mathbf{c}^T\mathbf{T1}(s) + \mathbf{d}^T\mathbf{T2}(s) \tag{24}$$

such that the m-vectors $\mathbf{c}$ and $\mathbf{d}$ are TFs coefficients of $x(s)$ that should be determined.

Substituting Eq. (24) into (23) follows:

$$\mathbf{c}^T\left(\mathbf{T1}(s)+\int_a^b k(s,t)\mathbf{T1}(t)dt\right) + \mathbf{d}^T\left(\mathbf{T2}(s)+\int_a^b k(s,t)\mathbf{T2}(t)dt\right) \simeq y(s) \tag{25}$$

Now, let s_i, $i = 0, 1, \dots, 2m-1$ be $2m$ appropriate points in interval $[a, b)$; putting $s = s_i$ in Eq. (25) follows:

$$\mathbf{c}^T\left(\mathbf{T1}(s_i) + \int_a^b k(s_i,t)\mathbf{T1}(t)dt\right) + \mathbf{d}^T\left(\mathbf{T2}(s_i) + \int_a^b k(s_i,t)\mathbf{T2}(t)dt\right) \simeq y(s_i), \qquad i = 0, 1, \dots, 2m-1 \tag{26}$$

or

$$\sum_{j=0}^{m-1}\left[c_j(\mathbf{T1}(s_i)+\int_a^b k(s_i,t)T1_j(t)dt)+ d_j(\mathbf{T2}(s_i)+\int_a^b k(s_i,t)T2_j(t)dt)\right] \simeq y(s_i), \qquad i = 0, 1, \dots, 2m-1 \tag{27}$$

Now, replace $\simeq$ with $=$, hence Eq. (27) is a linear system of $2m$ algebraic equations for $2m$ unknown components $c_0, c_1, \ldots, c_{m-1}$ and $d_0, d_1, \ldots, d_{m-1}$. So, an approximate solution $x(s) \simeq \mathbf{c}^T\mathbf{T1}(s) + \mathbf{d}^T\mathbf{T2}(s)$, is obtained for Eq. (23).

Note that using (20) follows:

$$d_i = c_{i+1}, \qquad \text{for} \ \ i = 0, 1, \ldots, m-2 \tag{28}$$

So, for this representation the count of unknown coefficients in algebraic system (27) can be reduced to $m+1$, therefore it should be considered just $m+1$ equations with selecting $m+1$ appropriate points in interval $[a, b)$.

4. NUMERICAL RESULTS

Now, we can solve the problem of electromagnetic scattering from resistive surfaces using the presented method. Two types of resistance distribution are considered here.

4.1. Uniform Resistance Distribution

Assume that the $R_s(x)$ has a uniform value in throughout of the surface of strip. Considering Eq. (5), $I(x)$ is computed for R_s of 0, 500, 1000 (Ω/m^2), $\phi_0 = \frac{\pi}{2}$, $a = 6\lambda\,(\text{m})$ and $f = 0.3\,\text{GHz}$, and then RCS is obtained of Eqs. (14) and (15). The current distributions of the resistive strip for these values of R_s are shown in Figs. 2–4.

In Fig. 5, the bistatic RCS of the $6-\lambda$ resistive strip, for R_s of 0, 500, 1000 (Ω/m^2), and for $\phi_0 = \frac{\pi}{2}$ has been shown. Also, in Fig. 6 the monostatic RCS of this strip is given. It is seen that the level of the first side lobe is nearly 13 dB down from the main lobe.

4.2. Quadratic Resistance Distribution

Consider a quadratic resistance distribution expressed by:

$$R_s(x) = 2\eta \left(\frac{2x}{a}\right)^2 \quad (\Omega/\text{m}^2) \tag{29}$$

After computing $I(x)$ by Eq. (5), the RCS of this strip can be obtained. For $\phi_o = \frac{\pi}{2}$, the magnitude, real part and imaginary part of strip current are shown in Figs. 7 and 8, and the bistatic radar cross section of this strip shown in Fig. 9 has been calculated for $\phi_0 = \frac{\pi}{2}$ and $f = 0.3\,\text{GHz}$. Fig. 10 shows the monostatic RCS. It is seen that the quadratic resistance distribution reduces the first side lobe to a level

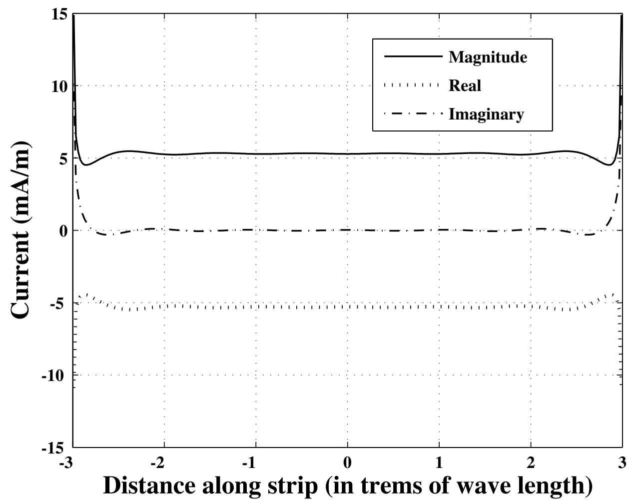

Figure 2. Current distribution across a $6 - \lambda$ strip created by a TM-polarized plane wave for $R_s = 0$ and $f = 0.3\,\mathrm{GHz}$.

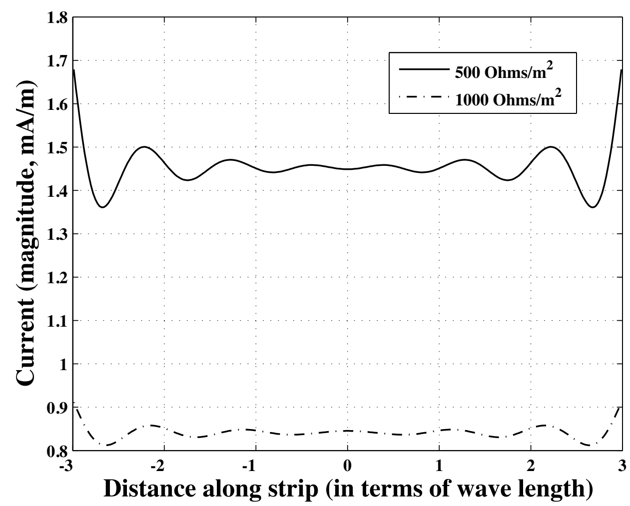

Figure 3. Current magnitude across the $6 - \lambda$ resistive strip for R_s of 500 and 1000 (Ω/m^2) and $f = 0.3\,\mathrm{GHz}$.

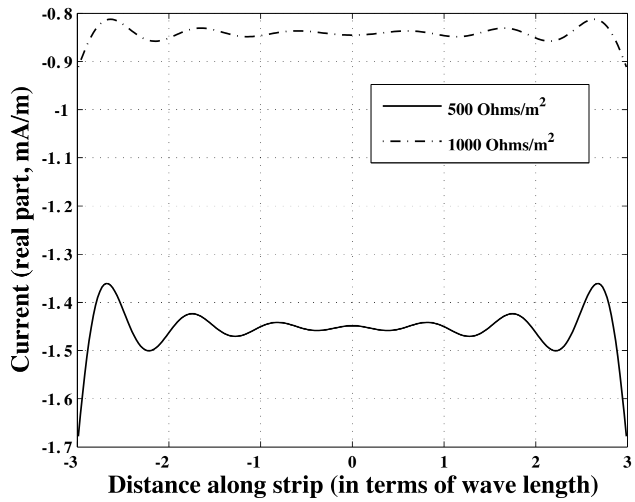

Figure 4. The real part of current across the $6-\lambda$ resistive strip for R_s of 500 and 1000 (Ω/m^2) and $f = 0.3\,\mathrm{GHz}$.

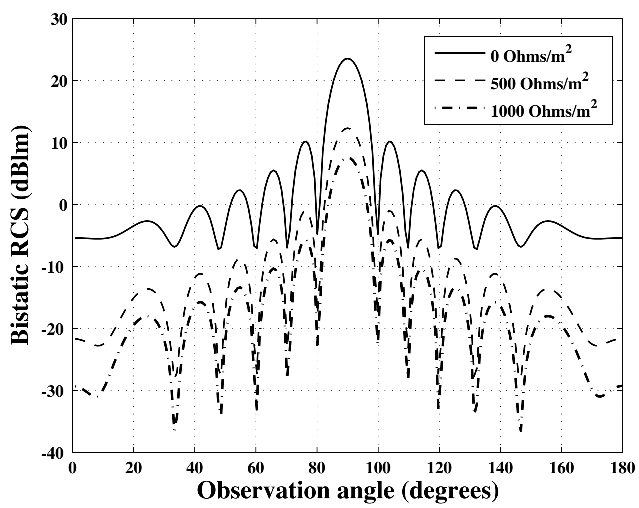

Figure 5. The bistatic RCS of the $6-\lambda$ resistive strip for R_s of 0, 500, 1000 (Ω/m^2) and $\phi_0 = \frac{\pi}{2}$.

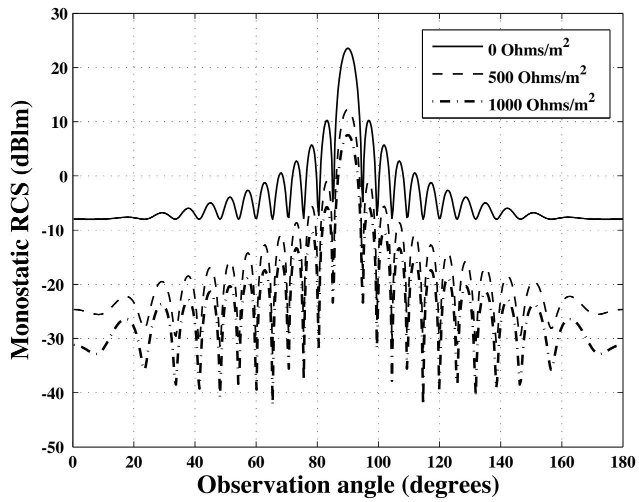

Figure 6. The monostatic RCS of the $6-\lambda$ resistive strip for R_s of 0, 500, 1000 (Ω/m^2).

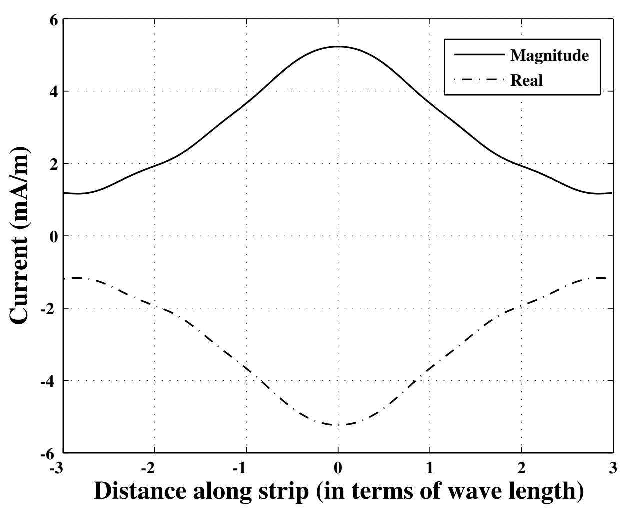

Figure 7. The magnitude and real part of current across the $6-\lambda$ quadratic resistive strip for $f = 0.3\,\mathrm{GHz}$.

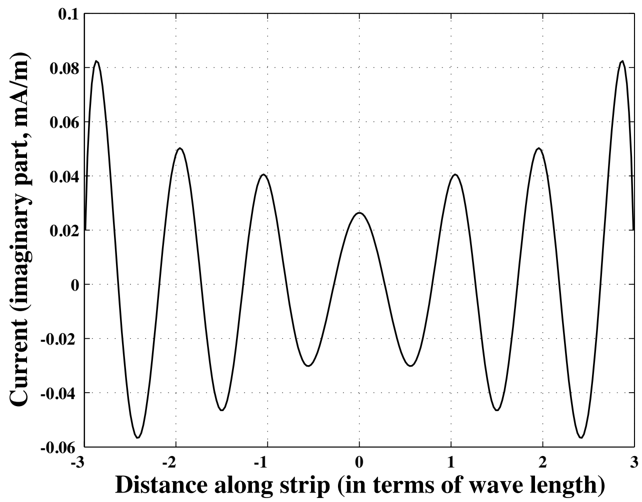

Figure 8. The imaginary part of current across the $6-\lambda$ quadratic resistive strip for $f = 0.3\,\mathrm{GHz}$.

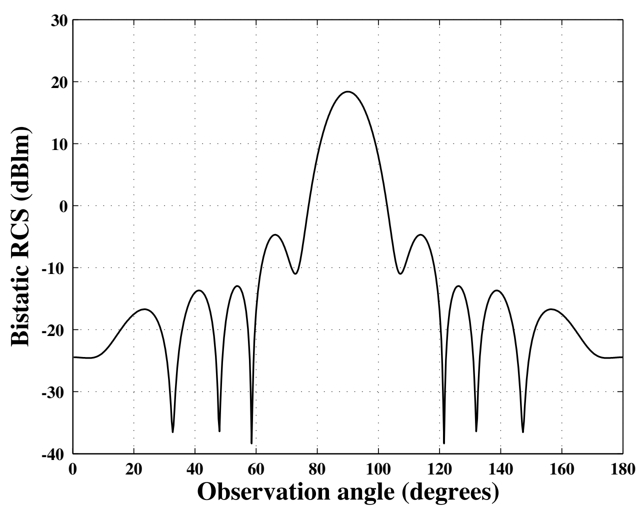

Figure 9. The bistatic RCS of the $6-\lambda$ quadratic resistive strip for $\phi_0 = \frac{\pi}{2}$.

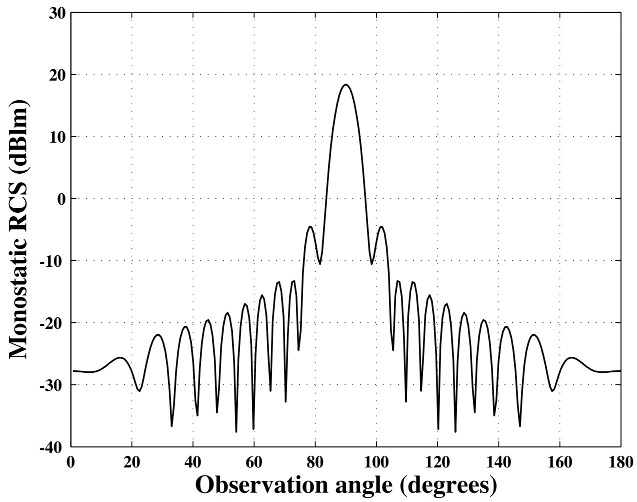

Figure 10. The monostatic RCS of the $6-\lambda$ quadratic resistive strip.

of -23 dB below the main lobe. This taper has reduced the first side lobe by 10 dB, compared with a uniform distribution.

5. CONCLUSION

The presented method in this paper is applied to solve the integral equations of the second kind arising in problem of electromagnetic scattering from resistive surfaces.

Reference to the obtained numerical results, this method reduces an integral equation of the second kind to a well-condition linear system of algebraic equations.

It is clear that we can easily generalize this approach to apply to arbitrary cases.

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