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# AN INTENSIONAL LOGIC OF PREDICATES AND PREDICATE MODIFIERS WITHOUT MODAL OPERATORS

JAMES ANDREW FULTON

*Introduction* There are certain features of our language which resist interpretation along the lines of traditional logic.* For example, suppose that the following were true:

1. All and only handball players are squash players.

Or, equivalently,

2. All and only those who play handball play squash.

From either of these it follows that

3. All and only female handball players are female squash players.

On the other hand, none of the following can legitimately be inferred from either sentence 1 or 2:

4. All and only reluctant handball players are reluctant squash players.
5. All and only those who play handball aggressively play squash aggressively.
6. All and only those who play handball in knickers play squash in knickers.

Thus the contribution of the adjective 'female' to the logical form[1] of sentence 3 differs from the respective contributions of the adjective 'reluctant,' the adverb 'aggressively,' and the prepositional phrase 'in knickers' to the logical forms of sentences 4, 5, and 6. Nonetheless,

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'reluctant,' 'aggressively,' and 'in knickers' do contribute to the logical form of sentences in which they are present, for from each of the following:

7. Henrietta is a reluctant handball player.
8. Henrietta plays handball aggressively.
9. Henrietta plays handball in knickers.

as well as from

10. Henrietta is a female handball player.

we can infer

11. Henrietta plays handball.

Partly because of these differences in logical function, these modifiers are classified differently in the terminology of contemporary logic and linguistics. The adjective 'female' in this usage is said to create an *extensional* context, while 'reluctant,' 'aggressively,' and 'in knickers' create *intensional* contexts: the first does while the last three do not permit substitution *salva veritate* of co-extensive predicates. Moreover, 'female' is taken to be a *predicative* adjective while 'reluctant' is an *attributive* adjective: the former can and the latter cannot be predicated independently and non-elliptically of its subject. For example, sentence 10 is equivalent to

12. Henrietta is female and Henrietta is a handball player.

while 7 is not equivalent to

13. Henrietta is reluctant and Henrietta is a handball player.

What is at issue is how to provide a theoretical account of the facts thus classified. The logician wants a theory which explains the inferences these modifiers permit (and which rules out those inferences not intuitively acceptable); the grammarian wants a theory which explains the syntactic transformations involved among the sentences. Both want theories that are mutually compatible. The result has been a joint effort to develop a theory of language whose formal semantics can provide the requisite rules of inference and whose formal syntax is consistent with the theories of the grammar of the natural language.

The semantic theories that logicians have developed to accommodate these phenomena seem to be of three types. One was developed by Donald Davidson primarily as a means of analyzing prepositional phrases occurring in sentences about human action. In essence, his proposal was to treat prepositional phrases, such as 'in knickers', as independently applicable predicates of a syntactically suppressed logical subject: the *action.* Action sentences were hypothesized to have a logical form containing at least one more relational place than was usually apparent in the surface structure of the sentence. For example, sentence 11 would be analyzed not as a two-place relation between Henrietta and handball but as a three-place relation among Henrietta, handball, and her action. Prepositional phrases would apply as predicates of the last of these. Thus, 11 would be paraphrased as

14. There are actions which are Henrietta's and which are actions of playing handball.

while 9 would be paraphrased as

15. There are actions which are Henrietta's and which are actions of playing handball, and which are performed in knickers.

The rules of the ordinary predicate calculus would then permit simplification corresponding to deletion of the phrase [2]. Although Davidson's approach seems formally adequate to the prepositional phrases he considers, there is no natural way to extend the method to treat adverbs or attributive adjectives. For example, suppose that in playing handball Henrietta was pleasing Buford. The standard form for expressing this using Davidson's method would seem to be

16. There are actions which are Henrietta's and which are actions of playing handball and which are actions of pleasing Buford.

Suppose further than Henrietta was reluctant to play handball. Treating 'reluctant' as a predicate of her action, we obtain

17. There are actions which are Henrietta's and which are actions of playing handball and which are reluctant actions of Henrietta's and which are actions of pleasing Buford.

From this and the same principles of translation we infer that Henrietta reluctantly pleased Buford. But this surely does not follow from the given premises. Thus the principles of translation applicable to prepositional phrases cannot be applied directly to attributive adjectives, and a similar line of reasoning will show them inapplicable to adverbs.

A further difficulty, most acutely felt by those whose background is in traditional philosophy, is the semantic requirement of a special kind of ontology. The satisfaction of sentences of the kind required presupposes a domain of discourse which includes necessarily a non-empty subset whose members are intuitively classified as *actions*, *acts*, *activities*, *events*, *states*, and the like. Moreover, this commitment does not seem limited to statements about actions, for Davidson's analysis seems effective with regard to prepositional phrases wherever they occur (cf. [1], p. 314). A logical commitment to such an ontology seems to violate the traditional strictures of economy. In conjunction with the difficulty of extending the method to other modifiers, it at least suggests that we look for another approach.

One alternative has been to employ the tools of modal logic, a well-developed method which is already known to supply intensional contexts. Additional motivation for such treatment is supplied by a general theoretical difficulty of non-modal methods which has been noted by Bas van Fraassen:

> Language can be syntactically analyzed so that each expression is formed from simpler component expressions in a certain systematic way. This syntactic structure has an exact parallel in the semantics: An interpretation gives each expression a value, which is determined in a systematic way by the values it gives to the component expressions. Now, a predicate modifier turns predicates into predicates; hence it is natural to take the value of a predicate modifier to be an operator that turns values of predicates into values of predicates. [van Fraassen remarks in a footnote, "Natural bnt [sic] not necessary; the general thesis implies only that $|\phi(F)|$ is some function of $|\phi|$ and $|F|$."] Designating the value of expression $E$ as $|E|$, the thesis has a simple formulation:
>
> (1) $|\phi(F)| = |\phi|(|F|)$
>
> for any predicate $F$ and predicate modifier $\phi$.
>
> But what values do expressions receive? As a first candidate, let us suppose that $|F|$ is the extension of $F$. Then equation (1) says that the semantic correlate of $\phi$ is an operator on sets (subsets of the domain of discourse; for convenience I shall restrict myself to monadic predicates for now). However, that candidate fails, for under this supposition the consequence
>
> (2) if $|F| = |G|$ then $|\phi(F)| = |\phi(G)|$
>
> has a corollary
>
> (2a) $(x)(Fx \equiv Gx) \supset (x)(\phi(F)x \equiv \phi(G)x)$
>
> which means that the slow drivers are the slow walkers if the drivers are exactly the walkers. ([8], p. 107)

This argument rules out for the analysis of intensional modifiers the conjunctive treatment that we would ordinarily give predicative adjectives like 'female'. It seems to prohibit virtually any assignment of a function on extensions to these modifiers as long as that function takes as its domain the actual assignment to the predicate to which the modifier is applied. If, however, the modifier is assigned a function whose domain is the set of *possible* assignments to predicates, then in the case of contingently equivalent predicates the modifier would operate in each case on different sets with the result that (2a) can be avoided. The formal details of this method are too complex to describe briefly. The formal semantics of each version of this method that I have seen requires an ontology invariably interpreted as consisting of "possible worlds."[2] For reasons analogous to those I offered against Davidson's proposal, I am reluctant to concede such a commitment as a necessary price of understanding ordinary sentences involving predicate modifiers.

A third, and I think metaphysically more promising, approach has been suggested by Romane Clark [1]. Instead of treating modifiers as predicates applying to a special kind of logical subject as in Davidson's system, or defining their assignments in terms of the possible assignments to the predicates they modify, Clark offers a natural set of syntactic rules allowing recursive composition of sentences and a corresponding set of rules of inference allowing deletion of those modifiers (at least in most cases).

This method faces at least three obstacles. First, in the ordinary predicate calculus, of which his method is to be an extension, the only sentences which have predicates on whose extensions modifiers can operate are atomic sentences. But in the natural language modifers can apply to molecular and general sentences as well:

18. Wisely Fischer moved his pawn and protected his Queen's Bishop.
19. Fortunately, someone had locked the gate.

Neither of these examples seems to permit distribution of the adverb to apply only to atomic components. Clark provides no clue concerning the extension of his method to encompass such sentences.

Secondly, Clark's description of the formal semantics presented as formal justification for his method contains reference to entities which are hardly less mystifying than *actions* or *possible worlds*, for he postulates a set of *states of affairs*:

> Let $\overline{P}$ be the set of all sets of $n$-tuples of $D$. For each $n$-tuple, $i^n$, of individuals of $D$, and for each set of $n$-tuples of individuals, $p_j$, belonging to $\overline{P}$, consider the states of affairs designated by the expression 'that-$i^n \epsilon p_j$', i.e., that $i^n$ exemplifies some attribute the extension of which is $p_j$. ([1], p. 331)

My difficulty with this semantics may simply have resulted from my failure to understand precisely how the referent of 'that-$i^n \epsilon\ p_j$' is related in the formal structure to $i^n$ and $p_j$, and this failure itself may be due to the perplexity surrounding the notion of a state of affairs. It may well be that what Clark intends is some ordinary set-theoretic pairing. Without such an interpretation, however, his system with the others introduces a strange ontological commitment.

This very fact raises a new objection of a more theoretical nature. If traditional principles of inference—the ordinary predicate calculus in the case of Davidson's method, or modal logic—can be extended to accommodate the inferences based on the presence of modifiers in sentences, then there must be very good reason to explain those inferences using new rules of inference (cf. [3]). The intuitive model that Clark gives for his semantics does not appear ontologically simpler than those of the alternative methods. Hence Ockham's Razor does not provide the necessary justification for the new rules.

In what follows I shall construct a formal system having some of the syntactic characteristics of Clark's: modifiers will form sentences from sentences; rules of inference will permit certain operations corresponding to intuitively acceptable inferences in the natural language. However, the syntax will be such that every sentence will have an associated predicate and a reference sequence, as do atomic sentences of the ordinary predicate calculus; and every sentence will be true just in case the referent of its reference sequence belongs to the extension of the predicate. Thus the extension of the predicate of a sentence formed by the application of a modifier will always be determined from the extension of the predicate of the sentence to which the modifier is applied. Moreover, the semantics of the system will not require any special kind of ontology, neither *actions* nor possible worlds. Nonetheless, the system will be shown to be adequate to all of the tasks of the predicate calculus, and the rules of inference will be shown to be sound and complete. Finally, the semantics of the logical constants corresponding to those of the ordinary predicate calculus will be seen as a special case of the semantics of modifiers. Thus the disadvantage of a requirement of new rules of inference will be to some extent offset by the twin advantages of ontological simplicity and a deeper theory of the nature of sentential operations.

*Notational preliminaries* In what follows I shall be using the term 'set' rather loosely. I shall invoke certain intuitive inferences regarding sets without regard for the degree of difficulty of the justification of those inferences within an axiomatic framework. However, with the exception of inferences based on sets discussed in the metalanguage of the ordinary predicate calculus, the inferences I shall make all concern finite sets. Thus their addition to the metalanguage should not result in any paradox.

Sets will be named in the usual way: braces—$\{\ \}$—for unordered sets; angle brackets—$\langle\ \rangle$—for ordered sets. To designate a set by description, I shall use the notation $\{x: \ldots\}$. I shall use the usual symbols for union, intersection, and difference. The null set and the null sequence will be taken to be identical (designated by $\emptyset$); and the unit sequence of an object will be identified with the unit set of that object. For any set $\Gamma$ and integer $n$, $\Gamma^n$ will be the set of all $n$-tuples from $\Gamma$.

The following notation is not to be found in ordinary discussions of relations but will prove of crucial importance in the system to follow: for any sets $\Gamma$, $\Delta$, and $\theta$, if $\Gamma \subseteq \theta^m$ and $\Delta \subseteq \theta^n$, then

$$\Gamma \wedge \Delta =_{df} \{\langle x_1, \ldots, x_{m+n}\rangle: \langle x_1, \ldots, x_m\rangle \in \Gamma \text{ and } \langle x_{m+1}, \ldots, x_{m+n}\rangle \in \Delta\}.$$

The effect of this operation of *adjunction* is to form from two relations a third relation in such a way that the initial segment of each member of the third is itself a member of the first and the final segment of each member of the third is itself a member of the second. It will also be useful to define *adjunction* between individual sequences:

$$\langle \alpha_1, \ldots, \alpha_m\rangle \wedge \langle \beta_1, \ldots, \beta_n\rangle =_{df} \langle \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n\rangle$$

to form a longer sequence whose initial members compose the first of the subsequences and whose final members compose the second.

Integers will frequently be used to index sequences. The following definition will permit a mode of indexing which will allow significant simplification of the statement of the syntax and semantics of quantification: Let $\langle 1, \ldots, m\rangle$ be the numerically ordered sequence of all positive integers not greater than $m$. Then the sequences $\langle i_1, \ldots, i_k\rangle$ and $\langle j_1, \ldots, j_{m-k}\rangle$ are *complementary subsequences* of $\langle 1, \ldots, m\rangle$ iff

a. $\{i_1, \ldots, i_k\} \cup \{j_1, \ldots, j_{m-k}\} = \{i, \ldots, m\}$
b. $\{i_1, \ldots, i_k\} \cap \{j_1, \ldots, j_{m-k}\} = \emptyset$
c. $1 \leq i_1 < \ldots < i_k \leq m$
d. $1 \leq j_1 < \ldots < j_{m-k} \leq m$.

Corners—$\ulcorner\ \urcorner$—will generally be used to introduce syntactic elements which are combinations of others and will thereafter usually be omitted. I shall interpret them as instructions to write one component after the other in the order given, but such an interpretation of the syntactic operation is not essential.

*Syntax I: Composition of expressions* I shall refer to this system of the logic of predicate modifiers, to this "adverbial logic", as $\mathfrak{A}$. The *vocabulary* will take as *predicate letters* the upper case Roman letters $A$ through $Z$ with or without subscripts from the non-negative integers, as *terms* the lower case Roman letters $a$ through $t$ with or without subscripts from the non-negative integers, and as *logical constants* the symbols ( ), &, $\neg$, $\forall$ together with the non-negative integers and the comma.

Each predicate letter is assigned a *type* from the set $\{0, 1, 2\}$ and a *degree* from the set of non-negative integers. A predicate letter $\lambda$ assigned type $t$ and degree $n$ will be written formally as $\ulcorner{}^{t}\lambda^{n}\urcorner$, but the type will invariably be determinable from context and thus be omitted from the notation. Occasionally the context will permit a similar omission of the degree. An *elementary predicate* is a predicate letter of type 0; an *elementary modifier*, a predicate letter of type 1; and an *intensifier*, a predicate letter of type 2. While there is no upper bound on the degree assignable to a predicate letter, and while it will be presumed at every stage of composition that for any degree there is a previously unused predicate letter of each type, I interpret this not as an existence assertion but as a denial of a restriction; similar remarks apply to syntactic categories to be defined below.

If $\rho^n$ is an intensifier and $\delta = \langle\beta_1, \ldots, \beta_n\rangle$ is a sequence of terms, then $\ulcorner\rho^n\beta_1 \ldots \beta_n\urcorner$ is an *intensifying phrase whose intensifier is* $\rho^n$ *and whose reference sequence is* $\delta$. If $\pi^m$ is an elementary modifier and $\gamma = \langle\alpha_1, \ldots, \alpha_m\rangle$ is a sequence of terms, then $\ulcorner\pi^m\alpha_1 \ldots \alpha_m\urcorner$ is an (*elementary*) (*atomic*) *modifying phrase whose modifier is* $\pi^m$ *and whose reference sequence is* $\gamma$. If $\phi$ is a modifying phrase whose modifier is $\pi^m$ and whose reference sequence is $\gamma$, and if $\psi$ is an intensifying phrase whose intensifier is $\rho^n$ and whose reference sequence is $\delta$, then $\ulcorner(\phi)\psi\urcorner$ is an (*atomic*) *modifying phrase whose modifier is* $\ulcorner(\pi^m)\rho^n\urcorner =_{df} \ulcorner((\pi)\rho)^{m+n}\urcorner$ *and whose reference sequence is* $\gamma \wedge \delta$. If $\phi$ and $\psi$ are modifying phrases whose respective modifiers are $\pi^m$ and $\rho^n$ and whose respective reference sequences are $\gamma$ and $\delta$, then $\ulcorner\phi\psi\urcorner$ is a *modifying phrase whose modifier is* $\ulcorner\pi^m\rho^n\urcorner =_{df} \ulcorner(\pi\rho)^{m+n}\urcorner$ *and whose reference sequence is* $\gamma \wedge \delta$.

If $\pi^m$ is an elementary predicate and $\gamma = \langle\alpha_1, \ldots, \alpha_m\rangle$ is a sequence of terms, then $\ulcorner\pi^m\alpha_1 \ldots \alpha_m\urcorner$ is a *sentence whose predicate is* $\pi^m$ *and whose reference sequence is* $\gamma$. If $\phi$ is a sentence whose predicate is $\pi^m$ and whose reference sequence is $\gamma$, and if $\psi$ is a modifying phrase whose modifier is $\rho^n$ and whose reference sequence is $\delta$, then $\ulcorner(\phi)\psi\urcorner$ is a *sentence whose predicate is* $\ulcorner(\pi^m)\rho^n\urcorner =_{df} \ulcorner((\pi)\rho)^{m+n}\urcorner$ *and whose reference sequence is* $\gamma \wedge \delta$. If $\phi$ and $\psi$ are sentences whose respective predicates are $\pi^m$ and $\rho^n$ and whose respective reference sequences are $\gamma = \langle\alpha_1, \ldots, \alpha_m\rangle$ and $\delta$, and if

$\langle i_1, \ldots, i_k\rangle$ and $\langle j_1, \ldots, j_{m-k}\rangle$ are complementary subsequences of $\langle 1, \ldots, m\rangle$, then

$\ulcorner \neg\phi \urcorner$ is a sentence whose predicate is $\ulcorner(\pi^m)\neg\urcorner =_{df} \ulcorner((\pi)\neg)^m\urcorner$ and whose reference sequence is $\gamma$

$\ulcorner(\phi \,\&\, \psi)\urcorner$ is a sentence whose predicate is $\ulcorner(\pi^m\rho^n)\&\urcorner =_{df} \ulcorner((\pi\rho)\&)^{m+n}\urcorner$ and whose reference sequence is $\gamma \wedge \delta$

$\ulcorner(\forall i_1, \ldots, i_k)\phi\urcorner$ is a sentence whose predicate is $\ulcorner(\pi^m)(\forall i_1, \ldots, i_k)\urcorner =_{df} \ulcorner(\pi)(\forall i_1, \ldots, i_k)^{m-k}\urcorner$ and whose reference sequence is $\langle \alpha_{j_1}, \ldots, \alpha_{j_{m-k}}\rangle$.

The expression $\ulcorner(\forall i_1, \ldots, i_k)\urcorner$ is the *universal quantifier* of $\mathfrak{A}$. The more familiar notation of quantification can be introduced as an abbreviation by including *variables* in the vocabulary (the lower case letters $u$ through $z$ were excluded from the primitive vocabulary for this purpose), by replacing the quantifier in $\ulcorner(\forall i_1 \ldots i_k)\phi\urcorner$ by $\ulcorner(\forall\beta)\urcorner$ where $\beta$ is a variable not appearing in $\phi$, and by replacing $\alpha_{i_1}$ in the reference sequence of $\phi$ by $\beta, \ldots,$ and $\alpha_{i_k}$ in the reference sequence of $\phi$ by $\beta$. In what follows, however, I shall adhere to the primitive notation.

For the purposes of the discussion to follow it will be convenient to introduce two abbreviations into the metalanguage: For any sentence $\phi$ whose predicate is $\pi^m$ and whose reference sequence is $\langle\alpha_1, \ldots, \alpha_m\rangle$, for any term $\beta$, and for any complementary subsequences $\langle i_1, \ldots, i_k\rangle$ and $\langle j_1, \ldots, j_{m-k}\rangle$ of $\langle 1, \ldots, m\rangle$, $\ulcorner(\phi)\beta/i_1, \ldots, i_k\urcorner$ is the sentence whose predicate is $\pi^m$ and whose reference sequence is $\langle\alpha'_1, \ldots, \alpha'_m\rangle$, where for each $i \in \langle i_1, \ldots, i_k\rangle$, $\alpha'_i = \beta$, and for each $j \in \langle j_1, \ldots, j_{m-k}\rangle$, $\alpha'_j = \alpha_j$. For any sentence $\phi = \ulcorner(\psi_0)\psi_1 \ldots \psi_k\urcorner$, where $\psi_0$ is a sentence and $\psi_1, \ldots, \psi_k$ are modifying phrases, and for any integer $i$, $1 \leq i < k$,

$$(\phi)/i =_{df} \ulcorner(\psi_0)\psi_1 \ldots \psi_{i-1}, \psi_{i+1}, \psi_i, \psi_{i+2}, \ldots, \psi_k\urcorner.$$

*Syntax II: Rules of inference* The *rules of inference* of $\mathfrak{A}$ are the following: for any sentences $\phi$ and $\psi$ and for any sets $\Gamma$ and $\Lambda$ of sentences,

*Premise* (**P.**) $\{\phi\} \vdash \phi$.
*Detachment* (**Det.**) If $\Gamma \vdash \neg(\phi \,\&\, \neg\psi)$ and $\Delta \vdash \phi$, then $\Gamma \cup \Delta \vdash \psi$.
*Simplification* (**Simp.**) If $\Gamma \vdash (\phi \,\&\, \psi)$, then $\Gamma \vdash \phi$.
*Conjunction* (**Conj.**) If $\Gamma \vdash \phi$ and $\Delta \vdash \psi$, then $\Gamma \cup \Delta \vdash (\phi \,\&\, \psi)$.
*Commutation* (**Comm.**) If $\Gamma \vdash (\phi \,\&\, \psi)$, then $\Gamma \vdash (\psi \,\&\, \phi)$.
*Contradiction* (**Cont.**) If $\Gamma \cup \{\phi\} \vdash (\psi \,\&\, \neg\psi)$, then $\Gamma \vdash \neg\phi$.

For any sentence $\phi$ whose predicate is of degree $m$, for any set $\Gamma$ of sentences, and for any *complementary subsequences* $\langle i_1, \ldots, i_k\rangle$ and $\langle j_1, \ldots, j_{m-k}\rangle$ of $\langle 1, \ldots, m\rangle$.

*Instantiation* (**Inst.**) If $\Gamma \vdash (\forall i_1, \ldots, i_k)\phi$, then for any term $\beta$, $\Gamma \vdash (\phi)\beta/i_1, \ldots, i_k$.
*Generalization* (**Gen.**) If $\Gamma \vdash (\phi)\beta/i_1, \ldots, i_k$, and if $\beta$ is a term not in the reference sequence of $\phi$ or of any sentence in $\Gamma$, then $\Gamma \vdash (\forall i_1, \ldots, i_k)\phi$.

For any sentence $\phi$, for any atomic modifying phrases $\psi_1, \ldots, \psi_k$ and for any set $\Gamma$ of sentences, *Deletion of Modifiers* (**D.M.**) If $\Gamma \vdash (\phi)\psi_1 \ldots \psi_k$, and if $\psi_k$ is an elementary modifying phrase, then $\Gamma \vdash (\phi)\psi_1 \ldots \psi_{k-1}$.

*Deletion of Intensifiers* (**D.I.**) If $\Gamma \vdash (\phi)\psi_1 \ldots \psi_k$, and if $\psi_k = (\chi)\sigma$, where $\chi$ is a modifying phrase and $\sigma$ is an intensifying phrase, then $\Gamma \vdash (\phi)\psi_1 \ldots \psi_{k-1}\chi$.

*Reordering* If $\Gamma \vdash (\phi)\psi_1 \ldots \psi_k$, then for any $i$, $1 \leq i < k$, $\Gamma \vdash ((\phi)\psi_1 \ldots \psi_k)/i$.

A sentence $\phi$ is *derivable* from a set ($\Gamma \vdash \phi$) iff there is a finite sequence of $n$ lines of the form $\Gamma_i \vdash \phi_i$, such that $\Gamma = \Gamma_n$, $\phi = \phi_n$, and for each line $\Gamma_i \vdash \phi_i$ in the sequence, either $\Gamma_i \vdash \phi_i$ is a rule of inference (i.e., $\Gamma_i = \{\phi_i\}$) or there are preceding lines in the sequence such that it is a rule of inference that if those lines are derivable then $\Gamma_i \vdash \phi_i$. *Derivability* so defined is applicable only to finite sets. Therefore the following is added: If $\Gamma$ is infinite then $\Gamma \vdash \phi$ iff there is a finite subset $\Delta$ of $\Gamma$ such that $\Delta \vdash \phi$.

My purposes here are not proof theoretical. I shall defer most discussion of the formal consequences of the rules of inference until later. However, in order to prepare for a proof of the adequacy of these rules, it will be useful to present the following theorems:

Theorem 1 *For any set $\Gamma$ and any sentence $\phi$, if $\Gamma \vdash \neg\neg\phi$, then $\Gamma \vdash \phi$.*

(I shall assume in what follows that $P$ is a 0-ary predicate, i.e., a sentence; it can of course be replaced by any such predicate that the vocabulary contains.)

*Proof:*

| | | |
|---|---|---|
| 1. | $\Gamma \vdash \neg\neg\phi$ | Assume |
| 2. | $\{(\neg\neg\phi \;\&\; \neg\phi)\} \vdash (\neg\neg\phi \;\&\; \neg\phi)$ | **P.** |
| 3. | $\{(\neg\neg\phi \;\&\; \neg\phi)\} \vdash (\neg\phi \;\&\; \neg\neg\phi)$ | 2, **Comm.** |
| 4. | $\emptyset \vdash \neg(\neg\neg\phi \;\&\; \neg\phi)$ | 3, **Cont.** |
| 5. | $\Gamma \vdash \phi$ | 1, 4, **Det.** |

Theorem 2 *For any sentence $\phi$ and set $\Gamma$, if $\Gamma \vdash (\phi \;\&\; \neg\phi)$, then $\Gamma \vdash (P \;\&\; \neg P)$.*

*Proof:*

| | | |
|---|---|---|
| 1. | $\Gamma \vdash \phi \;\&\; \neg\phi$ | Assume |
| 2. | $\{\neg(P \;\&\; \neg P)\} \vdash \neg(P \;\&\; \neg P)$ | **P.** |
| 3. | $\Gamma \cup \{\neg(P \;\&\; \neg P)\} \vdash ((\phi \;\&\; \neg\phi) \;\&\; (P \;\&\; \neg P))$ | 1, 2, **Conj.** |
| 4. | $\Gamma \cup \{\neg(P \;\&\; \neg P)\} \vdash (\phi \;\&\; \neg\phi)$ | 3, **Simp.** |
| 5. | $\Gamma \vdash \neg\neg(P \;\&\; \neg P)$ | 4, **Cont.** |
| 6. | $\Gamma \vdash (P \;\&\; \neg P)$ | 5, Th. 1. |

Theorem 3 *For any set $\Gamma$ and any sentence $\phi$, $\Gamma \vdash \phi$ iff $\Gamma \cup \{\neg\phi\} \vdash P \;\&\; \neg P$.*

*Proof:*

| | | |
|---|---|---|
| 1. | $\Gamma \vdash \phi$ | Assume |
| 2. | $\{\neg\phi\} \vdash \neg\phi$ | **P.** |
| 3. | $\Gamma \cup \{\neg\phi\} \vdash (\phi \;\&\; \neg\phi)$ | 1, 2, **Conj.** |
| 4. | $\Gamma \cup \{\neg\phi\} \vdash (P \;\&\; \neg P)$ | 3, Th. 2 |

Next

| | | |
|---|---|---|
| 1. | $\Gamma \cup \{\neg\phi\} \vdash (P \,\&\, \neg P)$ | Assume |
| 2. | $\Gamma \vdash \neg\neg\phi$ | 1, **Cont.** |
| 3. | $\Gamma \vdash \phi$ | 2, Th. 1 |

Theorem 4 *For any sets* $\Gamma$ *and* $\Delta$ *and any sentence* $\phi$, *if* $\Gamma \subseteq \Delta$ *and if* $\Gamma \vdash \phi$, *then* $\Delta \vdash \phi$.

*Proof:* If both sets are infinite, the theorem holds by transitivity of the subset relation and the definition of *derivability*. If $\Gamma$ is finite and $\Delta$ is infinite, the theorem holds directly by definition. If both are finite, then there is an obvious procedure of applying rules **P.**, **Conj.**, and **Simp.** to an ordering of $\Delta \sim \Gamma$ following the line $\Gamma \vdash \phi$.

Theorem 5 *For any set* $\Gamma$, *sentence* $\phi$, *and modifying phrases* $\psi_1, \ldots, \psi_k$, *if* $\Gamma \vdash (\phi)\psi_1 \ldots \psi_k$, *then* $\Gamma \vdash (\phi)\psi_1 \ldots \psi_{k-1}$.

*Proof:* The *depth* of a modifying phrase is defined as follows: An elementary modifying phrase is of depth 1. If $\psi$ is a modifying phrase of depth $m$ and $\psi'$ is a modifying phrase of depth $n$, then $\psi\psi'$ is a modifying phrase of depth $m + n$. If $\psi$ is a modifying phrase of depth $m$ and $\psi'$ is an intensifying phrase, then $(\psi)\psi'$ is a modifying phrase of depth $m + 1$.

Let $\psi_k$ be a modifying phrase of depth $m$, and assume the theorem for modifying phrases of depth $m - 1$. If $\psi_k$ is an elementary modifying phrase, then $\Gamma \vdash (\phi)\psi_1 \ldots \psi_{k-1}$ by Deletion of Modifiers. If $\psi_k = (\chi)\sigma$, where $\chi$ is a modifying phrase and $\sigma$ is an intensifying phrase, then $\Gamma \vdash (\phi)\psi_1 \ldots \psi_{k-1}\chi$ by Deletion of Intensifiers. But $\chi$ is a modifying phrase of depth $m - 1$. $\therefore$ $\Gamma \vdash (\phi)\psi_1 \ldots \psi_{k-1}$, by the induction hypothesis. If $\psi_k = \psi'\psi''$, where $\psi'$ is a modifying phrase and $\psi''$ is an atomic modifying phrase, then either $\psi''$ is an elementary modifying phrase, in which case $\Gamma \vdash (\phi)\psi_1 \ldots \psi_{k-1}\psi'$, by Deletion of Modifiers, in which case $\psi'$ is a modifying phrase of depth $m - 1$ and $\Gamma \vdash (\phi)\psi_1 \ldots \psi_{k-1}$ by the induction hypothesis; or $\psi'' = (\chi)\sigma$, where $\chi$ is a modifying phrase, $\sigma$ is an intensifier, and $\psi'(\chi)\sigma = \psi_k$, in which case $\Gamma \vdash (\phi)\psi_1 \ldots \psi_{k-1}\psi'\chi$, by Deletion of Intensifiers, where $\psi'\chi$ is a modifying phrase of depth $m - 1$, and thus $\Gamma \vdash (\phi)\psi_1 \ldots \psi_{k-1}$ by the induction hypothesis.

*Semantics I: assignments of values* A model $\mathfrak{M}$ of the logic $\mathfrak{A}$ is an ordered pair $\langle \mathfrak{D}, \mathfrak{f} \rangle$ such that $\mathfrak{D}$, the *domain* of $\mathfrak{M}$, is a non-empty set and $\mathfrak{f}$ is an *assignment function* from the syntax of $\mathfrak{A}$ into $\mathfrak{D}$ and relations on $\mathfrak{D}$, conforming to the following requirements:

For any term $\alpha$, $\mathfrak{f}(\alpha) \in \mathfrak{D}$. (For any sequence of terms $\delta = \langle \alpha_1, \ldots, \alpha_m \rangle$, $\ulcorner \mathfrak{f}(\delta) \urcorner =_{df} \ulcorner \langle \mathfrak{f}(\alpha_1), \ldots, \mathfrak{f}(\alpha_m) \rangle \urcorner$.) When $\mathfrak{M}$ is the only model under consideration in a given context, then for any term or sequences of terms $\gamma$, $\ulcorner \mathfrak{f}(\gamma) \urcorner$ will be abbreviated as $\ulcorner \overline{\gamma} \urcorner$.

For any sentence $\phi$ whose predicate is $\psi$,

if $\psi = \pi^n$ is an elementary predicate, then $\mathfrak{f}(\psi) \subseteq \mathfrak{D}^n$;

if $\psi = \ulcorner (\pi^{n_0})\rho_1^{n_1} \ldots \rho_k^{n_k} \urcorner$, where $\pi^{n_0}$ is a predicate, and where each $\rho_i^{n_i}$ is the modifier of an atomic modifying phrase, then

$$\mathfrak{f}(\psi) \subseteq \mathfrak{f}((\pi^{n_0})\rho_1^{n_1} \ldots \rho_{k-1}^{n_{k-1}}) \wedge \mathfrak{D}^{n_k}$$

such that for any integer $i$, $1 \leq i < k$, for any sequence $\delta_0 \wedge \ldots \wedge \delta_k \in \mathfrak{f}(\psi)$, where each $\delta_i$ is an $n_i$-tuple,

$$\delta_0 \wedge \ldots \wedge \delta_{i-1} \wedge \delta_{i+1} \wedge \delta_i \wedge \delta_{i+2} \wedge \ldots \wedge \delta_k$$
$$\in \mathfrak{f}((\pi^{n_0})\rho_1^{n_1} \ldots \rho_{i-1}^{n_{i-1}} \rho_{i+1}^{n_{i+1}} \rho_1^{n_i} \rho_{i+2}^{n_{i+2}} \ldots \rho_k^{n_k});$$

and such that if $\rho_k^{n_k} = \ulcorner(\chi^{m_0})\sigma^{m_1}\urcorner$, where $\chi^{m_0}$ is the modifier of some modifying phrase and $\sigma^{m_1}$ is an intensifier, then $\mathfrak{f}(\psi) \subseteq \mathfrak{f}((\pi^{n_0})\rho_1^{n_1} \ldots \rho_{k-1}^{n_{k-1}}\chi^{m_0}) \wedge \mathfrak{D}^{m_1}$;

if $\psi = \ulcorner(\pi^m)\urcorner\urcorner$, where $\pi^m$ is any predicate, then $\mathfrak{f}(\psi) = \mathfrak{D}^m \sim \mathfrak{f}(\pi^m)$—thus '$\urcorner$' forms the *complement* of a predicate;

if $\psi = \ulcorner(\pi^m\rho^n)\&\urcorner$, where $\pi^m$ and $\rho^n$ are predicates, then $\mathfrak{f}(\psi) = \mathfrak{f}(\pi^m) \wedge \mathfrak{f}(\rho^n)$—thus '&' forms the *adjunction* of predicates;

if $\psi = \ulcorner(\pi^m)(\forall i_1, \ldots, i_k)\urcorner$, where $\pi^m$ is a predicate and where $\langle i_1, \ldots, i_k\rangle$ and $\langle j_1, \ldots, j_{m-k}\rangle$ are complementary subsequences of $\langle 1, \ldots, m\rangle$, then $\mathfrak{f}(\psi) = \{\langle x_{j_1}, \ldots, x_{j_{m-k}}\rangle \in \mathfrak{D}^{m-k}$: for any $x_{i_1} = \ldots = x_{i_k} \in \mathfrak{D}$, $\langle x_1, \ldots, x_m\rangle \in \mathfrak{f}(\pi^m)\}$.

The following definitions are based on the above characterization of a model:

*Satisfaction* ($\models$) For any model $\mathfrak{M} = \langle\mathfrak{D}, \mathfrak{f}\rangle$ and sentence $\phi$ whose predicate is $\pi^m$ and whose reference sequence is $\delta$, $\mathfrak{M} \models \phi$ iff $\mathfrak{f}(\delta) \in \mathfrak{f}(\pi^m)$; $\mathfrak{M} \not\models \phi$ iff $\mathfrak{f}(\delta) \notin \mathfrak{f}(\pi^m)$. For any set $\Gamma$, $\mathfrak{M} \models \Gamma$ iff $\mathfrak{M} \models \psi$, for all $\psi \in \Gamma$.

*Entailment* ($\Vdash$) For any set $\Gamma$ and sentence $\phi$, $\Gamma \Vdash \phi$ iff for any model $\mathfrak{M}$, if $\mathfrak{M} \models \Gamma$ then $\mathfrak{M} \models \phi$; $\Gamma \nVdash \phi$ iff for some model $\mathfrak{M}$, $\mathfrak{M} \models \Gamma$ and $\mathfrak{M} \not\models \phi$.

*Consistency* For any set $\Gamma$, Cons. $\Gamma$ iff for some model $\mathfrak{M}$, $\mathfrak{M} \models \Gamma$.

*Semantics II: Truth-functional completeness* The semantics of $\mathfrak{A}$ does not contain any operations on truth values; nonetheless, $\mathfrak{A}$ is adequate to the expression of any sentence which is a truth function of other sentences. To prove this it will be sufficient to show that negation and conjunction are expressible in $\mathfrak{A}$, since these are known to be truth-functionally complete.

The Negation Theorem *For any sentence $\phi$ whose predicate is $\pi^n$ and whose reference sequence is $\delta$, for any model $\mathfrak{M} = \langle\mathfrak{D}, \mathfrak{f}\rangle$, $\mathfrak{M} \models \urcorner\phi$ iff $\mathfrak{M} \not\models \phi$.*

*Proof:* By definition of $\models$ and rules of construction

$$\mathfrak{M} \models \urcorner\phi \text{ iff } \bar{\delta} \in \mathfrak{f}((\pi^n)\urcorner);$$

i.e., by the definition of $\mathfrak{f}$, iff $\bar{\delta} \in \mathfrak{D}^n \sim \mathfrak{f}(\pi^n)$; i.e., since every such $\bar{\delta} \in \mathfrak{D}^n$, iff $\bar{\delta} \notin \mathfrak{f}(\pi^n)$; i.e., again by definition of $\models$, iff $\mathfrak{M} \not\models \phi$.

The Conjunction Theorem *For any sentences $\phi$ and $\psi$ whose respective predicates are $\pi^m$ and $\rho^n$ and whose respective reference sequences are $\gamma$ and $\delta$, and for any model $\mathfrak{M} = \langle\mathfrak{D}, \mathfrak{f}\rangle$, $\mathfrak{M} \models (\phi \,\&\, \psi)$ iff both $\mathfrak{M} \models \phi$ and $\mathfrak{M} \models \psi$.*

*Proof:* By definition of $\models$ and the rules of composition

$$\mathfrak{M} \models (\phi \,\&\, \psi) \text{ iff } \bar{\gamma} \wedge \bar{\delta} \in \mathfrak{f}((\pi^m\rho^n)\&);$$

i.e., by definition of $\mathfrak{f}$, iff $\bar{\gamma} \wedge \bar{\delta} \in \mathfrak{f}(\pi^m) \wedge \mathfrak{f}(\rho^n)$; i.e., by definition of $\wedge$, since $\bar{\gamma}$ is an $m$-tuple and $\bar{\delta}$ is an $n$-tuple, iff both $\bar{\gamma} \in \mathfrak{f}(\pi^m)$ and $\bar{\delta} \in \mathfrak{f}(\rho^n)$; i.e., again by definition of $\models$, iff both $\mathfrak{M} \models \phi$ and $\mathfrak{M} \models \psi$.

A similar but extended result could be obtained concerning the adequacy of $\mathfrak{A}$ to the expression of the semantic relations expressed by the ordinary predicate calculus, by adding here a theorem about the universal quantifier of $\mathfrak{A}$. Such a proof, however, overlaps exactly with what will be done in proving the deductive adequacy of the rules of inference of $\mathfrak{A}$, and thus the proof is omitted here.

*Semantics III: Soundness of the rules of inference* In this section I shall prove that if $\Gamma \vdash \phi$ then $\Gamma \Vdash \phi$. The following lemma will be useful in justifying the rule of Generalization:

Lemma I *Let $\phi$ be a sentence whose predicate is $\pi^m$ and whose reference sequence is $\gamma$. Let $\mathfrak{M} = \langle \mathfrak{D}, \mathfrak{f} \rangle$ and $\mathfrak{M}' = \langle \mathfrak{D}', \mathfrak{f}' \rangle$ be models such that $\mathfrak{D}' = \mathfrak{D}$, $\mathfrak{f}'(\gamma) = \mathfrak{f}(\gamma)$, and $\mathfrak{f}'(\pi^m) = \mathfrak{f}(\pi)$. Then $\mathfrak{M} \models \phi$ iff $\mathfrak{M}' \models \phi$.*

*Proof:* By definition of $\models$, $\mathfrak{M} \models \phi$ iff $\mathfrak{f}(\gamma) \in \mathfrak{f}(\pi^m)$; i.e., from the hypothesis, iff $\mathfrak{f}'(\gamma) \in \mathfrak{f}'(\pi^m)$; i.e., iff $\mathfrak{M}' \models \phi$.

Corollary I *Let $\Gamma$ be a set of sentences. Let $\mathfrak{M} = \langle \mathfrak{D}, \mathfrak{f} \rangle$ and $\mathfrak{M}' = \langle \mathfrak{D}', \mathfrak{f}' \rangle$ be models such that for each sentence $\phi \in \Gamma$, whose predicate is $\pi^m$ and whose reference sequence is $\gamma$, $\mathfrak{D}' = \mathfrak{D}$, $\mathfrak{f}'(\pi^m) = \mathfrak{f}(\pi^m)$, and $\mathfrak{f}'(\gamma) = \mathfrak{f}(\gamma)$. Then $\mathfrak{M} \models \Gamma$ iff $\mathfrak{M}' \models \Gamma$.*

*Proof:* Follows immediately from the Lemma and the fact that a model satisfies a set iff it satisfies every sentence in that set.

The soundness theorem itself will proceed as usual as an induction on the length of the derivation of $\phi$ from $\Gamma$. The hypothesis of induction will be that shorter derivations do yield entailments and hence so do previous lines of the derivation of $\phi$. Therefore if the line $\Gamma \vdash \phi$ is derived by the application of a rule of inference $\mathfrak{R}$ from previous lines then, since those lines express entailments by the hypothesis of induction, $\Gamma \Vdash \phi$ if $\mathfrak{R}$ is "entailment-preserving". Thus it will suffice for the proof of the soundness theorem to show that each of the rules of inference of $\mathfrak{A}$ is "entailment-preserving", i.e., that the result of replacing $\vdash$ by $\Vdash$ in each rule expresses a truth about the models of $\mathfrak{M}$. The following lemmas establish that the rules are "entailment-preserving" (for the first six rules, let $\phi$ and $\psi$ be sentences whose respective predicates are $\pi^m$ and $\rho^n$ and whose respective reference sequences are $\gamma$ and $\delta$):

*Premise* $\{\phi\} \Vdash \phi$.

*Proof:* Follows immediately from the definitions of $\Vdash$ and 'satisfaction of sets'.

*Detachment* If $\Gamma \Vdash \neg(\phi \ \& \ \neg\psi)$ and $\Delta \Vdash \phi$, then $\Gamma \cup \Delta \Vdash \psi$.

*Proof:* Assume protasis and let $\mathfrak{M} = \langle \mathfrak{D}, \mathfrak{f} \rangle \models \Gamma \cup \Delta$.

$\therefore$ $\mathfrak{M} \models \Gamma$ and $\mathfrak{M} \models \Delta$.

$\therefore$ $\mathfrak{M} \models \neg(\phi \ \& \ \neg\psi)$ and $\mathfrak{M} \models \phi$ (by def. of $\Vdash$).

∴ $\bar{\gamma} \wedge \bar{\delta} \in \mathfrak{f}(((\pi^m(\rho^n)\neg)\&)\neg)$ and $\bar{\gamma} \in \mathfrak{f}(\pi^m)$ (by def. of $\models$).
∴ $\bar{\gamma} \wedge \bar{\delta} \in \mathfrak{D}^{m+n} \sim \mathfrak{f}((\pi^m(\rho^n)\neg)\&)$ (by def. of $\mathfrak{f}$).
∴ $\bar{\gamma} \wedge \bar{\delta} \notin \mathfrak{f}((\pi^m(\rho^n)\neg)\&)$.
∴ $\bar{\gamma} \wedge \bar{\delta} \notin \mathfrak{f}(\pi^m) \wedge \mathfrak{f}((\rho^n)\neg)$ (by def. of $\mathfrak{f}$).
∴ Not both $\bar{\gamma} \in \mathfrak{f}(\pi^m)$ and $\bar{\delta} \in \mathfrak{f}((\rho^n)\neg)$ (by def. of $\wedge$).
∴ Since $\bar{\gamma} \in \mathfrak{f}(\pi^m)$, $\bar{\delta} \notin \mathfrak{f}((\rho^n)\neg)$.
∴ $\bar{\delta} \notin \mathfrak{D}^n \sim \mathfrak{f}(\rho^n)$ (by def. of $\mathfrak{f}$).
∴ Since $\bar{\delta} \in \mathfrak{D}^n$ (by def. of $\mathfrak{f}$), $\bar{\delta} \in \mathfrak{f}(\rho^n)$.
∴ $\mathfrak{M} \vdash \psi$ (by def. of $\models$).
∴ Any model that satisfies $\Gamma \cup \Delta$ satisfies $\psi$.
∴ $\Gamma \cup \Delta \Vdash \psi$.

*Simplification* If $\Gamma \Vdash (\phi \,\&\, \psi)$, then $\Gamma \Vdash \phi$.

*Proof:* Assume protasis and let $\mathfrak{M} = \langle \mathfrak{D}, \mathfrak{f} \rangle \models \Gamma$.
∴ $\mathfrak{M} \models (\phi \,\&\, \psi)$.
∴ $\bar{\gamma} \wedge \bar{\delta} \in \mathfrak{f}((\pi^m \rho^n)\&)$.
∴ $\bar{\gamma} \wedge \bar{\delta} \in \mathfrak{f}(\pi^m) \wedge \mathfrak{f}(\rho^n)$.
∴ $\bar{\gamma} \in \mathfrak{f}(\pi^m)$ and $\bar{\delta} \in \mathfrak{f}(\rho^n)$.
∴ $\mathfrak{M} \models \phi$.
∴ $\Gamma \Vdash \phi$.

*Conjunction* If $\Gamma \Vdash \phi$ and $\Delta \Vdash \psi$, then $\Gamma \cup \Delta \Vdash (\phi \,\&\, \psi)$.

*Proof:* Assume protasis and let $\mathfrak{M} = \langle \mathfrak{D}, \mathfrak{f} \rangle \models \Gamma \cup \Delta$.
∴ $\mathfrak{M} \models \Gamma$ and $\mathfrak{M} \models \Delta$.
∴ $\bar{\gamma} \in \mathfrak{f}(\pi^m)$ and $\bar{\delta} \in \mathfrak{f}(\rho^n)$.
∴ $\bar{\gamma} \wedge \bar{\delta} \in \mathfrak{f}(\pi^m) \wedge \mathfrak{f}(\rho^n)$.
∴ $\bar{\gamma} \wedge \bar{\delta} \in \mathfrak{f}((\pi^m \rho^n)\&)$.
∴ $\mathfrak{M} \models (\phi \,\&\, \psi)$.
∴ $\Gamma \cup \Delta \Vdash (\phi \,\&\, \psi)$.

*Commutation* If $\Gamma \Vdash (\phi \,\&\, \psi)$, then $\Gamma \Vdash (\psi \,\&\, \phi)$.

*Proof:* Assume protasis and let $\mathfrak{M} = \langle \mathfrak{D}, \mathfrak{f} \rangle \models \Gamma$.
∴ $\mathfrak{M} \models (\phi \,\&\, \psi)$.
∴ $\bar{\gamma} \wedge \bar{\delta} \in \mathfrak{f}((\pi^m \rho^n)\&)$.
∴ $\bar{\gamma} \wedge \bar{\delta} \in \mathfrak{f}(\pi^m) \wedge \mathfrak{f}(\rho^n)$.
∴ $\bar{\gamma} \in \mathfrak{f}(\pi^m)$ and $\bar{\delta} \in \mathfrak{f}(\rho^n)$.
∴ $\bar{\delta} \wedge \bar{\gamma} \in \mathfrak{f}(\rho^n) \wedge \mathfrak{f}(\pi^m)$.
∴ $\bar{\delta} \wedge \bar{\gamma} \in \mathfrak{f}((\rho^n \pi^m)\&)$.
∴ $\mathfrak{M} \models (\psi \,\&\, \phi)$.
∴ $\Gamma \Vdash (\psi \,\&\, \phi)$.

*Contradiction* If $\Gamma \cup \{\phi\} \Vdash (\psi \,\&\, \neg\psi)$, then $\Gamma \Vdash \neg\phi$.

*Proof:* Assume protasis and let $\mathfrak{M} \models \Gamma$.
If $\mathfrak{M} \models \phi$, then $\mathfrak{M} \models \Gamma \cup \{\phi\}$;
then $\mathfrak{M} \models (\psi \,\&\, \neg\psi)$;
then $\bar{\delta} \wedge \bar{\delta} \in \mathfrak{f}((\rho^n(\rho^n)\neg)\&)$;
then $\bar{\delta} \wedge \bar{\delta} \in \mathfrak{f}(\rho^n) \wedge \mathfrak{f}((\rho^n)\neg)$;

then $\bar{\delta} \in \mathfrak{f}(\rho^n)$ and $\bar{\delta} \in \mathfrak{f}((\rho^n)\neg)$;
then $\bar{\delta} \in \mathfrak{f}(\rho^n)$ and $\bar{\delta} \in \mathfrak{D}^n \sim \mathfrak{f}(\rho^n)$ contrary to the definition of $\sim$.

$\therefore$ $\mathfrak{M} \not\models \phi$.
$\therefore$ $\mathfrak{M} \models \neg\phi$, by the Negation Theorem.
$\therefore$ $\Gamma \Vdash \neg\phi$.

For the following two rules, let $\phi$ be any sentence whose predicate is $\pi^m$ and whose reference sequence is $\gamma = \langle \alpha_1, \ldots, \alpha_m \rangle$; and let $\langle i_1, \ldots, i_k \rangle$ and $\langle j_1, \ldots, j_{m-k} \rangle$ be complementary subsequences of $\langle 1, \ldots, m \rangle$.

*Instantiation* If $\Gamma \Vdash (\forall i_1, \ldots, i_k)\phi$, then for any term $\beta$, $\Gamma \Vdash (\phi)\beta/i_1, \ldots, i_k$.

*Proof:* Assume protasis and let $\mathfrak{M} = \langle \mathfrak{D}, \mathfrak{f} \rangle \models \Gamma$.

$\therefore$ $\mathfrak{M} \models (\forall i_1, \ldots, i_k)\phi$.
$\therefore$ $\langle \alpha_{j_1}, \ldots, \alpha_{j_{m-k}} \rangle \in \mathfrak{f}((\pi^m)(\forall i_1, \ldots, i_k))$.
$\therefore$ $\langle \alpha_{j_1}, \ldots, \alpha_{j_{m-k}} \rangle \in \{\langle x_{j_1}, \ldots, x_{j_{m-k}} \rangle \in \mathfrak{D}^{m-k}$: for any $x = x_{i_1} = \ldots = x_{i_k} \in \mathfrak{D}$, $\langle x_1, \ldots, x_m \rangle \in \mathfrak{f}(\pi^m)\}$.
$\therefore$ For any $x \in \mathfrak{D}$, $\langle \bar{\alpha}'_1, \ldots, \bar{\alpha}'_m \rangle \in \mathfrak{f}(\pi^m)$, where for all $i \in \langle i_1, \ldots, i_k \rangle$, $\bar{\alpha}'_i = x$, and for all $j \in \langle j_1, \ldots, j_{m-k} \rangle$, $\bar{\alpha}_j = \bar{\alpha}_j$.
$\therefore$ For any term $\beta$, $\langle \bar{\alpha}'_1, \ldots, \bar{\alpha}'_m \rangle \in \mathfrak{f}(\pi^m)$, where for all $i \in \langle i_1, \ldots, i_k \rangle$, $\bar{\alpha}'_i = \bar{\beta}$, and for all $j \in \langle j_1, \ldots, j_{m-k} \rangle$, $\bar{\alpha}'_j = \bar{\alpha}_j$.
$\therefore$ For any term $\beta$, $\mathfrak{M} \models (\phi)\beta/i_1, \ldots, i_k$.
$\therefore$ For any term $\beta$, $\Gamma \Vdash (\phi)\beta/i_1, \ldots, i_k$.

*Generalization* If $\Gamma \Vdash (\phi)\beta/i_1, \ldots, i_k$, and if $\beta$ is a term not in the reference sequence of $\phi$ or of any sentence in $\Gamma$, then $\Gamma \Vdash (\forall i_1, \ldots, i_k)\phi$.

*Proof:* Assume the protasis and let $\mathfrak{M} = \langle \mathfrak{D}, \mathfrak{f} \rangle \models \Gamma$. Let $x \in \mathfrak{D}$, and let $\mathfrak{M}' = \langle \mathfrak{D}', \mathfrak{f}' \rangle$ be a model such that $\mathfrak{D}' = \mathfrak{D}$, $\mathfrak{f}'(\beta) = x$, $\mathfrak{f}'(\pi) = \mathfrak{f}(\pi)$, $\mathfrak{f}'(\gamma) = \mathfrak{f}(\gamma)$, and for each sentence $\psi \in \Gamma$ whose predicate is $\rho$ and whose reference sequence is $\delta$, $\mathfrak{f}'(\rho) = \mathfrak{f}(\rho)$ and $\mathfrak{f}'(\delta) = \mathfrak{f}(\delta)$—since $\beta$ is neither in $\gamma$ nor in the reference sequence of any sentence in $\Gamma$, there is such a model for each member of $\mathfrak{D}$.

By Corollary I, $\mathfrak{M}' \models \Gamma$.
$\therefore$ $\mathfrak{M}' \models (\phi)\beta/i_1, \ldots, i_k$.
$\therefore$ $\langle \bar{\alpha}'_1, \ldots, \bar{\alpha}'_m \rangle \in \mathfrak{f}'(\pi^m) = \mathfrak{f}(\pi^m)$, where for each $i \in \langle i_1, \ldots, i_k \rangle$, $\bar{\alpha}'_i = \mathfrak{f}'(\beta) = x$, and for each $j \in \langle j_1, \ldots, j_{m-k} \rangle$, $\bar{\alpha}'_j = \bar{\alpha}_j$.
$\therefore$ For each $x \in \mathfrak{D}$, $\langle \bar{\alpha}'_1, \ldots, \bar{\alpha}_m \rangle \in \mathfrak{f}(\pi)$, where for each $i \in \langle i_1, \ldots, i_k \rangle$, $\bar{\alpha}'_i = x$, and for each $j \in \langle j_1, \ldots, j_{m-k} \rangle$, $\bar{\alpha}'_j = \bar{\alpha}_j$.
$\therefore$ $\langle \bar{\alpha}_{j_1}, \ldots, \bar{\alpha}_{j_{m-k}} \rangle \in \{\langle x_{j_1}, \ldots, x_{j_{m-k}} \rangle \in \mathfrak{D}^m$: for all $x = x_{i_1} = \ldots = x_{i_k} \in \mathfrak{D}$, $\langle x_1, \ldots, x_m \rangle \in \mathfrak{f}(\pi)\}$.
$\therefore$ $\langle \bar{\alpha}_{j_1}, \ldots, \bar{\alpha}_{j_{m-k}} \rangle \in \mathfrak{f}((\pi)(\forall i_1, \ldots, i_k))$.
$\therefore$ $\mathfrak{M} \models (\forall i_1, \ldots, i_k)\phi$.
$\therefore$ $\Gamma \Vdash (\forall i_1, \ldots, i_k)\phi$.

For the remaining three rules, let $\phi$ be a sentence whose predicate is $\pi^{n_0}$ and whose reference sequence is $\delta_0$, and let $\psi_1, \ldots, \psi_k$ be modifying phrases whose respective modifiers are $\rho_1^{n_1}, \ldots, \rho_k^{n_k}$ and whose respective reference sequences are $\delta_1, \ldots, \delta_k$.

*Deletion of Modifiers* If $\Gamma \Vdash (\phi)\psi_1 \ldots \psi_k$, and if $\psi_k$ is an elementary modifying phrase, then $\Gamma \Vdash (\phi)\psi_1 \ldots \psi_{k-1}$.

*Deletion of Intensifiers* If $\Gamma \Vdash (\phi)\psi_1 \ldots \psi_k$, and if $\psi_k = (\chi)\sigma$, where $\chi$ is a modifying phrase and $\sigma$ is an intensifying phrase, then $\Gamma \Vdash (\phi)\psi_1 \ldots \psi_k\chi$.

*Reordering* If $\Gamma \Vdash (\phi)\psi_1 \ldots \psi_k$, then for any $i$, $1 \leq i < k$, $\Gamma \Vdash ((\phi)\psi_1 \ldots \psi_k)/i$.

(Since the protases of these are identical, it will simplify the proofs to unite them, especially since the statement of the semantic assignments to complex predicates is itself rather complex.)

*Proof:* Assume the protasis and let $\mathfrak{M} = \langle \mathfrak{D}, \mathfrak{f} \rangle \models \Gamma$.
$\therefore\ \mathfrak{M} \models (\phi)\psi_1 \ldots \psi_k$.
$\therefore\ \delta_0 \wedge \ldots \wedge \delta_k \in \mathfrak{f}((\pi^{n_0})\rho_1^{n_1} \ldots \rho_k^{n_k})$.
$\therefore\ \delta_0 \wedge \ldots \wedge \delta_k \in \mathfrak{f}((\pi^{n_0})\rho_1^{n_1} \ldots \rho_{k-1}^{n_{k-1}}) \wedge \mathfrak{D}^{n_k}$,
and for each $i$, $1 \leq i < k$,

$$\begin{aligned}&\delta_0 \wedge \ldots \wedge \delta_{i-1} \wedge \delta_{i+1} \wedge \delta_i \wedge \delta_{i+2} \wedge \ldots \wedge \delta_k \\ &\quad \in \mathfrak{f}((\pi^{n_0})\rho_1^{n_1} \ldots \rho_{i-1}^{n_{i-1}}\,\rho_{i+1}^{n_{i+1}}\,\rho_i^{n_i}\,\rho_{i+2}^{n_{i+2}} \ldots \rho_k^{n_k}),\end{aligned}$$

and if $\rho_k^{n_k} = (\chi'^{m_0})\sigma'^{m_1}$, where $\chi'^{m_0}$ is the modifier of some modifying phrase $\chi$ and $\sigma'^{m_1}$ is the intensifier of some intensifying phrase $\sigma$, then

$$\delta_0 \wedge \ldots \wedge \delta_k \in \mathfrak{f}((\pi^{n_0})\rho_1^{n_1} \ldots \rho_{k-1}^{n_{k-1}}\chi'^{m_0}) \wedge \mathfrak{D}^{m_1}.$$

Since $\psi_k$ is an atomic modifying phrase, it is either an elementary modifying phrase, in which case it is already established that

$$\delta_0 \wedge \ldots \wedge \delta_k \in \mathfrak{f}((\pi^{n_0})\rho_1^{n_1} \ldots \rho_{k-1}^{n_{k-1}}) \wedge \mathfrak{D}^{n_k},$$

i.e., that $\delta_0 \wedge \ldots \wedge \delta_{k-1} \in \mathfrak{f}((\pi^{n_0})\rho_1^{n_1} \ldots \rho_{k-1}^{n_{k-1}})$ and $\delta_k \in \mathfrak{D}^{n_k}$, i.e., that $\mathfrak{M} \models (\phi)\psi_1 \ldots \psi_{k-1}$; or $\psi_k = \ulcorner(\chi)\sigma\urcorner$, where $\chi$ is a modifying phrase whose modifier is $\chi'^{m_0}$ and whose reference sequence is $\gamma_0$, and $\sigma$ is an intensifying phrase whose intensifier is $\sigma'^{m_1}$ and whose reference sequence is $\gamma_1$, and where $\delta_k = \gamma_0 \wedge \gamma_1$ and $n_k = m_0 + m_1$, in which case

$$\begin{aligned}&\delta_0 \wedge \ldots \wedge \delta_k = \delta_0 \wedge \ldots \wedge \delta_{k-1} \wedge \gamma_0 \wedge \gamma_1 \\ &\quad \in \mathfrak{f}((\pi^{n_0})\rho_1^{n_1} \ldots \rho_{k-1}^{n_{k-1}}\chi'^{m_0}) \wedge \mathfrak{D}^{m_1},\end{aligned}$$

i.e., $\delta_0 \wedge \ldots \wedge \delta_{k-1} \wedge \gamma_0 \in \mathfrak{f}((\pi^{n_0})\rho_1^{n_1} \ldots \rho_{k-1}^{n_{k-1}}\chi'^{m_0})$ and $\gamma_1 \in \mathfrak{D}^{m_1}$, i.e., $\mathfrak{M} \models (\phi)\psi_1 \ldots \psi_k\chi$.
$\therefore$ If $\psi_k$ is an elementary modifying phrase then $\Gamma \Vdash (\phi)\psi_1 \ldots \psi_{k-1}$, and Deletion of Modifiers is established; and if $\psi_k = (\chi)\sigma$, as specified above, then $\Gamma \Vdash (\phi)\psi_1 \ldots \psi_{k-1}\chi$, and Deletion of Intensifiers is established.
Moreover, we have directly that

$$\begin{aligned}&\delta_0 \wedge \ldots \wedge \delta_{i-1} \wedge \delta_{i+1} \wedge \delta_i \wedge \delta_{i+2} \wedge \ldots \wedge \delta_k \\ &\quad \in \mathfrak{f}((\pi^{n_0})\rho_1^{n_1} \ldots \rho_{i-1}^{n_{i-1}}\,\rho_{i+1}^{n_{i+1}}\,\rho_i^{n_i}\,\rho_{i+2}^{n_{i+2}} \ldots \rho_k^{n_k})\end{aligned}$$

for each $i$, $1 \leq i < k$.
$\therefore$ For each $i$, $1 \leq i < k$, $\mathfrak{M} \models ((\phi)\psi_1 \ldots \psi_k)/i$.
$\therefore$ For each $i$, $1 \leq i < k$, $\Gamma \Vdash ((\phi)\psi_1 \ldots \psi_k)/i$; and Reordering is established.

As remarked earlier, this suffices to establish the soundness of the rules of inference of $\mathfrak{A}$.

For simplicity of proofwork, it would be useful to introduce v., $\supset$, $\equiv$, and $\exists$ into $\mathfrak{A}$ via the usual definitions and to add rules of Existential Generalization, Existential Instantiation (à la Mates [4], pp. 120-123), and Conditional Proof. For my purposes here, however, I shall not make these introductions.

*Semantics IV: Deduction completeness* Deductive completeness will be established as a corollary to the following theorem:

The Consistency Theorem *If* $\Gamma \nvdash P \,\&\, \neg P$, *then Cons.* $\Gamma$.

*Proof:* Let $\Gamma$ be a set of sentences such that $\Gamma \nvdash P \,\&\, \neg P$. Let $\mathfrak{F} = \langle \phi_1, \ldots, \phi_n, \ldots \rangle$ be an ordering of all the sentences of $\mathfrak{A}$ such that for any $\phi_i \in \mathfrak{F}$, if

$$\phi_i = \ulcorner \neg (\forall i_1, \ldots, i_k) \psi \urcorner$$

then for some term $\beta$ such that $\beta$ is not in the reference sequence of any sentence in $\Gamma$ or of any previous ('previous' and 'later' will be used in what follows with the understanding that they are defined on $\mathfrak{F}$ in the natural way) sentence in $\mathfrak{F}$,

$$\phi_{i+1} = \ulcorner (\neg \psi) \beta / i_1, \ldots, i_k \urcorner.$$

Let $\Delta_0 = \Gamma$. For each $i$, $i > 0$, if

$$\Delta_{i-1} \cup \{\phi_i\} \vdash P \,\&\, \neg P,$$

then

$$\Delta_i = \Delta_{i-1};$$

and if

$$\Delta_{i-1} \cup \{\phi_i\} \nvdash P \,\&\, \neg P,$$

then

$$\Delta_i = \Delta_{i-1} \cup \{\phi_i\}.$$

Let $\Delta = \bigcup_{i=0}^{\omega} \Delta_i$. The following lemmas indicate facts about $\Delta$ which will be useful in the proof to follow:

Lemma II $\Delta \nvdash P \,\&\, \neg P$.

*Proof:* For any finite subset $\Delta'$ of $\Delta$ there is a sentence $\phi_n \in \mathfrak{F}$ such that $\phi_n \in \Delta'$ and there is no later sentence of $\mathfrak{F}$ in $\Delta'$. For each such subset $\Delta'$ and respective $\phi_n$, $\Delta' \subseteq \Delta_n$, and $\phi_n \in \Delta_n = \Delta_{n-1} \cup \{\phi_n\}$, for $\Delta_n$ is the set of all sentences in $\Delta$ not later than $\phi_n$ and $\Delta' \subseteq \Delta$.

$\therefore$ For each such $\Delta'$ and $\phi_n$, $\Delta_n = \Delta_{n-1} \cup \{\phi_n\} \nvdash P \,\&\, \neg P$, by construction of $\Delta_n$.

$\therefore$ Since $\Delta' \subset \Delta_n$, by Theorem 4, $\Delta' \nvdash P \,\&\, \neg P$.

$\therefore$ No finite subset of $\Delta$ permits the derivation of $P \,\&\, \neg P$.

$\therefore$ $\Delta \nvdash P \,\&\, \neg P$.

Lemma III *For any sentence* $\phi$, $\phi \in \Delta$ *iff* $\Delta \vdash \phi$.

*Proof:* First assume $\phi \in \Delta$.
$\therefore$ $\{\phi\} \subseteq \Delta$.
But $\{\phi\} \vdash \phi$ by Premise.
$\therefore$ $\Delta \vdash \phi$ by Theorem 4.
Next assume $\Delta \vdash \phi$, but suppose $\phi \notin \Delta$. $\phi = \phi_i$ for some $\phi_i \in \mathfrak{F}$, and $\phi \in \Delta$ iff $\phi_i \in \Delta_i$.
$\therefore$ $\phi_i \notin \Delta_i$.
$\therefore$ $\Delta_{i-1} \cup \{\phi_n\} \vdash P \,\&\, \neg P$.
$\therefore$ $\Delta_{i-1} \vdash \neg\phi_n$, by Contradiction.
$\therefore$ $\Delta \vdash \neg\phi$, since $\phi = \phi_n$ and $\Delta_{i-1} \subseteq \Delta$, by Theorem 4.
$\therefore$ Since $\Delta \vdash \phi$, $\Delta \vdash \phi \,\&\, \neg\phi$, by Conjunction.
$\therefore$ $\Delta \vdash P \,\&\, \neg P$, by Theorem 2.
But $\Delta \nvdash P \,\&\, \neg P$, by Lemma II.
$\therefore$ If $\Delta \vdash \phi$, then $\phi \in \Delta$.

Lemma IV *For any sentence* $\phi$, $\phi \in \Delta$ *iff* $\neg\phi \notin \Delta$.

*Proof:* First assume $\phi \in \Delta$.
If $\neg\phi \in \Delta$ as well, then $\Delta \vdash \phi$ and $\Delta \vdash \neg\phi$, by Lemma III;
then $\Delta \vdash \phi \,\&\, \neg\phi$, by Conjunction;
then $\Delta \vdash P \,\&\, \neg P$, by Theorem 2.
$\therefore$ Since $\Delta \nvdash P \,\&\, \neg P$, by Lemma II, if $\phi \in \Delta$ then $\neg\phi \notin \Delta$.
Next assume $\phi \notin \Delta$.
$\phi = \phi_i$, for some $\phi_i \in \mathfrak{F}$.
$\therefore$ $\phi_i \notin \Delta_i$.
$\therefore$ $\Delta_{i-1} \cup \{\phi_i\} \vdash P \,\&\, \neg P$, by construction of $\Delta_i$.
$\therefore$ $\Delta_{i-1} \vdash \neg\phi_i = \neg\phi$, by Contradiction since $\phi = \phi_i$.
$\therefore$ $\Delta \vdash \neg\phi$, by Theorem 4 since $\Delta_{i-1} \subseteq \Delta$.
$\therefore$ $\neg\phi \in \Delta$, by Lemma III.
Let $\mathfrak{M} = \langle \mathfrak{D}, \mathfrak{f} \rangle$ be as follows:

$$\mathfrak{D} = \{x\colon\ x \text{ is a term in the reference sequence of some sentence in } \Delta\}$$

$\mathfrak{f}$ is a function such that for any term $\beta$, $\mathfrak{f}(\beta) = \beta$, and for any sentence $\phi$ whose predicate is $\pi^m$,

$$\mathfrak{f}(\pi^m) = \{\bar{\delta} \in \mathfrak{D}^m\colon \text{ there is a sentence in } \Delta \text{ whose predicate is } \pi^m \text{ and whose reference sequence is } \delta\}.$$

Lemma V $\mathfrak{M}$, *so defined, is a model.*

*Proof:* Let $\mathfrak{M}$ be so defined. Then $\mathfrak{D}$ is a non-empty set, and for every term $\beta$, $\mathfrak{f}(\beta) \in \mathfrak{D}$. Therefore it will suffice to complete the proof of the lemma to show that for every sentence whose predicate is $\pi^m$, $\mathfrak{f}(\pi^m)$ conforms to the definition of a model.
Let $\phi$ be a sentence whose predicate is $\pi^m$.
A. Let $\pi^m$ be an elementary predicate. By construction of $\mathfrak{M}$, $\mathfrak{f}(\pi^m) = \{\bar{\delta} \in \mathfrak{D}^m$: there is a sentence in $\Delta$ whose predicate is $\pi^m$ and whose reference sequence is $\delta\}$.

$\therefore\ \mathfrak{f}(\pi^m) \subseteq \mathfrak{D}^m$.
$\therefore\ \mathfrak{M}$ conforms to the definition of models with respect to elementary predicates.
B. Let $\pi^m = \ulcorner(\rho_0^{n_0})\rho_1^{n_1} \ldots \rho_k^{n_k}\urcorner$, where $\rho_0^{n_0}$ is a predicate and $\rho_1^{n_1}, \ldots, \rho_k^{n_k}$ are modifiers of atomic modifying phrases, and where $m = \sum_{i=0}^{k} n_i$.
Let $\overline{\delta} \in \mathfrak{f}(\pi^m)$. By construction of $\mathfrak{M}$, there is a sentence $(\psi_0)\psi_1 \ldots \psi_k \in \Delta$ whose predicate is $(\rho_0^{n_0})\rho_1^{n_1} \ldots \rho_k^{n_k}$ and whose reference sequence is $\delta$, where $\psi_0$ is a sentence whose predicate is $\rho_0^{n_0}$ and whose reference sequence is $\delta_0$, where $\psi_1, \ldots, \psi_k$ are modifying phrases whose respective modifiers are $\rho_1^{n_1}, \ldots, \rho_k^{n_k}$, and whose respective reference sequences are $\delta_1, \ldots, \delta_k$, and where $\delta = \delta_0 \wedge \ldots \wedge \delta_k$.
$\therefore\ \Delta \vdash (\psi_0)\psi_1 \ldots \psi_k$, by Lemma III.
$\therefore\ \Delta \vdash (\psi_0)\psi_1 \ldots \psi_{k-1}$, by Theorem 5.
$\therefore$ Moreover, if $\psi_k = (\chi)\sigma$, where $\chi$ is a modifying phrase whose predicate is $\chi'^{m_0}$ and whose reference sequence is $\gamma_0$, and where $\sigma$ is an intensifying phrase whose intensifier is $\sigma'^{m_1}$ and whose reference sequence is $\gamma_1$, and where $\delta_k = \gamma_0 \wedge \gamma_1$, then $\Delta \vdash (\psi_0)\psi_1 \ldots \psi_{k-1}\chi$, by Deletion of Intensifiers.
$\therefore$ Moreover, for each $i$, $1 \leqslant i < k$, $\Delta \vdash ((\psi_0)\psi_1 \ldots \psi_k)/i$, by Reordering.
$\therefore$ By Lemma III, $(\psi_0)\psi_1 \ldots \psi_{k-1} \in \Delta$; for each $i$, $1 \leqslant i < k$, $((\psi_0)\psi_1 \ldots \psi_k)/i \in \Delta$; and if $\psi_k = (\chi)\sigma$, as above, then $(\psi_0)\psi_1 \ldots \psi_{k-1}\chi \in \Delta$.
$\therefore$ By construction of $\mathfrak{M}$,

$$\overline{\delta}_0 \wedge \ldots \wedge \overline{\delta}_{k-1} \in \mathfrak{f}((\rho_0^{n_0})\rho_1^{n_1} \ldots \rho_{k-1}^{n_{k-1}});$$

for each $i$, $1 \leqslant i < k$,

$$\overline{\delta}_0 \wedge \ldots \wedge \overline{\delta}_{i-1} \wedge \overline{\delta}_{i+1} \wedge \overline{\delta}_i \wedge \overline{\delta}_{i+2} \wedge \ldots \wedge \overline{\delta}_k \in \mathfrak{f}((\rho_0^{n_0})\rho_1^{n_1} \ldots \rho_{i-1}^{n_{i-1}}\, \rho_{i+1}^{n_{i+1}}\, \rho_i^{n_1}\, \rho_{i+2}^{n_{i+2}} \ldots \rho_k^{n_k});$$

and if $\psi_k = (\chi)\sigma$, as above, then

$$\overline{\delta}_0 \wedge \ldots \wedge \overline{\delta}_{k-1} \wedge \overline{\gamma}_0 \in \mathfrak{f}((\rho_0^{n_0})\rho_1^{n_1} \ldots \rho_{k-1}^{n_{k-1}}\chi'^{m_0}).$$

$\therefore\ \mathfrak{f}(\pi^m) \subseteq \mathfrak{f}((\rho_0^{n_0})\rho_1^{n_1} \ldots \rho_{k-1}^{n_{k-1}}) \wedge \mathfrak{D}^{n_k}$, such that for each $i$ $1 \leqslant i < k$, and for each sequence $\overline{\delta}_0 \wedge \ldots \wedge \overline{\delta}_k \in \mathfrak{f}(\pi^m)$, where each $\overline{\delta}_i$ is an $n_i$-tuple,

$$\overline{\delta}_0 \wedge \ldots \wedge \overline{\delta}_{i-1} \wedge \overline{\delta}_{i+1} \wedge \overline{\delta}_i \wedge \overline{\delta}_{i+2} \wedge \ldots \wedge \overline{\delta}_k \in \mathfrak{f}((\rho_0^{n_0})\rho_1^{n_1} \ldots \rho_{i-1}^{n_{i-1}}\, \rho_{i+1}^{n_{i+1}}\, \rho_i^{n_i}\, \rho_{i+1}^{n_{i+1}} \ldots \rho_k^{n_k});$$

and such that if $\rho_k = (\chi'^{m_0})\sigma'^{m_1}$, where $\chi'^{m_0}$ is the modifier of some modifying phrase and $\sigma'^{m_1}$ is an intensifier, then $\mathfrak{f}(\pi^m) \subseteq \mathfrak{f}((\rho_0^{n_0})\rho_1^{n_1} \ldots \rho_{k-1}^{n_{k-1}}\chi'^{m_0}) \wedge \mathfrak{D}^{m_1}$.
$\therefore\ \mathfrak{M}$ conforms to the definition of models with respect to modified predicates.
C. Let $\pi^m = \ulcorner(\rho^m)\neg\urcorner$. $\overline{\delta} \in \mathfrak{f}(\pi^m) = \mathfrak{f}((\rho^m)\neg)$ iff there is a sentence $\neg\psi \in \Delta$ whose predicate is $(\rho)\neg$ and whose reference sequence is $\delta$, by construction of $\mathfrak{M}$; i.e., iff there is no sentence $\psi \in \Delta$ whose predicate is $\rho$ and whose reference sequence is $\delta$, Lemma IV; i.e., iff $\overline{\delta} \notin \mathfrak{f}(\rho)$, by construction of $\mathfrak{M}$.
$\therefore\ \mathfrak{f}(\pi^m) = \mathfrak{f}((\rho^m)\neg) = \mathfrak{D}^m \sim \mathfrak{f}(\rho)$, since $\overline{\delta} \in \mathfrak{D}^m$.
$\therefore\ \mathfrak{M}$ conforms to the definition of models with respect to predicate complements.

D. Let $\pi^m = \ulcorner((\rho_1^{n_1}\rho_2^{n_2})\&)\urcorner$. $\bar{\delta} \in \mathfrak{f}(\pi^m)$ iff there is a sentence $(\psi_1 \& \psi_2) \in \Delta$ whose predicate is $(\rho_1^{n_1}\rho_2^{n_2})\&$ and whose reference sequence is $\delta = \gamma_1 \wedge \gamma_2$, where $\gamma_1$ is an $n_1$-tuple and $\gamma_2$ is an $n_2$-tuple; i.e., iff there are sentences $\psi_1 \in \Delta$ and $\psi_2 \in \Delta$ whose respective predicates are $\rho_1^{n_1}$ and $\rho_2^{n_2}$ and whose respective reference sequences are $\gamma_1$ and $\gamma_2$, by Lemma III in virtue of Simplification, Commutation, and Conjunction; i.e., iff $\bar{\gamma}_1 \in \mathfrak{f}(\rho_1^{n_1})$ and $\bar{\gamma}_2 \in \mathfrak{f}(\rho_2^{n_2})$.
$\therefore$ $\mathfrak{f}(\pi^m) = \mathfrak{f}((\rho_1^{n_1}\rho_2^{n_2})\ \&) = \mathfrak{f}(\rho_1^{n_1}) \wedge \mathfrak{f}(\rho_2^{n_2})$.
$\therefore$ $\mathfrak{M}$ conforms to the definition of models with respect to adjunctive predicates.
E. Let $\pi^m = (\rho^n)(\forall i_1 \ldots i_k)$, where $m = n - k$, and where $\langle i_1, \ldots, i_k\rangle$ and $\langle j_1, \ldots, j_{n-k}\rangle$ are complementary subsequences of $\langle 1, \ldots, n\rangle$. Let $\bar{\delta} = \langle \bar{\alpha}_{j_1}, \ldots, \bar{\alpha}_{j_{n-k}}\rangle \in \mathfrak{f}(\pi^m) = \mathfrak{f}((\rho^n)(\forall i_1, \ldots, i_k))$. Then there is a sentence $(\forall i_1, \ldots, i_k)\psi \in \Delta$ whose predicate is $(\rho^n)(\forall i_1, \ldots, i_k)$ and whose reference sequence is $\delta$. By Lemma III, $\Delta \vdash (\forall i_1, \ldots, i_k)\psi$.
$\therefore$ By Instantiation, $\Delta \vdash (\psi)\beta/i_1, \ldots, i_k$, for each term $\beta$.
$\therefore$ By Lemma III, for each $\beta$, $(\psi)\beta/i_1, \ldots, i_k \in \Delta$.
$\therefore$ By construction of $\mathfrak{M}$, since for each term $\beta$, $\bar{\beta} = \beta$, and by definition of $\ulcorner\beta/i_1, \ldots, i_k\urcorner$, for each term $\beta = \beta_{i_1} = \ldots = \beta_{i_k} \in \mathfrak{D}$, $\langle \bar{\alpha}'_1, \ldots, \bar{\alpha}_n\rangle \in \mathfrak{f}(\rho^n)$, where for each $i \in \langle i_1, \ldots, i_k\rangle$, $\bar{\alpha}'_i = \bar{\beta}$, and for each $j \in \langle j_1, \ldots, j_{n-k}\rangle$, $\bar{\alpha}'_j = \bar{\alpha}_j$.
$\therefore$ Since $\mathfrak{D}$ is the set of terms, $\bar{\delta} \in \{\langle x_{j_1}, \ldots, x_{j_{n-k}}\rangle$: for each $x = x_{i_1} = \ldots = x_{i_k} \in \mathfrak{D}$, $\langle x_1, \ldots, x_n\rangle \in \mathfrak{f}(\rho^n)\}$.
Next let $\bar{\delta} = \langle \bar{\alpha}_{j_1}, \ldots, \bar{\alpha}_{j_{n-k}}\rangle \notin \mathfrak{f}(\pi^m) = \mathfrak{f}((\rho^n)(\forall i_1, \ldots, i_k))$. Then there is no sentence $(\forall i_1, \ldots, i_k)\psi \in \Delta$ whose predicate is $(\rho^n)(\forall i_1, \ldots, i_k)$ and whose reference sequence is $\Delta$.
$\therefore$ By Lemma IV, $\neg(\forall i_1, \ldots, i_k)\psi \in \Delta$.
But $\neg(\forall i_1, \ldots, i_k)\psi = \phi_i$ for some $\phi_i \in \mathfrak{F}$. By construction of $\mathfrak{F}$, $\phi_{i+1} = (\neg\psi)\beta/i_1, \ldots, i_k$, where $\beta$ is a term not in the reference sequence of any sentence in $\Gamma$ or of any previous sentence in $\mathfrak{F}$, and therefore not in the reference sequence of any sentence in $\Delta_i$.
If $\Delta_i \cup \{\phi_{i+1}\} = \Delta_i \cup \{(\neg\psi)\beta/i_1, \ldots, i_k\} \vdash P \& \neg P$, then $\Delta_i \vdash (\neg\neg\psi)\beta/i_1, \ldots, i_k$, by Contradiction; then $\Delta_i \vdash (\psi)\beta/i_1, \ldots, i_k$, by Theorem 1; then $\Delta_i \vdash (\forall i_1, \ldots, i_k)\psi$, by Generalization; then $\Delta \vdash (\forall i_1, \ldots, i_k)\psi$, by Theorem 4; then $(\forall i_1, \ldots, i_k)\psi \in \Delta$, by Lemma III.
But $(\forall i_1, \ldots, i_k)\psi \notin \Delta$, by Lemma IV.
$\therefore$ $\Delta_i \cup \{(\neg\psi)\beta/i_1, \ldots, i_k\} \nvdash P \& \neg P$.
$\therefore$ $\Delta_{i+1} = \Delta_i \cup \{(\neg\psi)\beta/i_1, \ldots, i_k\}$, by construction of $\Delta_{i+1}$.
$\therefore$ $(\neg\psi)\beta/i_1, \ldots, i_k \in \Delta$, by construction of $\Delta$.
$\therefore$ $(\psi)\beta/i_1, \ldots, i_k \notin \Delta$, by Lemma IV.
$\therefore$ There is a term $\beta = \beta_{i_1} = \ldots \beta_{i_k} \in \mathfrak{D}$ such that $\langle \bar{\alpha}'_1, \ldots, \bar{\alpha}_n\rangle \notin \mathfrak{f}(\rho^n)$, where $\langle \bar{\alpha}'_{i_1}, \ldots, \bar{\alpha}'_i\rangle = \langle \bar{\beta}_{i_1}, \ldots, \bar{\beta}_{i_k}\rangle$ and where $\langle \bar{\alpha}'_{j_1}, \ldots, \bar{\alpha}_{j_{n-k}}\rangle = \delta$.
$\therefore$ $\delta \notin \{\langle x_{j_1}, \ldots, x_{j_{n-k}}\rangle$: for any $x = x_{i_1} = \ldots = x_{i_k} \in \mathfrak{D}$, $\langle x_1, \ldots, x_n\rangle \in \mathfrak{f}(\rho^n)\}$.
$\therefore$ $\mathfrak{f}(\pi^m) = \mathfrak{f}((\rho^n)(\forall i_1, \ldots, i_k)) = \{\langle x_{j_1}, \ldots, x_{j_{n-k}}\rangle \in \mathfrak{D}$: for any $x_{i_1} = \ldots = x_{i_k} \in \mathfrak{D}$, $\langle x_1, \ldots, x_n\rangle \in \mathfrak{f}(\rho^n)\}$.
$\therefore$ $\mathfrak{M}$ conforms to the definition of models with respect to general predicates.
Therefore, $\mathfrak{M}$ is a model.

Since $\mathfrak{M}$ is a model, it follows immediately that $\mathfrak{M} \models \Delta$ and therefore $\mathfrak{M} \models \Gamma$, since $\Gamma \subseteq \Delta$: Let $\phi \in \Delta$, where $\pi^m$ is the predicate of $\phi$ and $\delta$ is the reference sequence of $\phi$. By construction of $\mathfrak{M}$, $\mathfrak{f}(\pi^m) = \{\delta'$: there is a sentence $\phi' \in \Delta$ whose predicate is $\pi^m$ and whose reference sequence is $\delta\}$, and $\mathfrak{f}(\delta) = \delta$. But $\phi$ itself is such a sentence. Therefore, $\mathfrak{f}(\delta) \in \mathfrak{f}(\pi^m)$. Therefore, $\mathfrak{M} \models \phi$.

By Theorem 3, for any set $\Gamma$ and sentence $\phi$, $\Gamma \vdash \phi$ iff $\Gamma \cup \{\neg \phi\} \vdash P \,\&\, \neg P$. Therefore if $\Gamma \nvdash \phi$, then $\Gamma \cup \{\neg\phi\} \nvdash P \,\&\, \neg P$. Therefore, by the Consistency Theorem, if $\Gamma \nvdash \phi$, there is a model $\mathfrak{M}$ such that $\mathfrak{M} \models \Gamma \cup \{\neg \phi\}$. Therefore, if $\Gamma \nvdash \phi$, then $\Gamma \nVdash \phi$. Therefore, if $\Gamma \Vdash \phi$, then $\Gamma \vdash \phi$; and the deductive completeness of $\mathfrak{A}$ is established.

*Intensionality* The purpose of the construction of $\mathfrak{A}$ was to develop a formal language adequate not only to the inferences of the predicate calculus but also to those based on the presence of adverbs, prepositional phrases, and attributive adjectives in sentences of the natural language. As remarked in the introduction, such a system must be such that the formal counterparts of these linguistic entities formed *intensional contexts*: they must not in general permit substitution *salva veritate* of co-extensive predicates. In this section I shall show that $\mathfrak{A}$ is intensional in the required sense.

Let $\mathfrak{M} = \langle \mathfrak{D}, \mathfrak{f} \rangle$ be a model, let $\mathfrak{F}$ be an elementary predicate of degree 1, let a be a term, and let G be an elementary modifier of degree 0, such that

$$\mathfrak{D} = \{1, 2, 3\};$$
$$\mathfrak{f}(a) = 1;$$
$$\mathfrak{f}(F) = \{\langle 1\rangle, \langle 2\rangle\} \subseteq \mathfrak{D}^1;$$
$$\mathfrak{f}((F)G) = \{\langle 1\rangle\} \subseteq \mathfrak{f}(F) \wedge \mathfrak{D}^0;$$
$$\mathfrak{f}((F)\neg) = \{\langle 3\rangle\} = \mathfrak{D}^1 \sim \mathfrak{f}(F);$$
$$\mathfrak{f}(((F)\neg)\neg) = \{\langle 1\rangle, \langle 2\rangle\} = \mathfrak{D}^1 \sim \mathfrak{f}((F)\neg) = \mathfrak{f}(F);$$
$$\mathfrak{f}((((F)\neg)\neg)G) = \{\langle 2\rangle\} \subseteq \mathfrak{f}(((F)\neg)\neg) \wedge \mathfrak{D}^0.$$

The reference sequence of the sentence 'Fa' is $\langle a \rangle$; therefore $\mathfrak{f}(\langle a\rangle) = \langle \mathfrak{f}(a)\rangle = \langle 1 \rangle \in \mathfrak{f}(F)$, and $\mathfrak{M} \models$ Fa. The reference sequence of '(Fa)G' is also $\langle a \rangle$; and since $\langle 1 \rangle \in \mathfrak{f}((F)G)$, $\mathfrak{M} \models$ (Fa)G. But $\langle a \rangle$ is also the reference sequence of '$((\neg\neg$Fa)G)'; and since $\langle 1 \rangle \notin \mathfrak{f}((((F)\neg)\neg)G)$, $\mathfrak{M} \nvDash (\neg\neg$Fa)G, even though 'Fa' and '$\neg\neg$Fa' are logically equivalent, indeed, even though 'F' and '$((F)\neg)\neg$' must have identical extensions. Since the substitution of '$\neg\neg$Fa' for the logically equivalent 'Fa' in the context of the modifier 'G' changes the truth value of the sentence, that context is intensional.[3]

Although this is but one example, it indicates the general reason for the intensionality of $\mathfrak{A}$. The initial segment of each member of the extension of a modified predicate is itself a member of the extension of the predicate which is modified. In the limiting case of a modifier of degree 0, the extension of the complex predicate is a subset of the extension of the component predicate. But the semantic rules of $\mathfrak{A}$ do not require a modifier to pick the same subset when applied to distinct but co-extensive predicates. The results predicted by van Fraassen have been avoided by making predicate modifiers *syncategorematic*: The syntactic operation forming sentences from sentences, which is the function of modifying phrases, does not have as an analogue in the semantics an operation forming extensions from extensions (much less truth values from truth values). Nonetheless, each complex predicate, i.e., each predicate formed by applying a modifier to a component predicate, has an extension which is restricted by the extension of the component predicate. Moreover, these restrictions are sufficient to produce a picture of the truth conditions of sentences formed from these predicates and the entailment relations among these sentences.

It is of course one thing to show that non-logical constants can be introduced into a formal system as syncategorems with an adequate semantic structure; it is quite another to assert of expressions in the natural language that they are syncategorematic. In order to reinforce my suggestion that predicate modifiers behave in this way, let me offer for consideration the following sentences

*i.* That was a short speech.
*ii.* That was a short cigar.
*iii.* That was an ugly speech.
*iv.* That was an ugly cigar.

It would certainly be false to say that either 'short' or 'ugly' has no *meaning* apart from its application to some noun. Speakers of English know what kinds of characteristics to expect of a thing that has been described as 'short' or 'ugly' (for things of that kind), even though they may never have heard those adjectives before conjoined with the nouns used to name those things. But these expectations may result not from our knowledge of general rules for the use of attributive adjectives (or even more general rules for predicate modifiers), but rather from our knowledge of the specific rules of application of 'short' or 'ugly'. If the similarity of function between the uses of 'short' in sentences *i* and *ii*, and between the uses of 'ugly' in *iii* and *iv*, results from the former general rules, then it ought to be the case that *i* was to *ii* as *iii* was to *iv*; i.e., if we knew how to determine which cigars were short from our knowledge of which speeches were short, then we ought to be able to determine which cigars were ugly from our knowledge of which speeches were ugly without any further knowledge of the specific meaning of 'ugly'. But this is surely not possible. The sense in which 'short' in *i* is similar in meaning to 'short' in *ii* is not the same sense in which 'ugly' in *iii* is similar in meaning to 'ugly' in *iv*. The respective similarities result from the *content* of the words. Thus a formal system intended to explicate the general function of predicate modifiers can legitimately treat them as syncategorems.

*Application to the natural language* In order to illustrate the intended mode of application of $\mathfrak{A}$ to the natural language, I shall use a variant of a widely discussed example employed by Davidson (cf. [2], pp. 81-84): "Jones buttered the toast slowly, deliberately, in the bathroom, with a knife, at midnight." In order to simplify discussion by avoiding irrelevant quantification, I shall replace the 'a' preceding 'knife' with 'the', thereby permitting the use of a term. I shall also omit the word 'deliberately', not because I believe that words which express *intentions* have a grammar precluding their interpretation into $\mathfrak{A}$, but because that interpretation, because of their specific content, would require for adequate treatment more space than is presently available. Unlike Davidson, however, I shall leave the word 'slowly' in the example, for it is easily explicated by $\mathfrak{A}$. Indeed, I shall modify 'slowly' with the adverb 'very'. Thus the subject of this illustration will be the sentence

(A) Jones buttered the toast very slowly, in the bathroom, with the knife, at midnight.

The first level of analysis of (A) would be to translate it into the following sentence of $\mathfrak{A}$:

($A_1$) $(Bjt)(S)\,VIbWkAm$

whose predicate is $(B^2)(S^0)V^0I^1W^1A^1$ where '$V^0$' is an intensifier, and whose reference sequence is $\langle j, t, b, k, m\rangle$. Since the symbols of ($A_1$) are the initials of their syntactic correlates in (A) and thus indicate the intended assignments, I shall not explicitly state those assignments. From ($A_1$) the rule of Deletion of Modifiers yields

($B_1$) $(Bjt)(S)\,VIb\,Wk$

while the rule of Reordering yields progressively

($C_1$) $(Bjt)Ib(S)\,VWkAm$
($D_1$) $(Bjt)Ib\,Wk(S)\,VAm$
($E_1$) $(Bjt)Ib\,WkAm(S)\,V$.

From ($E_1$) the rule of Deletion of Intensifiers yields

($F_1$) $(Bjt)Ib\,WkAmS$.

Translating these back into English using the same principles of translation that produced ($A_1$) from (A), we obtain

(B) Jones buttered the toast very slowly in the bathroom with the knife.
(C) Jones buttered the toast in the bathroom, very slowly, with the knife, at midnight.
(D) Jones buttered the toast in the bathroom, with the knife, very slowly, at midnight.
(E) Jones buttered the toast in the bathroom, with the knife, at midnight, very slowly.
(F) Jones buttered the toast in the bathroom, with the knife, at midnight, slowly.

Each of these is an intuitively acceptable, although perhaps somewhat oddly ordered, consequence of (A).

Moreover, it is a consequence of the rules that any of the modifying phrases of ($A_1$) can be deleted by repeated application of the rule of Reordering until the phrase is in last place followed by an application of the appropriate rule of Deletion. Thus the following is an illustrative but not exhaustive list of formal implications of ($A_1$):

($G_1$) $(Bjt)(S)VAm$
($H_1$) $(Bjt)SIb$
($J_1$) $Bjt$

corresponding respectively to

(G) Jones buttered the toast very slowly at midnight.
(H) Jones buttered the toast slowly in the bathroom.
(J) Jones buttered the toast.

Each of these is an intuitive consequence of (A).

There are, however, intuitive consequences of (A) whose formal correlates according to the above principles of translation cannot be derived from ($A_1$):

(K) The toast was buttered in the bathroom.
(L) Jones buttered at midnight.
(M) Buttering was done in the bathroom.
(N) The knife was used very slowly.

Are these *formal* implications of (A)? The answer to that question depends on the rules of grammar. For example, the passive transformation on (A) yields

(O) The toast was buttered by Jones very slowly, in the bathroom, with the knife, at midnight.

A natural translation of this into $\mathfrak{A}$ would render the main predicate as one-place and put the translation of 'Jones' into a deletable phrase, enabling us to derive (K). If the rules of grammar contain transformations which when applied to (A) yield the sentence

(A′) An act of buttering was done by Jones, to the toast, very slowly, in the bathroom, using the knife, at midnight.

then the natural translation of this sentence would be

($A'_2$) $(A)BDjTt(S)VibUkWm$.

Here $A$ would effectively be a "dummy" sentence corresponding perhaps to the English 'There was an act'. $B$ would indicate that it was a particular kind of act: a *buttering*. $D$ would translate 'done by'. The remaining correlations are straightforward. Given this formal sentence we can derive by the rules of $\mathfrak{A}$ the formal correlates of the sentences (K) through (N).

A possibly interesting extension would be to attach certain grammatical categories to particular modifiers. Thus $D$ in the above would not be a specific translation of 'done by' but a universal *subject-indicator*; $T$, a *direct-object-indicator*; *U*, an *instrument-indicator*; *W*, a *time-indicator*; *I*, a *location-indicator*. We could also have an *indirect-object-indicator* and an *objective-complement-indicator*. This is of course all speculation at this point.

As this example suggests, if one's rules of grammar are sufficiently powerful, inferences of a wide variety can be formally justified by analysis within $\mathfrak{A}$. Whether they ought to be so justified depends on whether one takes them to be formal inferences or inferences based on the content of words. $\mathfrak{A}$ is a semantic structure which does not presuppose a particular set of rules of syntactic interpretation.

*Clark's categories of modifiers* The system $\mathfrak{A}$ has been developed as a means of extending the possible applications of a method proposed by Romane Clark (see *Introduction*). In his paper Clark was attempting to achieve some specific goals; i.e., he wanted his modifiers to be adequate to certain tasks:

> Let *M* be any pred-mod [= modifying phrase with a null reference sequence in $\mathfrak{A}$] and *P* any predicate. Consider then the possible ways in which the extensions, $^*MP$ and $^*P$, associated with the predicates *MP* and *P*, may be related. [Footnote omitted.] These cases will be helpful soon in considering how to specify semantic interpretations for complex predicates which embody the various kinds of pred-mods. Let '$\lambda$' be the name of the null-set. Relative to *P*, we have the following possibilities.
>
> Case 1. Modifiers which create a predicate with an extension identical to that of the predicate which came to be modified: $^*MP = {}^*P$. This divides into two further cases, as the extensions are empty or not.
>
> A. $^*MP = {}^*P = \lambda$. Example: vicious unicorn; unicorn. Type: standard.
> B. $^*MP = {}^*P \neq \lambda$. Example: extended surface; surface. Type: standard.
>
> Case 2. Modifiers which create a predicate the extension of which is distinct from that of the predicate which came to be modified: $^*MP \neq {}^*P$. Here there are a range of sub-cases: those in which one of the extensions is null; those in which one of the extensions includes the other; those in which neither of these things is the case.
> A. $^*MP = \lambda. {}^*P \neq \lambda$. Example: mythical beast; beast. Kind: fictionalizer.
> B. $^*MP \neq \lambda.\ {}^*P = \lambda$. Example: simulated (mock, artificial) griffin; griffin. Kind: Defictionalizer.
> C. $^*MP \neq \lambda\ \ {}^*P \neq \lambda$. Neither extension empty. Sub-cases then are:
> i) $^*MP \subset {}^*P$. Example: male clerk; clerk. Kind: standard.
> ii) $^*P \subset {}^*MP$. Example: possible addict; addict. Kind: enlarger.
> iii) Neither extension includes the other. Then the overlap of the extensions may or may not be empty.
> a) $^*MP \cap {}^*P = \lambda$. Example: fake Ming vase; Ming vase. Kind: negators.
> b) $^*MP \cap {}^*P \neq \lambda$. Example: alleged thief; thief. Kind: neutralizer.
>
> Evidently, the manner of classifying these distinct modifiers into distinct kinds anticipates that distinct inferential powers will be assigned to predicates of different kinds ([1], pp. 328-329).

This last remark seems to enforce the stronger of two possible interpretations of the various relations between $*MP$ and $*P$, i.e., between the extension of a complex predicate and the extension of the component predicate: The weaker interpretation would be that the relation is the result of contingent facts about this world; the stronger, that the relation is the result of grammatical requirements, the logical function, of the respective modifiers. This stronger interpretation seems problematic, even if Clark's method of typing is otherwise acceptable.

The "standard" modifiers—1-A, 1-B, and 2-C-ii—present no difficulties, either for interpretation of grammatical function or for interpretation into $\mathfrak{A}$: a 0-ary modifier applied to a predicate with a null extension will yield a predicate with a null extension; applied to a predicate with a non-null extension, it will yield a predicate whose extension is a subset of the former and may be identical with it. It may also produce an empty extension; thus "fictionalizers" in a sense may be included under this heading. However, under the strong interpretation of Clark's procedure, this may be impermissible: he may intend a fictionalizer necessarily to produce an empty extension (if this is his intention, I wonder whether his categorization ought not also to include modifiers which necessarily produce extensions identical to those on which they operate). Such a fictionalizer would seem to have little use, however; the sentence produced would necessarily be false unless the system is endowed with an alternative set of truth-conditions for non-referring terms, i.e., unless the system is made into a "free logic". But if one is serious about avoiding ontological commitment or about interpreting fictional entities, then it is not clear that the mere addition of a logical fictionalizer will be at all helpful. Therefore, I suggest that fictionalizers be subsumed, as they can under the weaker interpretation, under the category of standard modifiers.

Consider next the "defictionalizers" and the "negators": 2-B and 2-C-iii-a. Are these really distinct kinds of modifiers? Both produce extensions disjoint from those on which they operate. The only difference is that in one case the component extension is empty and in the other it is not. Certainly the English modifier of the example of a defictionalizer does not necessarily apply to empty predicates. Thus I suggest that defictionalizers be subsumed under the category of negators. This last class, in turn, can be analyzed as a combination of a standard modifier (in Clark's sense) and complementation (in the sense of $\mathfrak{A}$). That is, the fake Ming vases will compose a subset of the things that are not Ming vases; the simulated griffins will form a subset of the things that are not griffins. I am not sure this analysis is quite adequate to the usage of "impostor-words". Isn't a fake Ming vase *necessarily* a vase; isn't a simulated pearl necklace *necessarily* a necklace? I do not know the answer to the first of these questions (consider the solid vase-like form at Madame Tussaud's); the second is ambiguous: is it a necklace made of simulated pearls or a thing made to simulate a pearl necklace. In the latter case, like the case of the vase, I am inclined toward a negative answer; but even if the answer were positive, I do not think it would be on *grammatical* grounds. In the case of the necklace of simulated pearls, the treatment would be that of a subset of the necklaces which are not pearl necklaces: "$x$ is a simulated pearl necklace" would be translated with the sentence '$(Nx \mathbin{\&} \neg(Nx)P)S$'.

While it is true that there are complex predicates whose extensions stand to the extensions of their component predicates according to the requirements of Clark's category of "neutralizers", the very fact of this relation suggests that the conditions of application of the complex predicate should not be defined in terms of the conditions of application of the component predicate. The same seems to be true of "enlargers": if a complex predicate can apply whether or not the component predicate applies, then the usual recursive semantics is no longer appropriate.

$\mathfrak{A}$ does suggest some possible ways of developing these conditions. For example, we might define 'possible addict', Clark's example of an enlarger, as 'not necessarily a non-addict', i.e., as a combination of complementation with a standard modifier (which of course has important and formalizable lexical content). Another possibility is to treat *that-clauses* as modifiers of intentional verbs (or alternatively as intensifiers of intentional modifiers). For example, is not "Jones is a suspected criminal" equivalent to "Jones is such that someone suspects that he is a criminal"? Such a treatment would enable us to derive "Jones is suspected" from "Jones is a suspected criminal", which seems desirable. Again, this is mere speculation. In any case, to the extent that Clark's categories represent genuine grammatical operations, they seem to be interpretable into $\mathfrak{A}$.

*Conclusion* This paper is intended only as a tentative starting-point, a suggestion about how predicate modifiers might be treated in a semantic structure without presupposing any particular kind of primitive ontology. There are several areas where further investigation is in order, some of which have been indicated in the preceding sections.

In the first place, $\mathfrak{A}$ is only a special case of a general theory of modifications on extensions. Such a general theory would require not only the development of a syntax whose rules of composition corresponded to the rules of grammar of the natural language in a way that permitted relatively unambiguous rules of translation, it would also require a much more detailed examination of the formal characteristics of the structure, for example, of the relation between *logical co-extension and logical equivalence.*

$\mathfrak{A}$ also requires the introduction of the formal concepts of identity and functions. For one thing, these are essential for a general thesis about prepositional phrases, for these often serve—as in 'the father of John'—as modifiers of a referring expression rather than as modifiers of the predicate. For another, they are needed so that the various alternatives regarding *term-intensionality*—the failure of substitutivity of putative referring expressions in, for example, *intentional* contexts—can be expressed and examined.

Finally, since this method has the effect of making events into "second-order entities", of assimilating them to predicates, the fact that we quantify over events and actions in the natural language implies that the set-theoretic details of quantification over sets and subsets of sequences be investigated.

Whether this effort is worthwhile depends partly on whether there are any general objections to this methodology. It is for the purpose of eliciting those objections that it is being offered in its present form. For example, it is not clear to me how it stands with respect to Harman's criteria ([3], pp. 41-42). As $\mathfrak{A}$ stands it requires three rules of inference beyond those necessary to accommodate the predicate calculus. I believe that a general theory of modification along these lines will yield the operations of the predicate calculus as special cases of the operations of modifiers, but this remains to be seen. On the other hand, $\mathfrak{A}$ has no ontological commitment beyond that of the predicate calculus, and even that might be pared away by incorporating principles of free logic. It is somewhat premature to evaluate the method with respect to syntactic evidence or the need for non-logical axioms to accommodate inferential features of our language. Further testing is in order; and it is yet unclear whether syntactic theory can provide hard evidence without presupposing a semantic structure. Thus $\mathfrak{A}$ is offered quite tentatively as a somewhat less cumbersome method of handling adverbial modification than its modal alternatives.

## NOTES

1. The "logical form" in question, just as the validity of the various inferences mentioned, is intuitive. What is at stake is how to formalize these intuitions.

2. The method of modal logic has virtually exploded into the literature. In addition to van Fraassen its advocates include the late R. Montague [5]; T. Parsons [6]; and R. H. Thomason and R. C. Stalnaker [7].

3. It should be noted that *equivalence* and *logical equivalence* are respectively distinct concepts from *co-extension* and *logical co-extension*. $\mathfrak{A}$ is not like the ordinary predicate calculus in which sentences have truth values as extensions, where identity of sentential extension is necessary and sufficient for equivalence. Rather, in $\mathfrak{A}$ the co-extensiveness of two predicates is sufficient but not necessary for the equivalence of the sentences formed from them as co-extensive reference sequences. For example, the sentences

$$Pa \vee Qb$$

and

$$(Pa \,\&\, Qb) \vee ((Pa \,\&\, \neg Qb) \vee (\neg Pa \,\&\, Qb))$$

are logically equivalent but have predicates and reference sequences with distinct extensions: the former having a predicate assigned a set of pairs; the latter, a set of sextuples. I shall not devote any further space here to the relation between equivalence and co-extension, but it is an issue that a formal theory of this type must develop.

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*Wichita State University*
*Wichita, Kansas*