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AN INTEGRO-DIFFERENTIAL EQUATION WITH APPLICATION IN HEAT FLOW*

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Abstract. The problem

$$u_t(x, t) = \int_0^t a(t - \tau) \frac{\partial}{\partial x} \sigma(u_x(x, \tau)) \, d\tau + f(x, t), \qquad 0 < x < 1, \qquad t > 0,$$

$$u(0, t) \equiv u(1, t) \equiv 0 \qquad u(x, 0) = u_0(x)$$

is considered. Asymptotic stability theorems for the solution are established under appropriate conditions on a, σ and f. The conditions on a are of frequency domain type and are related to ones used previously in the study of Volterra integral equations,

$$\dot{u} = -\int_0^t a(t - \tau) g(u(\tau)) \, d\tau + f(t)$$

on a Hilbert space. An existence theorem for the problem is established under smallness assumptions on f and u_0 . This theorem is related to one by Nishida for the damped non-linear wave equation,

$$u_{tt} + \alpha u_t - \frac{\partial}{\partial x} \sigma(u_x) = 0.$$

It is shown that the problem is related to a theory of heat flow in materials with memory.

1. Introduction. This paper is concerned with the problem

$$u_t(x, t) = \int_0^t a(t - \tau) \frac{\partial}{\partial x} \sigma(u_x(x, \tau)) \, d\tau + f(x, t), \qquad 0 < x < 1, \qquad t > 0$$

$$u(0, t) \equiv u(1, t) \equiv 0 \qquad u(x, 0) = u_0(x). \qquad \text{(P)}$$

The problem serves as a very special model for one-dimensional heat flow in materials with memory [3]. This aspect is discussed in Sec. 7. It is also an example in the general theory of equations of the form,

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$$\dot{u}(t) = -\int_0^t a(t - \tau)g(u(\tau))\,d\tau + f(t), \qquad u(0) = u_0 \tag{E}$$

on a Hilbert space H, with g a non-linear unbounded operator.

There is an extensive literature on equation (E); see for example [5], [6], [9], [10] and [12]. The main concern has been asymptotic stability. On infinite-dimensional spaces with unbounded g's the results are incomplete in two respects. The stability results depend on a priori smoothness assumptions* and there are no existence theorems. (In [1] and [7] these difficulties are overcome for modified versions of (E).)

For our special model we partially remedy these defects. We establish an asymptotic stability result with no a priori assumptions other than existence. It will be seen in Sec. 4 that this result can easily be extended to a general class of equations (E). We also establish an existence theorem but this depends crucially on the special form.

This work was prompted by a preprint of a remarkable paper by Nishida [11]. Nishida uses some ideas of Lax [4] to establish the existence, for small data, of global classical solutions of the problem

$$\begin{gathered} u_{tt} + \alpha u_t - \frac{\partial}{\partial x}\sigma(u_x) = 0, \qquad -\infty < x < \infty, \qquad t > 0 \\ u(x, 0) = u_0(x), \qquad u_t(x, 0) = u_1(x) \end{gathered} \tag{P$_1$}$$

when $\alpha > 0$. This result is indeed remarkable since it is known that (P_1) cannot have classical solutions for all t if $\alpha = 0$ and σ is non-linear.

The major theme of our work is that the problem (P) is closely associated with the inhomogeneous form of (P_1) and that more generally (E) is associated with,

$$\ddot{u}(t) + \alpha\dot{u}(t) + g(u(t)) = \varphi(t). \tag{E$_1$}$$

If $a(t) = e^{-\alpha t}$ it is easy to see that (E) is equivalent to (E_1) $(\varphi = \dot{f} + \alpha f)$. In the general case (E) is equivalent to,

$$\ddot{u}(t) + k(0)\dot{u} + g(u(t)) + \int_0^t \dot{k}(t - \tau)\dot{u}(\tau)\,d\tau = \varphi(t) \tag{E$_k$}$$

for some function k. Eq. (E_1) admits of a number of energy estimates and the special case corresponding to (P) admits of a very detailed analysis by means of Riemann invariants.

The object of this paper is to show that (E_k) admits of the same treatment as (E_1) provided that the kernel a of equation (E) satisfies the conditions used in [9] to establish the provisional stability results. (There is a minor technical strengthening of these conditions.)

The transition from (P_1) to the more general equation (E_k) is of interest in the physical context. The inclusion of the term αu_t, although it stabilizes the equation, is rather ad hoc. The presence of the hereditary effect in (P) is, on the other hand, a natural consequence of the assumption of material memory. What we show is that the memory effect produces the same result as the term αu_t. On the other hand we want to observe that our results are incomplete in the physical context. We assume that $a \in L_1(0, \infty)$. As

* Specifically for problem (P) it would be required that for any $\eta \in C^1[0, 1]$, $\eta(0) = \eta(1) = 0$, $\int_0^1 \sigma(u_x(x, t))\eta_x(x)\,dx$ is bounded and uniformly continuous in $t > 0$. See [9].

we show in Sec. 7, this is natural in the heat flow situation. In the problem of non-linear viscoelasticity, which is formally the same as (P), one wants $a = a_\infty + b$, $a_\infty > 0$, $b \in L_1(0, \infty)$ and here our theory does not apply. It would be very instructive to have an extension to this case, especially so since Nishida's problem (P) has non-linear elasticity as a prototype.

2. Statement of results. We denote by $L_a[\varphi]$ the Volterra integral operator,

$$L_a[\varphi](t) = \int_0^t a(t - \tau)\varphi(\tau)\, d\tau, \tag{2.1}$$

and we consider the problem (P);

$$u_t(x, t) = L_a\left[\frac{\partial}{\partial x}\sigma(u_x(x, \cdot))\right](t) + f(x, t), \qquad 0 < x < 1, \qquad t > 0, \tag{2.2}$$

$$u(0, t) \equiv u(1, t) \equiv 0, \qquad t \geq 0, \tag{2.3}$$

$$u(x, 0) = u_0(x), \qquad 0 \leq x \leq 1. \tag{2.4}$$

By a *solution* we will mean a function $u(x, t) \in C^{(2)}([0, 1] \times [0, \infty))$ which satisfies (2.2)–(2.4).

We first list a set of hypotheses on a which will hold throughout the paper. We assume that $a \in C^{(2)}[0, \infty)$ and that,

$$a(0) > 0, \qquad \dot{a}(0) < 0 \tag{a_1}$$

$$a^{(k)} \in L_1(0, \infty), \qquad k = 0, 1, 2. \tag{a_2'}$$

It follows from $(a_2)'$ that a has a Laplace transform $\hat{a}(s)$ in Re $s \geq 0$. We require that,

$$\operatorname{Re} \hat{a}(i\eta) > 0 \quad \text{for all} \quad \eta. \tag{a_3}$$

Conditions (a_1), $(a_2)'$ and (a_3) were the ones previously used in [9] in the study of the general equation (E). (The condition,

$$(-1)^k a^{(k)}(t) > 0, \qquad k = 0, 1, 2, \tag{2.5}$$

was used in [5]. It is shown in [9] that, under conditions (a_1) and $(a_2)'$, (2.5) implies (a_3) but not conversely.)

For the purposes of this paper we need a strengthening of $(a_2)'$. We require that,

$$t^j a^{(k)}(t) \in L_1(0, \infty), \qquad k = 0, 1, 2; \qquad j \leq 3 + N \quad \text{for some} \quad N \geq 0. \tag{a_2}$$

Condition (a_2), $k = 0$, implies that $\hat{a}(s)$ is of class $C^{(N+3)}$ in Re $s \geq 0$.

We assume that $f \in C^{(3)}([0, 1] \times [0, \infty))$ and we define $\bar{f}(t)$ by,

$$\bar{f}(t) = \sup_{x \in [0,1]} (|f(x, t)|, |f_t(x, t)|). \tag{2.6}$$

A basic assumption on f is

$$\bar{f} \in L_1(0, \infty) \cap L_2(0, \infty) \cap L_\infty(0, \infty). \tag{f_1}$$

For σ we make the assumptions:

$$\sigma \in C^{(2)}(-\infty, \infty), \qquad \sigma(0) = 0, \qquad \sigma'(\xi) \geq \epsilon > 0 \quad \text{for all} \quad \xi. \tag{σ_1}$$

Conditions (a_1), (a_2) *(for some* N*)*, (a_3), (f_1), (σ_1) *hold throughout the paper.*

Theorem (I). (i) *If u is a solution of* (P) *then*

$$\lim_{t\to\infty}\int_0^1 u^2(x, t)\,dx = 0. \tag{2.7}$$

(ii) *If, in addition, σ satisfies the condition $|\sigma(\xi)| \le M(|\xi| + |\xi|^r)$ for some $M > 0$ and r, $1 < r < 2$ then for any $\eta \in C^{(1)}[0, 1]$ with $\eta(0) = \eta(1) = 0$,*

$$\lim_{t\to\infty}\int_0^1 u_t(x, t)\eta(x)\,dx = 0. \tag{2.8}$$

Remark 2.1. Theorem (I) is capable of generalization to the abstract equation (E) as will be clear from the proof in Sec. 4. It also extends to a result on approach to steady state. Suppose f in the right side of (2.2) is replaced by $f_0(x) + f(x, t)$ where f satisfies (f_1). It is not difficult to verify from (σ_1) that there exists a unique solution of the problem,

$$\sigma(U_0'(x))' = -f_0(x)/\hat{a}(0), \qquad 0 < x < 1, \qquad U_0(0) = U_0(1) = 0. \tag{2.9}$$

Then the methods of [7] can be used to show that if the solution of (P) is $U(x, t)$ and we write $U(x, t) = U_0(x) + u(x, t)$ then (2.7) and (2.8) hold.

We show that (2.2) can be solved for

$$\frac{\partial}{\partial x}\sigma(u_x)$$

in the form (E_k), that is,

$$u_{tt} + k(0)u_t - \frac{\partial}{\partial x}\sigma(u_x) + \int_0^t \dot{k}(t - \tau)u_\tau(x, \tau)\,d\tau = \varphi(x, t). \tag{2.10}$$

Our further results require condition (a_2) for some $N \ge 4$ and the following strengthened version of (f_1):

$$t^j\bar{f}(t) \in L_1(0, \infty) \cap L_2(0, \infty) \cap L_\infty(0, \infty), \qquad j \le N, \tag{f_2}$$

where N is the integer in (a_2).

With these strengthened assumptions we establish a preliminary result (Lemma 5.11) which shows that if σ satisfies

$$|\sigma(\xi)| \le \bar{\sigma}\,|\xi| \quad \text{for all} \quad \xi \tag{σ_2}$$

then much stronger energy estimates can be obtained for solutions of (P). Included in these are estimates of the form

$$\Big(\sup_{x\in[0,1]} |u(x, t)|\Big)^2,\ \int_0^1 u_t^2(x, t)\,dx = O(t^{-N}), \tag{2.11}$$

where N is the integer in (a_2). These strengthen (2.7) and (2.8).

The estimates (2.11) are used in Sec. 6 to establish pointwise bounds for solutions of (P). These results are strongly dependent on the special form of (P). They use the ideas of Lax and Nishida and require the conversion of (P) into an initial-value problem. This, in turn necessitates the further conditions

$$f(0, t) \equiv f(1, t) \equiv f_{xx}(0, t) \equiv f_{xx}(1, t) \tag{f_3}$$

$$u_0(x) \in C^{(3)}[0, 1], \qquad u_0(0) = u_0(1) = u_0''(0) = u_0''(1) = 0. \tag{u_0}$$

Finally the bounds are used to establish our existence theorem. This result needs a measure of "smallness of data". This is given by:

$$D = \sup_{x\in[0,1]} \sum_{j=0}^{2} \left|\frac{d^j u_0}{dx^j}(x)\right| + \sum_{j=0}^{N} ||t^j \bar{f}(t)||_{L_1(0,\infty)} + ||t^j \bar{f}(t)||_{L_2(0,\infty)} + ||t^j \bar{f}(t)||_{L_\infty(0,\infty)} . \qquad (2.12)$$

Theorem (II). Suppose (a_2) *for* $N \geq 4$, (f_2), (f_3) *and* (u_0) *all hold. Then if D is sufficiently small there exists a unique solution of* (P).

Note that (σ_2) is not required in Theorem (II).

3. Linear Volterra operators. We consider operators of the form (2.1). We are concerned here with the equation

$$\frac{d}{dt} L_a[\zeta](t) = \varphi(t). \qquad (3.1)$$

It is easy to verify, by successive approximations, that (3.1) has a unique solution ζ which can be written in the form,

$$\zeta(t) = \frac{1}{a(0)} \varphi(t) + L_k[\varphi](t). \qquad (3.2)$$

LEMMA 3.1. *The function k satisfies:*

(i) $k = k_\infty + K(t), \quad k_\infty = (a\hat{\ }(0))^{-1}, \quad K^{(j)} \in L_1(0, \infty), \quad j = 0, 1, 2,$

(ii) $t^j K^{(m)} \in L_1(0, \infty)$, B $= 0, 1$; $j \leq N$ *where N is the integer in* (a_2),

(iii) $$k(0) = -\frac{\dot{a}(0)}{a(0)^2}.$$

Moreover there exists an $\alpha > 0$ *such that for any* $T > 0$ *and any* $\varphi \in C[0, T]$,

(iv) $$\int_0^T \varphi(t) \frac{d}{dt} L_k[\varphi](t)\, dt \geq \alpha \int_0^T \varphi^2(t)\, dt.$$

Proof: We proceed formally by using Laplace transforms. If we transform (3.1) we obtain:

$$\zeta\hat{\ }(s) = \frac{1}{sa\hat{\ }(s)} \varphi\hat{\ }(s) = \frac{1}{a(0)} \varphi\hat{\ }(s) + k\hat{\ }(s)\varphi\hat{\ }(s), \qquad (3.3)$$

where

$$k\hat{\ }(s) = \frac{1}{sa\hat{\ }(s)} - \frac{1}{a(0)}. \qquad (3.4)$$

We study the properties of $k\hat{\ }$. We observe first that $k\hat{\ }$ is defined in Re $s \geq 0$, $s \neq 0$. For (a_3) and the maximum principle for harmonic functions guarantees that $a\hat{\ }(s) \neq 0$ in Re $s \geq 0$. $k\hat{\ }$ is analytic in Re $s > 0$. We have,

$$k\hat{\ }(s) = \frac{1}{sa\hat{\ }(0)} + K\hat{\ }(s), \qquad K\hat{\ }(s) = \frac{a(0) - a\hat{\ }(s)}{sa\hat{\ }(s)a\hat{\ }(0)}. \qquad (3.5)$$

It can be verified that (a_2) implies that $K^{\wedge}$ is $N + 2$ times continuously differentiable in $\operatorname{Re} s \geq 0$.

We next consider $k^{\wedge}(s)$ for large s. From (a_2) one finds, after integrating by parts,

$$a^{\wedge}(s) = a(0)s^{-1} + \dot{a}(0)s^{-2} + o(s^{-2}) \quad \text{as} \quad s \to \infty.$$

It follows from (3.4) that

$$k^{\wedge}(s) = -\frac{\dot{a}(0)}{a(0)^2 s} + O\left(\frac{1}{s^2}\right) \quad \text{as} \quad s \to \infty. \tag{3.6}$$

We now set

$$K(t) = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} e^{st} K^{\wedge}(s)\, ds, \qquad k(t) = k_\infty + K(t), \qquad k_\infty = \frac{1}{a^{\wedge}(0)}. \tag{3.7}$$

From (3.5) and (3.6) one has

$$K^{\wedge}(s) = \Gamma s^{-1} + O(s^{-2}), \qquad \Gamma = -\frac{\dot{a}(0)}{a(0)^2} - \frac{1}{a^{\wedge}(0)}. \tag{3.8}$$

It follows from (3.8) that the integral in (3.7) exists for $c \geq 0$, is independent of c and defines a continuous function $K(t)$ with $K(0) = \Gamma$. It can be verified (see [8]) that k satisfies (3.2) and $K(0) = \Gamma$ implies (iii). If we take $c = 0$ in (3.7) the Riemann-Lebesgue lemma implies that $K(t) \to 0$ as $t \to \infty$. We can say more. We have already indicated that $K^{\wedge}(s)$ is $N + 2$ time continuously differentiable in we $s \geq 0$. From (a_2) it can also be verified that the derivatives of $K^{\wedge}$ up to order $N + 2$ have estimates for large s which are those obtained by differentiating (3.8). Thus we can integrate by parts $N + 2$ times in (3.7) for $c = 0$ and obtain

$$K(t) = (2\pi)^{-1}(it)^{-N-2} \int_{-\infty}^{+\infty} e^{i\eta t} (K^{\wedge})^{(N+2)}(i\eta)\, d\eta. \tag{3.9}$$

Since $N \geq 0$, (3.9) and the Riemann-Lebesgue lemma yield $K(t) = O(t^{-2})$ so $K \in L_1(0, \infty)$* and also $t^N K \in L_1(0, \infty)$. Since the transform of $\dot{K}$ is $sK^{\wedge} - K(0)$ we can apply the same process to obtain the other estimates in (i) and (iii).

It remains to verify (iv). From (3.4) and (a_3) we have

$$-\eta \operatorname{Im} k^{\wedge}(i\eta) = (\operatorname{Re} a^{\wedge}(i\eta))\, |a^{\wedge}(i\eta)|^{-2} > 0 \quad \text{for all} \quad \eta \neq 0.$$

For sufficiently large η we have from (3.6)

$$-\eta \operatorname{Im} k^{\wedge}(i\eta) > -\frac{\dot{a}(0)}{2a(0)^2} > 0.$$

For sufficiently small η we have by (3.5)**

$$-\eta \operatorname{Im} k^{\wedge}(i\eta) > \frac{1}{2a^{\wedge}(0)} > 0.$$

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* This result is related to one in [2]. We conjecture that K is always in $L_1(0, \infty)$ as long as $a \in L_1$ and that one does not need the condition $t^j a \in L_1$. We have not been able to verify this.

** It is here that we use $a \in L_1$. If we had $a = a_\infty + b$ where $a_\infty > 0$ and $b \in L_1$ then we would have $a^{\wedge}(s) = a_\infty s^{-1} + b^{\wedge}(s)$ near zero. This would yield $(sa^{\wedge}(s))^{-1} = a_\infty + 0(1)$ near $s = 0$ and we would have $\eta \operatorname{Im} a^{\wedge}(i\eta) \to 0$ as $\eta \to 0$.

We conclude that there exists an $\alpha > 0$ such that

$$-\eta \operatorname{Im} \hat{k}(i\eta) \geq \alpha \quad \text{for all} \quad \eta. \tag{3.10}$$

It follows by a simple extension of Lemma 2.2 of [8] that for any $\varphi \in C^{(2)}[0, T]$ with $\varphi(0) = 0$ we have

$$\int_0^T \varphi(t) \frac{d}{dt} L_k[\varphi](t)\, dt = \frac{2}{\pi} \int_0^\infty -\eta \operatorname{Im} \hat{k}(i\eta)\{C(\varphi, \eta; T)^2 + S(\varphi, \eta; T)\}\, d\eta \tag{3.11}$$

where

$$C(\varphi, \eta; T) = \int_0^T \varphi(t) \cos \eta t\, dt, \qquad S(\varphi, \eta; T) = \int_0^T \varphi(t) \sin \eta t\, dt.$$

It follows from (3.10), (3.11) and Parseval's theorem that (iv) holds for the special φ's. Since these are dense in $L_2(0, T)$ it holds for arbitrary φ. This concludes the proof of Lemma 3.1.

4. Preliminary energy estimates. In this section we derive some integral estimates for solutions of (P). The hypotheses on a are (a_1), (a_2) for some N, and (a_3). We also require (f_1) and (σ_1). We carry out the calculations in detail only for problem (P) but we will phrase them in such a way as to try to make clear that they would hold for a quite large class of equations of type (E). (See [9].)

There are two norms associated in a natural way with problem (P). For functions $\varphi \in C^1[0, 1]$ we set

$$||\varphi|| = \left(\int_0^1 \varphi(x)^2\, dx\right)^{1/2}, \qquad ||\varphi||_1 = \left(\int_0^1 \varphi'(x)^2\, dx\right)^{1/2}. \tag{4.1}$$

For φ's in the domain* of g in (P) $||\ ||_1$ dominates $||\ ||$. ($||\varphi|| \leq ||\varphi||_1$.) Under condition (σ_1) the operator g in (P) is coercive in $||\ ||_1$ that is,

$$(g(u), u) = -\int_0^1 \frac{\partial}{\partial x} \sigma(u_x) u\, dx = \int_0^1 \sigma(u_x) u_x\, dx \geq \epsilon \int_0^1 u_x^2\, dx = \epsilon\, ||u||_1^2. \tag{4.2}$$

We observe also the following property of g. Define a functional $G(u)$ on Dg by

$$G(u) = \int_0^1 \int_0^{u_x} \sigma(\xi)\, d\xi\, dx. \tag{4.3}$$

Then if $u \in C^{(2)}$ and $u(0, t) \equiv u(1, t) \equiv 0$ we have

$$(\dot{u}, g(u)) = -\int_0^1 u_t \frac{\partial}{\partial x} \sigma(u_x)\, dx = \int_0^1 u_{tx} \sigma(u_x)\, dx = \frac{d}{dt} G(u(\cdot, t)). \tag{4.4}$$

By (σ_1) we have

$$G(u) \geq \frac{\epsilon}{2} \int_0^1 u_x^2\, dx = \frac{\epsilon}{2}\, ||u||_1^2. \tag{4.5}$$

For functions which are in $C^1([0, 1] \times [0, T])$ we write

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* For (P) we take domain of g as $Dg = \{ : \varphi \in C^{(2)}\, [0, 1], \varphi(0) = \varphi(1) = 0\}$.

$$||u||\,(t) = ||u(\cdot,\,t)||, \qquad ||u||_1\,(t) = ||u(\cdot,\,t)||_1$$
$$|||u|||^T = \left(\int_0^T ||u||^2\,(t)\,dt\right)^{1/2} \qquad |||u|||_1{}^T = \left(\int_0^T ||u||_1\,(t)^2\,dt\right)^{1/2}. \tag{4.6}$$

LEMMA 4.1. *There exists a constant M such that for any solution u of (P) and any $T > 0$,*

$$||u_t||\,(t), \qquad ||u||_1\,(t), \qquad |||u_t|||^T, \qquad ||u||\,(t) \le M \tag{4.7}$$

$$|||u|||_1{}^T, \qquad |||u|||^T \le M. \tag{4.8}$$

We observe that Lemma 4.1 yields the proof of Theorem I(i) immediately. (4.8) implies that $||u||\,(t) \in L_2(0,\,\infty)$ while (4.7) implies that this quantity is uniformly continuous. Hence $||u||\,(t) \to 0$ as $t \to \infty$ and this is the conclusion of Theorem I(i).

We begin the proof of Lemma 4.1 by differentiating (2.2), and solving for

$$\frac{\partial}{\partial x}\,\sigma(u_x).$$

From (3.2) we obtain

$$\frac{1}{a(0)}\,u_{tt} + L_k[u_{tt}] - \frac{\partial}{\partial x}\,\sigma(u_x) = \frac{1}{a(0)}\,f_t + L_k[f_t],$$

or

$$\frac{1}{a(0)}\,u_{tt} + \frac{d}{dt}\,L_k[u_t] - \frac{\partial}{\partial x}\,\sigma(u_x) = \Phi, \tag{4.9}$$

where

$$\Phi(x,\,t) = \frac{1}{a(0)}\,f_t(x,\,t) + L_k[f_t(x,\,\cdot)](t) + k(t)u_t(x,\,0)$$
$$= \frac{1}{a(0)}\,f_t(x,\,t) + k(0)f(x,\,t) + L_{\dot k}[f(x,\,\cdot)](t) + k(t)[u_t(x,\,0) - f(x,\,0)]$$

But by (2.2) $u_t(x,\,0) = f(x,\,0)$; hence

$$\Phi(x,\,t) = \frac{1}{a(0)}\,f_t(x,\,t) + k(0)f(x,\,t) + L_{\dot k}[f(x,\,\cdot)](t). \tag{4.10}$$

We recall that, by Lemma 3.1(i) $\dot k \in L_1(0,\,\infty)$. It follows from this and (f_1) that $|||L_{\dot k}[f(\cdot,\,\cdot)]|||^T$ is bounded independently of T. (f_1) shows that the same is true of the first two terms in (4.10). Thus we have

$$|||\Phi|||^T \le M_1 \quad \text{independently of } T. \tag{4.11}$$

We multiply (4.9) by u_t and integrate with respect to x from 0 to 1 and t from 0 to T. By (4.3), (4.4), (4.5), Lemma 3.1(iv) and (4.11) this yields

$$\frac{1}{2a(0)}\,||u_t||^2\,(T) + \alpha(|||u_t|||^T)^2 + \frac{\epsilon}{2}\,||u||_1{}^2\,(T) \le \frac{\alpha}{2}\,(|||u_t|||^T)^2 + \frac{1}{2\alpha}\,M_1{}^2 + M_2\,. \tag{4.12}$$

This yields the first three estimates in (4.7). The last follows from $||u|| \le ||u||_1$.

The estimates (4.8) are derived from another energy integral. We need two preliminary estimates. By (4.7) we have

$$\left| \int_0^T \int_0^1 uu_{tt} \, dx \, dt \right| = \left| \int_0^1 u_t(x, T)u(x, T) \, dx - \int_0^1 u_t(x, 0)u(x, 0) \, dx - (|||u_t|||^T)^2 \right| \leq c_1 . \quad (4.13)$$

Next we have, by (4.7) and the fact that $\dot{k} \in L_1(0, \infty)$,

$$\left| \int_0^T \int_0^1 u(x, t) L_{\dot{k}}[u_t(x, \cdot)](t) \, dx \, dt \right| \leq |||u|||^T \, ||\dot{k}||_{L_1(0,\infty)} \, |||u_t|||^T \leq c_2 \, |||u|||^T. \quad (4.14)$$

We write (4.9) as

$$\frac{1}{a(0)} u_{tt} + k(0)u_t - \frac{\partial}{\partial x} \sigma(u_x) = \Phi - L \, [u_t]. \quad (4.15)$$

We multiply by u and integrate. By (4.13), 4.2) and (4.14) this yields

$$\frac{k(0)}{2} \, ||u||^2 \, (T) + \epsilon(|||u|||_1{}^T)^2 \leq c_1 + c_2 \, |||u|||^T + \int_0^T \int_0^1 u(x, t)\Phi(x, t) \, dx \, dt + c_3 . \quad (4.16)$$

Since $|||u|||^T \leq |||u|||_1{}^T$ Eqs. (4.16) and (4.11) yield the first and hence also the second of estimates (4.8).

Proof of Theorem (I)(ii): Eq. (4.7) shows that $||u_t|| \in L_2(0, \infty)$. It follows that the quantity

$$\chi(t) = \int_0^1 u_t(x, t)\eta(x) \, dx$$

is in $L_2(0, \infty)$. Thus the conclusion will follow if we can show that, for the η's described, $\chi(t)$ is uniformly continuous.

Recall that σ is to satisfy here the estimate $|\sigma(\xi)| \leq M(|\xi| + |\xi|^r)$, $1 < r < 2$. We have then for the η's described

$$\left| \int_0^1 \frac{\partial}{\partial x} \sigma(u_x)\eta \, dx \right| = \left| \int_0^1 \sigma(u_x)\eta_x \, dx \right| \leq M \int_0^1 (|u_x| + |u_x|^r)\eta_x \, dx$$

$$\leq M \left\{ ||u||_1 \, ||\eta||_1 + ||u||_1{}^{r/2} \left(\int_0^1 |\eta_x|^{2/2-r} \, dx \right)^{2-r/2} \right\} \leq c. \quad (4.17)$$

We multiply (2.2) by η and integrate from 0 to 1. This yields

$$\chi(t) = \int_0^t a(t - \tau) \int_0^1 \frac{\partial}{\partial x} \sigma(u_x(x, \tau))\eta \, dx \, d\tau + \int_0^1 f(x, t)\eta(x) \, dx. \quad (4.18)$$

From (f_1) it follows that the second term is uniformly continuous. For the first we have for $t_2 > t_1$,

$$\left| \int_0^{t_2} a(t_2 - \tau) \int_0^1 \frac{\partial}{\partial x} \sigma(u_x(x, \tau))\eta \, dx \, d\tau - \int_0^{t_1} a(t_1 - \tau) \int_0^1 \frac{\partial}{\partial x} \sigma(u_x(x, \tau))\eta \, dx \, d\tau \right|$$

$$\leq c \int_0^{t_1} |a(t_2 - \tau) - a(t_1 - \tau)| \, d\tau + c \int_{t_1}^{t_2} |a(t_2 - \tau)| \, d\tau,$$

where we have used (4.17). Since $a \in L_1$ both terms on the right side are uniformly small with $t_2 - t_1$. Hence χ is indeed uniformly continuous.

5. Further energy estimates. In this section we establish some stronger integral estimates. These require stronger hypotheses. We demand that (a_2) hold with an $N > 0$. By Lemma 3.1 (ii) this means that $t^j k \in L_1(0, \infty)$, $0 \le j \le N$, that is,

$$k_j = \int_0^\infty t^j \, |k(t)| \, dt < \infty, \qquad 0 \le j \le N. \tag{5.1}$$

We require also that (f_2) and (σ_2) hold. We write for any $v(x, t)$,

$$^m|||v|||^T = \left(\int_0^T \int_0^1 t^m v^2(x, t) \, dx \, dt\right)^{1/2}; \quad ^m|||v|||_1{}^T = \left(\int_0^T \int_0^1 t^m v_x{}^2(x, t) \, dx \, dt\right)^{1/2}. \tag{5.2}$$

Lemma 5.1. *For any $m \le N$ there exists a constant M_m such that any solution of* (P) *satisfies,*

$$T^m \, ||u_t||^2 \, (T), \qquad T^m \, ||u||_1{}^2 \, (T), \qquad ^m|||u_t|||^T, \qquad T^m \, ||u||^2 \, (T) \le M_m \,, \tag{5.3}$$

$$^m|||u|||_1{}^T, \qquad ^m|||u|||^T \le M_m \,. \tag{5.4}$$

The proof is by induction. The case $m = 0$ is contained in Lemma 4.1. We assume the conclusion holds for $m - 1$. The induction step is accomplished by the same two calculations as in the last section except that we multiply by $t^m u_t$ and $t^m u$. We need some preliminary formulas.

We need an extension of (4.14). This is

$$\int_0^T \int_0^1 t^m p(x, t) L_k[q(x, \cdot)](t) \, dx \, dt \le {}^m|||p|||^T \, k_0{}^{1/2} \sum_{j=0}^m \binom{m}{j} ({}^j|||q|||^T) k_{m-j}{}^{1/2}. \tag{5.5}$$

We need also a somewhat more refined formula of the same type. This states

$$\left| \int_0^T \int_0^1 t^m p(x, t) \frac{d}{dt} L_k[p(x, \cdot)](t) \, dx \, dt - \int_0^T \int_0^1 t^{m/2} p(x, t) \frac{d}{dt} \right.$$
$$\left. \cdot \int_0^t k(t - \tau) \tau^{m/2} p(x, \tau) \, d\tau \, dt \right| \le K_m \, {}^m|||p|||^T \sum_{r=0}^{m-1} {}^r|||p|||^T, \tag{5.6}$$

for some constant K_m depending on m and k.

Formula (5.5) is quite straightforward while (5.6) is somewhat more delicate and we outline the proof at the end of the section. First, however, let us complete the proof of Lemma (5.1). From (4.3), (4.4), (4.5), and (σ_2) we have the following extension of (4.4):

$$-\int_0^T \int_0^1 \frac{\partial}{\partial x} \sigma(u_x) t^m u_t \, dx \, dt = \int_0^T t^m \frac{d}{dt} G(u(\cdot, t)) \, dt$$
$$= T^m G(u(\cdot, T)) - m \int_0^T t^{m-1} G(u(t)) \, dt$$
$$\ge \frac{\epsilon}{2} T^m \, ||u||_1{}^2 \, (T) - m \left| \int_0^T t^{m-1} \int_0^1 \int_0^{u_x} \sigma(\xi) \, d\xi \, dx \, dt \right|$$
$$\ge \frac{\epsilon}{2} T^m \, ||u||_1{}^2 \, (T) - \frac{m\bar{\sigma}}{2} ({}^{m-1}|||u|||_1{}^T)^2. \tag{5.7}$$

From (σ_1) we have a similar extension for (4.2):

$$-\int_0^T \int_0^1 \frac{\partial}{\partial x} \sigma(u_x) t^m u \, dx \, dt = \int_0^T \int_0^1 \sigma(u_x) t^m u_x \, dx \, dt \ge \epsilon ({}^m|||u|||_1{}^T)^2. \tag{5.8}$$

We set $\Phi_j = {}^j|||\Phi|||^T$ and note that (f_2), (4.10) and (5.1) imply that these are bounded, independently of T, for $j \le N$.

We multiply (4.9) by $t^m u_t$ and integrate. We integrate by parts in the first term. For the second we use (5.6) and Lemma (3.1)(iv). For the third we use (5.7). If we insert the bounds from the induction hypothesis the result is the inequality

$$\frac{1}{2a(0)} T^m \, ||u_t||^2 \, (T) + \alpha({}^m|||u_t|||^T)^2 + \frac{\epsilon}{2} T^m \, ||u||_1{}^2 \, (T) \le C_1({}^m|||u_t|||^T)$$

$$+ \int_0^T \int_0^1 t^m \, |u_t| \, |\Phi| \, dx \, dt + C_2 \le C_1({}^m|||u_t|||^T) + \frac{\alpha}{2} ({}^m|||u_t|||^T)^2 + C_2'.$$

This yields the first of three inequalities (5.3). The last follows from $||u|| \le ||u||_1$.

Next we multiply (4.15) by $t^m u$ and integrate. If we use (5.3), (5.5), (5.7) and the and the induction hypothesis, this produces the inequality

$$\frac{k(0)}{2} T^m \, ||u||^2 \, (T) + \epsilon({}^m|||u|||_1{}^T)^2 \le \int_0^T \int_0^1 t^m \, |u| \, |\Phi| \, dx \, dt$$

$$+ k_0{}^{1/2} \, {}^m|||u|||^T \sum_{j=0}^{m} c_j({}^m|||u_t|||^T) + C. \tag{5.9}$$

By (5.3) the sum on the right side is bounded. Hence we have

$$\epsilon({}^m|||u|||_1{}^T)^2 \le \frac{\epsilon}{2} ({}^m|||u|||^T)^2 + C'({}^m|||u|||^T) + C.$$

Since $|||u|||_1{}^T \ge |||u|||^T$ the estimates (5.4) follow.

We indicate the proof of (5.6). Denote the quantity whose absolute value is taken in (5.6) by I and suppose $m = 2n$ is even. Then we have

$$I = \int_0^T \int_0^1 t^n p(x, t) \int_0^t \dot{k}(t - \tau)(t^n - \tau^n) p(x, \tau) \, d\tau dx dt$$

$$= \sum_{j=0}^{n-1} \int_0^T \int_0^1 t^{n+j} p(x, t) \int_0^t \tilde{k}(t - \tau) \tau^{n-1-j} p(x, \tau) \, d\tau dx dt, \tag{5.10}$$

where $\tilde{k}(t) = t\dot{k}(t)$. We have the following estimates:

$$\left| \int_0^T \int_0^1 t^{n+j} p(x, t) \int_0^t \tilde{k}(t - \tau) \tau^{n-1-j} p(x, \tau) \, d\tau dx dt \right|$$

$$\le {}^{2n}|||p|||^T \left\{ \int_0^T \int_0^1 t^{2j} \left(\int_0^t \tilde{k}(t - \tau) \tau^{n-1-j} p(x, \tau) \, d\tau \right)^2 dx dt \right\}^{1/2}$$

$$\le {}^{2n}|||p|||^T \left\{ \int_0^T \int_0^1 t^{2j} \left(\int_0^t |\tilde{k}(t - \tau)| \, d\tau \right) \int_0^t |\tilde{k}(t - \tau)| \, \tau^{2n-2-2j} p^2(x, \tau) \, d\tau dx dt \right\}^{1/2}$$

$$\le {}^{2n}|||p|||^T \, k_1{}^{1/2} \left\{ \int_0^T \int_0^1 \tau^{2n-2-2j} p^2(x, \tau) \int_\tau^T t^{2j} \, |\tilde{k}(t - \tau)| \, dt dx d\tau \right\}^{1/2}$$

$$= {}^{2n}|||p|||^T \, k_1{}^{1/2} \left\{ \sum_{l=0}^{2j} \binom{2j}{l} \left(\int_0^T \int_0^1 \tau^{2n-2-2j+l} p^2(x, \tau) \, d\tau dx \right) \int_0^{T-\tau} |\tilde{k}(\eta)| \, \eta^{2j-l} \, d\eta \right\}^{1/2}$$

$$\le {}^{2n}|||p|||^T \, k_1{}^{1/2} \left\{ \sum_{l=0}^{2j} \binom{2j}{l} ({}^{2n-2-2j+l}|||p|||^T)^2 k_{2j-l+1} \right\}^{1/2}. \tag{5.11}$$

Since $j \leq n - 1$ we have $2j - l + 1 \leq 2j + 1 \leq 2n - 1 = m - 1 \leq N$; hence the k_{2j-l+1} are bounded. Thus the sum in (5.11) involves only constants times ${}^r|||p|||^T$ for $r \leq 2n - 2 = m - 2$. Inserting these estimates into (5.10) yields (5.6) with, in fact, ${}^r|||p|||^T$ for $r \leq m - 2$. The case of m odd proceeds the same way except that one uses $\tilde{k}(t) = t^{1/2}\dot{k}$. The sums corresponding to (5.11) then contain ${}^r|||p|||^T$ for r upto $m - 1$.

6. Pointwise estimates. In this section we derive some L_∞ bounds for u and its derivatives. These will suffice to prove Theorem II. Throughout this section, except in the proof of Theorem II, all the hypotheses in Sec. 5 hold and the constant N in (a_2) is greater than or equal to 4. We indicate immediately the significance of this last statement. We introduce the notation

$$|v|\,(t) = \sup_{x \in [0,1]} |v(x, t)|, \qquad |v|^T = \sup_{\substack{x \in [0,1] \\ t \in [0,T]}} |v(x, t)|. \tag{6.1}$$

LEMMA 6.1. *If $N \geq 4$ then for any solution of* (P) *we have $|u|\,(t) \in L_1(0, \infty)$.*

Proof. For a solution we have $u(0, t) \equiv 0$ hence $|u|\,(t) \leq ||u||_1\,(t)$. For $N \geq 4$, we have by (5.4),

$$|u|\,(T) = O(T^{-2}), \tag{6.2}$$

and the result follows.

We also assume throughout this section that conditions (f_3) and (u_0) are satisfied. We establish the following two results.

LEMMA 6.2. *There exists a constant M independent of T such that any solution of* (P) *satisfies*

$$|u_x|^T, \qquad |u_t|^T \leq M. \tag{6.3}$$

LEMMA 6.3. *If D (see* (2.8)) *is sufficiently small there exists a constant M independent of T such that*

$$|u_{xx}|^T, \qquad |u_{xt}|^T, \qquad |u_{tt}|^T \leq M. \tag{6.4}$$

We write (4.15) as,

$$\frac{1}{a(0)}\, u_{tt} + k(0)u_t - \frac{\partial}{\partial x}\, \sigma(u_x) = R(x, t), \tag{6.5}$$

where

$$R(x, t) = \Phi(x, t) - L_{\dot{k}}[u_t(x, \cdot)](t) = \Phi(x, t) - \dot{k}(0)u(x, t) + \dot{k}(t)u(x, 0) + L_{\ddot{k}}[u(x, \cdot)](t). \tag{6.6}$$

We view (6.5) as a non-linear hyperbolic equation with a forcing term R. Eq. (4.10) together with (f_1) and Lemmas 3.1 and 5.1 imply that

$$|R(\cdot, \cdot)|\,(t) \in L_1(0, \infty). \tag{6.7}$$

This is the crucial result of all our energy estimates. We solve (6.5) subject to the initial conditions (see (2.2))

$$u(x, 0) = u_0(x), \qquad u_t(x, 0) = f(x, 0) \tag{6.8}$$

and the boundary conditions

$$u(0, t) \equiv u(1, t) \equiv 0. \tag{6.9}$$

We show that, under conditions (f_3) and (u_0), (6.5) and (6.8) are equivalent to an initial-value problem. Let us extend $u^\circ(x)$ and $f(x, t)$ to $x \in (-\infty, \infty)$ as odd functions of x which are periodic of period two in x. Call these functions $\bar{u}^\circ(x)$ and $\bar{f}(x, t)$. Conditions (f_3) and (u_0) guarantee that $\bar{u}^\circ$ is of class $C^{(3)}$ and $\bar{f}$ is of class $C^{(2)}$. We extend the function Φ in (4.10) in a similar way to $\bar{\Phi}$. (f_3) insures that $\bar{\Phi} \in C^{(2)}$ and that $\Phi(0, t) \equiv \Phi(1, t) \equiv 0$. Then (4.15) and (6.9) show that

$$\frac{\partial}{\partial x} \sigma(u_x(0, t)) \equiv \frac{\partial}{\partial x} \sigma(u_x(1, t)) \equiv 0$$

from which we deduce that $u_{xx}(0, t) \equiv u_{xx}(1, t) \equiv 0$. Thus we can extend u to $\bar{u} \in C^{(2)}$. It is readily checked that $\bar{u}$ is a solution of (6.5) for all x, with R replaced by its extension $\bar{R}$ and with

$$\bar{u}(x, 0) = \bar{u}_0(x), \qquad \bar{u}_t(x, 0) = \bar{f}(x, 0). \tag{6.10}$$

Conversely if $\bar{u}$ is a solution of (6.5), with $\bar{R}$ replacing R, and (6.10) and if the solution of this problem is unique then $\bar{u}$ restricted to $0 < x < 1$ is a solution of (P). For one checks that $-\bar{u}(-x, t)$ is a solution of the same problem as $\bar{u}(x, t)$. Hence by uniqueness $-\bar{u}(x, t) \equiv \bar{u}(x, t)$ so that $\bar{u}(0, t) \equiv 0$. Similarly $\bar{u}(1, t) \equiv 0$.

From the above remarks we will henceforth treat Eq. (6.5) on $(-\infty, \infty) \times [0, \infty)$ with the initial condition (6.8). In (6.7) $|R(\cdot, \cdot)|\ (t)$ is then to be interpreted as $\sup_{(-\infty, \infty)}^{h} |R(\cdot, \cdot)|\ (t)$ and the initial data are bounded for all x.

We now proceed along the lines of Nishida. We write (6.5) as a system by letting $\Psi = u_t$, $\chi = u_x$. Then,

$$\begin{aligned} \chi_t &= \psi_x \\ \psi_t + k(0)\psi - \sigma'(\chi)\chi_x &= R. \end{aligned} \tag{6.11}$$

Since $\sigma'(\chi) \geq \epsilon > 0$ the system is always hyperbolic. We set

$$\begin{matrix} \lambda \\ \mu \end{matrix} = \mp \sqrt{\sigma'(\chi)} \tag{6.12}$$

and introduce the Riemann invariants,

$$\begin{matrix} r \\ s \end{matrix} = \psi \pm \Gamma(\chi), \qquad \Gamma(\chi) = \int_0^\chi \sqrt{\sigma'(\xi)}\, d\xi. \tag{6.13}$$

$\sigma'(\chi) \geq {}''$ guarantees that the map $(\psi, \chi) \to (r, s)$ is one-to-one from $R \times R$ onto $R \times R$. Eqs. (6.11) become

$$\begin{aligned} r_t + \lambda r_x + \alpha(r + s) &= R \\ s_t + \mu s_x + \alpha(r + s) &= R \end{aligned} \tag{6.14}$$

where we have set the positive constant

$$\frac{k(0)}{2} \quad \text{equal to} \quad \alpha.$$

We introduce the characteristic curves defined by

$$x = x_1(t, \beta) = \beta + \int_0^t \lambda \, d\tau, \qquad x = x_2(t, \gamma) = \gamma + \int_0^t \mu \, d\tau \tag{6.15}$$

and we denote by ′ and ‵ differentiation along these curves; that is

$$' = \frac{\partial}{\partial t} + \lambda \frac{\partial}{\partial x}, \qquad ` = \frac{\partial}{\partial t} + \mu \frac{\partial}{\partial x}. \tag{6.16}$$

Along the curve x_1 Eq. $(6.14)_1$ yields

$$\frac{d}{dt} r(x_1(t, \beta), t) + \alpha r(x_1(t, \beta), t) = -\alpha s(x_1(t, \beta), t) + R(x_1(t, \beta), t).$$

We can integrate this ordinary differential equation and obtain

$$e^{\alpha t} r(x_1(t, \beta), t) = r(\beta, 0) - \alpha \int_0^t e^{\alpha \tau} s(x_1(\tau, \beta), \tau) \, d\tau + \int_0^t e^{\alpha \tau} R(x_1(\tau, \beta), \tau) \, d\tau. \tag{6.17}$$

From $(6.14)_2$ we obtain in a similar way

$$e^{\alpha t} s(x_2(t, \gamma), t) = s(\gamma, 0) - \alpha \int_0^t e^{\alpha \tau} r(x_2(\tau, \gamma), \tau) \, d\tau + \int_0^t e^{\alpha \tau} R(x_2(\tau, \gamma), \tau) \, d\tau. \tag{6.18}$$

In (6.17) and (6.18) we let β and γ vary over $(-\infty, \infty)$. The characteristic curves x_1 and x_2 will exist as long as λ and μ, that is r and s, remain bounded. Let Ω denote the set of all (x, t) such that $x = x_1(t, \beta)$ and $x = x_2(t, \gamma)$ for some β and γ. Let

$$\rho(t) = \sup_{(x, t) \in \Omega} e^{\alpha t} (|r(x, t)| + |s(x, t)|).$$

Then (6.17) and (6.18) yield

$$\rho(t) \leq c + \alpha \int_0^t \rho(\tau) \, d\tau + \int_0^t e^{\alpha \tau} J(\tau) \, d\tau \tag{6.19}$$

where $c = \sup (|r(x, 0)| + |s(x, 0)|)$ and $J(\tau) = 2 \, |R(\cdot, \cdot)| \, (\tau)$. By (6.7) $J(\tau) \in L_1(0, \infty)$. Hence (6.19) yields

$$\rho(t) \leq c + \int_0^t e^{\alpha \tau} J(\tau) \, d\tau + \alpha \int_0^t \left\{ c + \alpha \int_0^\tau J(\xi) \, d\xi \right\} e^{\alpha (t - \tau)} \, d\tau$$

$$= c e^{\alpha t} + e^{\alpha t} \int_0^t J(\tau) \, d\tau \leq M e^{\alpha t}.$$

Thus we have proved that

$$\sup_{(x, t) \in \Omega} (|r(x, t)| + |s(x, t)|) \leq M. \tag{6.20}$$

From (6.20), (6.12), (6.13) and (6.15) it follows that the slopes dx/dt of the characteristic curves remain bounded and this shows that Ω is all of $t > 0$. Hence r and s remain bounded and, by (6.14), we obtain (6.3).

In order to prove Lemma 6.3 we again follow the ideas of Nishida. We differentiate $(6.14)_1$ with respect to x and obtain,

$$r_{xt} + \lambda r_{xx} = -\lambda_r r_x^2 - \lambda_s s_x r_x - \alpha(r_x + s_x) + R_x . \tag{6.21}$$

Now from (6.16), (6.12) and $(6.14)_2$ we have,

$$s' = s_t + \lambda s_x = s_t + \mu s_x + (\lambda - \mu)s_x = s^{\backprime} + 2\lambda s_x = -\alpha(r + s) + 2\lambda s_x + R,$$

or,

$$s_x = \frac{s'}{2\lambda} + \frac{\alpha}{2\lambda}(r + s) - \frac{R}{2\lambda}. \tag{6.22}$$

Observe that by (6.12) and (6.13) λ and μ are functions of $r - s$. Consider the function

$$h = \tfrac{1}{2}\log(-\lambda(r - s)).$$

We have, by (6.14),

$$h' = \frac{\lambda_r}{2\lambda} r' + \frac{\lambda_s}{2\lambda} s' = -\frac{\alpha\lambda_r}{2\lambda}(r + s) + \frac{\lambda_s}{2\lambda} s' + \frac{1}{2}\frac{\lambda_r}{\lambda} R. \tag{6.23}$$

We substitute (6.22) and (6.23) into (6.21) and obtain

$$r_x' + r_x\{\alpha + \lambda_r r_x + h'\} = -\frac{\alpha}{2\lambda} s' - \frac{\alpha 2}{2\lambda}(r + s) + \frac{\alpha R}{2\lambda} + R_x\ ,$$

or

$$(e^h r_x)' + e^h r_x\{\alpha + \lambda_r r_x\} = -\frac{\alpha}{2\lambda} s' e^h - \frac{\alpha^2}{2\lambda}(r + s)e^h + \frac{\alpha}{2\lambda} R e^h + R_x e^h. \tag{6.24}$$

We let

$$g = \int_0^{r-s} \frac{\alpha e^h(\xi)}{2\lambda(\xi)}\, d\xi. \tag{6.25}$$

Then by (6.14),

$$g' = \frac{\alpha e^h}{2\lambda} r' - \frac{\alpha e^h s'}{2\lambda} = -\frac{\alpha^2}{2\lambda} e^h(r + s) - \frac{\alpha e^h s'}{2\lambda} + \frac{\alpha e^h}{2\lambda} R.$$

Hence (6.24) can be written

$$(e^h r_x) + e^h r_x\{\alpha + \lambda_r r_x\} = g' + R_x e^h. \tag{6.26}$$

We are now ready to use the constant D. A careful study of all our estimates will show that all the bounds tend to zero as D tends to zero. We are particularly concerned with the quantity R. Since $|v|^T \to 0$ with D it follows from (6.6) that $|R|^T \to 0$ with D. Since $M \to 0$ with D in (6.3) it follows also that $|R_x|^T$, $|R_t|^T$ and hence $|R|^T$ tend to zero with D. From (6.25) we see also that $|g|^T \to 0$ with D.

We start by making D less than some fixed quantity D_1. Then the quantities

$$v_x\ , v_t\ , r, s, \lambda, \frac{\lambda_r}{\lambda}, \frac{\lambda_s}{\lambda} \quad \text{and} \quad e^h$$

are all bounded. We next choose $D < D_1$ so small that

$$|\lambda_r(r(x, 0) - s(x, 0))r_x(x, 0)| < \frac{\alpha}{2}. \tag{6.27}$$

We integrate (6.26) along the x_1 characteristic starting at $(\beta, 0)$. Let

$$\begin{aligned} \zeta(t) &= \alpha + \lambda_r r_x \\ z(t) &= e^h r_x \qquad\qquad \text{along } x_1(t, \beta) \\ \gamma(t) &= g' + R_x e^h. \end{aligned} \tag{6.28}$$

Then integration of (6.26) yields

$$\exp\left(\int_0^t \zeta(\tau)\, d\tau\right) z(t) = z(0) + \int_0^t \exp\left(\int_0^\tau \zeta(\xi)\, d\xi\right) \gamma(\tau)\, d\tau,$$

or

$$z(t) = z(0) \exp\left(-\int_0^t \zeta(\tau)\, d\tau\right) + \int_0^t \exp\left(-\int_\tau^t \zeta(\xi)\, d\xi\right) \gamma(\tau)\, d\tau. \tag{6.29}$$

Suppose we have

$$|\lambda_r r_x| \leq \frac{\alpha}{2}. \tag{6.30}$$

Then, by (6.28) we have

$$\zeta(t) > \frac{\alpha}{2}$$

and (6.29) yields

$$|z(t)| \leq |z(0)|\, e^{-(\alpha/2)t} + \int_0^t e^{-\alpha/2(t-\tau)}\, |\gamma(\tau)|\, d\tau,$$

or

$$\sup_{t \leq T} |z(t)| \leq |z(0)| + \frac{2}{\alpha} \sup_{t \leq T} |\gamma(t)|. \tag{6.31}$$

The right side of (6.31) tends to zero with D. Also we have $\lambda_r r_x = \lambda_r e^{-h} z$ and $\lambda_r e^{-h}$ uniformly bounded. Thus $\lambda_r r_x$ is uniformly bounded in terms of z. We choose D so small that the right-hand side of (6.31), after multiplication by the maximum of $\lambda_r e^{-h}$ is less than $\alpha/2$. Then by (6.27) $\lambda_r r_x$ starts out satisfying (6.30) and by the preceding remarks continues to do so for all t. It follows that r_x is bounded independently of T. A similar argument shows that s_x is bounded. By (6.13) we conclude that ψ_x and χ_x that is u_{xx} and u_{xt} are bounded. It follows from (6.5) that u_{tt} is bounded. This concludes the proof of Lemma 6.3.

Proof of Theorem II. We return to Eq. (6.5) but with R expressed in terms of the solution u according to (6.6). This is a linear perturbation of a non-linear hyperbolic equation and the local existence and uniqueness theory for the latter carries over with trivial modifications. This local solution yields a local solution of (P) to which all our estimates apply. Thus we have, by Lemma 6.3, a priori bounds for the second derivatives of the local solution of (6.5). But just as in the hyperbolic equation case this yields global existence and hence a global existence theorem for (P).

This argument does not quite prove Theorem II. The reason is that all our bounds in Sec. 5 and hence here were predicted on the estimate (σ_2), that is $|\sigma(\xi)| \leq \bar{\sigma}(\xi)$. We can overcome this difficulty for small data. Choose $\delta > 0$ and let $\tilde{\sigma}(\xi) = \sigma(\xi)$ for $|\xi| \leq \delta$ but with $|\tilde{\sigma}(\xi)| \leq M\, |\xi|$ for all ξ where M is sufficiently large but fixed constant. Now solve (P) with $\tilde{\sigma}$ replacing σ. The data are then made so small that the problem with $\tilde{\sigma}$ has a solution and for that solution $|u_x| < \delta$. Then $\tilde{\sigma}(u_x) = \sigma(u_x)$ so one has in fact a solution of the original problem.

7. One-dimensional heat flow in materials with memory. Consider a rigid heat conductor in which heat flows in only one dimension. Let $u(x, t)$, $\epsilon(x, t)$ and $r(x, t)$ denote the temperature, internal energy, heat flux and heat supply respectively. The balance of heat requires that the equation

$$\epsilon_t(x, t) = -\frac{\partial}{\partial x} q(x, t) + r(x, t) \tag{7.1}$$

should hold. The simple linear theory of heat flow, in a homogeneous material, is obtained by assuming that $\epsilon(x, t) = cu(x, t)$ and $q(x, t) = -Ku_x(x, t)$ then (7.1) becomes the one-dimensional heat equation. One can obtain a simple non-linear theory by retaining the assumption $\epsilon = cu$ but taking $q(x, t) = -K\sigma(u_x(x, t))\sigma$ non-linear. This results in the equation,

$$cu_t(x, t) = K \frac{\partial}{\partial x} \sigma(u_x(x, t)) + r(x, t). \tag{7.2}$$

Our condition (σ_1) is a natural one for Eq. (7.2). Using it one can show that (7.2) has a unique solution, say on $0 < x < 1$, with $u(0, t) \equiv u(1, t) \equiv 0$, and $u(x, 0) = u_0(x)$. If $r(x, t) = f_0(x) + f(x, t)$ where $|f(\cdot, \cdot)| (t) \in L_1$ one can show further that the solution tends to $u^\circ(x)$, a solution of the problem

$$\frac{d}{dx} \sigma(u^0(x)) = -K^{-1}f_0(x), \qquad u^0(0) = u^0(1) = 0. \tag{7.3}$$

Models of the form (7.2) for heat flow have the disadvantage that they yield infinite speeds of propagation. Gurtin and Pipkin [2] have suggested a heat flow model which is based on a memory effect in the material. The linear one-dimensional version of their theory assumes that

$$\epsilon(x, t) = bu(x, t) + \int_0^\infty B(\tau)u(x, t - \tau)\, d\tau \tag{7.4}$$

$$q(x, t) = \int_0^\infty c(\tau)u_x(x, t - \tau)\, d\tau. \tag{7.5}$$

Thus ϵ and q are functionals of the histories, $u(x, t - \tau)$ and $u_x(x, t - \tau)$, of temperature and temperature gradient respectively.

On the basis of the present paper we can treat a partially non-linear version of (7.4) and (7.5). We keep (7.4) and replace (7.5) by

$$q(x, t) = -\int_0^\infty K(\tau)\sigma(u_x(x, t - \tau))\, d\tau. \tag{7.6}$$

Assume that the material is at zero temperature and internal energy up to time $t = 0$.* Then (7.4), (7.6) and (7.1) lead to the equation

$$bu_t + \int_0^t B(t - \tau)u_\tau(x, \tau)\, d\tau = \int_0^t K(t - \tau) \frac{\partial}{\partial x} \sigma(u_x(x, \tau))\, dx + r(x, t). \tag{7.7}$$

* Non-zero initial histories can be incorporated into the forcing term.

This equation is equivalent to (2.2) with

$$a(t) = \frac{1}{b} K(t) + \int_0^t \rho(t - \tau)K(\tau)\, d\tau$$
$$f(x, t) = \frac{1}{b} r(x, t) + \int_0^t \rho(t - \tau) r(x, \tau)\, d\tau. \tag{7.8}$$

Here ρ is the resolvant for B, that is the function such that

$$u(x, t) = \frac{1}{b} \epsilon(x, t) + \int_0^t \rho(t - \tau)\epsilon(x, \tau)\, d\tau \tag{7.9}$$

solves the equation

$$\epsilon(x, t) = bu(x, t) + \int_0^t B(t - \tau)u(x, \tau)\, d\tau. \tag{7.10}$$

We assert that it is to be expected in this context that a will be in L_1. Indeed it is to be expected that for a steady temperature gradient, $u_x(x, t) = \varphi(x)$, the heat flux would remain bounded. From (7.5) thisjimplies that $c \in L_1$. Suppose that the energy, for some x, is zero up to time $t = 0$ and then assumes a constant value. Then one would expect that u would remain bounded. By Eq. (7.9) this implies that $\rho \in L_1$. Hence from (7.8) one should have $a \in L_1$.

Suppose $r(x, t) = r_0(x) + R(x, t)$, $R(x, \cdot) \in L_1(0, \infty)$. Then it can be verified from $(7.8)_2$ that $f(x, t) = f_0(x) + F(x, t)$, where $F(x, \cdot) \in L_1$ and

$$f_0(x) = \left(\frac{1}{b} + \frac{1}{b + B\hat{\ }(0)}\right) r_0(x).$$

From Remark 2.1 one would then have $u(x, t) \to \tilde{u}^\circ(x)$ (in $L_2(0, 1)$) where $\tilde{u}^\circ$ is a solution of,

$$\frac{d}{dx}\sigma\left(\frac{d\tilde{u}^0(x)}{dx}\right) = -\frac{f_0(x)}{a\hat{\ }(0)}, \qquad \tilde{u}^0(0) = \tilde{u}^0(1) = 0. \tag{7.11}$$

By (7.8)

$$a\hat{\ }(0) = \left(\frac{1}{b} + \rho\hat{\ }(0)\right) K\hat{\ }(0) = \left(\frac{1}{b} + \frac{1}{b + B\hat{\ }(0)}\right) K\hat{\ }(0).$$

Hence

$$\frac{1}{a\hat{\ }(0)} = \frac{1}{K\hat{\ }(0)}.$$

If we compare (7.11) with (7.3) we see that as regards approach to steady state (7.7) behaves like the parabolic model (7.3) if one takes

$$K = \int_0^\infty a(t)\, dt.$$

It is shown in [7] that Eq. (2.2) also arises in the theory of one-dimensional visco-elasticity. In that context, however, one wants to have an a such that $a(t) = a_\infty + b(t)$, $a_\infty > 0$, $b \in L_1$. Thus our present theory does not apply.

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