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AN INTERFACE TRACKING ALGORITHM FOR THE POROUS MEDIUM EQUATION[1]

BY

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ABSTRACT. We study the convergence of a finite difference scheme for the Cauchy problem for the porous medium equation $u_t = (u^m)_{xx}$, $m > 1$.

The scheme exhibits the following two features. The first is that it employs a discretization of the known interface condition for the propagation of the support of the solution. We thus generate approximate interfaces as well as an approximate solution.

The second feature is that it contains a vanishing viscosity term. This term permits an estimate of the form $\|(u^{m-1})_{xx}\|_{1,\mathbf{R}} \leqslant c/t$.

We prove that both the approximate solution and the approximate interfaces converge to the correct ones.

Finally error bounds for both solution and free boundaries are proved in terms of the mesh parameters.

1. Introduction. In this paper we derive and analyze a finite difference scheme for computing both the solution and the interfaces for the porous medium equation in one space dimension. We demonstrate that the approximate solutions and the approximate interface curves converge to the correct ones, and we obtain L^∞ bounds for the error in terms of the mesh parameter.

Consider the laminar flow of a polytropic fluid of density $(x, t) \to u(x, t)$ in a porous medium which is assumed to occupy the whole space, and suppose that at time $t = 0$ the fluid is contained in the slab $\zeta_l(0) \leqslant x \leqslant \zeta_r(0)$. The phenomenon can be modeled by

$$u_t = (u^m)_{xx}, \qquad (x, t) \in S_T \equiv \mathbf{R} \times (0, T], \quad 0 < T < \infty, \tag{1.1}$$

$$u(\cdot, 0) = u_0(\cdot) \quad \text{in } \mathbf{R}, \tag{1.2}$$

where $m > 1$ is a given constant, and u_0 is a given nonnegative function such that $u_0(x) > 0$ if $x \in (\zeta_l(0), \zeta_r(0))$ and $u_0(x) = 0$ elsewhere. We assume u_0 is continuous in $\mathbf{R}$.

Since the problem is degenerate, (1.1)–(1.2) is interpreted in a weak sense and the solution possesses a modest degree of regularity. Precisely $(x, t) \to u(x, t)$ is said to be a weak solution of (1.1)–(1.2) if

$$u \in C(S_T); \qquad (u^m)_x \in L^2(S_T)$$

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and

$$\int_{\mathbf{R}} u(x,\cdot)\phi(x,\cdot)\,dx\big|_{t_0}^{t} + \int_{t_0}^{t}\int_{\mathbf{R}} [-u\phi_t + (u^m)_x\phi_x]\,dx\,dt = 0 \tag{1.3}$$

for all ϕ satisfying

$$\begin{cases} \phi \in H^1(S_T) \cap L^\infty(S_T) \text{ and} \\ x \to \phi(x,t) \text{ is compactly supported} \\ \text{in } \mathbf{R} \text{ uniformly in } t, \end{cases} \tag{1.4}$$

and for all intervals $[t_0, t] \subseteq [0, T]$.

The pressure v in the fluid is connected to the density by

$$v = u^{m-1} \tag{1.5}$$

up to some multiplicative constant, and it satisfies

$$v_t = mvv_{xx} + \frac{m}{m-1}(v_x)^2 \quad \text{in } S_T, \tag{1.6}$$

$$v_0(\cdot) = v(\cdot,0) = u_0^{m-1}. \tag{1.7}$$

The Cauchy problem (1.6)–(1.7) is also interpreted in the weak sense

$$v \in C(S_T); \qquad v_x \in L^2(S_T)$$

and

(1.8)

$$\int_{\mathbf{R}} v(x,\cdot)\phi(x,\cdot)\big|_{t_0}^{t}\,dx + \int_{t_0}^{t}\int_{\mathbf{R}} \left[-v\phi_t + mvv_x\phi_x + \frac{m(m-2)}{m-1}(v_x)^2\phi\right] dx\,dt = 0$$

for all ϕ satisfying (1.4), and all intervals $[t_0, t] \subset [0, T]$.

Existence and uniqueness of weak solutions of (1.1)–(1.2) was first proved by Oleinik, Kalashnikov and Chzou Yui-Lin in [**15**], and the equivalence of (1.3) and (1.8) is due to Aronson [**2**].

A consequence of the degeneracy is that $u(\cdot, t)$ and $v(\cdot, t)$ are supported in a finite interval $[\zeta_l(t), \zeta_r(t)]$.

The curves $(t, \zeta_l(t))$ and $(t, \zeta_r(t))$, which we refer to as the left and right interfaces, are Lipschitz continuous and monotone decreasing and increasing respectively (see [**3**]). The interface curves and the pressure v are connected by the Stefan-like conditions (see [**3, 11**])

$$\begin{cases} \lim_{x \nearrow \zeta_r} v_x(x,t) = -\dfrac{m-1}{m}\zeta_r'(t), \\ \lim_{x \searrow \zeta_l(t)} v_x(x,t) = -\dfrac{m-1}{m}\zeta_l'(t). \end{cases} \tag{1.9}$$

It should be noted that conditions (1.9) are not part of the original problem, but rather are known to be satisfied by the unique solution of (1.1)–(1.2). Nevertheless our algorithm will be based upon suitable discretization of both (1.6) and (1.9).

We now give a detailed description of our algorithm. Let Δx and Δt denote increments in x and t, and let

$$x_k = k\Delta x, \quad k \in \mathbf{Z}; \qquad t_n = n\Delta t, \quad n \in \mathbf{N} \cup \{0\}.$$

The approximations to $v(x_k, t_n)$, $\zeta_l(t_n)$ and $\zeta_r(t_n)$ will be denoted by v_k^n, ζ_l^n and ζ_r^n respectively.

Actually we shall describe the computations only for the right-hand interface; the computations near ζ_l^n are completely analogous. We therefore suppress the subscript and denote ζ_r^n by ζ^n.

To start the scheme let $v_k^0 = v_0(x_k)$ and $\zeta^0 = \zeta(0)$. Next define $K(1) = \max\{k: x_{k+1} \leqslant \zeta^0\}$ and $s_0 = \zeta^0 - x_{K(1)}$. Then in analogy with (1.9) we compute ζ^1 from the equation

$$\zeta^1 = \zeta^0 + \frac{m}{m-1}\frac{v^0_{K(1)}}{s_0}\Delta t.$$

Observe that $s_0 \geqslant \Delta x$ and $\zeta^1 \geqslant \zeta^0$.

Now given $\zeta^{n+1} \geqslant \zeta^n$ and v_j^n for $j \in \mathbf{Z}$, we proceed as follows. First define

$$\text{(1.10)} \qquad \begin{cases} K(n+1) = \max\{k: x_{k+1} \leqslant \zeta_r^n\}, \\ \overline{K}(n+1) = \min\{k: x_{k-1} \geqslant \zeta_l^n\}, \end{cases}$$

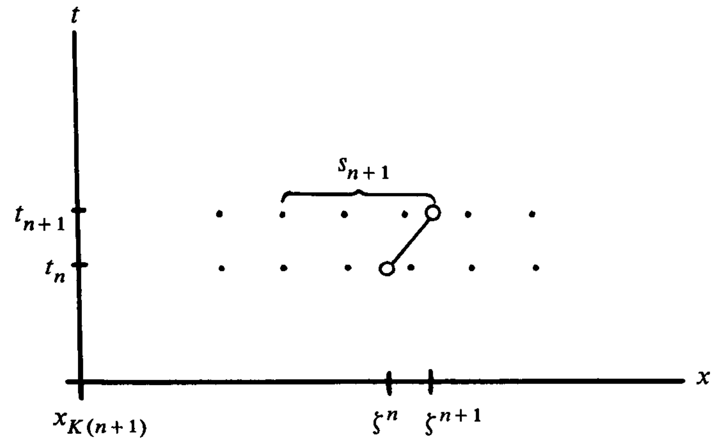

FIGURE 1

Then for $\overline{K}(n+1) \leqslant k \leqslant K(n+1)$ compute v_k^{n+1} from the finite difference equation

$$\text{(1.11)} \qquad \frac{v_k^{n+1} - v_k^n}{\Delta t} = m\left(v_k^n + \varepsilon\right)\frac{v_{k+1}^n - 2v_k^n + v_{k-1}^n}{(\Delta x)^2} + \frac{m}{m-1}\left(\frac{v_{k+1}^n - v_{k-1}^n}{2\Delta x}\right)^2,$$

where $\varepsilon > 0$ will be chosen later. Observe that we do not enforce the difference equation across the interface.

Next let

$$s_{n+1} = \zeta^{n+1} - x_{K(n+1)}, \tag{1.12}$$

and observe that

$$\Delta x \leqslant s_{n+1}. \tag{1.13}$$

Then for $x_{K(n+1)} \leqslant x_k \leqslant \zeta^{n+1}$, compute v_k^{n+1} from the linear interpolation

$$v_k^{n+1} = \frac{\zeta^{n+1} - x_k}{s_{n+1}} v_{K(n+1)}^{n+1}. \tag{1.14}$$

Finally set $v_k^{n+1} = 0$ for $x_k \geqslant \zeta^{n+1}$ and compute ζ^{n+2} from

$$\zeta^{n+2} = \zeta^{n+1} + \frac{m}{m-1} \frac{v_{K(n+1)}^{n+1}}{s_{n+1}} \Delta t. \tag{1.15}$$

We shall prove that $v_k^n \geqslant 0$ for all n and k so that by (1.15) $\zeta^{n+2} \geqslant \zeta^{n+1}$. Thus the support of the approximate solution increases monotonically in t.

In addition, the fact that $s_n \geqslant \Delta x$ insures that numerical instabilities are avoided in the computations (1.14) and (1.15).

Introducing the notations

$$\beta = \frac{\Delta t}{(\Delta x)^2}; \qquad A v_k = v_{k+1} - 2v_k + v_{k-1},$$

we can rewrite the difference scheme (1.11) in the form

$$v_k^{n+1} = v_k^n + m\beta(v_k^n + \varepsilon) A v_k^n + \frac{m\beta}{m-1}\left(\frac{v_{k+1}^n - v_{k-1}^n}{2}\right)^2. \tag{1.16}$$

We shall assume throughout that

[A1] $$0 \leqslant v_0(x) \leqslant M \quad \forall x \in \mathbf{R};$$

[A2] $$|v_0(x) - v_0(y)| \leqslant \gamma_0 |x - y| \quad \forall x, y \in \mathbf{R};$$

[A3] $$\varepsilon \text{ is of the order of } \Delta x \quad \text{and} \quad \varepsilon \geqslant 9\frac{m+2}{m-1}\gamma_0 \Delta x;$$

[A4] $$2m\beta\left[M + \varepsilon + \frac{m}{m-1}\gamma_0 \Delta x\right] \leqslant 1,$$

where M and γ_0 are given positive constants. Since $\varepsilon = O(\Delta x)$, condition [A4] on β is seen to be a slight strengthening of the usual parabolic stability condition.

We let h denote the pair $(\Delta x, \Delta t)$, and we construct approximate solutions v^h and approximate interface curves ζ_l^h and ζ_r^h by piecewise linear interpolation. Our results may be summarized as follows:

[I] $$\|v^h - v\|_{\infty, S_T} \leqslant C(T)(\Delta x)^p,$$

[II] $$v_x^h \to v_x \quad \text{in } L^q(S_T), \text{ for all } q \in [1, \infty),$$

[III] $$\left\|(\zeta_l^h, \zeta_r^h) - (\zeta_l, \zeta_r)\right\|_{\infty, [0, T]} \leqslant C(T)(\Delta x)^{p/2},$$

where p is defined in terms of m in Theorem 4.1 below. Further comments will be made in §6 about these rates of convergence, where we present and discuss the results of some numerical experiments.

The idea of exploiting an interface condition such as (1.9) for computational purposes seems to have been first used by Hüber [**10**] in connection with the one-phase Stefan problem (see also [7]).

We remark on the introduction of the vanishing viscosity ε. If ε were zero, the continuous analog of (1.11)–(1.15) would be overspecified. The artificial viscosity ε thus seems to stabilize our finite difference scheme. More specifically, the presence of the ε allows us to derive a lower bound for v^h_{xx} (in the sense of distributions). This in turn yields a uniform modulus of semicontinuity for v^h_x and, via the interface condition, for ζ^h_t. It is this semicontinuity which is crucial in proving the convergence of the approximate interface curves, as well as in estimating the rate of convergence.

We briefly comment on related, known results. In [**9**] Graveleau and Jamet obtained solutions of the porous medium equation and related equations by employing a difference scheme similar to ours. However their scheme is applied in all of $\{t \geqslant 0\}$ so that approximate interfaces are not computed. Moreover numerical evidence indicates that the supports of their approximate solutions spread out too rapidly in time. Thus computing the interfaces by "shock capturing" seems to be unsatisfactory.

While this paper was in preparation, Mimura and Tomoeda [**13**] informed us that they have recently derived an interface tracking algorithm for the porous medium equation. Numerical evidence suggests that the approximate interfaces computed by their scheme are accurate, but they are unable to prove this result. In addition, their scheme is somewhat complicated to implement, since it involves solving Riemann problems for the Burgers equation at each mesh point. Both their scheme and ours suffer from the parabolic stability condition $\Delta t = O[(\Delta x)^2]$.

The paper is organized as follows. §2 contains the derivation of basic estimates. Specifically we prove the finite difference analog of the following facts, which are known to hold for the exact solution v of (1.6)–(1.7):

$$0 \leqslant v \leqslant M, \tag{1.17}$$

$$\|v_x\|_{\infty, S_T} \leqslant \gamma_0 \quad \text{(see [2])}, \tag{1.18}$$

$$|v(x, t_2) - v(x, t_1)| \leqslant C|t_2 - t_1|^{1/2} \quad \text{(see [8, 12])}, \tag{1.19}$$

$$\begin{cases} v_{xx} \geqslant -\dfrac{m-1}{m(m+1)}\dfrac{1}{t} \quad \text{and} \\ \|v_{xx}(\cdot, t), v_t(\cdot, t)\|_{1,\mathbf{R}} \leqslant \dfrac{C}{t} \quad \text{(see [5])}. \end{cases} \tag{1.20}$$

In §3 we demonstrate the convergence of the approximate solutions and interfaces to the correct ones by making use of various compactness arguments. The error estimates are proved in §§4 and 5. Finally in §6 we present and discuss the results of several numerical experiments.

Throughout the paper we make the convention that C shall denote a generic positive constant depending only on m, M, γ_0 and some specified time T.

2. Basic estimates. We begin the analysis by establishing maximum principles for v_k^n and for the discrete space derivative

$$w_k^n = \frac{v_k^n - v_{k-1}^n}{\Delta x}.$$

We assume throughout that the initial function v_0 satisfies assumptions [A1] and [A2], and that the mesh parameters ε and β satisfy [A3] and [A4].

LEMMA 2.1. *The bounds*

$$0 \leqslant v_k^n \leqslant M \tag{2.1}$$

and

$$|w_k^n| \leqslant \gamma_0 \tag{2.2}$$

hold for all k and all $n \geqslant 0$.

PROOF. The results hold for $n = 0$ by hypothesis. Proceeding by induction on n, we rewrite the last term in the difference equation (1.16) as

$$\frac{m\Delta t}{m-1}\left(\frac{w_{k+1}^n + w_{k-1}^n}{2}\right)\left(\frac{v_{k+1}^n - v_{k-1}^n}{2\Delta x}\right).$$

Rearranging, (1.16) thus becomes

$$\begin{aligned} v_k^{n+1} = {} & \left[1 - 2m\beta(v_k^n + \varepsilon)\right] v_k^n \\ & + \left[m\beta(v_k^n + \varepsilon) + \frac{m}{m-1}\frac{\beta\Delta x}{4}(w_{k+1}^n + w_k^n)\right] v_{k+1}^n \\ & + \left[m\beta(v_k^n + \varepsilon) - \frac{m}{m-1}\frac{\beta\Delta x}{4}(w_{k+1}^n + w_k^n)\right] v_{k-1}^n. \end{aligned} \tag{2.3}$$

Using the induction hypotheses (2.1) and (2.2), we have that the coefficients of v_k^n and v_{k+1}^n in this expression are bounded below by $1 - 2m\beta(M + \varepsilon)$ and $m\beta(\varepsilon - \gamma_0\Delta x/2(m-1))$ respectively. Since these quantities are nonnegative by [A3] and [A4], (2.3) shows that v_k^{n+1} is a convex combination of v_k^n, v_{k-1}^n, and v_{k+1}^n. This proves that $0 \leqslant v_k^{n+1} \leqslant M$ for $k \leqslant K(n+1)$. When $k > K(n+1)$, these bounds follow from (1.14).

We prove (2.2) first for $k \leqslant K(n+1)$. Rewrite (1.16) as follows:

$$v_k^{n+1} = v_k^n + m\beta(v_k^n + \varepsilon)(w_{k+1}^n - w_k^n)\Delta x + \frac{m\Delta t}{m-1}\left(\frac{w_{k+1}^n + w_k^n}{2}\right)^2. \tag{2.4}$$

We subtract from this the equation corresponding to v_{k-1}^{n+1} and divide by Δx. Using the discrete product rule

$$a_k b_k - a_{k-1}b_{k-1} = \frac{a_k + a_{k-1}}{2}(b_k - b_{k-1}) + \frac{b_k + b_{k-1}}{2}(a_k - a_{k-1}),$$

we obtain

$$w_k^{n+1} = w_k^n + m\beta\left(\frac{v_k^n + v_{k-1}^n}{2} + \varepsilon\right)Aw_k^n + m\beta\Delta x w_k^n\left(\frac{w_{k+1}^n - w_{k-1}^n}{2}\right)$$
$$+ \frac{m}{m-1}\beta\Delta x\left(\frac{w_{k+1}^n + 2w_k^n + w_{k-1}^n}{4}\right)(w_{k+1}^n - w_{k-1}^n),$$

so that

$$w_k^{n+1} = w_k^n + m\beta\left(\frac{v_k^n + v_{k-1}^n}{2} + \varepsilon\right)Aw_k^n + \frac{m}{m-1}\frac{\beta\Delta x}{4}(w_{k+1}^n + 2mw_k^n + w_{k-1}^n)(w_{k+1}^n - w_{k-1}^n). \tag{2.5}$$

This equation has the form

$$w_k^{n+1} = (1-2a)w_k^n + (a+b)w_{k+1}^n + (a-b)w_{k-1}^n, \tag{2.6}$$

where

$$a = m\beta\left(\frac{v_k^n + v_{k-1}^n}{2} + \varepsilon\right) \tag{2.7}$$

and

$$b = \frac{m}{m-1}\frac{\beta\Delta x}{4}(w_{k+1}^n + 2mw_k^n + w_{k-1}^n). \tag{2.8}$$

By the induction hypotheses (2.1) and (2.2), a and b satisfy

$$m\beta\varepsilon \leqslant a \leqslant m\beta(M+\varepsilon) \tag{2.9}$$

and

$$|b| \leqslant \frac{m(m+1)}{m-1}\frac{\beta\gamma_0\Delta x}{2}. \tag{2.10}$$

Thus, using the mesh conditions [A3] and [A4], we obtain immediately that $1-2a$ and $a-|b|$ are nonnegative. Hence (2.6) shows that w_k^{n+1} is a convex combination of w_k^n, w_{k-1}^n and w_{k+1}^n, and so satisfies the bound (2.2).

Finally, we prove (2.2) for x_k near the interface. Thus let $k = K(n+1)$ and let $s_n' = \zeta^n - x_k$, so that

$$w_{k+1}^n = -v_k^n/s_n'.$$

Using the difference equation (1.16), we then have that

$$-w_{k+1}^{n+1} = \frac{v_k^{n+1}}{s_{n+1}} = \frac{1}{s_{n+1}}\left[v_k^n + m\beta(v_k^n+\varepsilon)(w_{k+1}^n - w_k^n)\Delta x + \frac{m\Delta t}{m-1}\left\{(w_{k+1}^n)^2 + \left(\frac{w_{k+1}^n + w_k^n}{2}\right)^2 - (w_{k+1}^n)^2\right\}\right]$$
$$= -\frac{w_{k+1}^n}{s_{n+1}}\left(s_n' - \frac{m}{m-1}w_{k+1}^n\Delta t\right)$$
$$+ (w_{k+1}^n - w_k^n)\frac{m}{s_{n+1}}\left[\beta\Delta x(v_k^n+\varepsilon) - \frac{\Delta t}{m-1}\left(\frac{3w_{k+1}^n + w_k^n}{4}\right)\right].$$

But $s_{n+1} = s'_n - m\Delta t w^n_{k+1}/(m-1)$ by (1.15). Therefore

$$w^{n+1}_{k+1} = w^n_{k+1} + c(w^n_k - w^n_{k+1}), \tag{2.11}$$

where

$$c = \frac{m}{s_{n+1}}\left[\beta\Delta x(v^n_k + \varepsilon) - \frac{\Delta t}{m-1}\left(\frac{3w^n_{k+1} + w^n_k}{4}\right)\right]. \tag{2.12}$$

Again, using the induction hypotheses (2.1) and (2.2), we obtain that

$$\frac{m\beta\Delta x}{s_{n+1}}\left(\varepsilon - \frac{\gamma_0\Delta x}{m-1}\right) \leqslant c \leqslant m\beta\left(M + \varepsilon + \frac{\gamma_0\Delta x}{m-1}\right). \tag{2.13}$$

The mesh conditions [A3] and [A4] then imply that $0 \leqslant c \leqslant 1$. Thus (2.11) shows that w^{n+1}_{k+1} is a convex combination of w^n_{k+1} and w^n_k, and so satisfies the bound (2.2).

Finally, when $k > K(n+1)+1$, w^{n+1}_k is between w^{n+1}_{k+1} and 0, and so again satisfies (2.2). □

The bound (2.2) for w^n_k, together with the interface condition (1.15), shows that $|(\zeta^{n+1} - \zeta^n)/\Delta t| \leqslant C$. Since $\Delta t = O(\Delta x^2)$ by [A4], it follows that $s_n \leqslant 2\Delta x + O(\Delta x^2)$ (see (1.10) and (1.12)). Combining this with (1.13), we therefore have

$$\Delta x \leqslant s_n \leqslant 3\Delta x \tag{2.14}$$

for small Δx. Actually, any upper bound on $s_n/\Delta x$ will suffice for our purposes. However, for the sake of simplicity, we shall make use of (2.14) without mentioning the precise conditions on Δx which justify it.

In the next lemma we establish a lower bound for the second spatial differences of v^n_k. This lower bound will provide a uniform modulus of semicontinuity for w^n_k and, via (1.15), for $(\zeta^{n+1} - \zeta^n)/\Delta t$ as well. This semicontinuity will be crucial later for obtaining error estimates for the approximate interfaces.

LEMMA 2.2. *Define K by $K = (m-1)/m(m+1)$. Then the bound*

$$\frac{Av^n_k}{\Delta x^2} = \frac{v^n_{k+1} - 2v^n_k + v^n_{k-1}}{\Delta x^2} \geqslant -\frac{K}{t_n} \tag{2.15}$$

holds for all k and all $n > 0$.

PROOF. Denote the variable in question by Z^n_k. That is,

$$Z^n_k = \frac{Av^n_k}{\Delta x^2} = \frac{w^n_{k+1} - w^n_k}{\Delta x}.$$

Now, if $0 < t_n \leqslant K\Delta x/2\gamma_0$, then

$$|Z^n_k| \leqslant \frac{|w^n_{k+1}| + |w^n_k|}{\Delta x} \leqslant \frac{2\gamma_0}{\Delta x} \leqslant \frac{K}{t_n}.$$

We proceed by induction, assuming that (2.15) holds at time level n, and that

$$t_{n+1} > K\Delta x/2\gamma_0. \tag{2.16}$$

The first case is that in which $k \leqslant K(n+1)-1$, so that both w_k^{n+1} and w_{k+1}^{n+1} satisfy (2.5). Subtracting and dividing by Δx, we thus obtain

$$\begin{aligned} Z_k^{n+1} = Z_k^n &+ m\beta\left(\frac{v_{k+1}^n + 2v_k^n + v_{k-1}^n}{4} + \varepsilon\right) A Z_k^n \\ &+ m\beta\left(\frac{v_{k+1}^n - v_{k-1}^n}{2\Delta x}\right)\left(\frac{Aw_{k+1}^n + Aw_k^n}{2}\right) \\ &+ \frac{m}{m-1}\frac{\beta\Delta x}{8}\left[w_{k+2}^n + (2m+1)(w_{k+1}^n + w_k^n) + w_{k-1}^n\right](Z_{k+1}^n - Z_{k-1}^n) \\ &+ \frac{m}{m-1}\frac{\beta\Delta x}{8}(Z_{k+1}^n + 2mZ_k^n + Z_{k-1}^n)(w_{k+2}^n - w_k^n + w_{k+1}^n - w_{k-1}^n). \end{aligned}$$

We rewrite the third term on the right as

$$\frac{m\beta\Delta x}{4}(w_{k+1}^n + w_k^n)(Z_{k+1}^n - Z_{k-1}^n),$$

and the last term on the right as

$$\frac{m}{m-1}\frac{\Delta t}{8}(Z_{k+1}^n + 2mZ_k^n + Z_{k-1}^n)(Z_{k+1}^n + 2Z_k^n + Z_{k-1}^n).$$

The result is that

$$\begin{aligned} Z_k^{n+1} &= (1-2p)Z_k^n + (p+q)Z_{k+1}^n + (p-q)Z_{k-1}^n \\ &\quad + r(Z_{k+1}^n + 2mZ_k^n + Z_{k-1}^n)(Z_{k+1}^n + 2Z_k^n + Z_{k-1}^n), \end{aligned} \tag{2.17}$$

where

$$p = m\beta\left(\frac{v_{k+1}^n + 2v_k^n + v_{k-1}^n}{4} + \varepsilon\right),$$

$$q = \frac{m}{m-1}\frac{\beta\Delta x}{8}\left[w_{k+2}^n + (4m-1)(w_{k+1}^n + w_k^n) + w_{k-1}^n\right]$$

and

$$r = \frac{m}{m-1}\frac{\Delta t}{8}.$$

We shall show that Z_k^{n+1} in (2.17) is an increasing function of each of the quantities Z_k^n, Z_{k-1}^n and Z_{k+1}^n. Using Lemma 2.1, we have

$$\begin{aligned} \frac{\partial Z_k^{n+1}}{\partial Z_k^n} &= (1-2p) + r\left[(2m+2)(Z_{k+1}^n + Z_{k-1}^n) + 8mZ_k^n\right] \\ &= (1-2p) + \frac{r}{\Delta x}\left[(2m+2)(w_{k+2}^n - w_{k-1}^n) + (6m-2)(w_{k+1}^n - w_k^n)\right] \\ &\geqslant 1 - 2m\beta(M+\varepsilon) - \frac{m}{m-1}\frac{\Delta t}{8\Delta x}\cdot 16m\gamma_0 \\ &= 1 - 2m\beta\left[M + \varepsilon + \frac{m}{m-1}\gamma_0\Delta x\right] \geqslant 0 \end{aligned}$$

by [A4]. And similarly,

$$\begin{aligned}\frac{\partial Z_k^{n+1}}{\partial Z_{k\pm 1}^n} &= p \pm q + 2r\left[Z_{k+1}^n + (m+1)Z_k^n + Z_{k-1}^n\right]\\ &= p \pm q + \frac{2r}{\Delta x}\left[w_{k+2}^n - w_{k-1}^n + m(w_{k+1}^n - w_k^n)\right]\\ &\geqslant m\beta\varepsilon - \frac{m}{m-1}\frac{\beta\Delta x}{8}\cdot 8m\gamma_0 - \frac{2m}{m-1}\frac{\Delta t}{8\Delta x}\cdot(2m+2)\gamma_0\\ &= m\beta\left[\varepsilon - \frac{3m+1}{2(m-1)}\gamma_0\Delta x\right] \geqslant 0\end{aligned}$$

by [A3].

Thus Z_k^{n+1} is bounded below by the right side of (2.17) with Z_k^n, Z_{k-1}^n and Z_{k+1}^n replaced by $-K/t_n$. That is,

$$\begin{aligned}Z_k^{n+1} &\geqslant -\frac{K}{t_n} + \frac{m(m+1)}{m-1}\left(\frac{K}{t_n}\right)^2\Delta t = -\frac{K}{t_n}\left[1 - \frac{m(m+1)}{m-1}\frac{K}{n}\right]\\ &= -\frac{K}{t_n}\frac{n-1}{n} \geqslant -\frac{K}{t_n}\frac{n}{n+1} = -\frac{K}{t_{n+1}}\end{aligned}$$

as required.

There are several cases to consider in order to establish the bound (2.15) for Z_k^{n+1} when x_k is near ζ^{n+1}. Now, when $Z_k^{n+1} \geqslant 0$, (2.15) is automatically satisfied. We may therefore assume that $Z_k^{n+1} < 0$; that is, that

$$\frac{v_{k+1}^{n+1} + v_{k-1}^{n+1}}{2} < v_k^{n+1}.$$

But this shows that v_k^{n+1} is positive and that v_k^{n+1} is not computed from the linear interpolation (1.14). Therefore it must be that $k \leqslant K(n+1)$. Since we already dealt with the case that $k \leqslant K(n+1) - 1$, we may therefore assume that $k = K(n+1)$.

Thus w_k^{n+1} and w_{k+1}^{n+1} satisfy (2.6) and (2.11) respectively. These equations may be rewritten

$$w_{k+1}^{n+1} = w_{k+1}^n - c\Delta x Z_k^n$$

and

$$w_k^{n+1} = w_k^n + (a+b)\Delta x Z_k^n - (a-b)\Delta x Z_{k-1}^n,$$

where a, b, and c are as in (2.7), (2.8), and (2.12). Subtracting and dividing by Δx, we thus obtain

$$Z_k^{n+1} = (1-a-b-c)Z_k^n + (a-b)Z_{k-1}^n. \tag{2.18}$$

We checked in the proof of Lemma 2.1 that $a - |b| \geqslant 0$. Using (2.9), (2.10), and (2.13), we have that

$$\begin{aligned}1-a-b-c &\geqslant 1 - m\beta(M+\varepsilon) - \frac{m(m+1)}{m-1}\frac{\beta\gamma_0\Delta x}{2} - m\beta\left(M + \varepsilon + \frac{\gamma_0\Delta x}{m-1}\right)\\ &= 1 - 2m\beta\left[M + \varepsilon + \frac{m+3}{4(m-1)}\gamma_0\Delta x\right],\end{aligned}$$

which is nonnegative by [A4].

Finally, using (2.9), (2.10), (2.13), and (2.14), we have that the sum of the coefficients in (2.18) is

$$1 - 2b - c \leqslant 1 + \frac{m(m+1)}{m-1}\beta\gamma_0\Delta x - \frac{m\beta}{3}\left(\varepsilon - \frac{\gamma_0\Delta x}{m-1}\right).$$

Using the definition of K, this bound may be rewritten as

$$1 + \frac{\beta\gamma_0\Delta x}{K}\left[1 + \frac{1}{3(m+1)}\right] - \frac{m\beta\varepsilon}{3}.$$

Using condition [A3], we then find that

$$1 - 2b - c \leqslant 1 - \frac{2\beta\gamma_0\Delta x}{K} = 1 - \frac{2\gamma_0}{K}\frac{\Delta t}{\Delta x}.$$

On the other hand, we have from (2.16) that $t_{n+1} \geqslant K\Delta x/2\gamma_0$, so that

$$\frac{1}{n+1} \leqslant \frac{2\gamma_0}{K}\frac{\Delta t}{\Delta x}.$$

Therefore $1 - 2b - c \leqslant 1 - 1/(n+1) = n/(n+1)$, and (2.18) shows that

$$Z_k^{n+1} \geqslant -\frac{K}{t_n}\frac{n}{n+1} = -\frac{K}{t_{n+1}}. \quad \square$$

REMARK. The Barenblatt-Pattle solution **[6, 16]** shows that the constant $K = (m-1)/m(m+1)$ is the best possible (see also **[5]**).

We can improve the bound (2.15) by imposing additional regularity conditions on v_0.

COROLLARY 2.3. (a) *If v_0 is a concave function, then*

$$Av_k^n/\Delta x^2 \leqslant 0 \tag{2.19}$$

for all k and all $n \geqslant 0$.

(b) *If there is a constant C_0 such that*

$$\frac{v_0(x+h) - 2v_0(x) + v_0(x-h)}{h^2} \geqslant -C_0 \tag{2.20}$$

holds for all x and all $h > 0$, then

$$Av_k^n/\Delta x^2 \geqslant -C_0 \tag{2.21}$$

for all k and all $n \geqslant 0$.

PROOF. We showed in the proof of Lemma 2.2 that, in all cases, $Av_k^{n+1}/\Delta x^2$ is an increasing function of $Av_k^n/\Delta x^2$ and $Av_{k\pm1}^n/\Delta x^2$. The bounds (2.19) and (2.21) then follow easily from (2.17) and (2.18). $\square$

Next, we obtain a bound for the discrete time derivative near the interface.

LEMMA 2.4. *There is a constant C such that*

$$\left|\frac{v_k^{n+1} - v_k^n}{\Delta t}\right| \leqslant C$$

holds for $n \geqslant 0$ and $k > K(n+1)$.

Proof. We use the symbol $O(\cdot)$ to denote dependence on mesh parameters. Let $k > K \equiv K(n+1)$ and let $s_n' = \zeta^n - x_K$. Then from (1.14) we have that

$$\frac{v_k^{n+1} - v_k^n}{\Delta t} = \frac{1}{\Delta t}\left[\frac{\zeta^{n+1} - x_k}{s_{n+1}} v_K^{n+1} - \frac{\zeta^n - x_k}{s_n'} v_K^n\right]$$

$$= O(1)\left[\frac{v_K^{n+1} - v_K^n}{\Delta t} + \frac{1}{\Delta t}\left(\frac{\zeta^{n+1} - x_k}{s_{n+1}} - \frac{\zeta^n - x_k}{s_n'}\right) v_K^n\right].$$

The first term on the right can be estimated by using the difference equation, (1.16):

$$\frac{v_K^{n+1} - v_K^n}{\Delta t} = m(v_K^n + \varepsilon)\frac{w_{K+1}^n - w_K^n}{\Delta x} + O(1) = O(1)$$

since $v_K^n + \varepsilon = O(\Delta x)$. And the second term on the right is

$$O(\Delta x) \cdot \frac{[\zeta^n + O(\Delta t) - x_k]s_n' - [s_n' + O(\Delta t)](\zeta^n - x_k)}{s_n' s_{n+1}\Delta t} = O(1). \quad \square$$

Lemmas 2.2 and 2.4, together with the difference equation (1.11), now imply the following bound for the discrete time derivative.

Corollary 2.5. *There is a constant C such that*

$$\frac{v_k^{n+1} - v_k^n}{\Delta t} \geqslant -C\left(1 + \frac{1}{t_n}\right) \tag{2.22}$$

holds for all k and all $n > 0$.

In the next lemma we use the one-sided bounds (2.15) and (2.22) to derive L^1 estimates for $Av_k^n/\Delta x^2$ and $(v_k^{n+1} - v_k^n)/\Delta t$.

Lemma 2.6. (a) *For a given $T > 0$ there is a constant C such that*

$$\sum_k \left|\frac{Av_k^n}{\Delta x^2}\right|\Delta x, \quad \sum_k \left|\frac{v_k^{n+1} - v_k^n}{\Delta t}\right|\Delta x \leqslant C\left(1 + \frac{1}{t_n}\right)$$

holds for $t_n \leqslant T$.

(b) *If the initial function v_0 satisfies the hypothesis* (2.20), *then*

$$\sum_k \left|\frac{Av_k^n}{\Delta x^2}\right|\Delta x, \quad \sum_k \left|\frac{v_k^{n+1} - v_k^n}{\Delta t}\right|\Delta x \leqslant C.$$

(c) *And if v_0 satisfies* (2.20) *and is concave, then*

$$\left|\frac{Av_k^n}{\Delta x^2}\right|, \quad \left|\frac{v_k^{n+1} - v_k^n}{\Delta t}\right| \leqslant C.$$

Proof. From Lemma 2.2 we have that

$$\left|\frac{Av_k^n}{\Delta x^2}\right| \leqslant \frac{Av_k^n}{\Delta x^2} + \frac{2K}{t_n}.$$

We multiply by Δx and sum over k. Since v_k^n is zero outside an interval of length $C(1 + t_n)$, we obtain that

$$\sum_k \left|\frac{Av_k^n}{\Delta x^2}\right|\Delta x \leqslant \frac{C}{t_n}(1 + t_n).$$

The other bounds in (a) and (b) are proved similarly. (c) follows from Corollary 2.3, the difference equation (1.11), and Lemma 2.4. □

In the final lemma of this section we establish the Hölder continuity in time of the sequence $\{v_k^n\}$.

LEMMA 2.7. *Let $T > 0$ be given. Then there is a constant C such that the inequality*

$$|v_k^n - v_k^m| \leqslant C\left(|t_n - t_m|^{1/2} + \Delta x\right) \leqslant \left(C + \beta^{-1/2}\right)|t_n - t_m|^{1/2}$$

holds for t_n and t_m in $[0, T]$ and for all k.

PROOF. Fix a point (x_{k_0}, t_{n_0}) and let $t_{n_1} > t_{n_0}$ be given. Let Q be the rectangle

$$Q = [x_{k_0} - \rho, x_{k_0} + \rho] \times \left[t_{n_0}, t_{n_1}\right],$$

where ρ is a multiple of Δx to be chosen later. Define the quantities

$$H = \max_{n_0 \leqslant n \leqslant n_1} |v_{k_0}^n - v_{k_0}^{n_0}|, \tag{2.23}$$

$$c = 2m(M + \varepsilon) + \frac{m}{m-1}\gamma_0\rho,$$

$$U_k^n = v_k^n - v_{k_0}^{n_0} - \gamma_0\rho - \frac{H}{\rho^2}\left[(x_k - x_{k_0})^2 + c\left(t_n - t_{n_0}\right)\right].$$

We shall show that $U_k^n \leqslant 0$ for $(x_k, t_n) \in Q$ by induction on n. When $n = n_0$ and $|x_{k_0} - x_k| \leqslant \rho$, we have, using Lemma 2.1, that $U_k^{n_0} \leqslant v_k^{n_0} - v_{k_0}^{n_0} - \gamma_0\rho \leqslant 0$.

For the induction step we consider the following three cases: $|x_k - x_{k_0}| = \rho$, $k > K(n+1)$ and $|x_k - x_{k_0}| < \rho$ with $k \leqslant K(n+1)$. In the first of these, we have that

$$U_k^{n+1} \leqslant \left(v_k^{n+1} - v_{k_0}^{n+1}\right) + \left(v_{k_0}^{n+1} - v_{k_0}^{n_0}\right) - \gamma_0\rho - H,$$

which is nonpositive by (2.2) and the definition of H. In the second case, we have $v_k^{n+1} \leqslant \gamma_0 s_{n+1} \leqslant 3\gamma_0\Delta x$, so that

$$U_k^{n+1} \leqslant 3\gamma_0\Delta x - \gamma_0\rho,$$

which is nonpositive provided that

$$\rho \geqslant 3\Delta x. \tag{2.24}$$

For the third case, we employ the linearized difference operator L, defined for a given sequence Z_k^n by

$$LZ_k^{n+1} = \frac{Z_k^{n+1} - Z_k^n}{\Delta t} - m\left(v_k^n + \varepsilon\right)\frac{AZ_k^n}{\Delta x^2} - \frac{m}{m-1}\left(\frac{v_{k+1}^n - v_{k-1}^n}{2\Delta x}\right)\left(\frac{Z_{k+1}^n - Z_{k-1}^n}{2\Delta x}\right).$$

Applying L to U_k^n and using (1.11), we find that

$$LU_k^{n+1} = \frac{H}{\rho^2}\left[-c + 2m\left(v_k^n + \varepsilon\right) + \frac{m}{m-1}\left(\frac{v_{k+1}^n - v_{k-1}^n}{2\Delta x}\right)\left(\frac{x_{k+1} + x_{k-1}}{2} - x_{k_0}\right)\right]$$

so that, by (2.1) and (2.2),

$$LU_k^{n+1} \leqslant \frac{H}{\rho^2}\left[-c + 2m(M+\varepsilon) + \frac{m}{m-1}\gamma_0\rho\right] = 0$$

by the definition of c, (2.23).

On the other hand, we can rewrite the inequality $LU_k^{n+1} \leqslant 0$ in the form

$$\begin{aligned} U_k^{n+1} \leqslant {} & [1 - 2m\beta(v_k^n + \varepsilon)]\, U_k^n \\ & + \left[m\beta(v_k^n + \varepsilon) - \frac{\beta}{4}\frac{m}{m-1}(v_{k+1}^n - v_{k-1}^n)\right] U_{k+1}^n \\ & + \left[m\beta(v_k^n + \varepsilon) + \frac{\beta}{4}\frac{m}{m-1}(v_{k+1}^n - v_{k-1}^n)\right] U_{k-1}^n. \end{aligned}$$

The coefficients of U_j^n on the right-hand side of this inequality are exactly the same as those of v_j^n in equation (2.3). We showed in the proof of Lemma 2.1 that these coefficients are nonnegative. We therefore have that U_k^{n+1} is a convex combination of U_{k-1}^n, U_k^n and U_{k+1}^n, and so is nonpositive by the induction hypothesis.

Setting $k = k_0$ in the result $U_k^n \leqslant 0$, we thus obtain that

$$v_{k_0}^n - v_{k_0}^{n_0} \leqslant \gamma_0\rho + \frac{Hc}{\rho^2}s,$$

where $s = t_{n_1} - t_{n_0}$. In a similar way, we can establish the same inequality for $v_{k_0}^{n_0} - v_{k_0}^n$. Taking the maximum over $n \in [n_0, n_1]$, we thus obtain

$$H \leqslant \gamma_0\rho + \frac{Hc}{\rho^2}s. \tag{2.25}$$

We shall choose ρ so that $cs/\rho^2 \leqslant 1/2$. Specifically, ρ should satisfy

$$\rho_1 + 3\Delta x \leqslant \rho \leqslant \rho_1 + 4\Delta x,$$

where ρ_1 is the larger root of the quadratic equation

$$\rho^2 - 2cs = \rho^2 - \frac{2m}{m-1}\gamma_0 s\rho - 4m(M+\varepsilon)s = 0.$$

An easy computation shows that $\rho_1 = O(s + s^{1/2}) = O(s^{1/2})$ for $t_{n_i} \leqslant T$, and $\rho \geqslant 3\Delta x$, as required by (2.24). Since $cs/\rho^2 \leqslant 1/2$, (2.25) becomes

$$H \leqslant 2\gamma_0\rho \leqslant 2\gamma_0(\rho_1 + 4\Delta x) \leqslant C(s^{1/2} + \Delta x).$$

In particular,

$$|v_{k_0}^{n_1} - v_{k_0}^{n_0}| \leqslant C\left(|t_{n_1} - t_{n_0}|^{1/2} + \Delta x\right). \quad \square$$

We remark that the proof of the above lemma is the discrete version of an argument given by Kruškov **[12]** and Gilding **[8]**.

3. Convergence of the approximate solutions. Let h denote a pair $(\Delta x, \Delta t)$ whose elements satisfy the mesh conditions [A3] and [A4]. We define approximate interface curves $t \mapsto \zeta_r^h(t)$ and $t \mapsto \zeta_l^h(t)$ by piecewise linear interpolation: for $t_n \leqslant t \leqslant t_{n+1}$,

$$\zeta_r^h(t) = \zeta_r^n + \frac{m}{m-1}\frac{v_{K(n)}^n}{s_n}(t - t_n),$$

where ζ_r^n, v_k^n, and s_n are as in §§1 and 2; and similarly for $\zeta_l^h(t)$. The estimate (2.2) for $v_{K(n)}^n/s_n$ then shows that the nets $\{\zeta_r^h(t)\}$ and $\{\zeta_l^h(t)\}$ are uniformly Lipschitz and uniformly bounded in finite time.

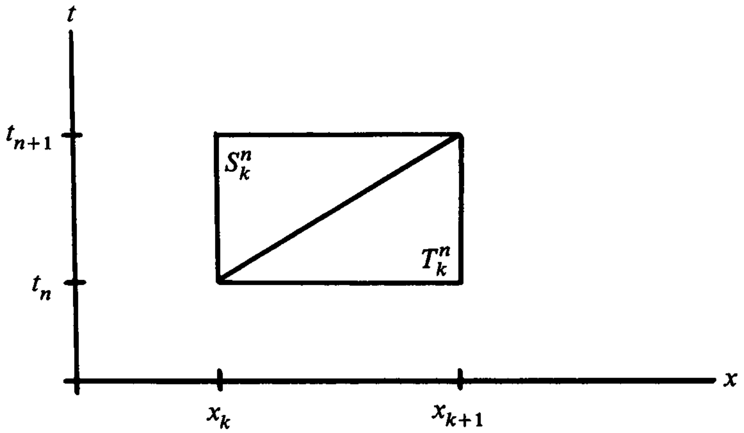

FIGURE 2

We construct approximate solutions $v^h(x, t)$ in an analogous way, as follows. If T_k^n and S_k^n are the triangles in Figure 3.1, then

$$v^h(x, t) = v_k^n + (x - x_k)w_{k+1}^n + (t - t_n)\sigma_{k+1}^n, \qquad (x, t) \in T_k^n,$$

and

$$v^h(x, t) = v_{k+1}^{n+1} + (x - x_{k+1})w_{k+1}^{n+1} + (t - t_{n+1})\sigma_k^n, \qquad (x, t) \in S_k^n.$$

Here

$$w_k^n = \frac{v_k^n - v_{k-1}^n}{\Delta x} \quad \text{and} \quad \sigma_k^n = \frac{v_k^{n+1} - v_k^n}{\Delta t}.$$

It follows immediately from Lemmas 2.1, 2.2, 2.6, and 2.7 that the functions v^h satisfy

$$0 \leqslant v^h(x, t) \leqslant M, \tag{3.1}$$

$$\left|\frac{\partial v^h}{\partial x}\right| \leqslant \gamma_0 \quad \text{a.e.,} \tag{3.2}$$

$$\left|\frac{\partial^2 v^h}{\partial x^2}(x, t)\right|, \quad \left|\frac{\partial v^h}{\partial t}(x, t)\right| \tag{3.3}$$

for $t > 0$ are finite measures in $\mathbf{R}$ with mass

$$\int_{\mathbf{R}}\left|\frac{\partial^2 v^h}{\partial x^2}\right|; \quad \int_{\mathbf{R}}\left|\frac{\partial}{\partial t}v^h\right| \leqslant C\left(1 + \frac{1}{t}\right)$$

and

$$|v^h(x, t + s) - v^h(x, t)| \leqslant C(T)s^{1/2}, \qquad 0 \leqslant t \leqslant t + s \leqslant T. \tag{3.4}$$

Throughout this section we fix a time T and a rectangle $Q = [a, b] \times [0, T]$, where $[a, b]$ is large enough to contain the supports of $v^h(\cdot, t)$ for all h and all $t \in [0, T]$ (see (1.15) and (2.2)).

The properties of ζ_r^h, ζ_l^h, and v^h described above insure that, for every sequence h_j tending to 0 subject to the mesh conditions [A3] and [A4], there is a subsequence, which we index simply by h, for which

$$v^h \to v^* \quad \text{uniformly in } Q,$$

$$\zeta_r^h, \zeta_l^h \to \zeta_r^*, \zeta_l^* \quad \text{uniformly in } [0, T],$$

and

$$v_x^h \to w^* \quad \text{weakly in } L^2(Q).$$

Our goal in this section will be to prove that v^*, ζ_r^*, and ζ_l^* coincide with the exact solution and interface curves for the problem (1.8). Actually, the convergence of v^h, ζ_r^h, and ζ_l^h also follows from the error bounds which we shall derive later in §§4 and 5. However, the arguments of the present section are much more direct. Moreover, we obtain here the convergence of v_x^h in $L^p(Q)$ for all $p < \infty$. As a byproduct of these arguments, we thus obtain in addition a constructive proof of the existence and regularity properties of the solution of (1.8).

We begin by showing that $v_x^h \to w^*$ strongly in $L^p(Q)$ and that, in fact, $w^* = v_x^*$.

LEMMA 3.1. *For any $t > 0$ the net $\{v_x^h(\cdot, t)\}$ is precompact in $L^1[a, b]$.*

PROOF. The proof consists of estimating the L^1 difference between $v_x^h(\cdot, t)$ and its spatial translates. Given $t > 0$, choose n so that $t_n \leqslant t < t_{n+1}$. Then when h is sufficiently small, $0 < t - \Delta t \leqslant t_n$.

If we take $\rho = l\Delta x$ where l is a positive integer, then it is easy to see that

$$\text{(3.5)} \quad \int_{\mathbf{R}} |v_x^h(x + \rho, t) - v_x^h(x, t)|\, dx \leqslant C\Big(\sum_k |w_{k+l}^n - w_k^n|\Delta x + \sum_k |w_{k+l}^{n+1} - w_k^{n+1}|\Delta x\Big).$$

The first of these sums is bounded by

$$\sum_k \sum_{j=k+1}^{k+l} \frac{|w_j^n - w_{j-1}^n|}{\Delta x}\Delta x^2 \leqslant \sum_j \sum_{k=j-l}^{j} \left|\frac{w_j^n - w_{j-1}^n}{\Delta x}\right|\Delta x^2 = \rho \sum_j \frac{|Av_j^n|}{\Delta x^2}\Delta x \leqslant \frac{C\rho}{t - \Delta t}$$

by Lemma 2.6. Dealing with the second sum in (3.5) in a similar way, we find that

$$\text{(3.6)} \qquad \int_{\mathbf{R}} |v_x^h(x + \rho, t) - v_x^h(x, t)|\, dx \leqslant \frac{C\rho}{t - \Delta t}$$

holds when $\rho/\Delta x$ is a positive integer.

When $\rho_1 < \Delta x$, $|v_x^h(x + \rho_1, t) - v_x^h(x, t)|$ will be zero except when x is within ρ_1 of the nonhorizontal sides of the triangles T_k^n and S_k^n. Thus

$$\int_{\mathbf{R}} |v_x^h(x + \rho_1, t) - v_x^h(x, t)|\, dx \leqslant C\rho_1\Big(\sum_k |w_{k+1}^n - w_k^n| + \sum_k |w_{k+1}^{n+1} - w_k^{n+1}|\Big).$$

But this expression is $\rho_1/\Delta x$ times the right side of (3.5) with l taken to be 1. The computations we made above therefore show that

$$\int_{\mathbf{R}} |v_x^h(x+\rho_1, t) - v_x^h(x,t)|\, dx \leqslant \frac{C\rho_1}{t - \Delta t}.$$

Combining this with (3.6), we see that (3.6) now holds for all $\rho > 0$. The conclusion of the lemma now follows from **[1]**. □

Lemma 3.1 thus shows that $\{v_x^h\}$ has strong L^1 limit points. In the next lemma, we prove that these limit points can be identified as the derivatives of limit points of $\{v^h\}$.

LEMMA 3.2. *Let v^h denote a sequence of approximate solutions which converge to v^* uniformly in Q. Then $v_x^h \to v_x^*$ in $L^p(Q)$ for every $p \in [1, \infty)$.*

PROOF. By Lemma 3.1, every subsequence of $\{v_x^h(\cdot, t)\}$ has a subsequence which converges in $L^1(\mathbf{R})$. Thus let $\{h'\} \subseteq \{h\}$ and let $v_x^{h'}(\cdot, t)$ converge to a function $\xi(x)$ in $L^1(\mathbf{R})$. We shall show that $\xi(x) = v_x^*(\cdot, t)$ a.e.

First, if $\psi \in H_0^1(a, b)$, then

$$\int_a^b [v_x^{h'}(x,t) - \psi_x(x,t)][v^{h'}(x,t) - \psi(x,t)] = 0.$$

We take $\psi = \eta\phi + (1-\eta)v^*(\cdot, t)$ in this relation, where $\phi \in H_0^1(a, b)$. Since $v^{h'} \to v^*$ uniformly and $v_x^{h'}(\cdot, t) \to \xi$ in L^1 we obtain, by letting $h \to 0$, that

$$\eta\int_a^b [\xi - \eta\phi_x - (1-\eta)v_x^*(\cdot, t)][\phi - v^*(\cdot, t)]\, dx = 0.$$

Dividing by η and letting $\eta \to 0$, we then find that

$$\int_a^b [\xi - v_x^*(\cdot, t)][\phi - v^*(\cdot, t)]\, dx = 0$$

for all $\phi \in H_0^1(a, b)$. This shows that $\xi = v_x^*(\cdot, t)$ a.e. Thus every subsequence of $\{v_x^h(\cdot, t)\}$ has in turn a subsequence which converges to $v_x^*(\cdot, t)$ in $L^1(\mathbf{R})$. Therefore the entire sequence converges to $v_x^*(\cdot, t)$ in $L^1(\mathbf{R})$. Finally, since $\|v_x^h\|_\infty \leqslant \gamma_0$ for every h, we have that

$$\|v_x^h - v_x^*\|_{p,Q} \leqslant C(p)\left[\int_0^T \|v_x^h(\cdot, t) - v_x^*(\cdot, t)\|_{1,\mathbf{R}}\, dt\right]^{1/p} \to 0$$

by the dominated convergence theorem. □

We remark that the proof of Lemma 3.2 is an adaptation of an argument given by Minty in **[14]**.

The next theorem contains the main results of this section.

THEOREM 3.3. *Let v, ζ_l, and ζ_r denote the exact solution and interface curves for the problem* (1.8). *Then*

$$v^h \to v \quad \textit{uniformly in } Q, \tag{3.7}$$

$$v_x^h \to v_x \quad \textit{in } L^p(Q), p < \infty, \tag{3.8}$$

and

$$\zeta_l^h, \zeta_r^h \to \zeta_l, \zeta_r \quad \textit{uniformly in } [0, T]. \tag{3.9}$$

We prove (3.7) by showing that limits of converging sequences from $\{v^h\}$ satisfy the weak equation (1.8), and so agree with its solution v, which is known to be unique. Thus let $\{v^h\}$ denote such a sequence and let $v^h \to v^*$ uniformly in Q, so that $v_x^h(\cdot, t) \to v_x^*(\cdot, t)$ in $L^1(\mathbf{R})$ for every $t > 0$. It will be sufficient to show that

(3.10)
$$\int_{\mathbf{R}} v^*(x,\cdot)\phi(x,\cdot)\Big|_{T_1}^{T_2} dx + \int_{T_1}^{T_2}\int_{\mathbf{R}}\left[-v^*\phi_t + mv^*v_x^*\phi_x + \frac{m(m-2)}{m-1}(v_x^*)^2\phi\right] dx\,dt = 0$$

for all C^∞ functions ϕ satisfying (1.4) and for $0 < T_1 < T_2 \leqslant T$.

Given such a function ϕ, let $\phi_k^n = \phi(x_k, t_n)$ and consider the quantity

(3.11)
$$\sum_{n=N_1}^{N_2-1}\left\{\sum_k\left[\frac{v_k^{n+1}-v_k^n}{\Delta t} - m(v_k^n+\varepsilon)\frac{Av_k^n}{\Delta x^2} - \frac{m}{m-1}\left(\frac{v_{k+1}^n - v_{k-1}^n}{2\Delta x}\right)^2\right]\phi_k^n\Delta x\right\}\Delta t$$

for appropriate N_1 and N_2. Now, the expression in brackets vanishes for $\overline{K}(n+1) \leqslant k \leqslant K(n+1)$. And for other values of k, we have that $(v_k^{n+1} - v_k^n)/\Delta t = O(1)$ by Lemma 2.4, and

$$(v_k^n + \varepsilon)\frac{Av_k^n}{\Delta x^2} = O(\Delta x)\frac{w_{k+1}^n - w_k^n}{\Delta x} = O(1)$$

by Lemma 2.1. Thus the quantity (3.11) approaches 0 as $h \to 0$.

On the other hand, we can sum by parts in (3.11) and match the resulting terms with the corresponding integrals in (3.10). We shall carry out the details only for the most complicated term. Using Lemma 2.2, we may rewrite the second two terms in (3.11) as follows:

(3.12)
$$\begin{aligned}
&-m\sum\sum\left[(v_k^n+\varepsilon)\phi_k^n\frac{w_{k+1}^n - w_k^n}{\Delta x} + \frac{1}{m-1}\left(\frac{w_{k+1}^n + w_k^n}{2}\right)^2\phi_k^n\right]\Delta x\Delta t \\
&= m\sum\sum\left[w_k^n\left\{\frac{(v_k^n+\varepsilon)\phi_k^n - (v_{k-1}^n+\varepsilon)\phi_{k-1}^n}{\Delta x}\right\} - \frac{1}{m-1}(w_k^n)^2\phi_k^n\right] \\
&\quad\times\Delta x\Delta t + O(\Delta x) \\
&= m\sum\sum\left[w_k^n\left\{\frac{v_k^n - v_{k-1}^n}{\Delta x}\phi_k^n + (v_{k-1}^n+\varepsilon)\frac{\phi_k^n - \phi_{k-1}^n}{\Delta x}\right\} - \frac{1}{m-1}(w_k^n)^2\phi_k^n\right] \\
&\quad\times\Delta x\Delta t + O(\Delta x) \\
&= \sum\sum\frac{m(m-2)}{m-1}(w_k^n)^2\phi_k^n\Delta x\Delta t + m\sum\sum v_k^n w_k^n\phi_x(\tilde{x}_k^n, t_n)\Delta x\Delta t + O(\Delta x)
\end{aligned}$$

for some $\tilde{x}_k^n \in [x_{k-1}, x_k]$. We shall show that the second sum here converges to the second term on the right in (3.10). First note that, since $v_x^h(x,t) = w_k^n$ on $T_{k-1}^n \cup S_{k-1}^{n-1}$,

$$w_k^n\Delta x\Delta t = \iint_{T_{k-1}^n\cup S_{k-1}^{n-1}} v_x^h(x,t)\,dx\,dt.$$

Since ϕ is smooth and v^h is Lipschitz in x and Hölder continuous in t,

$$v_k^n w_k^n\phi_x(\tilde{x}_k^n, t_n)\Delta x\Delta t = \iint_{T_{k-1}^n\cup S_{k-1}^{n-1}} v^h v_x^h\phi_x\,dx\,dt + O(\Delta x + \Delta t^{1/2})\Delta x\Delta t.$$

Therefore the second sum in (3.12) is

$$\int_{T_1}^{T_2}\int_{\mathbf{R}} m v^h v_x^h \phi_x \, dx\, dt + O(\Delta x),$$

which approaches

$$\int_{T_1}^{T_2}\int_{\mathbf{R}} m v^* v_x^* \phi_x \, dx\, dt$$

as $h \to 0$, since $v^h \to v^*$ in L^∞ and $v_x^h \to v_x^*$ in L^1, with v^h and v_x^h uniformly bounded. The other terms in (3.11) are handled in a similar manner. Thus v^* satisfies (3.10) for all appropriate test functions, and so coincides with the unique solution v of the problem (1.8).

The proof of (3.9) is based upon the following technical lemma, which will be used again in §4 for the derivation of error bounds for the approximate interface curves.

LEMMA 3.4. *Let $\{\zeta_r^h\}$ be a subsequence such that $\zeta_r^h \to \zeta_r^*$ uniformly in $[0, T]$ as $h \to 0$. Then for every $t > 0$ and for any positive numbers δ and η,*

$$\text{(3.13)} \quad \int_t^{t+\eta} v(\zeta_r^*(s) - \delta, s)\, ds \geqslant \frac{m-1}{m}\left\{\delta\left[\zeta_r^*(t+\eta) - \zeta_r^*(t)\right] - \frac{\delta^2\eta}{(m+1)t}\right\}$$

for $0 < t,\ t+\eta \leqslant T$. Moreover, if $d\zeta_r^*/dt$ exists and is positive at t, then there are positive numbers δ_0 and C such that

$$\text{(3.14)} \quad v(\zeta_r^*(t) - \delta, t) \geqslant C\delta$$

holds for $0 \leqslant \delta \leqslant \delta_0$.

PROOF. Let p, q, and N denote the largest integers in $\delta/\Delta x$, $\eta/\Delta t$, and $t/\Delta t$ respectively. Then

$$\text{(3.15)} \quad \sum_{n=N}^{N+q-1} v_{K(n)-p}^n \Delta t = \sum_{n=N}^{N+q-1}\left[v_{K(n)}^n - \sum_{j=K(n)-p+1}^{K(n)} w_j^n \Delta x\right]\Delta t.$$

Using Lemma 2.2 and the definition (1.15) of ζ_r^h, we have that

$$\begin{aligned} -\sum_{j=K(n)-p+1}^{K(n)} w_j^n \Delta x &= \sum_{j=K(n)-p+1}^{K(n)}\left[-w_{K(n)}^n + \sum_{l=j}^{K(n)-1}\frac{Av_l^n}{\Delta x^2}\Delta x\right]\Delta x \\ &\geqslant \frac{m-1}{m}\sum_{j=K(n)-p+1}^{K(n)}\left[\frac{\zeta_r^h(t_{n+1}) - \zeta_r^h(t_n)}{\Delta t} - \frac{1}{m+1}\frac{p\Delta x}{t_n}\right]\Delta x \\ &= \frac{m-1}{m}\left[\frac{\zeta_r^h(t_{n+1}) - \zeta_r^h(t_n)}{\Delta t}p\Delta x - \frac{1}{m+1}\frac{(p\Delta x)^2}{t_n}\right]. \end{aligned}$$

Substituting this into (3.15) and discarding the nonnegative term $v_{K(n)}^n$, we obtain

$$\text{(3.16)} \quad \sum_{n=N}^{N+q-1} v_{K(n)-p}^n \Delta t \geqslant \frac{m-1}{m}\left\{\left[\zeta_r^h(t_{N+q}) - \zeta_r^h(t_N)\right]p\Delta x - \frac{(p\Delta x)^2(q\Delta t)}{(m+1)t_N}\right\}.$$

Now (3.13) follows by letting $h \to 0$ and using the uniform convergence of v^h and ζ_r^h.

If $d\zeta_r^*(t)/dt > 0$, then there is a positive number ρ such that $\zeta_r^*(t+\eta) \geqslant \zeta_r^*(t) + \rho\eta$ for small η. For such η, then, (3.13) shows that

$$\int_t^{t+\eta} v(\zeta_r^*(s) - \delta, s)\, ds \geqslant \frac{m-1}{m}\left[\frac{\delta}{2}\rho\eta - \frac{\delta^2\eta}{(m+1)t}\right].$$

Dividing by η and letting $\eta \to 0$, we thus obtain that

$$v(\zeta_r^*(t) - \delta, t) \geqslant \frac{m-1}{m}\left[\frac{\delta\rho}{2} - \frac{\delta^2}{(m+1)t}\right] \geqslant \left(\frac{m-1}{m}\frac{\rho}{4}\right)\delta$$

if $\delta \leqslant \delta_0 \equiv (m+1)t\rho/4$. □

PROOF OF (3.9). Let $\{\zeta_r^h\}$ denote any subsequence converging to a curve ζ_r^* uniformly in $[0, T]$. We shall show that $\zeta_r^*(t) = \zeta_r(t)$ for every T. First observe that, since $v^h(x, t) = 0$ for $x \geqslant \zeta_r^h(t)$, $v(x, t)$ must be 0 for $x \geqslant \zeta_r^*(t)$; thus $\zeta_r(t) \leqslant \zeta_r^*(t)$.

Now suppose that $\zeta_r < \zeta_r^*$ on $(\bar{t}, \bar{t}+\eta)$ with $\zeta_r(\bar{t}) = \zeta_r^*(\bar{t})$. Then since ζ_r is increasing, there must be a time $t \in (\bar{t}, \bar{t}+\eta)$ at which $d\zeta_r^*/dt$ exists and is positive. But then (3.14) shows that $v(\zeta_r^*(t) - \delta, t)$ is positive for small δ. However, this implies that $\zeta_r(t) \geqslant \zeta_r^*(t)$, which is a contradiction. Therefore there is no maximal time $\bar{t}$ for which $\zeta_r^*(t) = \zeta_r(t)$ for $0 \leqslant t \leqslant \bar{t}$. Since ζ_r and ζ_r^* agree at $t = 0$, they agree for all t. Similar arguments hold for $\zeta_l^h(t)$. □

4. Error bounds for the approximate solution and interface curves. In this section we prove the following theorem.

THEOREM 4.1. *Fix $T > 0$. Then there is a constant C such that, for $0 \leqslant t \leqslant T$,*

$$\|v^h(\cdot, t) - v(\cdot, t)\|_{\infty,\mathbf{R}} \leqslant C\min\left[\left(\frac{\Delta x^\alpha|\log\Delta x|}{t}\right)^{1/(p+3)}, t^{1/2} + \Delta x\right] \tag{4.1}$$

and

$$|\zeta^h(t) - \zeta(t)| \leqslant Ct^{1/(m+1)}(\Delta x^\alpha|\log\Delta x|)^{1/(2(p+3))}, \tag{4.2}$$

where ζ is either ζ_l or ζ_r. Here

$$\alpha = \begin{cases} 1, & 1 < m < 2, \\ 1/(m-1), & 2 \leqslant m, \end{cases}$$

and

$$p = \begin{cases} (m+1)/(m-1), & 1 < m < 2, \\ m+1, & 2 \leqslant m. \end{cases}$$

We remark that, if the initial data v_0 satisfies the hypothesis (2.20), then the term $|\log\Delta x|$ may be omitted from the bounds in (4.1) and (4.2).

The proof of Theorem 4.1 will be given in a sequence of lemmas. First we introduce the weak truncation error associated with an approximate solution. If ϕ is a smooth function satisfying (1.4), define

$$J(v, \phi, t_1, t_2) = \int_{t_1}^{t_2}\int_{\mathbf{R}}\left[v_t\phi + mvv_x\phi_x + \frac{m(m-2)}{m-1}v_x^2\phi\right] dx\, dt. \tag{4.3}$$

Thus v is a weak solution of (1.6) if and only if $J(v, \phi, t_1, t_2) = 0$ for all ϕ and all intervals (t_1, t_2). The weak truncation error associated with the approximate solution v^h is then the functional $J(v^h + \varepsilon, \cdot, t_1, t_2)$. We have the following estimate for J.

LEMMA 4.2. *Let f satisfy* (1.4) *and assume that f_t and f_x are in $L^\infty(\mathbf{R} \times [0, T])$ with $f(x, T) = 0$ for all x. Then if $\phi = (v^h + \varepsilon)^{(2-m)/(m-1)} f$,*

$$(4.4) \qquad |J(v^h + \varepsilon, \phi, \delta, T)| \leqslant C|||f|||\Delta x^\alpha |\log \delta|,$$

where $|||f||| = \|f\|_\infty + \|f_t\|_\infty + \|f_x\|_\infty$.

The proof of Lemma 4.2 is quite technical and lengthy. We therefore postpone it to §5.

Next, we define functions u and u^h by the relations

$$u = v^{1/(m-1)} \quad \text{and} \quad u^h = (v^h + \varepsilon)^{1/(m-1)}.$$

In the following lemma we exploit the above estimate for J to obtain a bound for $u^h - u$ in L^{m+1}.

LEMMA 4.3. *There is a constant C such that*

$$(4.5) \quad \int_0^T \int_a^b |u^h - u|^{m+1}\, dx\, dt + \int_{\mathbf{R}} \left[\int_0^T ((u^h)^m - u^m)_x\, dt \right]^2 dx \leqslant C\left[L(u, u^h) \Delta x^\alpha |\log \Delta x| + (\Delta x)^{m/(m-1)} \right],$$

where

$$L(u, u^h) = \|(u^h)^m - u^m\|_{\infty, \mathbf{R}\times[0,T]} + \left\| \int_t^T ((u^h)^m - u^m)_x\, d\tau \right\|_{\infty, \mathbf{R}\times[0,T]}.$$

PROOF. Let ϕ and f be as in Lemma 4.2. Then after integrating by parts in (4.3) and substituting, we obtain

$$(4.6) \quad J(v^h + \varepsilon, \phi, \delta, T) = (m-1) \int_\delta^T \int_{\mathbf{R}} \left[u_t^h f + (u^h)_x^m f_x \right] dx\, dt$$
$$= (m-1) \left[-\int_{\mathbf{R}} u^h(x, \delta) f(x, \delta)\, dx - \int_\delta^T \int_{\mathbf{R}} \left(u^h f_t - (u^h)_x^m f_x \right) dx\, dt \right].$$

We shall replace δ in these integrals by 0. The resulting error in the second integral on the right will be no more than $C|||f|||\delta$. To estimate the change in the first integral, we use the fact that u^h is Hölder continuous in t, which may be established as follows. When $m \in (1, 2]$ we have from Lemma 2.7 that

$$|u^h(x, t_2) - u^h(x, t_1)| \leqslant C|v^h(x, t_2) - v^h(x, t_1)| \leqslant C|t_2 - t_1|^{1/2}.$$

And when $m > 2$,

$$|u^h(x, t_2) - u^h(x, t_1)|^{m-1} \leqslant |u^h(x, t_2)^{m-1} - u^h(x, t_1)^{m-1}|$$
$$= |v^h(x, t_2) - v^h(x, t_1)| \leqslant C|t_2 - t_1|^{1/2}.$$

Thus u^h is Hölder continuous in t with exponent $\alpha/2$, and

$$\left| \int_{\mathbf{R}} \left[u^h(x, \delta) f(x, \delta) - u^h(x, 0) f(x, 0) \right] dx \right| \leqslant C|||f|||\delta^{\alpha/2}.$$

We thus obtain from (4.6) and Lemma 4.2 that

$$\int_{\mathbf{R}} u^h(x,0)f(x,0)\,dx + \int_0^T\int_{\mathbf{R}}\left[u^h f_t - (u^h)_x^m f_x\right]dx\,dt \leqslant C|||f|||(\Delta x^\alpha|\log\delta| + \delta^{\alpha/2}).$$

Now subtract from this the weak form of the original equation for u, (1.3), and take $\delta = \Delta x^2$. The result is that

$$\int_0^T\int_{\mathbf{R}}\left[(u^h - u)f_t - ((u^h)^m - u^m)_x f_x\right]dx\,dt \leqslant C|||f|||\Delta x^\alpha|\log\Delta x|. \tag{4.7}$$

We have used here the fact that

$$\|u(\cdot,0) - u^h(\cdot,0)\|_{\infty,\mathbf{R}} \leqslant C\Delta x^\alpha,$$

which follows directly from the definitions of u and u^h and from the Lipschitz continuity of v_0.

In (4.7) choose

$$f(x,t) = \int_T^t\left[(u^h)^m - u^m\right](x,s)\,ds + (T-t)\varepsilon^{m/(m-1)}.$$

It is clear that f satisfies (1.4) and the conditions of Lemma 4.2. Observe also that

$$\int_0^T\int_{\mathbf{R}}(u^h - u)f_t = \int_0^T\int_{\mathbf{R}}(u^h - u)\left[(u^h)^m - u^m - \varepsilon^{m/(m-1)}\right]$$
$$\geqslant \int_0^T\int_a^b|u^h - u|^{m+1} - C\varepsilon^{\alpha m},$$

and that

$$\left[(u^h)^m - u^m\right]_x f_x = \frac{1}{2}\frac{\partial}{\partial t}\left[\int_T^t((u^h)^m - u^m)_x(x,s)\,ds\right]^2,$$

so that

$$\int_0^T\int_{\mathbf{R}}\left[(u^h)^m - u^m\right]_x f_x = -\frac{1}{2}\int_{\mathbf{R}}\left[\int_0^T((u^h)^m - u^m)_x(x,s)\,ds\right]^2 dx.$$

Now (4.5) follows by making these substitutions into (4.7). □

The next lemma contains the corresponding estimate for the error in v^h.

LEMMA 4.4. *There is a constant C such that*

$$\int_0^T\int_a^b|v^h - v|^p\,dx\,dt \leqslant C\left[L(u,u^h)\Delta x^\alpha|\log\Delta x| + \Delta x^{\alpha m}\right],$$

where L is as in Lemma 4.3 *and α and p are as in Theorem* 4.1.

PROOF. When $m \geqslant 2$, we have

$$|(v^h + \varepsilon) - v| = |(u^h)^{m-1} - u^{m-1}| \leqslant (m-1)M^{(m-2)/(m-1)}|u^h - u|.$$

And if $1 < m < 2$, then

$$|(v^h + \varepsilon) - v|^{1/(m-1)} \leqslant |(v^h + \varepsilon)^{1/(m-1)} - v^{1/(m-1)}| = |u^h - u|.$$

Since $\varepsilon = O(\Delta x)$, we have in either case that

$$|v^h - v|^p \leqslant C(|u^h - u|^{m+1} + \Delta x^p).$$

The conclusion then follows from Lemma 4.3. □

In order to deduce the L^∞ bound for $v - v^h$ from the above L^p bound, we shall require the following interpolation inequality.

LEMMA 4.5. *Let z be a continuous function on $S_T = \mathbf{R} \times [0, T]$ with support $z(\cdot, t) \subseteq [a, b]$ for $t \in [0, T]$. Assume also that*
(a) $z_x \in L^\infty(S_T)$, *and*
(b) $z_t(\cdot, t) \in L^1(\mathbf{R})$ *with* $\|z_t(\cdot, t)\|_{1,\mathbf{R}} \leqslant C_0/t$.
Let $p \in [1, \infty)$ and $l > 0$ be given. Then there is a constant C independent of l such that the inequality

$$\|z(\cdot, t)\|_{\infty,\mathbf{R}} \leqslant \frac{C}{(lt)^{1/(p+3)}} F(z) \|z\|_{p,S_T}^{p/(p+3)}$$

holds for $0 < t \leqslant T - l$. Here

$$F(z) = \|z_x\|_{\infty,S_T}^{1/(p+1)} \left[\sup_{t \leqslant s \leqslant T} \|z(\cdot, s)\|_{p,\mathbf{R}}^{2p/(p+1)} + C_0 \|z_x\|_{\infty,S_T}^{(p-1)/(p+1)} \right]^{1/(p+3)}.$$

PROOF. For $t > 0$ we have

$$\begin{aligned} |z(x, t)|^{p+1} &\leqslant (p + 1) \int_{\mathbf{R}} |z(x, t)|^p |z_x(x, t)| \, dx \\ &\leqslant C \|z_x\|_{\infty,Q} y(t)^p, \end{aligned} \tag{4.8}$$

where $y(t) = \|z(\cdot, t)\|_{p,\mathbf{R}}$. Differentiating the definition of $y(t)$, we obtain

$$y(t)^{p-1} \left| \frac{dy}{dt} \right| \leqslant C \|z(\cdot, t)\|_{\infty,\mathbf{R}}^{p-1} \|z_t(\cdot, t)\|_{1,\mathbf{R}}. \tag{4.9}$$

Combining (4.8) and (4.9), we thus obtain

$$y(t)^{p-1} \left| \frac{dy}{dt} \right| \leqslant C y(t)^{p(p-1)/(p+1)} \|z_x\|_{\infty,S_T}^{(p-1)/(p+1)} \|z_t(\cdot, t)\|_{1,\mathbf{R}},$$

so that

$$y(t)^{(p-1)/(p+1)} \left| \frac{dy}{dt} \right| \leqslant C \|z_x\|_{\infty,S_T}^{(p-1)/(p+1)} \|z_t(\cdot, t)\|_{1,\mathbf{R}}. \tag{4.10}$$

Now let $\chi(t)$ be the cut-off function

$$\chi(t) = \begin{cases} 1, & 0 \leqslant t \leqslant T - l, \\ 1 - \frac{1}{l}[t - (T - l)], & T - l < t \leqslant T. \end{cases}$$

Then for $q > 1$ and $0 < t \leqslant T - l$, we have

$$\begin{aligned} y(t)^q = (y\chi)^q |_T^t &\leqslant q \int_t^T (\chi y)^{q-1} \left(|\chi_t| y + \chi \left| \frac{dy}{dt} \right| \right) \\ &\leqslant \frac{q}{l} \int_t^T \left(y^q + y^{q-1} \left| \frac{dy}{dt} \right| \right) \\ &= \frac{q}{l} \int_t^T y^{q-1-(p-1)/(p+1)} \left(y^{1+(p-1)/(p+1)} + y^{(p-1)/(p+1)} \left| \frac{dy}{dt} \right| \right). \end{aligned}$$

Now choose $q = p(p+3)/(p+1) > 1$ and use (4.10) to estimate the second term in the above integrand. The result is that

$$y(t)^{p(p+3)/(p+1)} \leqslant \frac{C}{l}\left[\int_0^T y(t)^p\,dt\right] \sup_{t\leqslant s\leqslant T}\left[y(s)^{2p/(p+1)} + \|z_x\|_{\infty,S_T}^{(p-1)/(p+1)}\|z_t(\cdot,s)\|_{1,\mathbf{R}}\right].$$

Using hypothesis (b), this may be rewritten as

(4.11)
$$y(t)^{p(p+3)/(p+1)} \leqslant \frac{C}{lt}\|z\|_{p,S_T}^p\left[\sup_{t\leqslant s\leqslant T}\|z(\cdot,s)\|_{p,\mathbf{R}}^{2p/(p+1)} + C_0\|z_x\|_{\infty,S_T}^{(p-1)/(p+1)}\right].$$

Now by (4.8) we have that

$$\|z(\cdot,t)\|_{\infty,\mathbf{R}} \leqslant C\|z_x\|_{\infty,S_T}^{1/(p+1)}y(t)^{p/(p+1)}. \tag{4.12}$$

The result then follows by substituting (4.11) into (4.12). □

We can now prove the first half of Theorem 4.1 by applying the interpolation inequality of Lemma 4.5 to the L^p error bound in Lemma 4.4 as follows.

Proof of (4.1). It is clear that $v^h - v$ satisfies the first hypothesis of Lemma 4.5. In addition, hypothesis (b) is satisfied because of (3.3) and the results of [5]. (Alternatively, we can observe that the conclusion of Lemma 4.5 remains valid for limits of functions which satisfy hypotheses (a) and (b) uniformly. And v is such a function by the results of §3.) Replacing $T - l$ and T by T and $T + 1$ respectively, we therefore obtain that

$$\|v^h(\cdot,t) - v(\cdot,t)\|_{\infty,\mathbf{R}} \leqslant \frac{C}{t^{1/(p+3)}}\|v^h - v\|_{p,S_T'}^{p/(p+3)},$$

where $S_T' = \mathbf{R} \times [0, T+1]$ and $p \in [1, \infty)$. But it we choose p as in the statement of Theorem 4.1, then Lemma 4.4 shows that

$$\|v^h - v\|_{p,S_T'}^p \leqslant C\Delta x^\alpha|\log \Delta x|.$$

Therefore

$$\|v^h(\cdot,t) - v(\cdot,t)\|_{\infty,\mathbf{R}} \leqslant C\left(\frac{\Delta x^\alpha|\log \Delta x|}{t}\right)^{1/(p+3)}.$$

On the other hand, the Hölder continuity of v^h and v in time shows that

$$\|v^h(\cdot,t) - v(\cdot,t)\|_{\infty,\mathbf{R}} \leqslant C(t^{1/2} + \Delta x)$$

for any t. The estimate (4.1) follows from these last two inequalities. □

Next, we deduce the bound (4.2) for the error in the approximate interfaces from the above bound (4.1) for the error in v^h. Again we drop the subscripts and denote ζ_r^h and ζ_r by ζ^h and ζ.

Proof of (4.2). First we refine the result in Lemma 3.4. Divide (3.13) by η and let $\eta \to 0$. The result is that

$$v(\zeta(t) - y, t) \geqslant \frac{m-1}{m}\left[y\dot{\zeta}(t) - \frac{y^2}{(m+1)t}\right] \quad \text{a.e.} \tag{4.13}$$

A similar bound holds for v^h: taking $q = 1$ in (3.16), we have that

$$v^n_{K(n)-j} \geqslant \frac{m-1}{m}\left[(j\Delta x)\dot{\zeta}^h(t_n) - \frac{(j\Delta x)^2}{(m+1)t_n}\right].$$

Since v^h is Lipschitz in x and Hölder continuous in t, and since ζ^h is piecewise linear, we can conclude that

$$(4.14) \qquad v^h(\zeta^h(t) - y, t) \geqslant \frac{m-1}{m}\left[\dot{\zeta}^h(t)y - \frac{y^2}{(m+1)t}\right] - C\Delta x \quad \text{a.e.}$$

Now let $\{v_1, v_2\} = \{v, v^h\}$ and let ζ_1 and ζ_2 be the corresponding interface curves. Assume that, at time t, $0 \leqslant \zeta_2(t) - \zeta_1(t) = y$. Then we have, using either (4.13) or (4.14), that

$$\begin{aligned}(4.15) \quad \|v^h(\cdot, t) - v(\cdot, t)\|_{\infty,\mathbf{R}} &\geqslant |(v_2 - v_1)(\zeta_1(t), t)| \\ &= v_2(\zeta_1(t), t) = v_2(\zeta_2(t) - y, t) \\ &\geqslant \frac{m-1}{m}\left[\dot{\zeta}_2(t)y - \frac{y^2}{(m+1)t}\right] - C\Delta x \\ &\geqslant \frac{m-1}{m}\left[(\dot{\zeta}_2 - \dot{\zeta}_1)(\zeta_2 - \zeta_1) - \frac{(\zeta_2 - \zeta_1)^2}{(m+1)t}\right] - C\Delta x.\end{aligned}$$

Now let

$$F(t) = [\zeta_2(t) - \zeta_1(t)]^2$$

and let $E(t)$ be the bound in (4.1) for

$$\|v^h(\cdot, t) - v(\cdot, t)\|_{\infty,\mathbf{R}}.$$

Then (4.15) shows that

$$\dot{F}(t) - \frac{2}{(m+1)t}F(t) \leqslant E(t),$$

where we have subsumed the $C\Delta x$ term into $E(t)$. Integrating, we thus obtain that, for any $\delta > 0$,

$$(4.16) \qquad F(t) \leqslant t^{2/(m+1)}\left[\delta^{-2/(m+1)}F(\delta) + \int_\delta^T s^{-2/(m+1)}E(s)\,ds\right].$$

Now, since

$$F(\delta) = [\zeta^h(\delta) - \zeta(\delta)]^2 \leqslant (|\zeta^h(\delta) - \zeta^h(0)| + |\zeta(0) - \zeta(\delta)|)^2 \leqslant C\delta^2,$$

the first term on the right of (4.16) approaches 0 as $\delta \to 0$. In addition, the integrand in the second term of (4.16) is bounded by

$$Cs^{-(1/(m+1)+1/(p+3))}(\Delta x^\alpha|\log \Delta x|)^{1/(p+3)}.$$

A short computation shows that the exponent of s here is greater than -1, so that this term is integrable on $[0, t]$. We may therefore conclude that

$$|\zeta^h(t) - \zeta(t)| = F(t)^{1/2} \leqslant Ct^{1/(m+1)}(\Delta x^\alpha|\log \Delta x|)^{1/2(p+3)}. \quad \square$$

5. Estimate for the weak truncation error. This section is devoted to the proof of Lemma 4.2. Since the arguments are technically involved, even though simple, we begin by listing a few facts to which we will refer systematically. In the estimates to follow we will use repeatedly the bounds established in §2 without specific mention.

Let T_k^n and S_k^n be the triangles in Figure 3.1, and let v^h be the piecewise linear interpolation of v_k^n introduced in §3. In T_k^n, v^h can be written in any one of the equivalent forms

$$v^h(x,t) \equiv \begin{cases} v_k^n + w_{k+1}^n(x-x_k) + \sigma_{k+1}^n(t-t_n), \\ v_{k+1}^n + w_{k+1}^n(x-x_{k+1}) + \sigma_{k+1}^n(t-t_n), \\ v_{k+1}^{n+1} + w_{k+1}^n(x-x_{k+1}) + \sigma_{k+1}^n(t-t_{n+1}), \end{cases} \tag{5.1}$$

and in S_k^n it can be written in any one of the equivalent forms

$$v^h(x,t) = \begin{cases} v_k^n + w_{k+1}^{n+1}(x-x_k) + \sigma_k^n(t-t_n), \\ v_k^{n+1} + w_{k+1}^{n+1}(x-x_k) + \sigma_k^n(t-t_{n+1}), \\ v_{k+1}^{n+1} + w_{k+1}^{n+1}(x-x_{k+1}) + \sigma_k^n(t-t_{n+1}). \end{cases} \tag{5.2}$$

Let $\phi \in H^1(S_T)$ be a test function satisfying (1.4), denote with ϕ^h the piecewise linear interpolation of the values $\phi_k^n = \phi(x_k, t_n)$ and set

$$\psi_k^n = \frac{\phi_k^n - \phi_{k-1}^n}{\Delta x}, \quad \Phi_k^n = \frac{\phi_k^{n+1} - \phi_k^n}{\Delta t}.$$

In T_k^n, ϕ^h can be written in any of the equivalent forms

$$\phi^h(x,t) = \begin{cases} \phi_k^n + \psi_{k+1}^n(x-x_k) + \Phi_{k+1}^n(t-t_n), \\ \phi_{k+1}^n + \psi_{k+1}^n(x-x_{k+1}) + \Phi_{k+1}^n(t-t_n), \\ \phi_{k+1}^{n+1} + \psi_{k+1}^n(x-x_{k+1}) + \Phi_{k+1}^n(t-t_{n+1}), \end{cases} \tag{5.3}$$

and in S_k^n

$$\phi^h(x,t) = \begin{cases} \phi_k^n + \psi_k^{n+1}(x-x_k) + \Phi_k^n(t-t_n), \\ \phi_k^{n+1} + \psi_k^{n+1}(x-x_k) + \Phi_k^n(t-t_{n+1}), \\ \phi_{k+1}^{n+1} + \psi_k^{n+1}(x-x_{k+1}) + \Phi_k^n(t-t_{n+1}). \end{cases} \tag{5.4}$$

REMARK. From (5.1)–(5.2) it follows that v^h can be written as the value of v^h at any one of the corners of T_k^n (S_k^n respectively), plus terms of the order of Δx. An analogous fact holds for ϕ^h.

5.1. *The basic identity.* Consider the quantity

(5.5)

$$J(v^h+\varepsilon, \phi^h, \delta, T) = \iint_{S_{T,\delta}} \left\{ v_t^h \phi^h + m(v^h+\varepsilon) v_x^h \phi_x^h + \frac{m(m-2)}{m-1} (v_x^h)^2 \phi^h \right\} dx\, dt,$$

where δ is a fixed positive number. We assume for simplicity that $T = (N+1)\Delta t$ and $\delta = n_0 \Delta t$ for two positive integers $n_0 < N$, and calculate the various parts of (5.5) as follows:

(i)
$$J_1 = \iint_{S_{T,\delta}} v_t^h \phi^h \, dx \, d\tau = \sum_{n_0}^{N} \sum_{\mathbf{Z}} \left\{ \iint_{T_k^n} v_t^h \phi^h \, dx \, d\tau + \iint_{S_k^n} v_t^h \phi^h \, dx \, dt \right\}$$
$$= \sum_{n_0}^{N} \sum_{\mathbf{Z}} \sigma_k^n \phi_k^n \Delta x \Delta t + \sum_{n_0}^{N} \sum_{\mathbf{Z}} \frac{1}{6} \left[\sigma_k^n \psi_{k+1}^{n+1} - \sigma_k^n \psi_k^n \right] (\Delta x)^2 \Delta t,$$

(ii)
$$J_2 = m \iint_{S_{T,\delta}} \left\{ (v^h + \varepsilon) v_x^h \phi_x^h + (v_x^h)^2 \phi^h \right\} dx \, d\tau$$
$$= m \iint_{S_{T,\delta}} v_x^h \left[(v^h + \varepsilon) \phi^h \right]_x dx \, d\tau$$
$$= \sum_{n_0}^{N} \sum_{\mathbf{Z}} \left\{ \iint_{T_k^n} w_{k+1}^n \left[(v^h + \varepsilon) \phi^h \right]_x dx \, d\tau + \iint_{S_k^n} w_{k+1}^{n+1} \left[(v^h + \varepsilon) \phi^h \right]_x dx \, d\tau \right\}.$$

On T_k^n we have
$$\left[(v^h + \varepsilon) \phi^h \right]_x = (v_k^n + \varepsilon) \psi_{k+1}^n + w_{k+1}^n \phi_{k+1}^n + w_{k+1}^n \psi_{k+1}^n \left[(x - x_k) + (x - x_{k+1}) \right] + (\sigma_{k+1}^n \psi_{k+1}^n + w_{k+1}^n \Phi_{k+1}^n)(t - t_n).$$

Therefore
$$\iint_{T_k^n} w_{k+1}^n \left[(v^h + \varepsilon) \phi^h \right]_x dx \, d\tau = \frac{1}{2} \left[(v_k^n + \varepsilon) \psi_{k+1}^n + w_{k+1}^n \phi_{k+1}^n \right] \Delta x \Delta t + \frac{1}{6} \left[\sigma_{k+1}^n \psi_{k+1}^n + w_{k+1}^n \Phi_{k+1}^n \right] w_{k+1}^n \Delta x (\Delta t)^2.$$

By direct calculation
$$w_{k+1}^n \left[(v_k^n + \varepsilon) \psi_{k+1}^n + w_{k+1}^n \phi_{k+1}^n \right] = -(v_k^n + \varepsilon) \frac{[Av]_k^n}{(\Delta x)^2} \phi_k^n + (v_k^n + \varepsilon) \frac{w_{k+1}^n - w_k^n}{\Delta x} \phi_{k+1}^n + (v_k^n + \varepsilon) w_k^n \frac{\phi_{k+1}^n - \phi_k^n}{\Delta x} + (w_{k+1}^n)^2 \phi_{k+1}$$
$$= -(v_k^n + \varepsilon) \frac{[Av]_k^n}{(\Delta x)^2} \phi_k^n + \frac{1}{\Delta x} \left\{ (v_{k+1}^n + \varepsilon) w_{k+1}^n \phi_{k+1}^n - (v_k^n + \varepsilon) w_k^n \phi_k^n \right\}.$$

This implies that
$$\sum_{n_0}^{N} \sum_{\mathbf{Z}} \iint_{T_k^n} w_{k+1}^n \left[(v^h + \varepsilon) \phi^h \right]_x dx \, dt = -\frac{1}{2} \sum_{n_0}^{N} \sum_{\mathbf{Z}} (v_k^n + \varepsilon) \frac{[Av]_k^n}{(\Delta x)^2} \phi_k^n \Delta x \Delta t + \frac{1}{6} \sum_{n_0}^{N} \sum_{\mathbf{Z}} w_k^n \left[\sigma_k^n \psi_k^n + w_k^n \Phi_k^n \right] \Delta x (\Delta t)^2.$$

Analogous calculations on S_k^n give

$$\sum_{n_0}^{N}\sum_{\mathbf{Z}}\iint_{S_k^n} w_{k+1}^{n+1}\left[(v^h+\varepsilon)\phi^h\right]_x\,dx\,dt = -\frac{1}{2}\sum_{n_0}^{N}\sum_{\mathbf{Z}}\left(v_k^{n+1}+\varepsilon\right)\frac{[Av]_k^{n+1}}{(\Delta x)^2}\phi_k^{n+1}$$
$$-\frac{1}{6}\sum_{n_0}^{N}\sum_{\mathbf{Z}} w_{k+1}^{n+1}\left[\sigma_k^n\psi_{k+1}^{n+1}+w_{k+1}^{n+1}\Phi_k^n\right]\Delta x(\Delta t)^2.$$

Substituting there calculations in the expression of J_2 we obtain

$$\text{(ii)}'\quad J_2=\sum_{n_0+1}^{N}\sum_{\mathbf{Z}} m\left(v_k^n+\varepsilon\right)\frac{[Av]_k^n}{(\Delta x)^2}\phi_k^n$$
$$-\frac{m}{2}\Delta t\sum_{\mathbf{Z}}\Delta x\left\{\left(v_k^{n_0}+\varepsilon\right)\frac{[Av]_k^{n_0}}{(\Delta x)^2}\phi_k^{n_0}+\left(v_k^{N+1}+\varepsilon\right)\frac{[Av]_k^{N+1}}{(\Delta x)^2}\phi_k^{N+1}\right\}$$
$$+\frac{m}{6}\sum_{n_0}^{N}\sum_{\mathbf{Z}} w_k^n\left(\sigma_k^n\psi_k^n+w_k^n\Phi_k^n\right)\Delta x(\Delta t)^2$$
$$-\frac{m}{6}\sum_{n_0}^{N}\sum_{\mathbf{Z}} w_{k+1}^{n+1}\left(\sigma_k^n\psi_{k+1}^{n+1}+w_{k+1}^{n+1}\Phi_k^n\right)\Delta x(\Delta t)^2.$$

We finally transform the remaining integral in $J(v^h+\varepsilon,\phi^h,\delta,T)$.

$$\text{(iii)}\quad J_3=-\frac{m}{m-1}\iint_{S_{T,\delta}}\left(v_x^h\right)^2\phi^h\,dx\,dt$$
$$=-\frac{m}{m-1}\sum_{n_0}^{N}\sum_{\mathbf{Z}}\left\{\iint_{T_k^n}\left(w_{k+1}^n\right)^2\phi^h\,dx\,dt+\iint_{S_k^n}\left(w_{k+1}^{n+1}\right)^2\phi^h\,dx\,dt\right\}.$$

Using (5.3)–(5.4), by standard calculations we obtain

$$J_3=-\frac{m}{m-1}\sum_{n_0+1}^{N}\sum_{\mathbf{Z}}\left(w_k^n\right)^2\phi_k^n\Delta x\Delta t$$
$$+\frac{m}{3(m-1)}\sum_{n_0+1}^{N}\sum_{\mathbf{Z}}\left(w_k^n\right)^2\psi_k^n(\Delta x)^2\Delta t$$
$$-\frac{m}{6(m-1)}\sum_{n_0}^{N}\sum_{\mathbf{Z}}\left[\left(w_k^{n+1}\right)^2-\left(w_k^n\right)^2\right]\Phi_k^n\Delta x(\Delta t)^2$$
$$-\frac{m}{m-1}\Delta t\sum_{\mathbf{Z}}\left\{\frac{1}{2}\left[\left(w_k^{n_0}\right)^2\phi_k^{n_0}+\left(w_k^{N+1}\right)^2\phi_k^{N+1}\right]\right.$$
$$\left.-\frac{1}{6}\left[\left(w_k^{n_0}\right)^2\psi_k^{n_0}+\left(w_k^{N+1}\right)^2\psi_k^{N+1}\right]\right\}\Delta x.$$

We rewrite the first summand in J_3 as

$$\left(w_k^n\right)^2\phi_k^n=\left(\frac{w_{k+1}^n+w_k^n}{2}\right)^2\phi_k^n+\left[\left(w_k^n\right)^2-\left(\frac{w_{k+1}^n+w_k^n}{2}\right)^2\right]\phi_k^n,$$

and combine the expressions of J_i, $i = 1, 2, 3$, so obtained, as parts of $J(v^h + \varepsilon, \phi^h, \delta, T)$, to deduce the basic identity

$$
\begin{aligned}
&J(v^h + \varepsilon, \phi^h, \delta, T) \\
&\quad = \sum_{n_0+1}^{N} \sum_{\mathbf{Z}} \left\{ \frac{v_k^{n+1} - v_k^n}{\Delta t} - m(v_k^n + \varepsilon) \frac{[Av]_k^n}{(\Delta x)^2} \right. \\
&\qquad\qquad \left. - \frac{m}{m-1} \left(\frac{v_{k+1}^n - v_{k-1}^n}{2\Delta x} \right)^2 \right\} \phi_k^n \Delta x \Delta t \\
&\qquad + \Delta t \sum_{\mathbf{Z}} \sigma_k^{n_0} \phi_k^n \Delta x \\
&\qquad - \frac{m\Delta t}{2} \sum_{\mathbf{Z}} \left\{ (v_k^{n_0} + \varepsilon) \frac{[Av]_k^{n_0}}{(\Delta x)^2} \phi_k^{n_0} + (v_k^{N+1} + \varepsilon) \frac{[Av]_k^{N+1}}{(\Delta x)^2} \phi_k^{N+1} \right\} \Delta x \\
&\qquad + \frac{\Delta x}{6} \sum_{n_0}^{N} \sum_{\mathbf{Z}} [\sigma_k^n \psi_{k+1}^{n+1} - \sigma_k^n \psi_k^n] \Delta x \Delta t \\
&\qquad - \frac{m\Delta t}{6} \sum_{n_0}^{N} \sum_{\mathbf{Z}} w_{k+1}^{n+1} [\sigma_k^n \psi_{k+1}^{n+1} + w_{k+1}^{n+1} \Phi_k^n] \Delta x \Delta t \\
&\qquad + \frac{m\Delta x}{3(m-1)} \sum_{n_0}^{N} \sum_{\mathbf{Z}} (w_k^n)^2 \psi_k^n \Delta x \Delta t \\
&\qquad + \frac{m\Delta t}{6} \sum_{n_0}^{N} \sum_{\mathbf{Z}} w_k^n [\sigma_k^n \psi_k^n + w_k^n \Phi_k^n] \Delta x \Delta t \\
&\qquad - \frac{m\Delta t}{6(m-1)} \sum_{n_0}^{N} \sum_{\mathbf{Z}} \left[(w_k^{n+1})^2 - (w_k^n)^2 \right] \Phi_k^n \Delta x \Delta t \\
&\qquad - \frac{m\Delta t}{2(m-1)} \sum_{\mathbf{Z}} \left\{ \left[(w_k^{n_0})^2 \phi_k^{n_0} + (w_k^{N+1})^2 \phi_k^{N+1} \right] \right. \\
&\qquad\qquad \left. - \frac{1}{3} \left[(w_k^{n_0})^2 \psi_k^{n_0} + (w_k^{N+1})^2 \psi_k^{N+1} \right] \right\} \Delta x \\
&\qquad + \frac{m\Delta x}{4(m-1)} \sum_{n_0+1}^{N} \sum_{\mathbf{Z}} \frac{[Av]_k^n}{(\Delta x)^2} [3w_k^n + w_{k+1}^n] \phi_k^n \Delta x \Delta t \\
&\quad = \sum_{i=1}^{10} H_i.
\end{aligned}
\tag{5.6}
$$

From now on we will select test functions ϕ of the form

$$\phi = (v^h + \varepsilon)^{(2-m)/(m-1)} f, \qquad m > 1,$$

where f satisfies (1.4). We will estimate the H_i, $i = 1, 2, \ldots, 10$, in (5.6) in terms of f, f_x and f_t. For notational simplicity set $\|f, f_x, f_t\|_{\infty, S_T} \equiv \|f, f_x, f_t\|$ and $|||f||| = \|f\| + \|f_x\| + \|f_t\|$. We start by making elementary estimates of ϕ and its first derivatives in terms of f.

Denoting by ϕ^h the piecewise linear interpolation of ϕ and recalling that ε is of the order of Δx we have

$$\|\phi^h\|_\infty \leqslant \begin{cases} C\|f\| & \text{if } 1 < m \leqslant 2, \\ C\|f\|(\Delta x)^{(2-m)/(m-1)} & \text{if } m > 2. \end{cases} \tag{5.7}$$

Also

(a)

$$\begin{aligned} \psi_k^n &= \frac{\phi_k^n - \phi_{k-1}^n}{\Delta x} = \frac{1}{\Delta x}\left\{(v_k^n+\varepsilon)^{(2-m)/(m-1)} f_k^n - (v_{k-1}^n+\varepsilon)^{(2-m)/(m-1)} f_{k-1}^n\right\} \\ &= \frac{1}{\Delta x}\left\{\left[(v_k^n+\varepsilon)^{(2-m)/(m-1)} - (v_{k-1}^n+\varepsilon)^{(2-m)/(m-1)}\right]\right\} \\ &\quad \times f_k^n + (v_{k-1}^n+\varepsilon)^{(2-m)/(m-1)} \frac{f_k^n - f_{k-1}^n}{\Delta x} \\ &\leqslant \left|\frac{2-m}{m-1}\right| (v_{\tilde{k}}^n+\varepsilon)^{(2-m)/(m-1)-1} |w_k^n|\,\|f\| + (v_{k-1}^n+\varepsilon)^{(2-m)/(m-1)} \|f_x\|; \end{aligned}$$

(b)

$$\begin{aligned} \Phi_k^n = \frac{\phi_k^{n+1} - \phi_k^n}{\Delta t} &\leqslant \left|\frac{2-m}{m-1}\right| (v_k^{\tilde{n}}+\varepsilon)^{(2-m)/(m-1)-1} |\sigma_k^n|\,\|f\| \\ &\quad + (v_k^n+\varepsilon)^{(2-m)/(m-1)} \|f_t\|, \end{aligned}$$

where

$$v_{\tilde{k}}^n = \min\{v_k^n; v_{k-1}^n\}, \qquad v_k^{\tilde{n}} = \min\{v_k^{n+1}, v_k^n\}.$$

From (a)–(b) we deduce

$$|\psi_k^n| \leqslant \begin{cases} C(v_{\tilde{k}}^n+\varepsilon)^{-1}\|f\| + C\|f_x\| & \text{if } 1 < m \leqslant 2, \\ C\left[(v_{\tilde{k}}^n+\varepsilon)^{-1}\|f\| + \|f_x\|\right](\Delta x)^{(2-m)/(m-1)} & \text{if } m > 2, \end{cases} \tag{5.8}$$

$$|\Phi_k^n| \leqslant \begin{cases} C(v_k^{\tilde{n}}+\varepsilon)^{-1}|\sigma_k^n|\,\|f\| + C\|f_t\| & \text{if } 1 < m \leqslant 2, \\ C\left[(v_k^{\tilde{n}}+\varepsilon)^{-1}|\sigma_k^n|\,\|f\| + \|f_t\|\right](\Delta x)^{(2-m)/(m-1)} & \text{if } m > 2. \end{cases} \tag{5.9}$$

5.2. *An auxiliary lemma.*

LEMMA 5.1. *For all test functions ϕ of the form $\phi = (v^h+\varepsilon)^{(2-m)/(m-1)} f$, the estimate*

$$|J(v^h+\varepsilon, \phi^h, \delta, T)| \leqslant C|||f|||(\Delta x)^\alpha |\log \delta|$$

holds, where $\alpha = \min\{1; 1/(m-1)\}$.

PROOF. We estimate the H_i on the right side of (5.6) separately. At points where the difference equation (1.11) holds, the summand in H_1 vanishes, and by virtue of Lemma 2.4, the sum extended over the remaining (n, k) is of the order of Δx.

Therefore $|H_1| \leqslant C(\Delta x)\|\phi\|_\infty \leqslant C|||f|||(\Delta x)^\alpha$. We have also easily $|H_2| + |H_3| \leqslant C|||f|||(\Delta x)^\alpha$.

In estimating H_4 we only consider the term whose summand is $\sigma_k^n \psi_{k+1}^{n+1}$. The estimate for the term whose summand is $\sigma_k^n \psi_k^n$ is analogous, and in fact simpler.

We have

$$\Delta x \sum_{n_0}^{N} \sum_{\mathbf{Z}} \sigma_k^n \psi_{k+1}^{n+1} \Delta x \Delta t = \Delta x \sum_{n_0}^{N} \sum_{\{k < \bar{K}(n+1)\} \cup [k > K(n+1)]} \sigma_k^n \psi_{k+1}^{n+1} \Delta x \Delta t$$
$$+ \Delta x \sum_{n_0}^{N} \sum_{\bar{K}(n+1) \leqslant k \leqslant K(n+1)} \sigma_k^n \psi_{k+1}^{n+1} \Delta x \Delta t = \kappa^{(1)} + \kappa^{(2)}.$$

As for $\kappa^{(1)}$, using Lemma 2.4 and (5.8),

$$|\kappa^{(1)}| \leqslant C(\Delta x)^2 \max_{n,k} |\psi_k^n| \leqslant C |||f||| (\Delta x)^\alpha.$$

We estimate $\kappa^{(2)}$ by using the difference equation (1.11):

$$\kappa^{(2)} = \Delta x \sum_{n_0}^{N} \sum_{\bar{K}(n+1) \leqslant k \leqslant K(n+1)} \left[m(v_k^n + \varepsilon) \frac{[Av]_k^n}{(\Delta x)^2} + \frac{m}{m-1} \left(\frac{w_{k+1}^n + w_k^n}{2} \right)^2 \right] \psi_{k+1}^{n+1} \Delta x \Delta t$$
$$\leqslant \Delta x \sum_{n_0}^{N} \sum_{\bar{K}(n+1) \leqslant k \leqslant K(n+1)} m(v_k^n + \varepsilon) \left| \frac{[Av]_k^n}{(\Delta x)^2} \right| |\psi_{k+1}^{n+1}| \Delta x \Delta t$$
$$+ \Delta x \left| \sum_{n_0}^{N} \sum_{\bar{K}(n+1) \leqslant k \leqslant K(n+1)} \frac{m}{m-1} \left(\frac{w_{k+1} + w_k}{2} \right)^2 \psi_{k+1}^{n+1} \Delta x \Delta t \right|$$
$$= \kappa_a^{(2)} + \kappa_b^{(2)}.$$

Estimate the summand in $\kappa_a^{(2)}$ as follows:

$$m \Delta x (v_k^n + \varepsilon) \left| \frac{[Av]_k^n}{(\Delta x)^2} \right| |\psi_{k+1}^{n+1}| \leqslant C \|f_x\| (\Delta x)^\alpha \left| \frac{[Av]_k^n}{(\Delta x)^2} \right|$$
$$+ C \left(\frac{v_k^n + \varepsilon}{v_{\tilde{k}}^n + \varepsilon} \right) \|f\| (\Delta x)^\alpha \left| \frac{[Av]_k^n}{(\Delta x)^2} \right|.$$

Now observe that

$$v_k^n = v_{\widetilde{(k+1)}}^n + w_{\widetilde{(k+1)}} \Delta x = v_{\widetilde{(k+1)}}^{n+1} + \sigma_{\widetilde{(k+1)}}^n \Delta t + w_{\widetilde{(k+1)}} \Delta x,$$

where

$$w_{\widetilde{(k+1)}} = \frac{v_k^n - v_{\widetilde{(k+1)}}^n}{\Delta x} \quad \text{and} \quad \sigma_{\widetilde{(k+1)}}^n = \frac{v_{\widetilde{(k+1)}}^n - v_{\widetilde{(k+1)}}^{n+1}}{\Delta t}.$$

By the estimates in §2 $|v_{\widetilde{(k+1)}}^n - v_{\widetilde{(k+1)}}^{n+1}| \leqslant C \Delta x$, and therefore

$$\left(\frac{v_k^n + \varepsilon}{v_{\widetilde{(k+1)}}^{n+1} + \varepsilon} \right) \leqslant 1 + C \frac{\Delta x}{\varepsilon} \leqslant C.$$

Consequently for $\kappa_a^{(2)}$ we have

$$|\kappa_a^{(2)}| \leqslant C |||f||| (\Delta x)^\alpha \sum_{n_0}^{N} \sum_{\mathbf{Z}} \left| \frac{[Av]_k^n}{(\Delta x)^2} \right| \Delta x \Delta t.$$

By Lemma 2.6

$$\sum_{n_0}^{N}\sum_{\mathbf{Z}}\left|\frac{[Av]_k^n}{(\Delta x)^2}\right|\Delta x\Delta t \leqslant C\sum_{n_0}^{N}\left(1+\frac{1}{t_n}\right)\Delta t \leqslant C(1+|\log\delta|),$$

so that

$$|\kappa_a^{(2)}| \leqslant C|||f|||(\Delta x)^\alpha|\log\delta|.$$

We estimate $|\kappa_b^{(2)}|$ by performing a discrete integration by parts

$$\left|\frac{m}{m-1}\Delta x\sum_{n_0}^{N}\sum_{\bar{K}(n+1)\leqslant k\leqslant K(n+1)}\left(\frac{w_{k+1}^n+w_k^n}{2}\right)^2\left(\frac{\phi_{k+1}^{n+1}-\phi_k^{n+1}}{\Delta x}\right)\Delta x\Delta t\right|$$

$$\leqslant C\Delta x\sum_{n_0}^{N}\sum_{\mathbf{Z}}\left|\frac{[Av]_k^n}{(\Delta x)^2}\right||\phi_k^{n+1}|\Delta x\Delta t + C\Delta x\|\phi\|$$

$$\leqslant C|||f|||(\Delta x)^\alpha|\log\delta|.$$

Combining these estimates we obtain $|H_4| \leqslant C|||f|||(\Delta x)^\alpha|\log\delta|$. By similar calculations involving the use of Lemma 2.6 we deduce

$$|H_5| + |H_7| \leqslant C|||f|||(\Delta x)^\alpha|\log\delta|.$$

In estimating H_6 we first integrate by parts (discrete integration) and use the techniques above to obtain $|H_6| \leqslant C|||f|||(\Delta x)^\alpha|\log\delta|$. Analogous techniques give the desired estimates for H_8, H_9 and H_{10}. The proof is complete.

COROLLARY 5.2. *If $-L \leqslant v_{0xx} \leqslant 0$ for some positive constant L, then*

$$|J(v^h+\varepsilon,\phi^h,\delta,T)| \leqslant C|||f|||(\Delta x)^\alpha \quad \forall\delta>0.$$

5.3. PROOF OF LEMMA 4.2. Since $J(v^h+\varepsilon,\phi,\delta,T) = J(v^h+\varepsilon,\phi-\phi^h,\delta,T) + J(v^h+\varepsilon,\phi^h,\delta,T)$ and the last term has been estimated in Lemma 5.1, we have only to estimate $J(v^h+\varepsilon,\phi-\phi^h,\delta,T)$. To this end we will need the following preliminary fact.

LEMMA 5.3. *There exists a constant C independent of n, k, Δx and ϕ such that*

$$(\Delta x\cdot\Delta t)^{-1}\iint_{T_k^n}(\phi^h-\phi)\,dx\,dt \leqslant C|||f|||(\Delta x)^\alpha\left[1+(\tilde{v}_k^n+\varepsilon)^{-1}\right],$$

where $\tilde{v}_k^n = \min_{T_k^n} v^h$.

REMARK. An analogous statement holds for S_k^n.

PROOF OF LEMMA 5.3.

$$\iint_{T_k^n}(\phi^h-\phi)\,dx\,dt = \iint_{T_k^n}(\phi_k^n-\phi)\,dx\,dt$$

$$+\iint_{T_k^n}\left[\psi_k^n(x-x_k)+\Phi_k^n(t-t_n)\right]dx\,dt = I^{(1)}+I^{(2)}.$$

For $I^{(2)}$, using (5.8)–(5.9) we have

$$|I^{(2)}| \leqslant C|||f|||(\Delta x)^\alpha\left[1+(\tilde{v}_K^n+\varepsilon)^{-1}\right]\Delta x\Delta t.$$

Estimating $I^{(1)}$,

$$I^{(1)} = \iint_{T_k^n} \{[\phi_k^n - \phi(x, t_n)] + [\phi(x, t) - \phi(x, t_n)]\} \, dx \, dt$$
$$\leqslant \{\|\phi_x\|_{\infty, T_k^n} \Delta x + \|\phi_t\|_{\infty, T_k^n} \Delta t\} \Delta x \Delta t,$$

and the desired estimate follows from (5.8)–(5.9).

COROLLARY 5.4. *For all n and k, if* $\tilde{\bar{v}}_k^n = \max_{T_k^n} v^h$ *then*

$$\frac{(\tilde{\bar{v}}_k^n + \varepsilon)}{\Delta x \Delta t} \iint_{T_k^n} (\phi^h - \phi) \, dx \, dt \leqslant C|||f|||(\Delta x)^\alpha,$$

$$\frac{\Delta x}{\Delta x \Delta t} \iint_{T_k^n} (\phi^h - \phi) \, dx \, dt \leqslant C|||f|||(\Delta x)^\alpha.$$

REMARK. Corollary 5.4 holds if T_k^n is replaced by S_k^n.

We are now in the position to estimate $J(v^h + \varepsilon, \phi - \phi^h, \delta, T)$.

(5.10)

$$\begin{aligned}
J(v^h + \varepsilon, \phi - \phi^h, \delta, T) &= \iint_{S_{T,\delta}} \Big\{ v_t^h(\phi - \phi^h) + m(v^h + \varepsilon)(v^h)_x(\phi - \phi^h)_x \\
&\qquad + \Big(m(v_x^h)^2 - \frac{m}{m-1}(v_x^h)^2\Big)(\phi - \phi^h)\Big\} \, dx \, dt \\
&= \sum_{n_0}^{N} \sum_{\mathbf{Z}} \left[\sigma_k^n - \frac{m}{m-1}\left(\frac{w_{k+1}^n + w_k^n}{2}\right)^2\right] \iint_{S_k^n} (\phi - \phi^h) \, dx \, dt \\
&\quad + \sum_{n_0}^{N} \sum_{\mathbf{Z}} \left[\sigma_{k+1}^n - \frac{m}{m-1}\left(\frac{w_{k+2}^n + w_{k+1}^n}{2}\right)^2\right] \iint_{T_k^n} (\phi - \phi^h) \, dx \, dt \\
&\quad + \frac{1}{4} \sum_{n_0}^{N} \sum_{\mathbf{Z}} (3w_{k+1}^n + w_{k+2}^n)(w_{k+1}^n - w_{k+2}^n) \iint_{T_k^n} (\phi - \phi^h) \, dx \, dt \\
&\quad + \frac{1}{4} \sum_{n_0}^{N} \sum_{\mathbf{Z}} (3w_{k+1}^{n+1} + w_k^{n+1})(w_k^{n+1} - w_{k+1}^{n+1}) \iint_{S_k^n} (\phi - \phi^h) \, dx \, dt \\
&\quad + m \iint_{S_{T,\delta}} \left[(v^h + \varepsilon)(v^h)_x(\phi - \phi^h)_x + (v_x^h)^2(\phi - \phi^h)\right] dx \, dt \\
&= \sum_{i=1}^{5} P_i.
\end{aligned}$$

Estimate of $|P_1| + |P_2|$.

$$\begin{aligned}
P_1 &= \sum_{n_0}^{N} \sum_{\bar{K}(n+1) \leqslant k \leqslant K(n+1)} \left[\sigma_k^n - \frac{m}{m-1}\left(\frac{w_{k+1}^n + w_k^n}{2}\right)^2\right] \iint_{S_k^n} (\phi - \phi^h) \, dx \, dt \\
&\quad + \sum_{n_0}^{N} \sum_{[k < \bar{K}(n+1)] \cup [k > K(n+1)]} \left[\sigma_k^n - \frac{m}{m-1}\left(\frac{w_{k+1}^n + w_k^n}{2}\right)^2\right] \iint_{S_k^n} (\phi - \phi^h) \, dx \, dt \\
&= p_1^* + p_1^{**}.
\end{aligned}$$

For p_1^{**} by the estimates in §2 and Corollary 5.4 we obtain easily $|p_1^{**}| \leqslant C|||f|||(\Delta x)^\alpha$. In estimating p_1^* we use the difference equation (1.11).

$$|p_1^*| \leqslant C\sum_{n_0}^{N}\sum_{\mathbf{Z}}\left|\frac{[Av]_k^n}{(\Delta x)^2}\right|\Delta x\Delta t\left\{\frac{(v_k^n+\varepsilon)}{\Delta x\Delta t}\left|\iint_{S_k^n}(\phi-\phi^h)\,dx\,dt\right|\right\}$$
$$\leqslant C|||f|||(\Delta x)^\alpha|\log\delta|.$$

The estimate of $|P_2|$ is analogous.

Estimate of $|P_3|+|P_4|$.

$$|P_3|+|P_4| \leqslant C\sum_{n_0}^{N}\sum_{\mathbf{Z}}\left|\frac{[Av]_k^n}{(\Delta x)^2}\right|\Delta x\Delta t\left\{\frac{\Delta x}{\Delta x\Delta t}\left|\iint_{S_k^n\cup T_k^n}(\phi-\phi^h)\,dx\,dt\right|\right\}.$$

By Corollary 5.4 and Lemma 2.6, $|P_3|+|P_4| \leqslant C|||f|||(\Delta x)^\alpha|\log\delta|$.

Estimate of P_5.

$$P_5 = m\sum_{n_0}^{N}\sum_{\mathbf{Z}}\left\{\iint_{T_k^n}\left[(v^h+\varepsilon)v_x^h(\phi-\phi^h)_x+(v_x^h)^2(\phi-\phi^h)\right]dx\,dt\right.$$
$$\left.+\iint_{S_k^n}\left[(v^h+\varepsilon)v_x^h(\phi-\phi^h)+(v_x^h)^2(\phi-\phi^h)\right]dx\,dt\right\}$$
$$= Q^{(1)}+Q^{(2)}.$$

We estimate $Q^{(1)}$ by performing an integration by parts in x over each T_k^n.

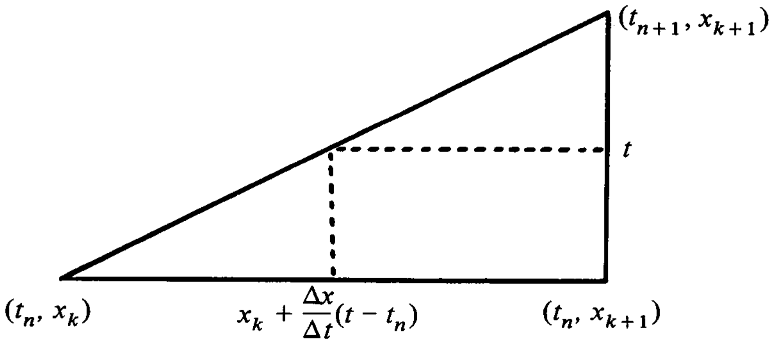

FIGURE 3

We have

$$\iint_{T_k^n}(v^h+\varepsilon)v_x^h(\phi-\phi^h)_x\,dx\,dt = \int_{t_n}^{t_{n+1}}\int_{x_k+\Delta x(t-t_n)/\Delta t}^{x_{k+1}}(v^h+\varepsilon)(v_x^h)(\phi-\phi^h)_x\,dx\,dt$$
$$=\int_{t_n}^{t_{n+1}}(v^h+\varepsilon)v_x^h(\phi-\phi^h)(t,x_{k+1})\,dt$$
$$-\int_{t_n}^{t_{n+1}}(v^h+\varepsilon)v_x^h(\phi-\phi^h)\left(t,x_k+\frac{\Delta x}{\Delta t}(t-t_n)\right)dt$$
$$-\iint_{T_k^n}(v_x^h)^2(\phi-\phi^h)\,dx\,dt.$$

Therefore

$$Q^{(1)} = m\sum_{n_0}^{N}\sum_{\mathbf{Z}}\left\{\int_{t_n}^{t_{n+1}}(v^h+\varepsilon)w_{k+1}^n(\phi-\phi^h)(t, x_{k+1})\,dt\right.$$
$$\left.-\int_{t_n}^{t_{n+1}}(v^h+\varepsilon)w_{k+1}^n(\phi-\phi^h)\left(t, x_k+\frac{\Delta x}{\Delta t}(t-t_n)\right)dt\right\}.$$

Similar calculations for $Q^{(2)}$ give

$$Q^{(2)} = m\sum_{n_0}^{N}\sum_{\mathbf{Z}}\left\{-\int_{t_n}^{t_{n+1}}(v^h+\varepsilon)w_{k+1}^{n+1}(\phi-\phi^h)(t, x_k)\,dt\right.$$
$$\left.+\int_{t_n}^{t_{n+1}}(v^h+\varepsilon)w_{k+1}^{n+1}(\phi-\phi^h)\left(t, x_k+\frac{\Delta x}{\Delta t}(t-t_n)\right)dt\right\}.$$

For all $k \in \mathbf{Z}$ and $t \in [t_n, t_{n+1}]$,

$$|(\phi-\phi^h)(t, x_k)| \leqslant C|||f|||(\Delta x)^\alpha(|\sigma_k^n|+1)\Delta x$$

so that

$$\sum_{n_0}^{N}\sum_{\mathbf{Z}}\int_{t_n}^{t_{n+1}}(v^h+\varepsilon)w_k^n(\phi-\phi^h)(t, x_k)\,dt$$
$$\leqslant C|||f|||(\Delta x)^\alpha\sum_{n_0}^{N}\sum_{\mathbf{Z}}(1+|\sigma_k^n|)\Delta x\Delta t$$
$$\leqslant C|||f|||(\Delta x)^\alpha|\log\delta|.$$

Consequently

$$|Q^{(1)}+Q^{(2)}| \leqslant C|||f|||(\Delta x)^\alpha|\log\delta|$$
$$+\left|m\sum_{n_0}^{N}\sum_{\mathbf{Z}}\int_{t_n}^{t_{n+1}}\left[w_{k+1}^{n+1}-w_{k+1}^n\right](v^h+\varepsilon)(\phi-\phi^h)\left(t, x_k+\frac{\Delta x}{\Delta t}(t-t_n)\right)dt\right|.$$

Now from calculations analogous to the ones leading to (5.8)–(5.9), we have in T_k^n (and in S_k^n)

$$|(v^h+\varepsilon)(\phi-\phi^h)| \leqslant (v^h+\varepsilon)\left(\|\phi_x\|_{\infty, T_k^n}\Delta x+\|\phi_t\|_{\infty, T_k^n}\Delta t\right)$$
$$\leqslant C(v^h+\varepsilon)\left[1+(v_k^n+\varepsilon)^{-1}+(v_k^{\tilde{n}}+\varepsilon)^{-1}\right]|||f|||(\Delta x)^\alpha$$
$$\leqslant C|||f|||(\Delta x)^\alpha.$$

Moreover

$$w_k^{n+1}-w_k^n = \frac{1}{\beta}\left[\frac{v_k^{n+1}-v_k^n}{\Delta t}+\frac{v_{k-1}^{n+1}-v_k^n}{\Delta t}\right]\Delta x = \beta^{-1}\left[\sigma_k^n+\sigma_{k-1}^n\right].$$

Hence

$$|Q^{(1)}+Q^{(2)}| \leqslant C|||f|||(\Delta x)^\alpha|\log\delta| + C|||f|||(\Delta x)^\alpha\sum_{n_0}^{N}\sum_{\mathbf{Z}}|\sigma_k^n|\Delta x\Delta t$$
$$\leqslant C|||f|||(\Delta x)^\alpha|\log\delta|.$$

This completes the proof of Lemma 4.2.

6. Numerical results. In this section we discuss briefly the results of our numerical experiments with the scheme (1.11). All computations were carried out on the CDC 6600 at Indiana University. In each case we specify only the value of Δx used. The values of ε and Δt were always chosen to be the smallest and largest convenient values, respectively, consistent with the mesh conditions [A3] and [A4].

Not surprisingly, the condition [A4] on $\Delta t/\Delta x^2$ is in fact necessary in practice. But the condition [A3] on ε is probably overly restrictive. For example, for the specific problems discussed below, [A3] requires that $\varepsilon \geqslant 13.8\Delta x$, which is not "small" when, say, $\Delta x = .05$. Indeed, for the second problem discussed below, we found that the accuracy increased noticeably as ε decreased. A practical (but not theoretically justified) alternative to [A3] is the condition

$$\varepsilon \geqslant \frac{m+1}{2(m-1)}\gamma_0 \Delta x. \tag{6.1}$$

Such a condition is sufficient for the bounds (2.1) and (2.2) for v^h and $\partial v^h/\partial x$ to remain in effect. (The more stringent condition [A3] was required only for bound (2.15) for $\partial^2 v^h/\partial x^2$.) For the specific problems discussed below, condition (6.1) requires only that $\varepsilon \geqslant .58\Delta x$.

For purposes of comparison, we used the Barenblatt-Pattle solution v, which for $m = 2$ is defined by (see [**16**])

$$v(x,t) = \begin{cases} \dfrac{1}{\zeta(t)}\left[1 - \left(\dfrac{x}{\zeta(t)}\right)^2\right], & |x| \leqslant \zeta(t), \\ 0, & |x| > \zeta(t), \end{cases} \tag{6.2}$$

where $\zeta(t) = [12(t+1)]^{1/3}$.

First, we applied the scheme (1.11) taking $v_k^0 = v(x_k, 0)$, $\zeta_l^0 = -\zeta(0)$, and $\zeta_r^0 = \zeta(0)$. The computations were performed with three different sets of mesh parameters. Comparing the exact and computed solutions, at $t = \frac{1}{2}$, we found the following:

Δx	$\lvert\zeta(\frac{1}{2}) - \zeta^h(\frac{1}{2})\rvert$	$\lVert v(\cdot,\frac{1}{2}) - v^h(\cdot,\frac{1}{2})\rVert_{\infty,\mathbf{R}}$
.1	.0202	.00666
.05	.0106	.00340
.025	.00551	.00173

Quite clearly, the observed errors in both ζ^h and v^h are $O(\Delta x)$. This is significantly better than the rates predicted by Theorem 4.1, which are $O(|\Delta x \log \Delta x|^{1/12})$ and $O(|\Delta x \log \Delta x|^{1/6})$ for ζ^h and v^h respectively. These discrepancies are explained by the fact that the solution (6.2) has derivatives of all orders which are uniformly bounded on its support, whereas the bounds (4.1) and (4.2) were derived under the minimal smoothness conditions which all solutions are known to satisfy. Another difference between the observed and theoretical results is that the observed rate of convergence for ζ^h is the same as that for v^h, whereas Theorem 4.1 predicts roughly that $|\zeta - \zeta^h| = O(\|v - v^h\|_\infty^{1/2})$. However, the computation (4.15) shows that, as long as $|\zeta(t) - \zeta^h(t)|$ is small,

$$|\zeta(t) - \zeta^h(t)| \leqslant \frac{C}{\dot{\zeta}_2(t)} \|v(\cdot, t) - v^h(\cdot, t)\|_{\infty,\mathbf{R}} + C\Delta x.$$

Thus when the interfaces are known to be moving with speeds bounded away from 0, the rate of convergence of ζ^h will in fact coincide with that of v^h.

In the second example we took the same initial function v_0 as before, but now with $\zeta_l^0 = -3$ and $\zeta_r^0 = 3$. Thus v_0 is neither concave nor continuously differentiable on $[\zeta_0^l, \zeta_0^r]$. The results were as follows:

Δx	$\lvert\zeta(\frac{1}{2}) - \zeta^h(\frac{1}{2})\rvert$	$\|v(\cdot, \frac{1}{2}) - v^h(\cdot, \frac{1}{2})\|_{\infty,\mathbf{R}}$
.1	.1107	.06905
.05	.1073	.06697
.025	.1055	.06592

We are uncertain as to whether meaningful comparisons can be made between these data and (4.1) and (4.2). Nevertheless, it is clear that, at least qualitatively, Theorem 4.1 gives the correct result: in the absence of smoothness, the convergence may be quite slow.

The scheme (1.11) is thus seen to have two shortcomings. The first is that the parabolic stability condition [A4] makes it impractical to apply the scheme with small values of Δx. This difficulty can probably be overcome by employing instead a suitable implicit variant of (1.11). We intend to discuss such a scheme elsewhere. The other shortcoming of the present method is the unsatisfactory rate of convergence. While this phenomenon is partly due to the coarseness of the exact solutions themselves, it may be possible to effect some improvement by a more sophisticated treatment near the interfaces.

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