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An Integral Representation, Complete Monotonicity, and Inequalities of the Catalan Numbers

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Abstract. In the paper, by virtue of the Cauchy integral formula in the theory of complex functions, the authors establish an integral representation for the generating function of the Catalan numbers in combinatorics. From this, the authors derive an alternative integral representation, complete monotonicity, determinantal and product inequalities for the Catalan numbers.

1. Introduction

It is known [7, 48] in combinatorial analysis that the Catalan numbers C_n for $n \ge 0$ form a sequence of natural numbers that occur in tree enumeration problems such as "In how many ways can a regular n-gon be divided into $n-2$ triangles if different orientations are counted separately?" whose solution is the Catalan number C_{n-2}. The Catalan numbers C_n can be generated by

$$\frac{2}{1+\sqrt{1-4x}} = \frac{1-\sqrt{1-4x}}{2x} = \sum_{n=0}^{\infty} C_n x^n = 1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + 132x^6 + 429x^7 + \cdots \tag{1.1}$$

Explicit formulas of C_n for $n \ge 0$ include

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!(n+1)!} = \frac{2^n(2n-1)!!}{(n+1)!} = (-1)^n 2^{2n+1}\binom{\frac{1}{2}}{n+1} = \frac{1}{n}\binom{2n}{n-1} = {}_2F_1(1-n,-n;2;1)$$

and

$$C_n = \frac{4^n\Gamma(n+1/2)}{\sqrt{\pi}\,\Gamma(n+2)}, \tag{1.2}$$

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where

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}\,t,\quad \Re(z)>0$$

is the classical Euler gamma function and

$${}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z)=\sum_{n=0}^\infty\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{z^n}{n!}$$

is the generalized hypergeometric series defined for complex numbers $a_i\in\mathbb{C}$ and $b_i\in\mathbb{C}\setminus\{0,-1,-2,\dots\}$, for positive integers $p,q\in\mathbb{N}$, and in terms of the rising factorial

$$(x)_n=\begin{cases}x(x+1)(x+2)\dots(x+n-1), & n\ge1;\\ 1, & n=0.\end{cases}$$

There are two integral representations

$$C_n=\frac{2^{2n+5}}{\pi}\int_0^1\frac{x^2\left(1-x^2\right)^{2n}}{(1+x^2)^{2n+3}}\,\mathrm{d}\,x\quad\text{and}\quad C_n=\frac{1}{2\pi}\int_0^4x^n\sqrt{\frac{4-x}{x}}\,\mathrm{d}\,x$$

for the Catalan numbers C_n in [2, p. 413, Proposition 2.1], [15, p. 10], and [17]. In [4, 7, 48, 51], the asymptotic expansion

$$C_x\triangleq\frac{4^x\Gamma\left(x+\frac12\right)}{\sqrt{\pi}\,\Gamma(x+2)}\sim\frac{4^x}{\sqrt{\pi}}\left(\frac{1}{x^{3/2}}-\frac98\frac{1}{x^{5/2}}+\frac{145}{128}\frac{1}{x^{7/2}}+\cdots\right)$$

for the Catalan function C_x was given. Recently, among other things, the formula

$$C_n=(-1)^n\frac{2^n}{n!}\sum_{k=0}^n\frac{1}{2^k}\sum_{\ell=0}^k(-1)^\ell\binom{k}{\ell}\prod_{m=0}^{n-1}(\ell-2m)=\frac{2^n}{n!}\sum_{k=0}^n\frac{k!}{2^k}\binom{2n-k-1}{2(n-k)}[2(n-k)-1]!!$$

has been established in [41, Theorem 3]. For more information on the Catalan numbers C_n, please refer to the monographs [1, 4] and references cited therein.

In the paper [47], motivated by the explicit expression (1.2) and by virtue of an integral representation of the gamma function $\Gamma(x)$, the authors established an integral representation of the Catalan function C_x for $x\ge0$.

Theorem 1.1 ([47, Theorem 1]). *For $x\ge0$, we have*

$$C_x=\frac{e^{3/2}4^x(x+1/2)^x}{\sqrt{\pi}\,(x+2)^{x+3/2}}\exp\left[\int_0^\infty\left(\frac{1}{e^t-1}-\frac1t+\frac12\right)\frac{e^{-t/2}-e^{-2t}}{t}e^{-xt}\,\mathrm{d}\,t\right].\tag{1.3}$$

Recall from [14, Chapter XIII], [46, Chapter 1], and [53, Chapter IV] that an infinitely differentiable function f is said to be completely (or absolutely, respectively) monotonic on an interval I if it satisfies $0\le(-1)^kf^{(k)}(x)<\infty$ or $0\le f^{(k)}(x)<\infty$, respectively on I for all $k\ge0$ and that a function $f(x)$ is completely monotonic on I if and only if the function $f(-x)$ is absolutely monotonic on $-I$. Recall from [29] that an infinitely differentiable and positive function f is said to be logarithmically completely monotonic on an interval I if $0\le(-1)^k[\ln f(x)]^{(k)}<\infty$ hold on I for all $k\in\mathbb{N}$. For more information on logarithmically completely monotonic functions, please refer to [33, 34, 37, 44].

The formula (1.3) can be rearranged as

$$\ln\left[\frac{\sqrt{\pi}\,(x+2)^{x+3/2}}{e^{3/2}4^x(x+1/2)^x}C_x\right]=\int_0^\infty\frac1t\left(\frac{1}{e^t-1}-\frac1t+\frac12\right)\left(e^{-t/2}-e^{-2t}\right)e^{-xt}\,\mathrm{d}\,t.\tag{1.4}$$

Since the function $\frac{1}{t}\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right)$ is positive on $(0,\infty)$ (see [8, 38, 54] and references therein), the right hand side of (1.4) is a completely monotonic function on $(0,\infty)$. This means that the function

$$\frac{(x+2)^{x+3/2}}{4^x(x+1/2)^x}C_x \tag{1.5}$$

is logarithmically completely monotonic on $(0,\infty)$. Because any logarithmically completely monotonic function must be completely monotonic (see [34, Eq. (1.4)] and references therein), the function (1.5) is also completely monotonic on $(0,\infty)$.

By virtue of (1.2), the function (1.5) can be rewritten as

$$\frac{(x+2)^{x+3/2}\Gamma(x+1/2)}{(x+1/2)^x\Gamma(x+2)}, \quad x>0. \tag{1.6}$$

Hence, the logarithmically complete monotonicity of (1.5) implies the logarithmically complete monotonicity of (1.6). The function (1.6) is a special case $F_{1/2,2}(x)$ of the general function

$$F_{a,b}(x)=\frac{\Gamma(x+a)}{(x+a)^x}\frac{(x+b)^{x+b-a}}{\Gamma(x+b)}, \quad a,b\in\mathbb{R}, \quad a\ne b, \quad x>-\min\{a,b\}. \tag{1.7}$$

We note that the function $F_{a,b}(x)$ does not appear in the expository and survey articles [20, 21, 34–36] and plenty of references therein. This naturally motivated us to pose an open problem in our previous work, which is recapitulated below.

Open Problem 1.1 ([47, Open Problem 1]). *What are the necessary and sufficient conditions on $a,b\in\mathbb{R}$ such that the function $F_{a,b}(x)$ defined by (1.7) is (logarithmically) completely monotonic in $x\in(-\min\{a,b\},\infty)$?*

This problem was solved in [43] as follows.

Theorem 1.2 ([43, Theorem 1.3]). *For $a,b>0$, the function $F_{a,b}(x)$ defined by (1.7) has the exponential representation*

$$F_{a,b}(x)=\exp\left[b-a+\int_0^\infty\frac{1}{t}\left(\frac{1}{1-e^{-t}}-\frac{1}{t}-a\right)\left(e^{-at}-e^{-bt}\right)e^{-xt}\,\mathrm{d}\,t\right]$$

on $[0,\infty)$ and the function $[F_{a,b}(x)]^{\pm1}$ is logarithmically completely monotonic on $[0,\infty)$ if and only if $(a,b)\in D_\pm(a,b)$, where

$$D_\pm(a,b)=\{(a,b):a\gtrless b,a\ge1\}\cup\left\{(a,b):a\lessgtr b,a\le\frac{1}{2}\right\}.$$

Recall from monographs [14, pp. 372–373] and [53, p. 108, Definition 4] that a sequence $\{\mu_n\}_{0\le n\le\infty}$ is said to be completely monotonic if its elements are non-negative and its successive differences are alternatively non-negative, that is, $(-1)^k\Delta^k\mu_n\ge0$ for $n,k\ge0$, where

$$\Delta^k\mu_n=\sum_{m=0}^k(-1)^m\binom{k}{m}\mu_{n+k-m}.$$

Recall from [53, p. 163, Definition 14a] that a completely monotonic sequence $\{a_n\}_{n\ge0}$ is minimal if it ceases to be completely monotonic when a_0 is decreased.

Let $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)\in\mathbb{R}^n$ and $\mu=(\mu_1,\mu_2,\dots,\mu_n)\in\mathbb{R}^n$. A sequence λ is said to be majorized by μ (in symbols $\lambda\preceq\mu$) if

$$\sum_{\ell=1}^k\lambda_{[\ell]}\le\sum_{\ell=1}^k\mu_{[\ell]}, \quad k=1,2,\dots,n-1 \quad\text{and}\quad \sum_{\ell=1}^n\lambda_\ell=\sum_{\ell=1}^n\mu_\ell,$$

where $\lambda_{[1]} \ge \lambda_{[2]} \ge \cdots \ge \lambda_{[n]}$ and $\mu_{[1]} \ge \mu_{[2]} \ge \cdots \ge \mu_{[n]}$ are respectively the components of λ and μ in decreasing order. A sequence λ is said to be strictly majorized by μ (in symbols $\lambda \prec \mu$) if λ is not a permutation of μ. For example,

$$\Bigl(\underbrace{\frac{1}{n},\dots,\frac{1}{n}}_{n}\Bigr) \prec \Bigl(\underbrace{\frac{1}{n-1},\dots,\frac{1}{n-1}}_{n-1},0\Bigr) \prec \Bigl(\underbrace{\frac{1}{n-2},\dots,\frac{1}{n-2}}_{n-2},0,0\Bigr) \prec \cdots \\ \prec \Bigl(\frac{1}{3},\frac{1}{3},\frac{1}{3},0,\dots,0\Bigr) \prec \Bigl(\frac{1}{2},\frac{1}{2},0,\dots,0\Bigr) \prec (1,0,\dots,0).$$

For more information on the theory of majorization and its applications, please refer to monographs [6, 11] and closely related references therein.

In this paper, by virtue of the Cauchy integral formula in the theory of complex functions, we establish an integral representation for the generating function $\frac{1}{1+\sqrt{1-4x}}$ of the Catalan numbers C_n. From this, we derive an alternative integral representation, complete monotonicity, determinantal and product inequalities for the Catalan numbers C_n for $n \ge 0$ and the sequences $\bigl\{\frac{C_n}{4^n}\bigr\}_{n\ge0}$ and $\{n!C_n\}_{n\ge0}$.

Our main results in this paper can be stated as the following theorems.

Theorem 1.3. *For $x \in \bigl(-\infty, \frac{1}{4}\bigr]$, we have*

$$\frac{1}{1+\sqrt{1-4x}} = \frac{1}{2\pi}\int_0^\infty \frac{\sqrt{t}}{(t+1/4)(t-x+1/4)}\,\mathrm{d}\,t. \tag{1.8}$$

Consequently, the Catalan numbers C_n for $n \ge 0$ can be represented by

$$C_n = \frac{1}{\pi}\int_0^\infty \frac{\sqrt{t}}{(t+1/4)^{n+2}}\,\mathrm{d}\,t = \frac{2}{\pi}\int_0^\infty \frac{t^2}{(t^2+1/4)^{n+2}}\,\mathrm{d}\,t \tag{1.9}$$

and the sequence $\bigl\{\frac{C_n}{4^n}\bigr\}_{n\ge0}$ is completely monotonic and minimal.

Theorem 1.4. *If $m \ge 1$ and $a_0, a_1, \dots, a_m$ are non-negative integers, then*

$$\biggl(\frac{C_{a_0}}{4^{a_0}}\biggr)^{m-1} \frac{C_{\sum_{k=0}^m a_k}}{4^{\sum_{k=0}^m a_k}} \ge \prod_{k=1}^m \frac{C_{a_0+a_k}}{4^{a_0+a_k}} \tag{1.10}$$

and

$$\left|\frac{C_{a_i+a_j}}{4^{a_i+a_j}}\right|_m \ge 0, \tag{1.11}$$

where $|e_{kj}|_m$ denotes a determinant of order m with elements e_{kj}.

Theorem 1.5. *Let $m \in \mathbb{N}$ and let n and a_k for $1 \le k \le m$ be non-negative integers. Then the Catalan numbers C_n satisfy*

$$\left|(-1)^{a_i+a_j}\mathcal{C}_{n+a_i+a_j}\right|_m \ge 0 \tag{1.12}$$

and

$$\left|\mathcal{C}_{n+a_i+a_j}\right|_m \ge 0, \tag{1.13}$$

where

$$\mathcal{C}_\ell = \ell! C_\ell, \quad \ell \ge 0. \tag{1.14}$$

Theorem 1.6. *Let $m \in \mathbb{N}$ and let λ and μ be two m-tuples of non-negative integers such that $\lambda \le \mu$. Then*

$$\left|\prod_{i=1}^{m} C_{n+\lambda_i}\right| \le \left|\prod_{i=1}^{m} C_{n+\mu_i}\right|, \tag{1.15}$$

where C_ℓ is defined by (1.14). *Consequently,*

1. *the infinite sequence $\{C_n\}_{n\ge0}$ is logarithmically convex,*
2. *the inequality*

$$C_{\ell+k}^{n} \le C_{\ell+n}^{k} C_{\ell}^{n-k} \tag{1.16}$$

is valid for $\ell \ge 0$ and $n > k > 0$.

Theorem 1.7. *If $\ell \ge 0$, $n \ge k \ge m$, $k \ge n-k$, and $m \ge n-m$, then*

$$\frac{C_{\ell+k}C_{\ell+n-k}}{C_{\ell+m}C_{\ell+n-m}} \ge \frac{(\ell+m)!(\ell+n-m)!}{(\ell+k)!(\ell+n-k)!}. \tag{1.17}$$

For $n, m \in \mathbb{N}$ and $\ell \ge 0$, let

$$\begin{aligned}
\mathcal{G}_{n,m,\ell} &= C_{\ell+n+2m}(C_\ell)^2 - C_{\ell+n+m}C_{\ell+m}C_\ell - C_{\ell+n}C_{\ell+2m}C_\ell + C_{\ell+n}(C_{\ell+m})^2,\\
\mathcal{H}_{n,m,\ell} &= C_{\ell+n+2m}(C_\ell)^2 - 2C_{\ell+n+m}C_{\ell+m}C_\ell + C_{\ell+n}(C_{\ell+m})^2,\\
\mathcal{I}_{n,m,\ell} &= C_{\ell+n+2m}(C_\ell)^2 - 2C_{\ell+n}C_{\ell+2m}C_\ell + C_{\ell+n}(C_{\ell+m})^2,
\end{aligned}$$

where C_ℓ is defined by (1.14). *Then*

$$\mathcal{G}_{n,m,\ell} \ge 0, \quad \mathcal{H}_{n,m,\ell} \ge 0, \tag{1.18}$$

$$\mathcal{H}_{n,m,\ell} \lesseqqgtr \mathcal{G}_{n,m,\ell} \quad \text{when } m \lesseqgtr n, \tag{1.19}$$

and

$$\mathcal{I}_{n,m,\ell} \ge \mathcal{G}_{n,m,\ell} \ge 0 \quad \text{when } n \ge m. \tag{1.20}$$

2. Proofs of Theorems 1.3 to 1.7

We are now in a position to prove Theorems 1.3 to 1.7.

Proof. [Proof of Theorem 1.3] As usual, we use $\ln x$ for the logarithmic function having base e and applied to the positive argument $x > 0$. Further, the principal branch of the holomorphic extension of $\ln x$ from the open half-line $(0, \infty)$ to the cut plane $\mathbb{C} \setminus (-\infty, 0]$ is denoted by $\ln z = \ln|z| + i \arg z$, where $i = \sqrt{-1}$ is the imaginary unit and the principal value $\arg z$ of the argument of z satisfies $|\arg z| < \pi$.

Let

$$f(z) = \frac{2}{1 + \exp\frac{\ln(1-4z)}{2}}, \quad z \in \mathbb{C} \setminus \left[\frac14, \infty\right), \quad \arg\left(z - \frac14\right) \in (0, 2\pi)$$

and

$$F(z) = \frac{2}{1 + \exp\frac{\ln(-4z)}{2}}, \quad z \in \mathbb{C} \setminus [0, \infty), \quad \arg z \in (0, 2\pi).$$

By virtue of the Cauchy integral formula in the theory of complex functions, for any fixed point $z_0 = x_0 + iy_0 \in \mathbb{C} \setminus [0, \infty)$, we have

$$F(z_0) = \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{F(\xi)}{\xi - z_0} \,\mathrm{d}\,\xi,$$

where $\mathcal{L}$ is a positively oriented contour $\mathcal{L}(r, R)$ in $\mathbb{C} \setminus [0, \infty)$, such that

1. $\begin{cases} 0 < r < |y_0| \le |z_0| < R, & y_0 \ne 0, \\ 0 < r < -x_0 = |z_0| < r, & y_0 = 0, \end{cases}$
2. $\mathcal{L}(r,R)$ consists of the half circle $z = re^{i\theta}$ for $\theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$,
3. $\mathcal{L}(r,R)$ consists of the half lines $z = x \pm ir$ for $x > 0$,
4. $\mathcal{L}(r,R)$ consists of the circle $|z| = R$,
5. the half lines $z = x \pm ir$ for $x \ge 0$ cut the circle $|z| = R$ at the points $R(r) \pm ir$ and $0 < R(r) = \sqrt{R^2 - r^2} \to R$ as $r \to 0$.

Then the integral on the circle with radius R equals

$$\int_{\arcsin(r/R)}^{2\pi-\arcsin(r/R)} \frac{2Rie^{i\theta}}{\left(Re^{i\theta} - z_0\right)\left[1 + \exp\frac{\ln(-4Re^{i\theta})}{2}\right]} \,\mathrm{d}\,\theta = 2i \int_{\arcsin(r/R)}^{2\pi-\arcsin(r/R)} \frac{1}{\left(1 - \frac{z_0}{Re^{i\theta}}\right)\left[1 + \exp\frac{\ln(4R)+i\arg(-4Re^{i\theta})}{2}\right]} \,\mathrm{d}\,\theta$$

which tends uniformly to 0 as $R \to \infty$.

Further, the integral on the half circle $z = re^{i\theta}$ for $\theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ is

$$-\frac{1}{2\pi i} \int_{\pi/2}^{3\pi/2} \frac{2rie^{i\theta}}{\left(re^{i\theta} - z_0\right)\left[1 + \exp\frac{\ln(-4re^{i\theta})}{2}\right]} \,\mathrm{d}\,\theta = -\frac{1}{\pi} \int_{\pi/2}^{3\pi/2} \frac{1}{\left(1 - \frac{z_0}{re^{i\theta}}\right)\left[1 + 2\sqrt{r}\,\exp\frac{i\arg(-4re^{i\theta})}{2}\right]} \,\mathrm{d}\,\theta$$

which tends uniformly to 0 as $r \to 0^+$.

Continuously, because

$$F(x+ir) = \frac{2}{1 + \exp\frac{\ln(-4x-4ri)}{2}} = \frac{2\left(1 + 2\sqrt[4]{x^2+r^2}\,\sin\frac{\arctan(r/x)}{2}\right) + i4\sqrt[4]{x^2+r^2}\,\cos\frac{\arctan(r/x)}{2}}{\left[1 + 2\sqrt[4]{x^2+r^2}\,\sin\frac{\arctan(r/x)}{2}\right]^2 + \left[2\sqrt[4]{x^2+r^2}\,\cos\frac{\arctan(r/x)}{2}\right]^2} \to \frac{2 + i4\sqrt{x}}{1+4x}$$

as $r \to 0^+$, the integral on the half lines $z = x \pm ir$ for $x > 0$ is equal to

$$\int_0^{R(r)} \frac{F(x+ir)}{x+ir-z_0} \,\mathrm{d}\,x + \int_{R(r)}^0 \frac{F(x-ir)}{x-ir-z_0} \,\mathrm{d}\,x = \int_0^{R(r)} \frac{(x-z_0)[2i\Im(F(x+ir))] - ir[2\Re(F(x+ir))]}{(x+ir-z_0)(x-ir-z_0)} \,\mathrm{d}\,x$$
$$\to \int_0^\infty \frac{2i}{x-z_0} \frac{4\sqrt{x}}{1+4x} \,\mathrm{d}\,x = 8i \int_0^\infty \frac{\sqrt{x}}{(1+4x)(x-z_0)} \,\mathrm{d}\,x$$

as $r \to 0^+$ and $R \to \infty$. Consequently, it follows that

$$\frac{2}{1 + \exp\frac{\ln(-4z_0)}{2}} = \frac{4}{\pi} \int_0^\infty \frac{\sqrt{x}}{(1+4x)(x-z_0)} \,\mathrm{d}\,x \tag{2.1}$$

for $z_0 \in \mathbb{C} \setminus [0,\infty)$ and $\arg z_0 \in (0, 2\pi)$.

Furthermore, since $F(z) = f(z + \frac{1}{4})$ and the point z_0 in (2.1) is arbitrary, making use of (2.1) arrives at

$$\frac{2}{1 + \exp\frac{\ln(1-4z)}{2}} = \frac{4}{\pi} \int_0^\infty \frac{\sqrt{x}}{(1+4x)(x-z+1/4)} \,\mathrm{d}\,x \tag{2.2}$$

for $z \in \mathbb{C} \setminus \left[\frac{1}{4}, \infty\right)$ and $\arg\left(z - \frac{1}{4}\right) \in (0, 2\pi)$. In particular, when taking $z = x \in \left(-\infty, \frac{1}{4}\right)$, the formula (2.2) becomes the integral representation (1.8) on $\left(-\infty, \frac{1}{4}\right)$. When taking $x \to \left(\frac{1}{4}\right)^-$, the integral in the right hand side of (1.8) converges, consequently, the integral representation (1.8) holds on $\left(-\infty, \frac{1}{4}\right]$.

Differentiating $n \ge 0$ times with respect to x on both sides of (1.8) and taking the limit $x \to 0$ yield

$$\lim_{x\to0}\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\biggl(\frac{1}{1+\sqrt{1-4x}}\biggr)=\frac{1}{2\pi}\lim_{x\to0}\int_0^\infty\frac{n!\sqrt{t}}{(t+1/4)(t-x+1/4)^{n+1}}\,\mathrm{d}\,t$$
$$=\frac{n!}{2\pi}\int_0^\infty\frac{\sqrt{t}}{(t+1/4)(t+1/4)^{n+1}}\,\mathrm{d}\,t=\frac{n!}{2\pi}\int_0^\infty\frac{\sqrt{t}}{(t+1/4)^{n+2}}\,\mathrm{d}\,t=\frac{n!}{\pi}\int_0^\infty\frac{s^2}{(s^2+1/4)^{n+2}}\,\mathrm{d}\,s. \tag{2.3}$$

As a result, by virtue of (1.1), the integral representations in (1.9) follow.

We can rewrite the first integral representation in (1.9) as

$$\frac{C_n}{4^{n+2}}=\frac{1}{\pi}\int_0^\infty\frac{\sqrt{t}}{(1+4t)^{n+2}}\,\mathrm{d}\,t,\quad n\ge0.$$

It is easy to see that the function a^x for $0<a<1$ is completely monotonic in $x\in[0,\infty)$. Hence, the function $\frac{1}{(1+4t)^{x+2}}$ for any fixed $t>0$ is completely monotonic in $x\in[0,\infty)$. Equivalently, the function $\frac{C_x}{4^x}$ is completely monotonic in $x\in[0,\infty)$. Theorem 14b in [53, p. 164] states that "a necessary and sufficient condition that there should exist a function $f(x)$ completely monotonic in $0\le x<\infty$ such that $f(n)=a_n$ for $n\ge0$ is that $\{a_n\}_0^\infty$ should be a minimal completely monotonic sequence". As a result, the sequence $\frac{C_n}{4^n}$ for $n\ge0$ is minimal completely monotonic. The proof of Theorem 1.3 is complete. □

Proof. [Proof of Theorem 1.4] In [12] and [14, pp. 369 and 374], it was obtained that if f is completely monotonic on $[0,\infty)$ and $m\ge1$, then

$$[f(x_0)]^{m-1}f\Biggl(\sum_{k=0}^m x_k\Biggr)\ge\prod_{k=1}^m f(x_0+x_k) \tag{2.4}$$

and

$$|f(x_i+x_j)|_m\ge0. \tag{2.5}$$

Now we consider the function

$$\mathfrak{h}(x,s)=\frac{16}{\pi}\int_0^\infty\frac{\sqrt{t}}{(1+4t)(4x+4t+1)^{s+1}}\,\mathrm{d}\,t,\quad x,s\ge0.$$

It is easy to see that the function $\mathfrak{h}(x,s)$ for any fixed $x>0$ is completely monotonic in $s\in[0,\infty)$. By virtue of (2.3), we obtain

$$\lim_{x\to0^+}\mathfrak{h}(x,a_i)=\frac{C_{a_i}}{4^{a_i}},$$

where a_i for $i\ge0$ are non-negative integers. Replacing the function f and non-negative numbers $x_0,x_1,\dots,x_m$ in (2.4) and (2.5) by the function $\mathfrak{h}(x,s)$ and non-negative integers $a_0,a_1,\dots,a_m$ respectively yields

$$[\mathfrak{h}(x,a_0)]^{m-1}\mathfrak{h}\Biggl(x,\sum_{k=0}^m a_k\Biggr)\ge\prod_{k=1}^m\mathfrak{h}(x,a_0+a_k) \tag{2.6}$$

and

$$|\mathfrak{h}(x,a_i+a_j)|_m\ge0. \tag{2.7}$$

Therefore, taking $x\to0^+$ in (2.6) and (2.7) leads to (1.10) and (1.11). The proof of Theorem 1.4 is complete. □

Proof. [Proof of Theorem 1.5] From (2.3), we observe that $C_n = \lim_{x\to 0} h_n(x)$, where

$$h_n(x) = \frac{4}{\pi}\int_0^\infty \frac{\sqrt{t}}{(1+4t)(t-x+1/4)^{n+1}}\,\mathrm{d}\,t$$

and $h_n(x)$ is absolutely monotonic on $\left(-\infty, \frac{1}{4}\right)$. This means that the function

$$H_n(x) = h_n(-x) = \frac{4}{\pi}\int_0^\infty \frac{\sqrt{t}}{(1+4t)(x+t+1/4)^{n+1}}\,\mathrm{d}\,t \tag{2.8}$$

is completely monotonic on $\left(-\frac{1}{4}, \infty\right) \supset [0,\infty)$ and that $C_n = \lim_{x\to 0} H_n(x)$. A direct computation gives

$$H_n^{(k)}(x) = (-1)^k \frac{(n+k)!}{n!} H_{n+k}(x) \to (-1)^k \frac{(n+k)!}{n!} C_{n+k} = (-1)^k \frac{C_{n+k}}{n!} \tag{2.9}$$

for $k \ge 0$ as $x \to 0$. From either [13] or [14, p. 367], we know that if f is a completely monotonic function on $[0,\infty)$, then

$$\left|f^{(a_i+a_j)}(x)\right|_m \ge 0 \tag{2.10}$$

and

$$\left|(-1)^{a_i+a_j} f^{(a_i+a_j)}(x)\right|_m \ge 0. \tag{2.11}$$

Applying f in (2.10) and (2.11) to the function $H_n(x)$, we arrive at

$$\left|H_n^{(a_i+a_j)}(x)\right|_m \ge 0 \tag{2.12}$$

and

$$\left|(-1)^{a_i+a_j} H_n^{(a_i+a_j)}(x)\right|_m \ge 0. \tag{2.13}$$

Letting $x \to 0$ in (2.12) and (2.13) and making use of (2.9) arrive at

$$\left|(-1)^{a_i+a_j} \frac{C_{n+a_i+a_j}}{n!}\right|_m \ge 0 \tag{2.14}$$

and

$$\left|\frac{C_{n+a_i+a_j}}{n!}\right|_m \ge 0. \tag{2.15}$$

Further simplifications of (2.14) and (2.15) lead us to (1.12) and (1.13). The proof of Theorem 1.5 is complete. □

Proof. [Proof of Theorem 1.6] In [14, p. 367, Theorem 2] and [50, p. 106, Theorem A], which are a minor correction of [3, Theorem 1], it was obtained that if f is a completely monotonic function on $(0,\infty)$ and $\lambda \preceq \mu$, then

$$\left|\prod_{i=1}^{n} f^{(\lambda_i)}(x)\right| \le \left|\prod_{i=1}^{n} f^{(\mu_i)}(x)\right|. \tag{2.16}$$

The equality in (2.16) is valid only when λ and μ are identical or when $f(x) = e^{-cx}$ for $c \ge 0$. An application of the inequality (2.16) to $H_n(x)$ defined in (2.8) leaves us

$$\left|\prod_{i=1}^{m} H_n^{(\lambda_i)}(x)\right| \le \left|\prod_{i=1}^{m} H_n^{(\mu_i)}(x)\right|.$$

Taking the limit $x \to 0$ on both sides of the above inequality and making use of (2.9) reveal

$$\left|\prod_{i=1}^{m}(-1)^{\lambda_i}\frac{C_{n+\lambda_i}}{n!}\right| \le \left|\prod_{i=1}^{m}(-1)^{\mu_i}\frac{C_{n+\mu_i}}{n!}\right|$$

which is equivalent to (1.15).

From the majorization relation $(i+2,i) \succeq (i+1,i+1)$ for $i \ge 0$ and the inequality (1.15), the logarithmic convexity of the sequence $\{C_n\}_{n\ge0}$ follows immediately.

In [14, p. 369] and [16, p. 429, Remark], it was formulated that if $f(t)$ is a completely monotonic function such that $f^{(k)}(t) \ne 0$ for $k \ge 0$, then the sequence

$$s_i(t) = \ln\Big[(-1)^{i-1}f^{(i-1)}(t)\Big], \quad i \ge 1$$

is convex. Applying this result to the function $H_n(x)$ and making use of (2.9) figures out that the sequence

$$\ln\Big[(-1)^{i-1}H_n^{(i-1)}(x)\Big] \to \ln\frac{C_{n+i-1}}{n!}, \quad x \to 0$$

for $i \ge 1$ is convex. Hence, the sequence $\{C_n\}_{n\ge0}$ is logarithmically convex.

As in [3], we consider the majorization relation

$$(\underbrace{k,k,\dots,k}_{n}) \prec (\underbrace{n,\dots,n}_{k},\underbrace{0,\dots,0}_{n-k})$$

for $n > k$, so that the inequality (2.16) becomes

$$(-1)^{nk}\Big[f^{(k)}(x)\Big]^n \le (-1)^{nk}\Big[f^{(n)}(x)\Big]^k[f(x)]^{n-k}, \quad n > k > 0.$$

Substituting $H_\ell(x)$ for $f(x)$ in the above inequality, letting $t \to 0$, and utilizing (2.9), we obtain

$$(-1)^{nk}\Big[H_\ell^{(k)}(x)\Big]^n \le (-1)^{nk}\Big[H_\ell^{(n)}(x)\Big]^k[H_\ell(x)]^{n-k}$$

which is equivalent to

$$(-1)^{nk}\left[(-1)^k\frac{C_{\ell+k}}{\ell!}\right]^n \le (-1)^{nk}\left[(-1)^n\frac{C_{\ell+n}}{\ell!}\right]^k(C_\ell)^{n-k}$$

for $n > k > 0$ and $\ell \ge 0$. This may be simplified as (1.16). The proof of Theorem 1.6 is complete. □

Proof. [Proof of Theorem 1.7] In [49, p. 397, Theorem D], it was revealed that if $f(x)$ is completely monotonic on $(0,\infty)$ and if $n \ge k \ge m$, $k \ge n-k$, and $m \ge n-m$, then

$$(-1)^n f^{(k)}(x)f^{(n-k)}(x) \ge (-1)^n f^{(m)}(x)f^{(n-m)}(x).$$

Replacing $f(x)$ by the function $H_\ell(x)$ defined by (2.8) in the above inequality, we are led to

$$(-1)^n H_\ell^{(k)}(x)H_\ell^{(n-k)}(x) \ge (-1)^n H_\ell^{(m)}(x)H_\ell^{(n-m)}(x).$$

Further taking $x \to 0$ and employing (2.9) find

$$(-1)^n(-1)^k\frac{C_{\ell+k}}{\ell!}(-1)^{n-k}\frac{C_{\ell+n-k}}{\ell!} \ge (-1)^n(-1)^m\frac{C_{\ell+m}}{\ell!}(-1)^{n-m}\frac{C_{\ell+n-m}}{\ell!}.$$

Simplifying this inequality, we are led to (1.17).

In [50, Theorem 1 and Remark 2], it was obtained that if f is completely monotonic on $(0,\infty)$ and

$$\begin{aligned} G_{n,m} &= (-1)^n\Big\{f^{(n+2m)}f^2 - f^{(n+m)}f^{(m)}f - f^{(n)}f^{(2m)}f + f^{(n)}\big[f^{(m)}\big]^2\Big\},\\ H_{n,m} &= (-1)^n\Big\{f^{(n+2m)}f^2 - 2f^{(n+m)}f^{(m)}f + f^{(n)}\big[f^{(m)}\big]^2\Big\},\\ I_{n,m} &= (-1)^n\Big\{f^{(n+2m)}f^2 - 2f^{(n)}f^{(2m)}f + f^{(n)}\big[f^{(m)}\big]^2\Big\} \end{aligned}$$

for $n,m\in\mathbb{N}$, then $G_{n,m}\ge 0$, $H_{n,m}\ge 0$, and

$$\begin{aligned} &H_{n,m} \lesseqgtr G_{n,m} \quad \text{when } m \lesseqslant n,\\ &I_{n,m} \ge G_{n,m} \ge 0 \quad \text{when } n \ge m. \end{aligned}$$

Replacing $f(x)$ by $H_\ell(x)$ in $G_{n,m}$, $H_{n,m}$, and $I_{n,m}$, taking $x\to 0$, and employing (2.9) produce

$$\begin{aligned} G_{n,m} &= (-1)^n\Big\{H_\ell^{(n+2m)}H_\ell^2 - H_\ell^{(n+m)}H_\ell^{(m)}H_\ell - H_\ell^{(n)}H_\ell^{(2m)}H_\ell + H_\ell^{(n)}\big[H_\ell^{(m)}\big]^2\Big\}\\ &= \frac{1}{(\ell!)^3}\Big[C_{\ell+n+2m}(C_\ell)^2 - C_{\ell+n+m}C_{\ell+m}C_\ell - C_{\ell+n}C_{\ell+2m}C_\ell + C_{\ell+n}(C_{\ell+m})^2\Big] = \frac{\mathcal{G}_{n,m,\ell}}{(\ell!)^3},\\ H_{n,m} &= (-1)^n\Big\{H_\ell^{(n+2m)}H_\ell^2 - 2H_\ell^{(n+m)}H_\ell^{(m)}H_\ell + H_\ell^{(n)}\big[H_\ell^{(m)}\big]^2\Big\} = \frac{\mathcal{H}_{n,m,\ell}}{(\ell!)^3},\\ I_{n,m} &= (-1)^n\Big\{H_\ell^{(n+2m)}H_\ell^2 - 2H_\ell^{(n)}H_\ell^{(2m)}H_\ell + H_\ell^{(n)}\big[H_\ell^{(m)}\big]^2\Big\} = \frac{\mathcal{I}_{n,m,\ell}}{(\ell!)^3}. \end{aligned}$$

Accordingly, the inequalities in (1.18), (1.19), and (1.20) are proved. The proof of Theorem 1.7 is complete. □

3. Remarks

Remark 3.1. The integral representation (1.8) can be rearranged as

$$\frac{1}{1+\sqrt{1+4x}} = \frac{1}{2\pi}\int_0^\infty \frac{\sqrt{t}}{(t+1/4)(x+t+1/4)}\,\mathrm{d}\,t$$

for $x\in\big[-\frac14,\infty\big)$. This means that

$$\frac{1}{1+2\sqrt{x}} = \frac{2}{\pi}\int_0^\infty \frac{\sqrt{t}}{(4t+1)(x+t)}\,\mathrm{d}\,t \tag{3.1}$$

for $x\in[0,\infty)$. In [46, p. 16, Definition 2.1], a (non-negative) Stieltjes function is defined as a function $f:(0,\infty)\to[0,\infty)$ which can be written in the form

$$f(\lambda) = \frac{a}{\lambda} + b + \int_{(0,\infty)} \frac{1}{\lambda+t}\sigma(\mathrm{d}\,t),$$

where $a,b\ge 0$ are non-negative constants and σ is a measure on $(0,\infty)$ such that $\int_{(0,\infty)}\frac{1}{1+t}\sigma(\mathrm{d}\,t)<\infty$. Therefore, the function $\frac{1}{1+2\sqrt{x}}$ is a positive Stieltjes function with $a=b=0$ and the measure

$$\sigma(\mathrm{d}\,t) = \frac{2}{\pi}\frac{\sqrt{t}}{4t+1}\,\mathrm{d}\,t.$$

In [46, p. 21, Definition 3.1], a Bernstein function f is defined as a function $f:(0,\infty)\to\mathbb{R}$ which is of class C^∞, $f(\lambda)\ge 0$ for all $\lambda>0$, and $(-1)^{n-1}f^{(n)}(\lambda)\ge 0$ for all $n\in\mathbb{N}$ and $\lambda>0$. Theorem 3.2 in [46, p. 21] states that a function $f:(0,\infty)\to\mathbb{R}$ is a Bernstein function if and only if it admits the presentation

$$f(\lambda) = a + b\lambda + \int_{(0,\infty)}\big(1-e^{-\lambda t}\big)\mu(\mathrm{d}\,t), \tag{3.2}$$

where $a, b \geq 0$ and μ is a measure on $(0, \infty)$ satisfying $\int_{(0,\infty)} \min\{1, t\}\mu(\mathrm{d}\, t) < \infty$. In [46, p. 69, Definition 6.1], it is defined that, if the measure μ in (3.2) has a completely monotonic density $m(t)$ with respect to Lebesgue measure, that is,

$$f(\lambda) = a + b\lambda + \int_{(0,\infty)} \left(1 - e^{-\lambda t}\right) m(t) \,\mathrm{d}\, t,$$

then f is said to be a complete Bernstein function. In [46, p. 93, Theorem 7.3], it is stated that a (non-trivial) function f is a complete Bernstein function if and only if $\frac{1}{f}$ is a (non-trivial) Stieltjes function. Hence, the function $1 + 2\sqrt{x}$ is a complete Bernstein function and has the integral representation

$$1 + 2\sqrt{x} = 1 + \frac{1}{\sqrt{\pi}} \int_{(0,\infty)} \left(1 - e^{-xt}\right) \frac{1}{t^{3/2}} \,\mathrm{d}\, t.$$

Remark 3.2. The anonymous referee points out that the integral representation (3.1) can be derived from the formulas

$$\frac{1}{1 + 2\sqrt{s}} = \frac{1}{2} \int_0^\infty \left[\frac{1}{\sqrt{\pi t}} - \frac{e^{t/4}}{2} \operatorname{erfc}\left(\frac{\sqrt{t}}{2}\right)\right] e^{-st} \,\mathrm{d}\, t \quad \text{and} \quad \frac{1}{1 + 2\sqrt{s}} = \frac{2}{\pi} \int_0^\infty \frac{\sqrt{u}}{1 + 4u} \frac{\mathrm{d}\, u}{s + u}$$

in [18, p. 11, Eq. 1; p. 43, Eq. 22; and p. 152, Eq. 12].

In the papers [27, 30], new conclusions and integral representations of the Catalan numbers C_n were reviewed and surveyed.

Remark 3.3. It is clear that the integral representation (1.9) is simpler and more significant than (1.3) for C_n. For more information, please read [2, 25] and closely-related references therein.

Remark 3.4. In the proof of [19, Theorem 4.1], the inequality (4.11) should be $\left|\frac{c_{n+a_i+a_j}}{n!}\right|_m \geq 0$. Consequently, the inequality (4.2) in [19, p. 248, Theorem 4.1] should be corrected as $|c_{n+a_i+a_j}|_m \geq 0$.

Remark 3.5. In recent years, the first author of this paper and his coauthors established integral representations, complete monotonicity, inequalities, closed expressions for the Lah numbers, tangent numbers, Bernoulli numbers of the first and second kinds, Stirling numbers of the first and second kind, Bernoulli polynomials, Euler numbers and polynomials. For detailed information, please refer to [5, 22–24, 28, 45, 52] and closely related references therein.

Remark 3.6. This paper is a companion of the formally published papers [9, 10, 26, 31, 32, 39, 41–43, 47] and closely related references therein. This paper is a revised version of the preprint [40].

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