An Integrative Theory of Circular Colourings of Hypergraphs

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Abstract

Colouring is one of the important branches of graph theory and has attracted the attention of almost all graph theorists, mainly because of the four colour theorem. This paper is concerned with circular colourings of hypergraphs and it is based on the results of Bondy and Hell. This paper also includes, few examples for this circular colourings of hypergraphs.

Keywords: *Chromatic number, Circular Chromatic number, Clique number, Hypergraph, (k, d) - colourings, k-uniform hypergraph, Pigeonhole Principle, universal vertex.*

1.Indroduction

In what follows hypergraphs are simple and finite unless otherwise specified. A hypergraph is an ordered triple $H = (V, E, \phi)$, where V is a set of objects called vertices, E is a set of objects called edges, and the incidence function ϕ maps E to the set of non-empty subsets of V. If ϕ is one-to-one, and no one-subset of V is in its range, then the hypergraph H is called simple. The hypergraph H is called finite if its vertex set V is finite. A finite, simple hypergraph can be defined as an ordered pair $H = (V, E)$, where V is a finite set of objects called vertices and E is a set of subsets of V, each of which contains at least two vertices.

A subhypergraph of a hypergraph H is a hypergraph H' such that $V(H') \subseteq V(H)$ and $E(H') \subseteq E(H)$. I $X \subseteq V(H)$ and H' is the hypergraph with vertex set $V(H') = X$ and edge set $E(H') = \{e : e \in E(H) \text{ and } e \subseteq X\}$, then H' is the subgraph of H induced by X. A hypergraph is called k-uniform if every edge has cardinality k . A simple graph $G = (V, E)$ is therefore a 2-uniform hypergraph and will sometimes be regarded as such.

Let $H = (V, E)$ be a hypergraph. A k-colouring of H is a function $c : V \to Z_k$ such that no edge of H is monochromatic. Thus, if H has a k-colouring, then every edge $e \in E$ contains vertices x and y such that, $c(x) \neq c(y)$. For 2-uniform hypergraphs (i.e. undirected graphs), this is a restatement of the definition of a k-colouring of a graph. A k-colourable hypergraph is one that has a k-colouring. Every hypergraph $H = (V, E)$ is $|V|$- colourable: assign a different colour to every vertex of H. The smallest integer k for which H is k-colourable is called the chromatic number of H, and is denoted by $\chi(H)$.

An independent set in a hypergraph $H = (V, E)$ is a subset $X \subseteq V$ such that no edge $e \in E$ is a subset of X. In a k-colouring of H, the set V_i of vertices that are assigned colour i is an independent set.

Let H_1 and H_2 be hypergraphs. A homomorphism of H_1 to H_2 is a function $f : V(H_1) \to V(H_2)$ such that $f(e) \in E(H_2)$ for every edge $e \in E(H_1)$. If there exists a homomorphism of H_1 to H_2, then it is denoted as $H_1 \to H_2$, or $f : H_1 \to H_2$ to emphasize the name, f, of the mapping. If $f : H_1 \to H_2$ then, for every vertex $h \in V(H_2)$, the set $f^{-1}(h)$ is an independent set. Let k and d be positive integers such that $k \geq 2d$.

A (k, d)-colouring of a hypergraph $H = (V, E)$ is a function $c : V \to Z_k$ such that for each edge e there exist a pair of vertices $\{u, v\} \in e$ such that $|c(u) - c(v)|_k \geq d$. The infimum of the set of ratios k/d such that H has a (k, d)-colouring is called the circular chromatic number of H and is denoted by $\chi_c(H)$. A (k, d)-colouring of a graph G is a function $c : V(G) \to Z_k$ such that if $uv \in E(G)$ then $|c(u) - c(v)|_k \geq d$. The circular chromatic number of G, denoted $\chi_c(H)$ is the infimum of the set of ratios k/d such that G has a (k, d)-colouring.

For positive integers k and d with $k \geq 2d$, we define H_d^k to be the hypergraph with vertex set Z_k and edge set consisting of the subsets X of Z_k for which there exist vertices $u, v \in X$ such that $|u - v|_k \geq d$. This generalizes the corresponding concept for graphs: for positive integers k and d with $k \geq 2d$, the graph G_d^k is defined to have vertex set $V = Z_k$ and edge set $E = \{xy : |x - y|_k \geq d\}$. Here G_d^k, considered as a 2-uniform hypergraph, is a subhypergraph of H_d^k. In the case of 2-uniform hypergraphs, a graph G has a (k, d)-colouring if and only if $G \to G_d^k$

and $\chi_c(H)$ is known to be the minimum of the set of ratios k/d such that $G \to G_d^k$.

Definition(Pigeonhole Principle): Suppose P, k and n are integers such that $P > k + n$. If P pigeons are placed into n pigeonholes, then the pigeonhole principle states there must be (at least) one pigeonhole which contains at least $k+1$ pigeons.

1. Circular Colourings Of Hypergraphs

This section computes the circular chromatic number of hypergraphs in various families. Some basic results concerning circular colourings are proved along the way.

Theorem 2.1 Let $H = (V, E)$ be a hypergraph. If H has a (k,d)-colourings and $k/d \leq k'/d'$, where k' and d' positive intergers, then H has a (k', d')-colouring.

Proof Let $c : V \to Z_k$ be a (k,d)-colouring of H. Define a mapping $c' : V \to Z_{k'}$ by

$$c'(v) = [d'/d.c(v)].$$

Consider an edge $e \in E$, and in particular the two vertices u and v of e such that $|c(u) - c(v)|_k \geq d$. Assume that $c(u) \geq c(v)$.

Since c is a (k,d)-colourings of H, and

$$d \leq c(u) - c(v) \leq k - d.$$

Therefore,

$$c'(v) + d' = [d'/d.c(v) + (dd')/d] = c'(u)$$

Since $c(u) \leq c(v) + k - d$ and

$$c'(u) = [d'/d.c(u)]$$

Therefore $c'(u) \leq c'(v) + k' - d'$

(i.e.) $d' \leq c'(u) - c'(v) \leq k' - d'$

Thus c' is a (k', d')-colouring of H. ■

Corollary 2.2 If H has a (k,d)-colouring, then it has a (k', d')-colouring with $k/d = k'/d'$ and $gcd\,(k', d') = 1$.

Theorem 2.3 Let H be a hypergraph on n vertices. If H has a (k,d)- colouring c which is not onto Z_k, then H has a (k', d')-colouring with $k' \leq k$ and $k'/d' \leq k/d$.

Proof Let c be a (k,d)-colouring of H.Using *Corollary 2.2* and assume that $gcd\,(k,d) = 1$. Since c is not onto, some colour is not used. Assume that this colour is d. Now remove the colours that are a multiple of d (taken modulo k) by recolouring the vertices as follows. Recolour vertices of colour $2d$ with colour $2d - 1$. Since no vertex is coloured d this is still a (k,d)-colouring.

Similarly, recolour the vertices of colour $3d$ with colour $3d - 1$, as above this is still a (k,d)-colouring since there are no vertices of colour $2d$. As $gcd(k,d) = 1$, there exists a unique such that $\alpha d \equiv 1 \ (mod\ k)$, Then it will continue in this way with colours $4d, 5d, \ldots\ldots, \alpha d \equiv 1\,(mod\ k)$ to obtain a new (k,d)- colouring c'. Formally,

$$c'(u) = \begin{cases} c(u) - 1, & if\ c(u) = t.d\ for\ 2 \leq t \leq \alpha \\ c(u), & otherwise \end{cases}$$

Denote by S the set $\{d, 2d, \ldots., \alpha d\}$ and define X to be the set $Z_k - S$.

Again recolour any vertex $x \in X$ with the colour $x - |\{y \in S : y \leq x\}$.

Since $|X| = k - |S| = k - \alpha$. By relabeling the colours of X to be from the set $\{0, 1, \ldots, (k-1) - \alpha\}$. Set $k' = k - \alpha$ .Thus, the modified mappings c' is now a mapping $c' : V \to Z_{k'}$. As $\alpha d \equiv 1\,(mod\ k)$. Here $\alpha d - 1 = \beta k$ for some $\beta \in Z$.

Define $d' = d - \beta$. Next part of this theorem is to show that c' is a (k', d')-colouring of H. Consider the interval

$$I_j = \{j, j+1, \ldots., j + d - 1\} \text{ of } Z_k.$$

Each interval I_j contains the same number of elements of S, by construction, with the exception of I_1 which begins and ends with as element of S, namely αd and d . Thus I_1 contains one more elements of S than do the other intetvals. As S contains exactly $\alpha = \frac{\beta_{k+1}}{d}$ elements of Z_k and $|\,I_j\,| = d$, each interval I_j $(j \neq 1)$ contains β elements of S. Since I_1 contains the colour one, I_1 contains β+1 elements of S.

Consider any two vertices x and y such that $|c(x) - c(y)| \geq d$. As each interval $I_j (j \neq 1)$, has had β elements removed. Since $|c'(x) - c'(y)| \geq d - \beta = d'$. The interval I_1 as the colour d is not used in the c colouring and thus a colour difference of less than d' is not possible.

Thus, $d' \leq |c'(u) - c'(v)| \leq k' - d'$. Hence, c' is a (k', d')-colouring of H with $k'/d' = (k - \alpha)/(d - \beta) < k/d$. ■

Corollary 2.4 Let $H = (V, E)$ be a hypergraph. If H has a (k,d)-colouring $c : V \to Z_k$ then we may assume c is onto.

The following corollary implies that the circular chromatic number of a hypergraph is rational.

Corollary 2.5 If H is a hypergraph on n vertices, then

$$\chi_c(H) = min\{k/d : H\ has\ a\ (k,d) - colouring\ and\ k \leq n\}.$$

Proof Using *Theorem 2.3* and *Corollary 2.4***.** If H has a (k,d) - colouring, then it has a

(k', d') - colouring with $k' \le$ n and $k'/d' \le k/d$. Thus,

$$\chi_c(H) = \inf\{k/d : H\ has\ a\ (k,d) - colouring\ and\ k \le n\}.$$

Since this set is finite, the result follows. ■

Lemma 2.6 The maximum size of an independent set in the hypergraph H_q^p is q.

Proof Clearly the set $\{0,1,\ldots\ldots,q-1\}$ is independent, therefore the maximum size of an independent set in the hypergraph H_q^p is atleast q.

Let X be a $(q+1)$-subset of $V(H_q^p)$. Claim that X is not an independent set. Suppose the contrary. By symmetry, it can be assumed that $q \in X$. If X contains a vertex y not in $\{1, 2, \ldots, 2q - 1\}$, then X is not an independent set as $|q - y|_p \ge q$. Therefore, $X \subseteq \{1, 2, \ldots, 2q - 1\}$. Since $|X| = q + 1$, by the Pigeonhole Principle, some of the two elements $a, b \in X$ are congruent modulo q. But then $|a - b|_p = q$, so X is not an independent set. This completes the Proof. ■

Proposition 2.7 The hypergraph H_q^p has circular chromatic number p/q.

Proof Suppose $c : H_q^p \to H_d^k$ is a homomorphism. Since the maximum size of an independent set in the hypergraph H_q^p is q and also the inverse image of an independent, set in H_d^k is an independent set in H_q^p, it follows that for any integer $i \in Z_k$,

$$\sum_{j=i}^{|i+d-1|_k} |c^{-1}(j)| \le q$$

Hence,

$$pd = \sum_{i=0}^{k-1} \sum_{j=i}^{|i+d-1|_k} |c^{-1}(j)| \le kq$$

Therefore, $p/q \le k/d$ ■

Theorem 2.8 For any hypergraph H,

$$\chi(H) - 1 < \chi_c(H) \le \chi(H).$$

Proof Let H be a hypergraph with chromatic number $\chi(H)$.

Suppose $\chi_c(H) = k/d \le \chi(H) - 1$. Therefore, there is a homomorphism $H \to H_1^{\chi(H)-1}$, and thus a $(\chi(H) - 1, 1)$-colouring, contradicting the definition of the chromatic number.

Thus, $\chi(H) - 1 < \chi_c(H)$.On the other hand, here $H \to H_1^{\chi(H)}$ thus, $\chi_c(H) \le \chi(H)$. Finally $\chi(H) - 1 < \chi_c(H) \le \chi(H)$. ■

Corollary 2.9 Let H be a hypergraph. Then $\chi(H) = [\chi_c(H)]$

A hypergraph $H = (V, E)$ with at least one edge is bipartite if V can be partitioned into two independent sets. Clearly, H is bipartite if and only if $\chi(H) = 2$.

Proposition 2.10 Let $H = (V, E)$ be a hypergraph with at least one edge. Then $\chi_c(H) \ge 2$ with equality if and only if H is bipartite.

Proof It follows directly from the definition of the circular chromatic number that $\chi_c(H) \ge 2$.

If H is bipartite, then using Theorem 2.8 gives $\chi(H) \ge \chi\, c(H) = 2$. Therefore, $\chi_c(H) = 2$. Conversely, if $\chi_c(H) = 2$ then using Corollary 2.9 gives $\chi(H) = [\chi_c(H)] = 2$. Therefore H is bipartite. ■

A clique of a hypergraph is a complete subhypergraph. The clique number of H, denoted $\omega(H)$, is the maximum size of a clique in H. In a k-colouring of a clique each vertex must receive a different colour. Hence for any H, $\chi(H) \ge \omega(H)$. Using *Proposition 2.7* gives $\chi_c(H_d^k) = k/d$. Therefore $\chi_c(H_1^n) = n = \omega(H)$.

Let $H = (V, E)$ be a hypergraph. Vertex $v \in V$ is universal if every non-singleton subset of V containing v is an edge of H.

Theorem 2.11 If the hypergraph $H = (V, E)$ has a universal vertex v, then $\chi_c(H) = \chi(H)$.

Proof Suppose $\chi_c(H) = k/d$ and let $c : H \to H_d^k$ be a (k, d)-colouring of H. Then, for every edge $e \in E$ there exist vertices $x, y \in e$ such that $d \le |c(x) - c(y)|_k \le k - d$. Since v is a universal vertex, $\{v, x\} \in E$ for every vertex $x \in V$-$\{v\}$.

Assume $c(v) = k - d$. Thus, $c(x) \in \{0,1, \ldots, k - 2d\}$ for every vertex $x \in$ V-$\{v\}$.

Define $'$: $V \to \{0,1,\ldots,[k/d] -1\}$ by $c'(w) = [c(w)/d]$. Then, since c was a (k, d)-colouring of H, for every edge $e \in E$ there exist vertices $x, y \in e$ such that $c'(x) \ne c'(y)$. Therefore, $\chi(H) \le [k/d] = [\chi_c(H)]$. It follows that $\chi(H) = \chi_c(H)$ ■

Lemma 2.12 Let H be a hypergraph, then $\chi_c(H) \ge \omega(H)$

Proof Suppose H' is a clique in H of size $\omega(H)$. Since every vertex of H' is universal in H', It follows that $\chi_c(H') = \chi(H') = \omega(H)$. Thus, $\chi_c(H) \ge \chi_c(H') = \omega(H)$. ■

Corollary 2.13 Let H be a hypergraph. If $\chi(H) = \omega(H)$, then $\chi_c(H) = \chi(H)$

Proof Using *Lemma 2.12* $\chi(H) \ge \chi_c(H) \ge \omega(H)$ .The result follows. ■

Corollary 2.14 Let $H = (V, E)$ be a complete k-partite hypergraph. Then $\chi(H) = \chi_c(H) = k$.

Proof Suppose that $V_1, V_2, \dots, V_k$ is a partition of V such that

$E = \{X : X \subseteq V, and\ X \nsubseteq V_i, 1 \le i \le k\}$. Since any 2-subset of V not contained in any block of the partition is an edge, it is clear that $\chi(H) = k$. The subhypergraph induced by a set of k vertices, one from each block of the partition, is a clique of size k.

Thus $\omega(H) = k = \chi(H)$. It now follows that $\chi_c(H) = k$. ■

Theorem 2.15 Let $H = (V, E)$ be a complete k-uniform hypergraph on n vertices. Then

$\chi_c(H) = n/(k-1)$ and $\chi(H) = [n/(k-1)]$

Proof Suppose : $H \to H_q^p$, where $p \le n$, is a homomorphism. Since any k-subset of V is an edge, the maximum size of an independent set in H is $k - 1$ and the inverse image of an independent set in H_q^p is independent. Thus, for any integer $i \in Z_p$

$$\sum_{j=i}^{|i+q-1|_p} |c^{-1}(j)| \le k-1$$

Hence,

$$qn = \sum_{i=0}^{p-1} \sum_{j=i}^{|i+q-1|_p} |c^{-1}(j)| \le p(k-1)$$

Therefore, $n/(k-1) \le p/q$.

Claim that the function $'$: $V \to H_{k-1}^n$ defined by $c'(x) = x$ is an $(n,k\text{-}1)$-colouring of H. Let X be a k-subset of V. By symmetry, it can be assumed without loss of generality that $k-1 \in X$. If X contains a vertex y which is not in the set $\{1, 2, \dots, 2(k-1)-1\}$, then

$|(k-1) - y|_n \ge k-1$.

If $X \subseteq \{1, 2, \dots, 2(k-1)-1\}$ then, since $|X| = k$, by the Pigeonhole principle, some two elements $a, b \in X$ are congruent modulo n.

Thus, $|a-b|n = k-1$. Hence, any k-set X contains a pair of vertices x and y such that $|c'(x) - c'(y)|_n = k-1$. This proves the claim. Therefore, $\chi_c(H) = n/(k-1)$ and Using *Corollary 2.9* Finally $\chi(H) = [n/(k-1)]$. ■

Proposition 2.16 For every rational number $p/q \ge 2$ there exists a k-uniform hypergraph $H = (V, E)$ such that $\chi_c(H) = p/q$.

Proof Let r be an integer such that $rq \ge k$. Consider the hypergraph H_{qr}^{pr} (k). Since H_{qr}^{pr} (k) is the subhypergraph of H_{qr}^{pr} induced by the edges of cardinality k, $\chi_c(H_{qr}^{pr}(k)) \le (pr/qr) = p/q$. Suppose $c : H_{qr}^{pr}(k) \to H_t^s$ is a (s,t)-colouring of H_{qr}^{pr} (k).

Using *Lemma 2.6* gives the maximum size of an independent set in the hypergraph H_{qr}^{pr} (k) is , it follows that for any integer $i \in Z_s$,

$$\sum_{j=i}^{|i+t-1|_s} |c^{-1}(j)| \le qr$$

Hence

$$prt = \sum_{i=0}^{s-1} \sum_{j=i}^{i+t-1} |c^{-1}(i)| \le qrs$$

Therefore, $p/q \le s/t$. ■

3.CONCLUSION

Finally this paper concludes, if the hypergraph H has a universal vertex v, then the chromatic number's and the circular chromatic number's are equal. And also if H is a hypergraph then the chromatic number's, the circular chromatic number's and the Clique number's are all equal.

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