AN INTEGRO-DIFFERENTIAL EQUATION
ARISING FROM AN ELECTROCHEMISTRY MODEL

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Abstract. In this paper, we prove the existence and uniqueness of steady-state solutions for a system of equations arising from a model in electrochemistry. The same result was established by the authors in an earlier paper under the additional assumptions that the space-dimension $N = 2$ and the concentrations of the charged ions satisfy an electro-neutrality condition.

1. Introduction. Let $\Omega^*, \omega_1, \omega_2$ be open, bounded, connected subsets of $\mathbf{R}^N$, $N \geq 2$ such that $\overline{\omega_1}, \overline{\omega_2} \subset \Omega^*$ and $\overline{\omega_1} \cap \overline{\omega_2} = \varnothing$. Let $\Gamma_1, \Gamma_2, \Gamma_3$ be the $C^{2+\gamma}$ boundaries of ω_1, ω_2 and Ω^*, respectively, where $0 < \gamma < 1$ and define $\Omega = \Omega^* \setminus (\overline{\omega_1} \cup \overline{\omega_2})$. The purpose of this paper is to prove the following theorem:

THEOREM 1.1. Let Ω be defined as above and let ε, C_i be positive and z_i be nonzero constants for $i = 1, \ldots, m$. Suppose $Q \in C^\gamma(\overline{\Omega})$ where $0 < \gamma < 1$. Then the equation

$$\varepsilon \Delta \phi = - \sum \frac{z_i C_i e^{-z_i \phi(x)}}{\int_\Omega e^{-z_i \phi(y)} \, dy} + Q(x) \tag{1.1}$$

subject to the boundary conditions

$$\begin{cases} \phi(x) = 0 \text{ for } x \in \Gamma_1, \quad \phi(x) = \alpha \text{ for } x \in \Gamma_2, \\ \dfrac{\partial \phi}{\partial n}(x) = 0 \text{ for } x \in \Gamma_3 \end{cases} \tag{1.2}$$

have a unique solution $\phi \in C^{2+\gamma}(\overline{\Omega})$.

The above equations arise from the study of an electrochemistry model [1], [2], [3]. Consider a container holding some electrolyte, which is a mixture of charged particles in a solution. Two electrodes are inserted into the electrolyte and are subject to prescribed electric potentials. The densities of the charged particles are denoted by $u_1, \ldots, u_m$ and

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the electric potential is denoted by ϕ. Each species has a characteristic mobility constant Ω_i such that its velocity under the influence of the electric field is given by $-\Omega_i z_i \nabla \phi$ where z_i is its charge. The charge z_i must be integer-valued, and may be positive, negative, or zero. The species also diffuse with diffusion constant $d_i > 0$. It is frequently assumed that $\Omega_i = \mu\, d_i$ for $i = 1, \ldots, m$ where $\mu > 0$. This is called the Einstein relation. Under the above assumptions, the flux of species i is given by $-d_i(\nabla u_i + \mu z_i u_i \nabla \phi)$ and, after non-dimensionalization, u_i and ϕ satisfy the following system of equations:

$$\begin{cases} \dfrac{\partial u_i}{\partial t} = d_i \operatorname{div}(\nabla u_i + \mu z_i u_i \nabla \phi), \quad i = 1, \ldots, m, \\[2mm] \varepsilon \Delta \phi = -e \sum z_k u_k + Q(x). \end{cases} \tag{1.3}$$

In the above equations, $\varepsilon > 0$ is the permittivity of the solvent, $e > 0$ is the molar charge, and Q may be thought of as a charge density distributed inside the container.

Equations (1.3) are solved subject to the following boundary conditions:

$$\begin{cases} (\nabla u_i + \mu z_i u_i \nabla \phi) \cdot \mathbf{n} = 0, \quad \text{for } x \in \partial\Omega, \ t > 0, \\[2mm] \phi(x,t) = 0 \quad \text{for } x \in \Gamma_1, \qquad \phi(x,t) = \alpha \quad \text{for } x \in \Gamma_2, \ t > 0, \\[2mm] \dfrac{\partial \phi}{\partial n}(x,t) = 0 \quad \text{for } x \in \Gamma_3, \ t > 0, \end{cases} \tag{1.4}$$

and initial conditions

$$u_i(x,0) = u_i^0(x) \geq 0, \quad i = 1, \ldots, m. \tag{1.5}$$

In (1.4), $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, $\mathbf{n}$ is the unit outward normal at $\partial\Omega$, and $\partial\phi/\partial n = \nabla\phi \cdot \mathbf{n}$. Without loss of generality, we assume that α in (1.4) is positive.

The initial condition $\phi(x,0) = \phi^0(x)$ is not prescribed because it can be obtained from solving the equation $\varepsilon \Delta \phi^0 = -e \sum z_k u_k^0 + Q(x)$ together with the boundary conditions $\phi^0|_{\Gamma_1} = 0, \phi^0|_{\Gamma_2} = \alpha, \partial \phi^0/\partial n|_{\Gamma_3} = 0$. It is obvious that by absorbing μ into ϕ and e into ε, we may assume that $\mu = e = 1$. We also assume that $z_i \neq 0$ since if $z_i = 0$, then u_i is decoupled from the system (1.3) and may be solved independently.

The evolution problem (1.3), (1.4) has been studied in [3] with $Q(x) \equiv 0$. Using the techniques developed in [3], one can show that given $T > 0$, there exist unique, smooth solutions $u_i(x,t)$, $i = 1, \ldots, m$ and $\phi(x,t)$ to equations (1.3), (1.4), and (1.5) for $0 \leq t < T$. Furthermore, $u_i > 0$ if $t > 0$. Also, from (1.3), and (1.4a), $\int_\Omega u_i(x,t)\,dx = C_i$ is independent of t and depends only on the initial data u_i^0.

The steady-state solutions $u_i(x)$, $i = 1, \ldots, m$ and $\phi(x)$ satisfy the equations

$$\begin{cases} \operatorname{div}(\nabla u_i + z_i u_i \nabla \phi) = 0, \quad i = 1, \ldots, m, \\[2mm] \qquad\qquad \varepsilon \Delta \phi = -\sum z_k u_k + Q(x), \\[2mm] \displaystyle\int_\Omega u_i(x)\,dx = C_i > 0, \quad i = 1, \ldots, m, \\[2mm] \phi(x) = 0 \text{ for } x \in \Gamma_1, \qquad \phi(x) = \alpha \text{ for } x \in \Gamma_2, \\[2mm] \frac{\partial \phi}{\partial n}(x) = 0 \text{ for } x \in \Gamma_3 \end{cases} \tag{1.6}$$

where C_i, $i = 1, \ldots, m$ are known constants. Let $v_i = u_i e^{z_i \phi}$ for $i = 1, \ldots, m$. Then v_i satisfies the equation $\operatorname{div}(e^{-z_i \phi} \nabla v_i) = 0$ and the boundary conditions $\partial v_i/\partial n|_{\partial\Omega} = 0$.

Multiplying this equation by v_i and integrating by parts, we see that v_i is a constant. This constant may be found from condition (1.6c). Doing so, we obtain

$$u_i(x) = \frac{C_i e^{-z_i \phi(x)}}{\int_\Omega e^{-z_i \phi(y)} dy} \quad \text{for } i = 1, \dots, m. \tag{1.7}$$

Substituting this into Eq. (1.6b), we obtain Eq. (1.1). It is clear from above that proving the existence and uniqueness of solutions to Eqs. (1.6) is equivalent to proving the existence and uniqueness of solutions to Eq. (1.1) subject to the boundary conditions (1.2).

In the paper [3], existence and uniqueness of steady-state solutions to Eqs. (1.6) were established under the assumption $Q \equiv 0$ and the electro-neutrality condition $\sum z_i C_i = 0$. In the same paper, global stability of the steady-state solutions was also established when $N = 2$. This result is equivalent to saying that ions in any initial distribution will eventually settle down to their unique steady-states. Due to the lack of time-independent a priori bounds for $\|u_i(\cdot, t)\|_{L^2(\Omega)}$, the global stability result cannot be extended to the case $N = 3$.

The purpose of this paper is to prove the existence and uniqueness of steady-state solutions to Eqs. (1.6) without the assumption $Q \equiv 0$ or $\sum z_i C_i = 0$. The result is stated in Theorem 1.1. As a corollary of this theorem and Theorem 8.1 of [3], we have the following stability result.

THEOREM 1.2. Let $u_i^0 \geq 0$ be smooth initial data defined on $\Omega \subset \mathbf{R}^2$ for $i = 1, \dots, m$ and let $C_i := \int_0^1 u_i^0(x) \, dx > 0$. For $N = 2$, there exist unique solutions U_i, $i = 1, \dots, m$ and Φ to the steady-state equations (1.6). Moreover, the solutions u_i, $i = 1, \dots, m$ and ϕ of equations (1.3), (1.4), and (1.5) satisfy $\lim_{t \to \infty} u_i(x, t) = U_i(x)$ and $\lim_{t \to \infty} \phi(x, t) = \Phi(x)$ uniformly on $\overline{\Omega}$.

The same result holds for $\Omega \subset \mathbf{R}^3$ provided that we have a time-dependent $L^2(\Omega)$-norm bound of the solutions u_i, $i = 1, \dots, m$.

The organization of this paper is as follows. Section 2 contains the proof of a priori bounds for the steady-state solutions while Sec. 3 contains the proof of Theorem 1.1.

2. A priori estimates. Let $\Omega_+ = \{x \in \overline{\Omega} \mid u \geq 0\}$ and $\Omega_- = \{x \in \overline{\Omega} \mid u < 0\}$. We first establish two lemmas.

LEMMA 2.1. Let $w, u \in C(\overline{\Omega})$, and $w \geq 0$ on $\overline{\Omega}$. Then for any nonnegative integer n

$$z_i \frac{\int_{\Omega_+} w(x) e^{-z_i u} u^{n+1} \, dx}{\int_{\Omega_+} w(x) e^{-z_i u} u^n \, dx} \leq z_i \frac{\int_{\Omega_+} w(x) u^{n+1} \, dx}{\int_{\Omega_+} w(x) u^n \, dx}, \quad \text{if } z_i > 0. \tag{2.1}$$

Proof. Define positive measures M_i and m_i on Ω_+ such that their Radon-Nikodym derivatives satisfy $dM_i/dx = e^{-z_i u} w(x) u^n / \int_{\Omega_+} e^{-z_i u} w(x) u^n \, dx$ and $dm_i/dx = w(x) u^n / \int_{\Omega_+} w(x) u^n \, dx$, respectively. Then, $\int_{\Omega_+} dM_i = 1$ and $\int_{\Omega_+} dm_i = 1$.

Suppose $z_i > 0$. From Jensen's inequality

$$\exp\left(\frac{z_i \int_{\Omega_+} w(x)e^{-z_i u}u^{n+1}\,dx}{\int_{\Omega_+} w(x)e^{-z_i u}u^n\,dx}\right) = \exp\left(\int_{\Omega_+} z_i u\,dM_i\right)$$

$$\leq \int_{\Omega_+} e^{z_i u}\,dM_i$$

$$= \frac{\int_{\Omega_+} w(x)u^n\,dx}{\int_{\Omega_+} w(x)e^{-z_i u}u^n\,dx}$$

$$= 1/\int_{\Omega_+} e^{-z_i u}\,dm_i.$$

Taking the logarithm on both sides, we have

$$\frac{z_i \int_{\Omega_+} w(x)e^{-z_i u}u^{n+1}\,dx}{\int_{\Omega_+} w(x)e^{-z_i u}u^n\,dx} \leq -\log\left(\int_{\Omega_+} e^{-z_i u}\,dm_i\right)$$

$$\leq \int_{\Omega_+} z_i u\,dm_i \tag{2.2}$$

$$= \frac{z_i \int_{\Omega_+} w(x)u^{n+1}\,dx}{\int_{\Omega_+} w(x)u^n\,dx}.$$

The proof of the lemma is complete.

LEMMA 2.2. Let n be a nonnegative integer and let $\phi_0, u \in C(\overline{\Omega})$. Then

$$z_i C_i \frac{\int_\Omega e^{-z_i \phi_0}e^{-z_i u}u^{2n+1}\,dx}{\int_\Omega e^{-z_i \phi_0}e^{-z_i u}\,dx} \leq |z_i|C_i \frac{\int_\Omega e^{-z_i \phi_0}|u|^{2n+1}\,dx}{\int_\Omega e^{-z_i \phi_0}\,dx}. \tag{2.3}$$

Proof. Suppose $z_i > 0$. From Eq. (2.1), we have

$$z_i C_i \frac{\int_\Omega e^{-z_i \phi_0}e^{-z_i u}u^{2n+1}\,dx}{\int_\Omega e^{-z_i \phi_0}e^{-z_i u}\,dx}$$

$$\leq z_i C_i \frac{\int_{\Omega_+} e^{-z_i \phi_0}e^{-z_i u}u^{2n+1}\,dx}{\int_\Omega e^{-z_i \phi_0}e^{-z_i u}\,dx}$$

$$= z_i C_i \frac{\int_{\Omega_+} e^{-z_i \phi_0}e^{-z_i u}\,dx}{\int_\Omega e^{-z_i \phi_0}e^{-z_i u}\,dx} \frac{\int_{\Omega_+} e^{-z_i \phi_0}e^{-z_i u}u\,dx}{\int_{\Omega_+} e^{-z_i \phi_0}e^{-z_i u}\,dx} \cdots \frac{\int_{\Omega_+} e^{-z_i \phi_0}e^{-z_i u}u^{2n+1}\,dx}{\int_{\Omega_+} e^{-z_i \phi_0}e^{-z_i u}u^{2n}\,dx}$$

$$\leq z_i C_i \frac{\int_{\Omega_+} e^{-z_i \phi_0}e^{-z_i u}\,dx}{\int_\Omega e^{-z_i \phi_0}e^{-z_i u}\,dx} \frac{\int_{\Omega_+} e^{-z_i \phi_0}u\,dx}{\int_{\Omega_+} e^{-z_i \phi_0}\,dx} \cdots \frac{\int_{\Omega_+} e^{-z_i \phi_0}u^{2n+1}\,dx}{\int_{\Omega_+} e^{-z_i \phi_0}u^{2n}\,dx}$$

$$= z_i C_i \frac{\int_{\Omega_+} e^{-z_i \phi_0}e^{-z_i u}\,dx}{\int_\Omega e^{-z_i \phi_0}e^{-z_i u}\,dx} \frac{\int_{\Omega_+} e^{-z_i \phi_0}u^{2n+1}\,dx}{\int_{\Omega_+} e^{-z_i \phi_0}\,dx}$$

$$\leq z_i C_i \frac{\int_{\Omega_+} e^{-z_i \phi_0}u^{2n+1}\,dx}{\int_\Omega e^{-z_i \phi_0}\,dx}$$

$$\leq |z_i|C_i \frac{\int_\Omega e^{-z_i \phi_0}|u|^{2n+1}\,dx}{\int_\Omega e^{-z_i \phi_0}\,dx}.$$

The second to last of the above inequalities follows from the fact that

$$\left(\int_{\Omega_+} e^{-z_i\phi_0} e^{-z_i u}\, dx \right) \left(\int_{\Omega_+} e^{-z_i\phi_0}\, dx + \int_{\Omega_-} e^{-z_i\phi_0}\, dx \right)$$

$$\leq \left(\int_{\Omega_+} e^{-z_i\phi_0} e^{-z_i u}\, dx + \int_{\Omega_-} e^{-z_i\phi_0} e^{-z_i u}\, dx \right) \left(\int_{\Omega_+} e^{-z_i\phi_0}\, dx \right).$$

Hence, inequality (2.3) is valid for $z_i > 0$.

If $z_i < 0$, let $v = z_i u$. From (2.3), with $z_i = 1$ and ϕ_0 there replaced by $z_i \phi_0$,

$$z_i C_i \frac{\int_\Omega e^{-z_i\phi_0} e^{-z_i u} u^{2n+1}\, dx}{\int_\Omega e^{-z_i\phi_0} e^{-z_i u}\, dx}$$

$$= \frac{C_i}{z_i^{2n}} \frac{\int_\Omega e^{-z_i\phi_0} e^{-v} v^{2n+1}\, dx}{\int_\Omega e^{-z_i\phi_0} e^{-v}\, dx}$$

$$\leq \frac{C_i}{z_i^{2n}} \frac{\int_\Omega e^{-z_i\phi_0} |v|^{2n+1}\, dx}{\int_\Omega e^{-z_i\phi_0}\, dx}$$

$$= |z_i| C_i \frac{\int_\Omega e^{-z_i\phi_0} |u|^{2n+1}\, dx}{\int_\Omega e^{-z_i\phi_0}\, dx}.$$

The proof of the lemma is complete.

We now define $\phi_0 \in C^{2+\gamma}(\overline{\Omega})$ as the solution to the equation

$$\varepsilon \Delta \phi_0 = Q(x) \tag{2.4}$$

subject to the boundary conditions (1.2). Let $u = \phi - \phi_0$. Then u satisfies

$$\varepsilon \Delta u = -\sum \frac{z_i C_i e^{-z_i\phi_0} e^{-z_i u}}{\int_\Omega e^{-z_i\phi_0(y)} e^{-z_i u(y)}\, dy} \tag{2.5}$$

and the homogeneous boundary conditions

$$\begin{cases} u(x) = 0 \quad \text{for } x \in \Gamma_1, \qquad u(x) = 0 \quad \text{for } x \in \Gamma_2, \\ \dfrac{\partial u}{\partial n}(x) = 0 \quad \text{for } x \in \Gamma_3. \end{cases} \tag{2.6}$$

Proving Theorem 1.1 is equivalent to showing that Eqs. (2.5) and (2.6) have a unique solution in $C^{2+\gamma}(\overline{\Omega})$.

In what follows, $k, k_1, k_2, \ldots$, and $M, M_1, M_2, \ldots$ will denote generic positive constants. From the definition of ϕ_0, we have $\|\phi_0\|_\infty \leq k_1$.

Let $\overline{z} = \max |z_i|$ and $|\Omega| = \text{meas}(\Omega)$. Multiplying Eq. (2.5) by $-u^{2n+1}$ and integrating over Ω, we have, from (2.3),

$$\varepsilon \frac{2n+1}{(n+1)^2} \int_\Omega \nabla(u^{n+1}) \cdot \nabla(u^{n+1})\, dx = \sum z_i C_i \frac{\int_\Omega e^{-z_i\phi_0} e^{-z_i u} u^{2n+1}\, dx}{\int_\Omega e^{-z_i\phi_0} e^{-z_i u}\, dy}$$

$$\leq \sum |z_i| C_i \frac{\int_\Omega e^{-z_i\phi_0} |u|^{2n+1}\, dx}{\int_\Omega e^{-z_i\phi_0}\, dy} \tag{2.7}$$

$$\leq M \int_\Omega |u|^{2n+1}\, dx$$

where $M = \sum |z_i| C_i e^{2\bar{z}k_1}/|\Omega|$. In particular, taking $n = 0$ in Eq. (2.7) and employing Poincaré's inequality, we have

$$\|u\|_{H^1}^2 \le \frac{M}{\varepsilon}\|u\|_1 \le \frac{M}{\varepsilon}k_2\|u\|_2 \le \frac{M}{\varepsilon}k_3\|u\|_{H^1}.$$

Thus $\|u\|_{H^1} \le k_4$ and we conclude that

$$\|u\|_2 \le k_5. \tag{2.8}$$

We now bootstrap this estimate to obtain an L^∞ bound on u. We concentrate on the case $N \ge 3$ since the case $N = 2$ is easier.

LEMMA 2.3. Let $N \ge 3$ and let $n \ge 1$ be a positive integer. Define

$$p = 2N(n + 1)/(N - 2). \tag{2.9}$$

Then there exists a constant M_1, independent of n, such that

$$\|u\|_p \le M_1 p^{(N+1)/(N+4)}. \tag{2.10}$$

Proof. For any $w \in H^1(\Omega)$ with $w = 0$ on part of the boundary $\partial\Omega$, Sobolev's inequality implies that

$$\|w\|_{2N/(N-2)} \le k\|w\|_{H^1}.$$

From (2.7),

$$\|u\|_{2N(n+1)/(N-2)}^{2n+2} = \|u^{n+1}\|_{2N/(N-2)}^2 \le k^2\|u^{n+1}\|_{H^1}^2 \le \frac{k^2 M}{\varepsilon}\frac{(n+1)^2}{(2n+1)}\|u\|_{2n+1}^{2n+1}.$$

Now we employ the interpolation equation (7.9) on p. 146 of [4] to obtain

$$\|u\|_{2N(n+1)/(N-2)}^{2n+2} \le \frac{k^2 M}{\varepsilon}\frac{(n+1)^2}{(2n+1)}(\|u\|_2^\lambda \|u\|_{2N(n+1)/(N-2)}^{1-\lambda})^{2n+1} \tag{2.11}$$

where λ satisfies

$$\frac{1}{2n+1} = \frac{\lambda}{2} + \frac{1-\lambda}{2N(n+1)/(N-2)}.$$

It can be checked that

$$\lambda = \frac{N + 4n + 2}{(2n+1)(nN+2)}, \tag{2.12}$$

$$1 + \lambda(2n+1) = \frac{(n+1)(N+4)}{nN+2}, \tag{2.13}$$

and

$$\frac{1}{1 + \lambda(2n+1)} \to \frac{N}{N+4} \quad \text{as } n \to \infty. \tag{2.14}$$

From (2.11),

$$\|u\|_{2N(n+1)/(N-2)}^{1+\lambda(2n+1)} \le \frac{k^2 M}{\varepsilon}\frac{(n+1)^2}{(2n+1)}\|u\|_2^{(2n+1)\lambda} \le \frac{2k^2 nM}{\varepsilon}\|u\|_2^{(2n+1)\lambda}.$$

From (2.8),

$$\|u\|_{2N(n+1)/(N-2)} \le k_1 n^{1/(1+\lambda(2n+1))}$$

where $k_1 \geq 1$ is independent of n. From (2.14),

$$\|u\|_{2N(n+1)/(N-2)} \leq k_1 n^{(N+1)/(N+4)}$$

for sufficiently large n. By increasing k_1 if necessary, the above inequality holds for all n. From (2.9),

$$\|u\|_p \leq k_1 \left(\frac{p(N-2)}{2N} - 1 \right)^{(N+1)/(N+4)} \leq M_1 p^{(N+1)/(N+4)}.$$

The proof of the lemma is complete.

COROLLARY 2.4. For any $1 \leq p < \infty$,

$$\|u\|_p < \infty.$$

LEMMA 2.5. Let $N \geq 3$. For any integer $r \geq 4N/(N-2)$, there exists $M_2 > 0$, independent of r, such that

$$\|u\|_r \leq M_2 r^{(N+1)/(N+4)}. \tag{2.15}$$

Proof. Given $r \geq 4N/(N-2)$, choose an integer s such that

$$\frac{2Ns}{N-2} < r \leq \frac{2N(s+1)}{N-2}.$$

Define $p = 2N(s+1)/(N-2)$. Then it is easy to see that $0 \leq 1/r - 1/p < 1$. Hence $|\Omega|^{1/r-1/p} \leq k_1$. From Hölder's inequality,

$$\|u\|_r \leq |\Omega|^{1/r-1/p}\|u\|_p. \tag{2.16}$$

From Lemma 2.3,

$$\|u\|_r \leq k_1 M_1 p^{(N+1)/(N+4)}$$
$$\leq k_1 M_1 \left(\frac{4Ns}{N-2} \right)^{(N+1)/(N+4)}$$
$$\leq M_2 r^{(N+1)/(N+4)}.$$

This completes the proof of the lemma.

LEMMA 2.6. For any constant $\beta \in \mathbf{R}$, there exists $M_\beta > 0$ such that

$$\int_\Omega e^{\beta u} \, dx \leq M_\beta. \tag{2.17}$$

Proof. Since

$$\left| \int_\Omega e^{\beta u} \, dx \right| \leq \sum_{r=0}^{\infty} \int_\Omega \frac{(|\beta u|)^4}{r!} \, dx$$

and $u \in L^r$ for any positive integer r because of Corollary 2.4, it suffices to show that the above series converges.

From (2.15), we have, for any $r_0 > 4N/(N-2)$

$$\sum_{r=r_0}^{\infty} \int_\Omega \frac{|\beta u|^r}{r!} \, dx \leq \sum_{r=r_0}^{\infty} \frac{|\beta|^r}{r!} \|u\|_r^r \leq \sum_{r=r_0}^{\infty} \frac{|\beta|^r}{r!} M_2^r r^{(N+1)r/(N+4)}. \tag{2.18}$$

Stirling's formula implies that

$$\lim_{r \to \infty} \frac{r^r \sqrt{2\pi r}}{e^r r!} = 1.$$

Thus by increasing r_0 if necessary, for $r \geq r_0$,

$$\frac{1}{r!} \leq \frac{2e^r}{r^r \sqrt{2\pi r}}. \tag{2.19}$$

Therefore,

$$\sum_{r=r_0}^{\infty} \int_\Omega \frac{|\beta u|^r}{r!} \, dx \leq \sum_{r=r_0}^{\infty} \frac{2e^r M_2^r |\beta|^r}{r^r \sqrt{2\pi r}} r^{(N+1)r/(N+4)} = \sum_{r=r_0}^{\infty} \frac{2(eM_2|\beta|)^r}{r^{3r/(N+4)} \sqrt{2\pi r}}. \tag{2.20}$$

This series converges because of the root test and (2.17) follows. The lemma is also valid if $N = 2$ since in this case there exist constants $c_1, c_2 > 0$ such that $\int_\Omega \exp(c_1|u|/\|u\|_{H^1})^2 \, dx \leq c_2$ (see p. 162 in [4]) and $\|u\|_{H^1}$ has been shown to be bounded.

LEMMA 2.7. For any $1 \leq p < \infty$, there exists an $M_p > 0$ such that

$$\frac{1}{\int_\Omega e^{-z_i \phi_0(y)} e^{-z_i u(y)} \, dy} \|e^{-z_i \phi_0} e^{-z_i u}\|_p \leq M_p. \tag{2.21}$$

Proof. Recall that $\|\phi_0\|_\infty \leq k_1$ and $\bar{z} = \max |z_i|$. From Lemma 2.6,

$$\|e^{-z_i u}\|_p \leq k_2 \tag{2.22}$$

where k_2 may depend on p. Hence

$$\|e^{-z_i \phi_0} e^{-z_i u}\|_p \leq k_2 e^{\bar{z} k_1}. \tag{2.23}$$

The function $f(v) = 1/v$ is convex so that Jensen's inequality implies that

$$\left(\frac{1}{|\Omega|} \int_\Omega e^{-z_i u(y)} \, dy\right)^{-1} \leq \frac{1}{|\Omega|} \int_\Omega e^{z_i u(y)} \, dy. \tag{2.24}$$

Since

$$\int_\Omega e^{-z_i \phi_0(y)} e^{-z_i u(y)} \, dy \geq e^{-\bar{z} k_1} \int_\Omega e^{-z_i u(y)} \, dy, \tag{2.25}$$

we have

$$\frac{1}{\int_\Omega e^{-z_i \phi_0(y)} e^{-z_i u(y)} \, dy} \leq \frac{e^{\bar{z} k_1}}{|\Omega|^2} \int_\Omega e^{z_i u(y)} \, dy \leq k_3. \tag{2.26}$$

Inequality (2.21) follows from (2.23) and (2.26).

LEMMA 2.8. There exists an $M_3 > 0$ such that

$$\|u\|_{C^{2+\gamma}(\overline{\Omega})} \leq M_3. \tag{2.27}$$

Proof. From Lemma 2.7, the right-hand side of (2.5) is bounded in the L^p norm for any $1 \leq p < \infty$. Thus $u \in W^{2,p}$ by regularity estimates. By taking p sufficiently large, Sobolev's imbedding theorem implies that $u \in C^{1+\gamma}(\overline{\Omega})$. Since $\phi_0 \in C^{2+\gamma}(\overline{\Omega})$, the right-hand side of Eq. (2.5) is bounded in the $C^{1+\gamma}(\overline{\Omega})$ norm. Schauder's estimate then gives a bound in the $C^{2+\gamma}(\overline{\Omega})$ norm. The proof of the lemma is complete.

REMARK. Suppose the right-hand side of Eq. (2.5) is multiplied by $\lambda \in [0, 1]$ and that the same a priori bounds (2.27) for this new equation subject to the same boundary condition (2.6) also hold. In other words, M_3 can be taken to be independent of $\lambda \in [0, 1]$. This is important in our proof in the next section.

3. Proof of Theorem 1.1.

Let M_3 be defined as in Lemma 2.8 and let

$$S = \{u \in C^\gamma(\overline{\Omega}) : u = 0 \text{ on } \Gamma_1 \cup \Gamma_2, \partial u / \partial n = 0 \text{ on } \Gamma_3, \|u\|_{C^\gamma(\overline{\Omega})} \leq M_3\}. \tag{3.1}$$

We employ the notation $\Delta^{-1}(f)$ to denote the solution w of $\Delta w = f$ satisfying the boundary conditions (2.6).

Consider the map $T_\lambda : S \to C^\gamma(\overline{\Omega})$ such that for any $u \in S$,

$$T_\lambda(u) = u + \Delta^{-1}\left(\lambda \sum \frac{z_i C_i e^{-z_i \phi_0} e^{-z_i u}}{\int_\Omega e^{-z_i \phi_0(y)} e^{-z_i u(y)}\, dy}\right). \tag{3.2}$$

From Lemma 2.8 and its following remark, there is no solution to $T_\lambda(u) = 0$ on ∂S for any $0 \leq \lambda \leq 1$. Thus $\deg(T_\lambda, S, 0)$ is well-defined for $0 \leq \lambda \leq 1$. By the homotopy property, $\deg(T_0, S, 0) = \deg(T_1, S, 0)$. Since T_0 is the identity map, $\deg(T_0, S, 0) = 1$. Therefore $\deg(T_1, S, 0) = 1$ and a solution of the equation $T_1(u) = 0$ exists. By the regularity estimate, $u \in C^{2+\gamma}(\overline{\Omega})$. With $\phi = \phi_0 + u$ and $\phi_0 \in C^{2+\gamma}(\overline{\Omega})$, we have $\phi \in C^{2+\gamma}(\overline{\Omega})$. This proves the existence of a solution to Eqs. (1.1)–(1.2).

Next define

$$U = \{u \in C^1(\overline{\Omega}) : u = 0 \text{ on } \Gamma_1 \cup \Gamma_2, \partial u / \partial n = 0 \text{ on } \Gamma_3\}, \tag{3.3}$$

and $I : U \to \mathbf{R}$ by

$$I(u) = \int_\Omega \frac{\varepsilon}{2} |\nabla u|^2\, dx + \sum C_i \log \int_\Omega e^{-z_i \phi_0} e^{-z_i u}\, dx. \tag{3.4}$$

It can be checked that I is Fréchet differentiable, and for any $u, v \in U$,

$$I'(u)v = \int_\Omega \varepsilon \nabla u \cdot \nabla v\, dx - \sum z_i C_i \frac{\int_\Omega e^{-z_i \phi_0} e^{-z_i u} v\, dx}{\int_\Omega e^{-z_i \phi_0} e^{-z_i u}\, dy}. \tag{3.5}$$

In fact, $I \in C^2$. For any $u, v, w \in U$,

$$I''(u)(v, w) = \int_\Omega \varepsilon \nabla w \cdot \nabla v\, dx + \sum z_i^2 C_i \frac{\int_\Omega e^{-z_i \phi_0} e^{-z_i u} wv\, dx}{\int_\Omega e^{-z_i \phi_0} e^{-z_i u}\, dy}$$
$$- \sum z_i^2 C_i \frac{\int_\Omega e^{-z_i \phi_0} e^{-z_i u} v\, dx \int_\Omega e^{-z_i \phi_0} e^{-z_i u} w\, dx}{(\int_\Omega e^{-z_i \phi_0} e^{-z_i u}\, dy)^2}. \tag{3.6}$$

Cauchy-Schwarz's inequality implies that

$$\int_\Omega e^{-z_i \phi_0} e^{-z_i u}\, dx \int_\Omega e^{-z_i \phi_0} e^{-z_i u} v^2\, dx \geq \left(\int_\Omega e^{-z_i \phi_0} e^{-z_i u} v\, dx\right)^2. \tag{3.7}$$

Let $v = w$ in Eq. (3.6). Then (3.7) implies that

$$I''(u)(v, v) \geq \varepsilon \int_\Omega |\nabla v|^2\, dx \geq 0 \tag{3.8}$$

and

$$I''(u)(v, v) = 0 \quad \text{if and only if} \quad v = 0. \tag{3.9}$$

We are now in a position to prove the uniqueness result in Theorem 1.1. Let u_1 and u_2 be two solutions of Eqs. (2.5)–(2.6). We can multiply Eq. (2.5) by any $v \in U$, and integrate by parts to obtain

$$I'(u_i)v = 0 \quad \text{for } i = 1, 2. \tag{3.10}$$

Now

$$I'(u_2)v - I'(u_1)v = \int_0^1 I''(u_1 + t(u_2 - u_1))(u_2 - u_1, v) \, dt.$$

Take $v = u_2 - u_1$. Employing Eqs. (3.8) and (3.10), we have

$$0 = \int_0^1 I''(u_1 + tv)(v, v) \, dt \geq 0. \tag{3.11}$$

Since $I \in C^2$, so $I''(u_1 + tv)(v, v) = 0$ for all $t \in [0, 1]$. Hence $v = u_2 - u_1 = 0$ by Eq. (3.9). Thus, the solution u to Eqs. (2.5)–(2.6) is unique. The proof of Theorem 1.1 is complete.

REMARK. For $N = 2$, we can define

$$V = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_1 \cup \Gamma_2, \partial u/\partial n = 0 \text{ on } \Gamma_3\},$$

and $I : V \to \mathbf{R}$ by Eq. (3.4). Then we can proceed using variational calculus to show the existence and uniqueness of the solution to Eqs. (2.5)–(2.6). This approach fails for $N \geq 3$ since $e^{-z_i u}$ may not be integrable for $u \in V$. Moreover, the Fréchet derivative of I may not exist, making it difficult to relate the critical points of I to the weak solutions of Eqs. (2.5)–(2.6).

REFERENCES

[1] Y. S. Choi and R. Lui, *Uniqueness of steady-state solutions for an electrochemistry model with multiple species*, J. Diff. Equations **108**, 424–437 (1994)
[2] Y. S. Choi and R. Lui, *Global stability of solutions of an electrochemistry model with multiple species*, J. Diff. Equations (to appear)
[3] Y. S. Choi and R. Lui, *Multi-dimensional electrochemistry model*, Arch. Rational Mech. Anal. (to appear)
[4] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd edition, Springer-Verlag, 1983