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RESEARCH NOTE

An interpretation of the Nusselt–Rayleigh number relationship for convection in a spherical shell

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SUMMARY

Recent studies on the relationship between the Nusselt (Nu) and Rayleigh (Ra) numbers for base-heated convection in a spherical shell with a constant viscosity show that the power-law index is around 1/4, which is different from the value of 1/3 predicted by a simple boundary layer theory. We show that such a difference may be caused by the flow pattern due to the geometry. The flow pattern of the convection in a spherical shell at relatively low Ra, at least, less than 10^6, is characterized by narrow upwelling and broad downwelling, which is similar to the opposite flow pattern of internally heated convection. Convection in the internally heated case predicts the power-law index of 1/4. We demonstrate this relationship based on the concept of 'local' Rayleigh (Ra_l) and Nusselt (Nu_l) numbers

Key words: mantle convection, Nusselt number, Rayleigh number, spherical shell.

INTRODUCTION

Large-scale mantle convection is undoubtedly 3-D in character partly because of its geometry. Thus, it is important to understand how such a geometrical constraint influences various aspects of mantle convection. Numerical studies of the mantle convection in 3-D spherical shells (e.g. Baumgardner 1985; Glatzmaier 1988; Bercovici *et al.* 1989; Ratcliff, Schubert & Zebib 1995; Iwase 1996) give solutions that are clearly different from those obtained by 2-D or 3-D Cartesian convection. For example, for constant viscosity the upwelling flows are cylindrical and in the steady state are surrounded by sheet-like downwellings. The heat transport efficiency is also different from that of Cartesian convection. It is well known that the power-law index (β) of the Nusselt–Rayleigh numbers relationship is around 1/3 (e.g. Christensen 1984). $\beta = 1/3$ is predicted by the boundary layer theory of the box model without internal heating (Turcotte & Oxburgh 1967). However, $\beta \sim 1/4$ has been reported for spherical shell geometry models (Ratcliff, Schubert & Zebib 1996). This paper presents an interpretation of the Nu–Ra relationship of the convection in a spherical shell based on the recently introduced idea of 'local' Rayleigh and Nusselt numbers (Honda 1996).

DEFINITION OF Nu AND Ra

The Nusselt (Nu) and Rayleigh (Ra) numbers are generally defined by

$$\begin{cases} Nu \equiv \dfrac{Q_{\mathrm{cond+conv}}}{Q_{\mathrm{cond}}} \\ Ra \equiv \dfrac{g\alpha d^3 \Delta T}{\nu\kappa} \end{cases} \tag{1}$$

where $Q_{\mathrm{cond+conv}}$ is the heat flux of the convective state, and Q_{cond} that when the heat is transported only by conduction. ΔT, d, g, α, ν and κ are respectively the temperature drop across the convection layer, the depth of the convective layer, the acceleration of the gravity, the coefficient of the thermal expansion, kinematic viscosity, and thermal diffusivity. For the spherical shell model, the Nusselt number (eq. 1) at $r = r$ is given by

$$Nu \equiv \frac{Q(r)d}{k\Delta T}\frac{r^2}{r_1 r_0}, \tag{2}$$

where r_0, r_1, k and $Q(r)$ are the radii of the outer and inner shell, the thermal conductivity and the heat flux at $r = r$, respectively.

Honda (1996) recently introduced the 'local' Rayleigh (Ra_l) and Nusselt (Nu_l) numbers for interpreting the results of the convection with a strongly temperature-dependent viscosity. They are defined by

$$\begin{cases} Nu_l \equiv \dfrac{Q_l d}{k_l \Delta T_l}, \\ Ra_l \equiv \dfrac{g\alpha_l d^3 \Delta T_l}{\nu_l \kappa_l}, \end{cases} \tag{3}$$

where Q_l is the heat flux and ΔT_l is the temperature drop within a boundary layer. The suffix l stands for the 'local' value (l is either top or bottom). There are no apparent physical meanings for Nu_l and Ra_l. However, they are found to be convenient parameters to compare the various types of convection. Honda (1996) showed that Nu_l–Ra_l relationship for temperature-dependent viscosity can be interpreted by the results of constant-viscosity convection with free or rigid surfaces.

RESULTS OF 3-D SPHERICAL SHELL AND AXISYMMETRIC CONVECTION

Fig. 1 shows the *Nu–Ra* relationship for base-heated isoviscous convection in a spherical shell (full 3-D: filled and open circles; axisymmetric geometry: open triangles). The results of the full 3-D calculations were obtained by Ratcliff *et al.* (1996) and Iwase (1996). In both calculations, the convection equations (Boussinesq approximation) were solved by the control-volume (finite-volume) method, and the number of control volumes was either 40×50×100 (r, θ, ϕ) (Ratcliff *et al.* 1996) or 32×32×64 (some with 64×64×128) (Iwase 1996). The computation of the axisymmetric geometry model was performed using the full 3-D code described in Iwase (1996) by stipulating that the physical parameters, such as the velocity, temperature and pressure do not change in longitudinal direction and the longitudinal velocity is zero. The number of control volumes was 64×128 (r, θ). All the calculations were carried out under the condition of constant viscosity and the ratio of the inner to outer shell radius being 0.55, which is appropriate for the mantle of the Earth. In the full 3-D models, the results for both the tetrahedral and cubic flow patterns are shown for $Ra \leq 2\times10^5$. For $Ra > 2\times10^5$, time-dependent solutions are obtained. For the axisymmetric model, solutions of two or four cells with both rising or descending currents at poles are found. In this case, the Nusselt number and global mean temperature oscillate with time at higher Rayleigh numbers ($Ra > 10^5$). At $Ra = 5\times10^5$, the variations of the Nussult number and the globally averaged temperature with time are respectively $\sim$1.5 and $\sim$0.1 per cent for the axisymmetric model and $\sim$5 and $\sim$0.6 per cent for the full 3-D model. In this study, we take their time averages.

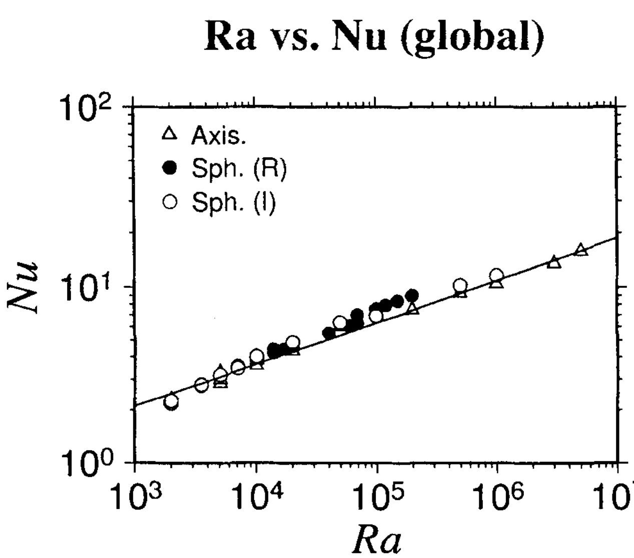

Figure 1. Relationship between Nusselt (*Nu*) and Rayleigh (*Ra*) numbers for the 3-D spherical shell convection. Triangles are the results for the axisymmetric model. Circles are those for the full 3-D model (filled circles: after Ratcliff *et al.* 1996; open circles: after Iwase 1996). The solid line is the least-squares fit.

We show the least-squares fit of the data assuming the power-law relationship between Nusselt and Rayleigh numbers ($Nu \propto Ra^\beta$) in Fig. 1. The gradient of this line (β) is 0.24, which is close to 1/4.

The flow pattern obtained for both full 3-D spherical and axisymmetric convections shows generally narrow upwellings and broad downwellings (e.g. Iwase 1996). This flow pattern is similar to the opposite flow pattern for internally heated convection. We show that this analogy works for the Nu_l–Ra_l relationship.

Nu_l–Ra_l FOR INTERNALLY HEATED CONVECTION

McKenzie, Roberts & Weiss (1974) studied the convection heated from within and introduced the following modified Nusselt (Nu_M) and Rayleigh (Ra_M) numbers,

$$\begin{cases} Nu_M \equiv \dfrac{Qd}{k\langle\Delta T\rangle}\left(1-\dfrac{\mu}{2}\right), \\ Ra_M \equiv \dfrac{g\alpha d^3}{\nu\kappa}\dfrac{Qd}{k}, \end{cases} \tag{4}$$

where $\langle\Delta T\rangle$ is the temperature difference between the horizontally averaged temperature at the bottom surface and the temperature at the top surface. μ is the fraction of heat produced within. Based on a simple boundary layer theory, McKenzie *et al.* (1974) derived the relationship between these parameters. They are given by

$$Nu_M \propto Ra_M^\beta. \tag{5}$$

The exponent β is 1/4 when the convection is totally heated from below (i.e. $\mu=0$). As the fraction of heat produced from within increases, β approaches 1/5. This difference occurs because of the breakdown of the symmetry of the down- and upwelling flows. It can be shown that the former case (i.e. $\beta=1/4$) predicts the commonly used exponent of 1/3 in the power-law *Nu–Ra* relationship. When the convection is totally driven from within (i.e. $\mu=1$), the upwelling becomes broad and weak, while the downwelling becomes narrow and strong. We can also assume that $\langle\Delta T\rangle \simeq \Delta T_{top}$. Thus, Nu_M and Ra_M can be written in terms of Ra_{top} and Nu_{top} as

$$\begin{cases} Nu_M \simeq \dfrac{Nu_{top}}{2} \\ Ra_M \simeq Ra_{top} Nu_{top}. \end{cases} \tag{6}$$

Thus, using eq. (5) with $\beta=1/5$, we obtain the following relationship between Nu_{top} and Ra_{top},

$$Nu_{top} \propto Ra_{top}^{1/4} . \tag{7}$$

We can determine the proportional constant in eqs (5) or (7) based on the results given by Schubert & Anderson (1985). They are given in Fig. 2, where we also include the cases with non-zero μ. The line, with a gradient of 1/5, is drawn by eye in the figure. This relationship is given by

$$Nu_M \simeq 2.1 \left(\frac{Ra_M}{Ra_c}\right)^{1/5} , \tag{8}$$

where Ra_c is the critical Rayleigh number (=1296 for $\mu=1$). Using eqs (6) and (8), we obtain

$$Nu_{top} \simeq 1.0 Ra_{top}^{1/4} . \tag{9}$$

Nu_l–Ra_l FOR 3-D SPHERICAL AND AXISYMMETRIC CONVECTION

We study the relationship between local Nusselt (Nu_l) and Rayleigh (Ra_l) numbers for the spherical shell geometry. We make the assumption that ΔT_l at the top or bottom boundary is the difference between the mean temperature and the top or bottom temperature, respectively. The relationship between the local Nusselt and Rayleigh numbers for the spherical shell geometry is shown in Fig. 3. Fig. 3(a) shows the relationship of Nu_{top}–Ra_{top} and Nu_{bottom}–Ra_{bottom}, and Figs 3(b) and (c) show the relationship of Nu_l–Ra_{top} and Nu_l–Ra_{bottom} (l = either top or bottom), respectively. The relationship obtained by 2-D convection for the internally heated case (eq. (9)) is also shown. In Fig. 3(a), the results are dispersed around the line for the 2-D internally heated case. However, if we use the local Rayleigh number defined at either the top or bottom, the scatter of the data becomes small (Figs 3b and c). The points are distributed close to the power-law relationship for the 2-D

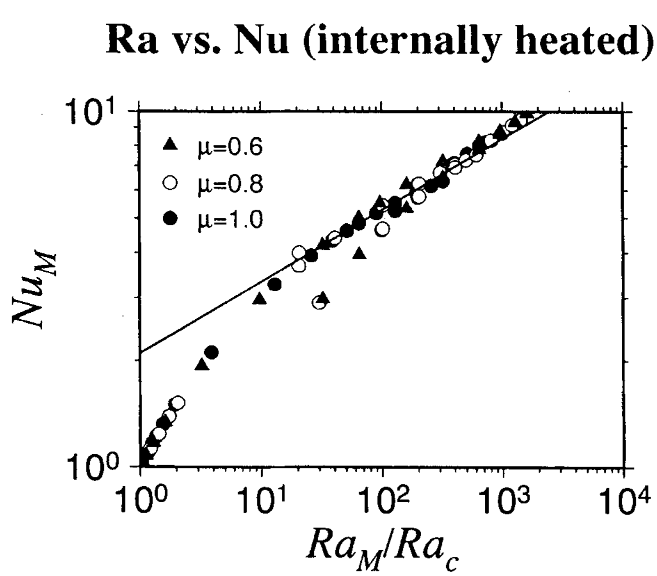

Figure 2. Relationship between modified Nusselt (Nu_M) and Rayleigh (Ra_M) numbers for 2-D convection with internal heating (after Schubert & Anderson 1985). μ is the ratio of the internal heat to the total heat source. The gradient of the solid line is 1/5.

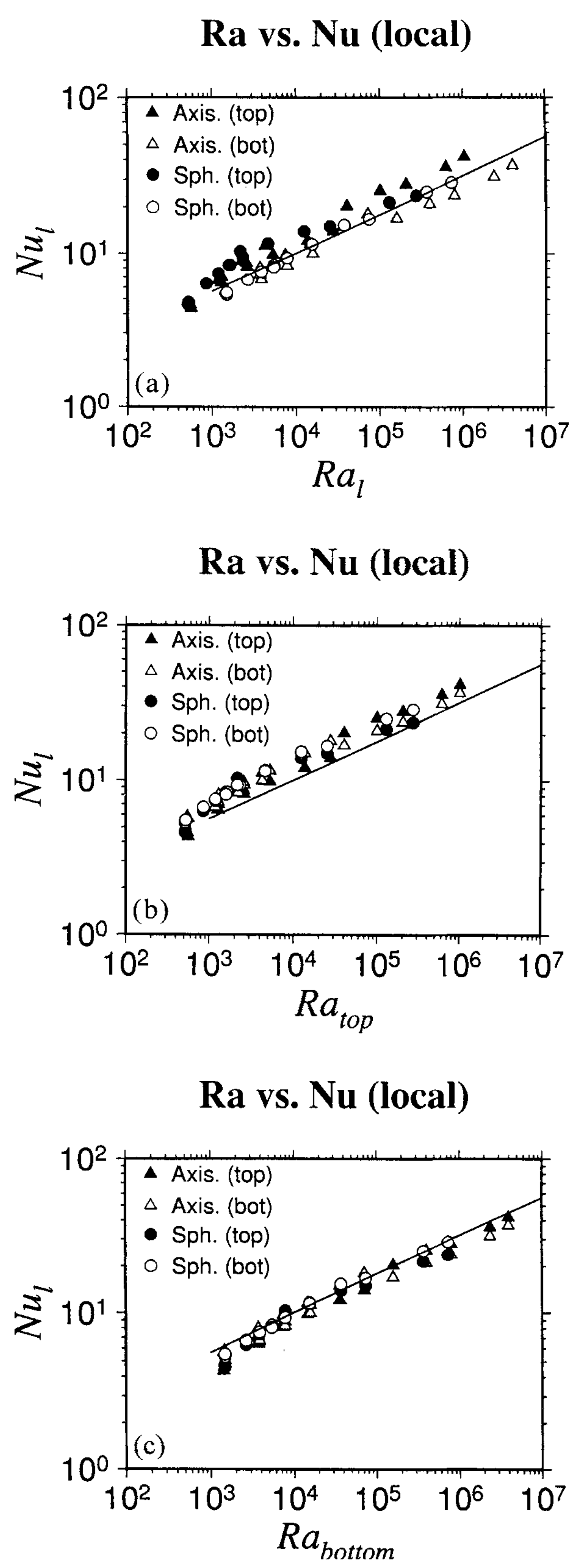

Figure 3. Relationship between local Nusselt (Nu_l) and Rayleigh (Ra_l) numbers. (a) Top Nusselt vs. top Rayleigh and bottom Nusselt vs. bottom Rayleigh numbers. (b) Local Nusselt vs. top local Rayleigh numbers. (c) Local Nusselt vs. bottom local Rayleigh numbers. The result of the 2-D convection heated from within is shown by the solid line.

internally heated case. In particular, the relationship between Nu_l and Ra_{bottom} (Fig. 3(c)) is almost the same as that obtained by the 2-D box convection with internal heating. This implies that $Nu_{bottom} \simeq Nu_{top}$ at the same Ra_{bottom}.

Davies (1993) argued that, if the core is cooled forcibly by the plate-scale flow, Q_{bottom} and Q_{top} are related by the following equations (eq. (E3) in his Appendix E):

$$Q_{bottom} = \gamma Q_{top} \frac{\Delta T_{bottom}}{\Delta T_{top}} \tag{10}$$

or, in terms of our local Nusselt number,

$$Nu_{bottom} = \gamma Nu_{top} \tag{11}$$

where γ is a constant (in his model, thermal conductivity is constant). This relationship suggests the possibility that both local Nusselt numbers are controlled by a local Rayleigh number at either the top or bottom. Since the mean mantle temperature is approximately 0.2 for the spherical shell geometry model, the local Rayleigh number at the bottom is larger than that at the top. Also, the Nu_l–Ra_l relationship is almost coincident with that for the 2-D internally heated case, if we use the Rayleigh number at the bottom (Fig. 3c). This may suggest that the convection is controlled mainly by the instability at the bottom thermal boundary layer.

CONCLUSION AND DISCUSSION

For moderate Rayleigh numbers, i.e. less than $\sim 10^6$, the Nu_l–Ra_{bottom} or Nu_l–Ra_{top} (l is either top or bottom) relationship with the power-law index of 1/4 is obtained for the spherical shell geometry model of mantle convection. In the spherical shell model, the symmetric nature of the flow pattern is broken. Generally, it is characterized by narrow upwellings and broad downwellings. A similar asymmetrical flow pattern is also found in the convection heated within for the box convection model. For these cases, we have shown that the power-law index of the Nu_l–Ra_l relationship is 1/4 based on the simple thermal boundary layer theory (McKenzie *et al.* 1974). We find that the Nu_l–Ra_l relationship for the 3-D spherical and axisymmetric convection shows the power-law index of 1/4. Qualitative studies suggest that both Nu_{top} and Nu_{bottom} may depend on Ra_{bottom} rather than Ra_{top}.

At very high Rayleigh numbers, the thicknesses of the boundary layers at the top and bottom become thin, and both upwellings and downwellings become narrower. In such cases, we can expect the breakdown of broad-scale features, and the prediction ($\beta = 1/3$) of the simple boundary layer theory (Turcotte & Oxburgh 1967) or the mean field theory for the spherical shell model (Olson 1981) may apply again (see Solheim & Peltier 1990). When the inner shell radius approaches the outer shell radius, the spherical shell geometry becomes close to 3-D box geometry. Thus, we also expect $\beta = 1/3$ in this situation. Temperature- and/or depth-dependent viscosity may also influence the heat transport efficiency. These cases will be looked at in the future.

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