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AN INTRINSIC FORMULATION OF THE PROBLEM ON ROLLING MANIFOLDS

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ABSTRACT. We present an intrinsic formulation of the kinematic problem of two n-dimensional manifolds rolling one on another without twisting or slipping. We determine the configuration space of the system, which is an $n(n+3)/2$-dimensional manifold. The conditions of no-twisting and no-slipping are encoded by means of a distribution of rank n. We compare the intrinsic point of view versus the extrinsic one. We also show that the kinematic system of rolling the n-dimensional sphere over $\mathbb{R}^n$ is controllable. In contrast with this, we show that in the case of SE(3) rolling over $\mathfrak{se}(3)$ the system is not controllable, since the configuration space of dimension 27 is foliated by submanifolds of dimension 12.

1. INTRODUCTION

Rolling surfaces without slipping or twisting is one of the classical kinematic problems that in recent years has again attracted the attention of mathematicians due to its geometric and analytic richness. The kinematic conditions of rolling without slipping or twisting are described by means of motion on a configuration space being tangential to a smooth subbundle that we call a distribution. The precise definition of the mentioned motion in the case of two n-dimensional manifolds imbedded in $\mathbb{R}^N$, given for example in [14], involves studying the behavior of the tangent bundles of the manifolds and the normal bundles induced by the imbeddings. This approach leads to significant simplifications, for instance, to study the trajectories the rolling manifolds follow it suffices to study the case in which the still manifold is the n-dimensional Euclidean space. This extrinsic point of view, which depends on the imbeddings, has been successfully applied, for example in [7, 8, 9]. The drawback of the extrinsic approach is that the

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geometric descriptions depend strongly on the imbedding under consideration.

So far, however, little attempts have been made to formulate this problem intrinsically. An early enlightening formulation is given in [2], in which the authors study the case of two abstract surfaces rolling in the above described manner. This is achieved by means of an intrinsic version of the moving frame method of Élie Cartan which, for this case, coincides with the classical intrinsic study of surfaces (see [15]). One of the important results established in [2] is the nonintegrability property of the rank two distribution corresponding to no-twisting and no-slipping restrictions, namely, if the two surfaces have different Gaussian curvature, then the distribution is of Cartan-type (see [3]). A control theoretic approach to the same problem, studied in [1], has the advantage that the kinematic restrictions are written explicitly as vector fields on appropriate bundles.

This article presents a generalization of the kinematic problem for two n-dimensional abstract manifolds rolling without twisting or slipping via an intrinsic formulation. Some of the results in the present paper were announced in [5]. We define the configuration space of the system, which is an $n(n+3)/2$-dimensional manifold and a direct analog to the one found in [1, 2]. We present an extrinsic definition of rolling for manifolds imbedded in Euclidean spaces, which is an adaptation of that presented in [14], and several equivalent definitions of rolling, involving intrinsic characteristics, and discuss their relations. The intrinsic approach permits to determine the imbedding-independent information contained in the extrinsic definition.

Moreover, we relax the smoothness condition of the rolling map to absolutely continuity. This allows to enlarge the class of mappings under consideration, still giving the possibility to apply the fundamental theorems of differential geometry and control theory without changing drastically the main classical ideas of rolling maps.

The conditions of no-twisting and no-slipping define a distribution of rank n in the tangent bundle of the configuration space. The distribution is written explicitly as a local span of vector fields defined on the configuration space. We also test the bracket generating condition of the above mentioned distribution on the known example [17] of rolling the n-dimensional sphere over the n-dimensional Euclidean space and the special group of Euclidean rigid motions SE(3) rolling over $\mathfrak{se}(3)$. As a result we obtain controllability of the first system and non controllability of the latter.

The structure of the present paper is the following. Section 2 starts with a definition of extrinsic rolling which is more appropriate for later developments than that in [14], and explain why we have adopted a slightly different definition. Here we also reformulate the classical no-twisting and no-slipping conditions for the rolling problem. In Sec. 3 we give the main formulation of extrinsic rolling as a curve on a configuration space defined as a direct sum of fiber bundles over the Cartesian product of the two rolling manifolds and we prove the equivalence of the new extrinsic definition of rolling with the previous ones and deduce the *intrinsic* definition of a rolling map. We also prove a theorem distinguishing the imbedding independent information contained in the definition of extrinsic rolling, and discuss the advantages of this novel approach. Section 4 is devoted to the construction of two distributions in the tangent bundle of the configuration space. These distributions encode the no-twisting and no-slipping kinematic conditions of the extrinsic and intrinsic rollings. These rollings can be written as curves in the configuration spaces tangent to the corresponding distributions. In Sec. 5 we present two aforementioned examples: rolling the n-dimensional sphere over the n-dimensional Euclidean space and rolling SE(3) over $\mathfrak{se}(3)$. In the first case the distribution is bracket generating, coinciding with the result obtained in [17]. In the second case, we obtain that the configuration space, of dimension 27, is foliated by 12 dimensional submanifolds.

2. Extrinsic rolling

The aim of this section is to present a definition of rolling, without slipping or twisting for Riemannian manifolds imbedded in some Euclidean space $\mathbb{R}^N$. These manifolds, hereafter denoted by M and $\widehat{M}$, are assumed to be oriented, connected, and having the same dimension $n < N$, while $\mathbb{R}^N$ is equipped with the standard Euclidean metric and standard orientation. The group $\mathrm{SE}(N)$ of orientation preserving Riemannian isometries of $\mathbb{R}^N$ will play an important role.

Objects (points, curves, ...) related to the manifold $\widehat{M}$ will be marked by a hat ($\widehat{\ }$) on top, objects related to M will be free of it, while those related to the ambient space $\mathbb{R}^N$ will carry a bar ($^-$).

The definition presented here is a reformulation of the definition of a rolling contained in [14, Appendix B], that turns out to be more fruitful for future considerations.

Let M and $\widehat{M}$ be abstract manifolds. By the well known result of Nash [11], there are isometric imbeddings of M and $\widehat{M}$, denoted by ι and $\widehat{\iota}$ respectively, into $\mathbb{R}^{n+\nu}$ for an appropriate choice of $\nu \geq 1$. So, as long as there is no possibility for confusion, the abstract manifolds M and $\widehat{M}$ will be identifies with their images under the corresponding imbeddings.

Note that for any manifold M imbedded in $\mathbb{R}^{n+\nu}$, there is a natural splitting of the tangent space of $\mathbb{R}^{n+\nu}$ into a direct sum:

$$T_x\mathbb{R}^{n+\nu} = T_xM \oplus T_xM^{\perp}, \quad x \in M, \tag{1}$$

where T_xM is the tangent space to M at the point x and $T_xM^{\perp}$ is normal space to M at x.

According to splitting (1), any vector $v \in T_x\mathbb{R}^{n+\nu}$, $x \in M$, can be written uniquely as the sum $v = v^{\top} + v^{\perp}$, where $v^{\top} \in T_xM$, $v^{\perp} \in T_xM^{\perp}$.

Similar projections can be defined for $\widehat{M}$. Before presenting the definition of extrinsic rolling, one needs to introduce some notation.

Let ∇ denote the Levi-Civita connection on M or on $\widehat{M}$. The context will indicate on which manifold the connection is defined. The "ambient" Levi-Civita connection on $\mathbb{R}^{n+\nu}$ is denoted by $\overline{\nabla}$. Note that if X and Y are tangent vector fields to M, and Υ is a normal vector field to M, then

$$\nabla_X Y(x) = \left(\overline{\nabla}_{\bar{X}} \bar{Y}(x)\right)^{\top}, \quad \nabla^{\perp}_X \Upsilon(x) = \left(\overline{\nabla}_{\bar{X}} \bar{\Upsilon}(x)\right)^{\perp}, \quad x \in M,$$

where $\bar{X}$, $\bar{Y}$, and $\bar{\Upsilon}$ are any local extensions to $\mathbb{R}^{n+\nu}$ of the vector fields X, Y, and Υ, respectively. Equivalent statements hold for $\widehat{M}$. We will use capital Latin letters X, Y, and Z to denote tangent vector fields and capital Greek letters Υ, Ψ for normal vector fields.

If $Z(t)$ and $\Psi(t)$ are vector fields along a curve $x(t)$, we use $\dfrac{D}{dt}Z(t)$ to denote the covariant derivative of $Z(t)$ along $x(t)$ and $\dfrac{D^{\perp}}{dt}\Psi(t)$ for the normal covariant derivative of $\Psi(t)$ along $x(t)$ (for this notation, see [10, p. 119]). Observe that an isometric imbedding of M into $\mathbb{R}^{n+\nu}$ induces the equalities

$$\frac{D}{dt}\, Z(t) = \left(\frac{d}{dt}\, Z(t)\right)^{\top}, \quad \frac{D^{\perp}}{dt}\, \Psi(t) = \left(\frac{d}{dt}\, \Psi(t)\right)^{\perp}.$$

A tangent vector $Y(t)$ along an absolutely continuous curve $x(t)$ is parallel if

$$\frac{D}{dt} Z(t) = 0$$

for almost every t. Note that it is possible to define the notion of parallel transport even though the derivative $\dot{x}(t)$ exists only almost everywhere. More precisely, let $x : [0, \tau] \to M$ be an absolutely continuous curve and let $v \in T_{x(t_0)}M$, where $0 \le t_0 \le \tau$, then there exists a unique absolutely continuous tangent vector field $Z(t)$ along $x(t)$, such that $Z(t)$ is parallel and satisfies $Z(t_0) = v$. This follows from a strong version of the theorem of existence and uniqueness of ODEs with initial data (see, e.g., [13, p. 476]).

We say that a normal vector field $\Psi(t)$ along $x(t)$ is *normal parallel* if $\dfrac{D^{\perp}}{dt}\Psi(t) = 0$ for almost every t. A normal analogue of parallel transport is defined likewise.

We are now ready to give a new formulation of the rolling map.

Definition 1. A rolling of M on $\widehat{M}$ without slipping or twisting is an absolutely continuous curve $(x, g) : [0, \tau] \to M \times \mathrm{SE}(n + \nu)$ satisfying the following conditions:

(i) $\widehat{x}(t) := g(t), x(t) \in \widehat{M}$;

(ii) $d_{x(t)} g(t)\, T_{x(t)}M = T_{\widehat{x}(t)}\widehat{M}$;

(iii) $d_{x(t)}g(t)|_{T_{x(t)}M} : T_{x(t)}M \to T_{\widehat{x}(t)}\widehat{M}$ is orientation preserving;
(iv) no-slip condition: $\dot{\widehat{x}}(t) = d_{x(t)}g(t)\,\dot{x}(t)$ for almost every t;
(v) no-twist condition (tangential part):

$$d_{x(t)}g(t)\,\frac{D}{dt}\,Z(t) = \frac{D}{dt}\,d_{x(t)}g(t)\,Z(t)$$

for any tangent vector field $Z(t)$ along $x(t)$ and almost every t;
(vi) no-twist condition (normal part):

$$d_{x(t)}g(t)\,\frac{D^{\perp}}{dt}\,\Psi(t) = \frac{D^{\perp}}{dt}\,d_{x(t)}g(t)\,\Psi(t)$$

for any normal vector field $\Psi(t)$ along $x(t)$ and almost every t.

From now on, we omit the words "without slipping or twisting" just writing "a rolling."

Remark 1. Condition (v) is equivalent to the requirement that any tangent vector field $Z(t)$ is parallel along $x(t)$ if and only if $d_{x(t)}g(t)\,Z(t)$ is parallel along $\widehat{x}(t)$. As a consequence, this condition is automatically satisfied in the case of manifolds of dimension one. Similarly, condition (vi) is equivalent to the statement that any normal vector field $\Psi(t)$ is normal parallel along $x(t)$ if and only if $d_{x(t)}g(t)\,\Psi(t)$ is normal parallel vector field along $\widehat{x}(t)$. Thus, for imbeddings of codimension one, condition (vi) holds automatically.

Remark 2. We now explain what are the points of contact between Definition 1 and the definition of rolling contained in [14, Appendix B], followed by a list of minor differences and our reasons for having adopted a new definition of extrinsic rolling.

1. Conditions (i) and (ii) are the rolling conditions in [14].

Apart from notations, condition (i) is formulated in exactly the same terms as the first rolling condition in [14].

Restricting the action of $g(t)$ to M, the differential $d_{x(t)}g(t)$ maps $T_{x(t)}M$ into $T_{g(t)\,x(t)}\,(g(t)\,M)$ by definition. Hence, the second rolling condition in [14], which reads as

$$T_{\widehat{x}(t)}(g(t)M) = T_{\widehat{x}(t)}\widehat{M},$$

holds if and only if (ii) holds.

2. Condition (iv) is the no-slip condition in [14].

To prove the equivalence between $\dot{g}(t) \circ g^{-1}(t)\widehat{x}(t) = 0$, which is the no-slip condition in [14], and condition (iv) above, we write a curve $g(t)$ in $\mathrm{SE}(n+\nu)$ as follows:

$$g(t) : \bar{x} \mapsto \bar{A}(t)\,\bar{x} + \bar{r}(t), \quad \bar{x} \in \mathbb{R}^{n+\nu},$$

where $\bar{A} : [0,\tau] \to \mathrm{SO}(n+\nu)$ and $\bar{r} : [0,\tau] \to \mathbb{R}^{n+\nu}$. Thus

$$d_{\bar{x}}g(t)\,v = \bar{A}(t)\,v, \quad v \in T_{\bar{x}}\mathbb{R}^{n+\nu},$$

and we get

$$\dot{g}(t) \circ g^{-1}(t)\, \widehat{x}(t) = \dot{g}(t)\, x(t) = \dot{\bar{A}}(t)\, x(t) + \dot{\bar{r}}(t)$$
$$= \frac{d}{dt}\big(\bar{A}(t)\, x(t) + \bar{r}(t)\big) - \bar{A}(t)\, \dot{x}(t) = \dot{\widehat{x}}(t) - d_{x(t)}g(t)\, \dot{x}(t)$$

whenever $\dot{x}(t)$ is defined. Hence

$$\dot{g}(t) \circ g^{-1}(t)\, \widehat{x}(t) = 0$$

if and only if

$$\dot{\widehat{x}}(t) = d_{x(t)}g(t)\, \dot{x}(t).$$

3. Conditions (v) and (vi) are the no-twist conditions in [14].

Note that (ii) and the splitting (1) imply that the equalities

$$d_{x(t)}g(t)(T_{x(t)}M) = T_{\widehat{x}(t)}\widehat{M} \quad \text{and} \quad d_{x(t)}g(t)(T_{x(t)}M^{\perp}) = T_{\widehat{x}(t)}\widehat{M}^{\perp}$$

hold. Hence, the map

$$d_{\widehat{x}(t)}g^{-1}(t) = (d_{x(t)}g(t))^{-1}$$

maps tangent vectors to tangent vectors and normal vectors to normal vectors. This allows us to restate the no twist conditions in [14],

$$d_{\widehat{x}(t)}(\dot{g}(t) \circ g^{-1}(t))(T_{\widehat{x}(t)}\widehat{M}) \subseteq T_0(\dot{g}(t) \circ g^{-1}(t)\widehat{M})^{\perp}, \quad \text{(tangential part)},$$

$$d_{\widehat{x}(t)}(\dot{g}(t) \circ g^{-1}(t))(T_{\widehat{x}(t)}\widehat{M}^{\perp}) \subseteq T_0(\dot{g}(t) \circ g^{-1}(t)\widehat{M}), \quad \text{(normal part)},$$

as the conditions

$$\big(d_{x(t)}\dot{g}(t)\, v^{\top}\big)^{\top} = 0 \quad \text{and} \quad \big(d_{x(t)}\dot{g}(t)\, v^{\perp}\big)^{\perp} = 0$$

holding for any $v = v^{\top} + v^{\perp} \in T_{x(t)}\mathbb{R}^{n+\nu}$. For any tangent vector field $Z(t)$ along $x(t)$ and for any value of t, where $\dot{x}(t)$ is defined, the equality

$$0 = \big(d_{x(t)}\dot{g}(t)\, Z(t)\big)^{\top} = \left(\frac{d}{dt}\big(d_{x(t)}g(t)\, Z(t)\big) - d_{x(t)}g(t)\left(\frac{d}{dt}\, Z(t)\right)\right)^{\top}$$
$$= \frac{D}{dt}\, d_{x(t)}g(t)\, Z(t) - d_{x(t)}g(t)\, \frac{D}{dt}\, Z(t)$$

holds, thus condition (v) follows. Similar calculations show the equivalence between the normal part of the no-twist condition in [14] and condition (vi).

And now the differences.

4. While Sharpe considers curves which are piecewise smooth, we relax these differentiability conditions to absolute continuous. By allowing a more general class of rollings we give a first step in employing results from control theory and stochastic analysis. Further applications will be studied in forthcoming papers.

5. Sharpe does not consider any orientability assumptions. Our orientability requirements on M, $\widehat{M}$, and on the rolling itself, will ensure later,

in the next section on intrinsic rolling, that we have a connected configuration space. Nevertheless, in case any of the manifolds is not orientable, we can restrict our attention to sufficiently small neighborhoods of the contact point that can be oriented. Similarly, if the differential of g is not orientation preserving, we can change the orientation of the neighborhoods to make it orientation preserving. The fact that the definition of rolling can be interpreted locally, as shown in Sec. 3, implies that the dynamic is well defined, regardless of the global orientability of the manifolds or of the rolling map.

6. We make x part of the data, while in [14] the map g is the rolling and the rolling curve x is not part of the definition. And although it is proved there that for any piecewise smooth curve x on M there exists a unique isometry curve g on $\mathrm{SE}(n+\nu)$ that rolls x onto $\widehat{x}$ with fixed initial configuration, the rolling may depend not just on the isometry g but also on the curve x along which the rolling of M on $\widehat{M}$ is performed. This is illustrated in the following example.

Example 1. Consider the submanifolds of $\mathbb{R}^3$, defined by

$$M = \left\{(\bar{x}_1, \sin\theta, 1-\cos\theta) \in \mathbb{R}^3 \mid \bar{x}_1 \in \mathbb{R},\ \theta \in [0, 2\pi)\right\},$$

$$\widehat{M} = \left\{(\bar{x}_1, \bar{x}_2, 0) \in \mathbb{R}^3 \mid \bar{x}_1, \bar{x}_2 \in \mathbb{R}, \right\}.$$

The rolling map

$$g(t) : \bar{x} = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{pmatrix} \mapsto \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \cos t + (\bar{x}_3 - 1)\sin t + t \\ -\bar{x}_2 \sin t + (\bar{x}_3 - 1)\cos t + 1 \end{pmatrix},$$

describes the rolling of the infinite cylinder M on $\widehat{M}$ along the $\bar{x}_2$-axis with constant speed 1. Then there is an infinite choice of curves $x(t) \in M$, given by

$$x(t) = (\bar{x}_1, \sin t, 1 - \cos t), \quad \bar{x}_1 \in \mathbb{R},$$

along which the rolling g can be realized. However, if we make $x(t)$ as part of the data, then each choice of the curve $x(t)$ will correspond to different rollings $\big(x(t), g(t)\big)$ (see Fig. 1).

In spite of these differences, our definition of extrinsic rolling still gives the possibility of applying the fundamental theorems of differential geometry and control theory without changing drastically the main classical ideas of rolling maps as presented in [14].

Remark 3. Definition 1 ignores physical restrictions given by the actual shapes of the imbedded manifolds. Thinking M and $\widehat{M}$ as touching along the curves $x(t)$ and $\hat{x}(t)$ and rolling according to the isometry $g(t)$, then we cannot rule out the possibility that there might be transverse intersections between the manifolds other than the contact points.

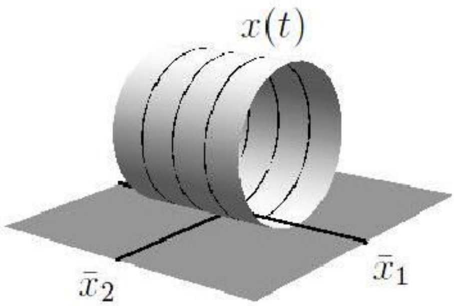

Fig. 1. Different choices of paths $x(t)$ in Example 1.

3. Intrinsic rolling

In this section, we introduce a new object called intrinsic rolling. We discuss its main properties and establish the fact that it encloses all of the fundamental information for a rolling, as presented in Sec. 2. In addition, we discuss the advantages of this new approach via several examples.

3.1. **Frame bundles and bundles of isometries.** Let V and $\widehat{V}$ be two oriented inner product spaces. We denote by $\mathrm{SO}(V, \widehat{V})$ the collection of all linear orientation preserving isometries between V and $\widehat{V}$. When $V = \widehat{V}$, we write $\mathrm{SO}(V)$ instead of $\mathrm{SO}(V, V)$. Note that $\mathrm{SO}(V)$ is a group.

For any pair M and $\widehat{M}$, we introduce the space Q of all relative positions in which M can be tangent to $\widehat{M}$

$$Q = \left\{ q \in \mathrm{SO}(T_x M, T_{\widehat{x}} \widehat{M}) \;\middle|\; x \in M, \widehat{x} \in \widehat{M} \right\}. \tag{2}$$

This space is a manifold with the structure of an $\mathrm{SO}(n)$-fiber bundle over $M \times \widehat{M}$ and can be considered as the configuration space of the rolling. Its dimension is $n(n+3)/2$.

The space Q can equivalently be described in terms of frame bundles. Let F and $\widehat{F}$ be the oriented orthonormal frame bundles of M and $\widehat{M}$, respectively. An oriented orthonormal frame $\{f_1, \dots, f_n\}$ defines a map $f \in \mathrm{SO}(\mathbb{R}^n, T_x M)$ as follows:

$$f\big(\underbrace{0, \dots, 0, 1, 0, \dots, 0}_{1 \text{ at the } j\text{th place}}\big) = f_j. \tag{3}$$

This gives an action of $\mathrm{SO}(\mathbb{R}^n) = \mathrm{SO}(n)$ on the right, inducing a principal $\mathrm{SO}(n)$-bundle structure on F. On the fiber over each point $x \in M$, there is also a left action by $\mathrm{SO}(T_x M)$. This group is isomorphic to $\mathrm{SO}(n)$, although not canonically when $n \geq 3$. Therefore, in general, there is no natural left action of $\mathrm{SO}(n)$ on Q. Similar considerations holds for $\widehat{F}$.

Consider $F \times \widehat{F}$ as a bundle over $M \times \widehat{M}$ with $\mathrm{SO}(n)$ acting diagonally on the fibers. Then, we can identify Q with $(F \times \widehat{F})/\mathrm{SO}(n)$ by the map assigning to each equivalence class $(f, \hat{f}) \cdot \mathrm{SO}(n)$ the mapping $q \in Q$, so that

$$\hat{f}_j = q\, f_j \tag{4}$$

for $j = 1, \ldots, n$. Clearly, this construction does not depend on the choice of a representative of an equivalence class of $(F \times \widehat{F})/\mathrm{SO}(n)$. Conversely, given an isometry $q \in Q$, there exists a unique equivalence class of frames satisfying (4).

As was mentioned above, except for the case when $n = 2$, the configuration space Q does not have the structure of a principal $\mathrm{SO}(n)$-bundle in a natural way. However, since Q is an $\mathrm{SO}(n)$-fiber bundle, it looks locally like the product $M \times \widehat{M} \times \mathrm{SO}(n)$. Let U be a neighborhood in M such that $F|_U$ is trivial and let e be a section of $F|_U$, that is, a smooth function on such that $e(x) \in \mathrm{SO}(\mathbb{R}^n, T_xM)$, for all $x \in U$. As in (3), the section e is uniquely determined by n vector fields $e_j : x \mapsto e_j(x)$ on U such that

$$e_1(x), \ldots, e_n(x), \quad x \in U,$$

is a positively oriented orthonormal basis of T_xM. Each section determines a left action of $\mathrm{SO}(n)$ on $F|_U$. To see this, recall that for each $x \in U$, the frame $e(x)$ can be considered as an isometry $e(x) : \mathbb{R}^n \to T_xM$. The map $e(x)$ induces an isomorphism of $\mathrm{SO}(n)$ and $\mathrm{SO}(T_xM)$. The aforementioned left action takes the following form: if $f \in F_x$ is any other frame at $x \in U$, written in terms of the frame e as

$$f_j = \sum_{i=1}^{n} f_{ij} e_i(x),$$

then $A = (a_{ij})_{i,j=1}^n \in \mathrm{SO}(n)$ acts on f via the equation

$$A \cdot f_j = \sum_{i,k=1}^{n} f_{ij} a_{ki} e_k, \quad j = 1, \ldots, n.$$

Observe that this action depends on the choice of the frame e.

From this, we can define locally a left and a right action of $\mathrm{SO}(n)$ on Q. Let U and $\widehat{U}$ be neighborhoods in M and $\widehat{M}$ respectively, so that both frame bundles trivialize over these neighborhoods. Let $e : U \to F|_U$ and $\hat{e} : \widehat{U} \to \widehat{F}|_{\widehat{U}}$ be sections. We define the left action of $A \in \mathrm{SO}(n)$ on Q with respect to $\hat{e}$ by

$$A \cdot \hat{f}_j = (A \cdot q) f_j,$$

where the left action of A on $\hat{f}_j$ is defined with respect to $\hat{e}$ and $\hat{f}_j = q f_j$ for $j = 1, \ldots, n$. Similarly, the right action of $\mathrm{SO}(n)$ on Q with respect to

e is defined by

$$\hat{f}_j = (q \cdot A)\big(A^{-1} \cdot f_j\big).$$

Note that if $A_0 = (\langle \hat{e}_i, qe_j\rangle)_{i,j=1}^n$, then we have

$$(\langle \hat{e}_i, (A \cdot q)e_j\rangle)_{i,j=1}^n = AA_0 \quad \text{and} \quad (\langle \hat{e}_i, (q \cdot A)e_j\rangle)_{i,j=1}^n = A_0 A.$$

3.2. **Reformulation of rolling in terms of bundles.** Both formulations of rolling surfaces given in [1, 2] define the configuration space as a manifold of isometries of tangent spaces of M and $\widehat{M}$, as in Sec. 3.1, without taking into account the imbedding into an ambient space. However, neither of these descriptions attempt to give any justifications for why the ambient space may be ignored, nor do they attempt to compare the intrinsic definition and the extrinsic definition given for imbedded manifolds in [14]. We would like to find a reformulation of Definition 1 in such a way that the conditions (i)–(vi) are stated both in terms of intrinsic conditions given on Q and some additional conditions given on another bundle, that carries the information on imbedding.

The conditions imposed over a rolling (x, g) by Definitions 1 and 2 are nontrivial whenever the codimension ν of the imbedded manifolds is greater than 1. So, it is natural to assume that the total configuration space of the rolling system will have a normal component which takes care of the action of g on the normal bundle. Therefore, in analogy to the construction of Q, we define a fiber bundle over $M \times \widehat{M}$ of isometries of the normal tangent space. Let

$$\iota : M \to \mathbb{R}^{n+\nu}, \quad \widehat{\iota} : \widehat{M} \to \mathbb{R}^{n+\nu}$$

be two imbeddings given as initial data. Let Φ be the principal $\mathrm{SO}(\nu)$-bundle over M, such that the fiber over a point $x \in M$ consists of all positively oriented orthonormal frames $\{\epsilon_\lambda(x)\}_{\lambda=1}^{\nu}$ spanning $T_xM^\perp$. Let $\widehat{\Phi}$ be the principal $\mathrm{SO}(\nu)$-bundle similarly defined on $\widehat{M}$. As in Sec. 3.1, we can identify the manifold $(\Phi \times \widehat{\Phi})/\,\mathrm{SO}(\nu)$ with

$$P_{\iota,\widehat{\iota}} := \Big\{ p \in \mathrm{SO}(T_xM^\perp, T_{\widehat{x}}\widehat{M}^\perp) \;\Big|\; x \in M, \widehat{x} \in \widehat{M} \Big\}. \tag{5}$$

As before, the space $P_{\iota,\widehat{\iota}}$ is not in general a principal $\mathrm{SO}(\nu)$-bundle, but there are local left and right actions defined similarly as on Q in Sec. 3.1. We note and reflect it in notations that Q is invariant of imbeddings, while $P_{\iota,\widehat{\iota}}$ is not. The dimension of $P_{\iota,\widehat{\iota}}$ is $2n + \nu(\nu - 1)/2$.

By abuse of notation, we will use $Q \oplus P_{\iota,\widehat{\iota}}$ for the fiber bundle over $M \times \widehat{M}$, so that the fiber over $(x, \widehat{x}) \in M \times \widehat{M}$, is $Q_{(x,\widehat{x})} \times P_{\iota,\widehat{\iota}(x,\widehat{x})}$. The dimension of $Q \oplus P_{\iota,\widehat{\iota}}$ is $(n(n + 3) + \nu(\nu - 1))/2$.

Proposition 1. *If a curve*

$$(x, g) : [0, \tau] \to M \times \mathrm{SE}(n + \nu)$$

satisfies the conditions (i)–(vi) *in Definition 1, then the mapping*

$$t \mapsto (d_{x(t)}g(t)|_{T_{x(t)}M}, d_{x(t)}g(t)|_{T_{x(t)}M^\perp}) =: (q(t), p(t)),$$

defines a curve in $Q \oplus P_{\iota,\widehat{\iota}}$ *with the following properties*:

(I) *no slip condition:*

$$\dot{\widehat{x}}(t) = q(t)\dot{x}(t)$$

for almost every t*;*

(II) *no-twist condition (tangential part):*

$$q(t)\frac{D}{dt}Z(t) = \frac{D}{dt}q(t)Z(t)$$

for any tangent vector field $Z(t)$ *along* $x(t)$ *and almost every* t*;*

(III) *no twist condition (normal part):*

$$p(t)\frac{D^\perp}{dt}\Psi(t) = \frac{D^\perp}{dt}p(t)\Psi(t)$$

for any normal vector field $\Psi(t)$ *along* $x(t)$ *and almost every* t*.*

Conversely, if

$$(q, p) : [0, \tau] \to Q \oplus P_{\iota,\widehat{\iota}}$$

is an absolutely continuous curve satisfying (I)–(III)*, then there exists a unique rolling*

$$(x, g) : [0, \tau] \to M \times \mathrm{SE}(n + \nu),$$

such that

$$d_{x(t)}g(t)|_{T_{x(t)}M} = q(t), \quad d_{x(t)}g(t)|_{T_{x(t)}M^\perp} = p(t).$$

Proof. Assume that $(x, g) : [0, \tau] \to M \times \mathrm{SE}(n+\nu)$ is a rolling map satisfying (i)–(vi). Conditions (i) and (ii) assure that

$$\begin{aligned} &d_{x(t)}g(t)|_{T_{x(t)}M} \in \mathrm{SO}(T_{x(t)}M, T_{\widehat{x}(t)}\widehat{M}), \\ &d_{x(t)}g(t)|_{T_{x(t)}M^\perp} \in \mathrm{SO}(T_{x(t)}M^\perp, T_{\widehat{x}(t)}\widehat{M}^\perp). \end{aligned} \tag{6}$$

Since $d_{x(t)}g(t)$ must be orientation preserving in $\mathbb{R}^{n+\nu}$, we conclude that both of the mappings (6) are either orientation reversing or orientation preserving. The additional requirement (iii) implies that (q, p) is orientation preserving. The conditions (I)-(III) correspond to the conditions (iv)–(vi).

Conversely, if we have a curve

$$(q(t), p(t)) \quad \text{in } Q \oplus P_{\iota,\widehat{\iota}}$$

with projection $(x(t), \widehat{x}(t))$ into $M \times \widehat{M}$, then we have the following isometry $g \in \mathrm{SE}(n + \nu)$. We write

$$g(t) : \bar{x} \mapsto \bar{A}(t)\bar{x} + \bar{r}(t), \quad \bar{A}(t) \in \mathrm{SO}(n + \nu),$$

where $\bar{A}(t) = d_{x(t)}g(t)$ is determined by the conditions

$$d_{x(t)}g(t)|_{T_{x(t)}M} = q(t)|_{T_{x(t)}M}, \quad d_{x(t)}g(t)|_{T_{x(t)}M^\perp} = p(t)|_{T_{x(t)}M^\perp}.$$

Then

$$\operatorname{Image} d_{x(t)}g(t)|_{T_{x(t)}M} = T_{\widehat{x}(t)}\widehat{M}, \quad \operatorname{Image} d_{x(t)}g(t)|_{T_{x(t)}M^{\perp}} = T_{\widehat{x}(t)}\widehat{M}^{\perp}.$$

The vector $\bar{r}(t)$ is determined by

$$\bar{r}(t) = \widehat{x}(t) - \bar{A}(t)x(t).$$

□

The one-to-one correspondence between rolling maps and absolutely continuous curves in $Q \oplus P_{\iota,\widehat{\iota}}$, satisfying (I)–(III), naturally leads to a definition of a rolling map in terms of these bundles.

Definition 2. A rolling of M on $\widehat{M}$ without slipping or twisting is an absolutely continuous curve

$$(q,p) : [0,\tau] \to Q \oplus P_{\iota,\widehat{\iota}}$$

such that $(q(t), p(t))$ satisfies

(I) no slip condition:

$$\dot{\widehat{x}}(t) = q(t)\dot{x}(t)$$

for almost every t;

(II) no twist condition (tangential part):

$$q(t)\frac{D}{dt}Z(t) = \frac{D}{dt}q(t)Z(t)$$

for any tangent vector field $Z(t)$ along $x(t)$ and almost every t,

(III) no twist condition (normal part):

$$p(t)\frac{D^{\perp}}{dt}\Psi(t) = \frac{D^{\perp}}{dt}p(t)\Psi(t)$$

for any normal vector field $\Psi(t)$ along $x(t)$ and almost every t.

Remark 4. Proposition 1 implies that the bundle $Q \oplus P_{\iota,\widehat{\iota}}$ can be seen as the configuration space for a rolling. According to [14], the dimension $\dfrac{n(n+3)+\nu(\nu-1)}{2}$ corresponds to the degrees of freedom of the system.

A purely intrinsic definition of a rolling is deduced from Definition 2, by restricting it to the bundle Q. This concept naturally generalizes the definition given in [1] for two-dimensional Riemannian manifolds imbedded into $\mathbb{R}^3$ and we use the term *intrinsic rolling* for this object.

Definition 3. An intrinsic rolling of two n-dimensional oriented Riemannian manifolds M and $\widehat{M}$ without slipping or twisting is an absolutely continuous curve $q : [0,\tau] \to Q$, with projections

$$x(t) = \operatorname{pr}_M q(t), \quad \widehat{x}(t) = \operatorname{pr}_{\widehat{M}} q(t),$$

satisfying the following conditions:

(I') no-slip condition:

$$\dot{\widehat{x}}(t) = q(t)\dot{x}(t)$$

for almost all t;

(II') no-twist condition: $Z(t)$ is a parallel tangent vector field along $x(t)$ if and only if $q(t)Z(t)$ is parallel along $\widehat{x}(t)$ for almost all t.

Remark 5. If the manifolds are imbedded into Euclidean space $\mathbb{R}^{n+1}$, then for each pair of points $(x, \widehat{x}) \in M \times \widehat{M}$, there is a unique orientation preserving isometry

$$p : T_x M^\perp \to T_{\widehat{x}} \widehat{M}^\perp.$$

Hence, since $P_{\iota,\widehat{\iota}}$ is an SO(1)-bundle, it can be identified with $M \times \widehat{M}$, and so $Q \oplus P_{\iota,\widehat{\iota}} \cong Q$. In this case we see that the notion of rolling in Definition 2 coincides with the intrinsic rolling in Definition 3.

3.3. **Extrinsic and intrinsic rollings along the same curves.** Let

$$(x, \widehat{x}) : [0, \tau] \to M \times \widehat{M}$$

be a fixed pair of curves which are projections of a rolling map. We aim to give an answer to the following questions:

(i) If $(q_1(t), p_1(t))$ and $(q_2(t), p_2(t))$ are two rollings of M on $\widehat{M}$, along $x(t)$ and $\widehat{x}(t)$, how are they related? What properties of the rolling are defined by fixing the paths x and $\widehat{x}$?

(ii) Assume that an intrinsic rolling $q(t)$ and imbeddings of M and $\widehat{M}$ into $\mathbb{R}^{n+\nu}$ are given. Is it possible to extend $q(t)$ to a rolling $(q(t), p(t))$? Is this extension unique?

The following example clarifies the situation for one dimensional manifolds, where different imbeddings are easy to describe.

Example 2. Consider $\widehat{M} = \mathbb{R}$, with the usual Euclidean structure, and $M = S^1$, with the usual round metric and positive orientation counterclockwise. Let $x : [0, \tau] \to S^1$ be written as $x(t) = e^{i\varphi(t)}$, where $\varphi : [0, \tau] \to \mathbb{R}$ is an absolutely continuous function. Since $\mathrm{SO}(1) = \{\mathbf{1}\}$, the configuration space for the intrinsic rolling is just $M \times \widehat{M}$. The no-slipping condition implies that

$$\widehat{x}(t) = \widehat{x}(0) + \varphi(t) - \varphi(0),$$

and we can assume that

$$\widehat{x}(0) = \varphi(0) = 0.$$

We consider different rollings of M on $\widehat{M}$ under various imbeddings. Without loss of generality, we can assume that

$$g(0) = \mathrm{id}_{\mathbb{R}^{1+\nu}}$$

is the identity map in $\mathbb{R}^{1+\nu}$. We will use $r = (r_1, \dots, r_{1+\nu})$ for coordinates of $\mathbb{R}^{1+\nu}$.

Case 1. Consider the imbeddings

$$\iota_1 : M \to \mathbb{R}^2, \quad e^{i\varphi} \mapsto (\sin\varphi, 1 - \cos\varphi),$$
$$\widehat{\iota}_1 : \widehat{M} \to \mathbb{R}^2, \quad \widehat{x} \mapsto (\widehat{x}, 0).$$

Simple calculations show that there is only one possible rolling.

Case 2. Consider the imbeddings

$$\iota_2 : M \to \mathbb{R}^3, \quad e^{i\varphi} \mapsto (\sin\varphi, (1-\cos\varphi)\cos\theta_0, (1-\cos\varphi)\sin\theta_0),$$
$$\widehat{\iota}_2 : \widehat{M} \to \mathbb{R}^3, \quad \widehat{x} \mapsto (\widehat{x}, 0, 0)$$

where θ_0 is any fixed angle from $(0, \pi/2)$. Conditions (ii), (iii), and (iv) of Definition 1 imply that the differential $d_{x(t)}g(t)$ of $g(t)$ on matrix form can be written uniquely as follows:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varkappa(t) & \sin\varkappa(t) \\ 0 & -\sin\varkappa(t) & \cos\varkappa(t) \end{pmatrix} \begin{pmatrix} \cos\varphi(t) & \sin\varphi(t) & 0 \\ -\sin\varphi(t) & \cos\varphi(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_0 & \sin\theta_0 \\ 0 & -\sin\theta_0 & \cos\theta_0 \end{pmatrix}$$

for some absolutely continuous function $\varkappa\colon [0,\tau] \to \mathbb{R}$. To satisfy the normal no-twist condition, $d_{x(t)}g(t)$ must map the normal parallel vector fields on M

$$\epsilon_1 = -\sin\varphi(t)\frac{\partial}{\partial r_1} + \cos\varphi(t)\cos\theta_0\frac{\partial}{\partial r_2} + \cos\varphi(t)\sin\theta_0\frac{\partial}{\partial r_3},$$
$$\epsilon_2 = -\sin\theta_0\frac{\partial}{\partial r_2} + \cos\theta_0\frac{\partial}{\partial r_3},$$

to normal parallel vector fields on $\widehat{M}$. Calculating the covariant derivative of $d_{x(t)}g(t)\epsilon_1$ and $d_{x(t)}g(t)\epsilon_2$, we conclude that $\varkappa(t)$ is constant and the assumption

$$g(0) = \mathrm{id}_{\mathbb{R}^{1+\nu}}$$

implies that the constant is 0. Hence, the circle will roll along the line with a constant tilt given by θ_0 (see Fig. 2.

Case 3. Consider the isometric imbedding of $\widehat{M}$ as a spiral:

$$\widehat{\iota}_3 : \widehat{M} \to \mathbb{R}^3, \quad \widehat{x} \mapsto \frac{1}{\sqrt{2}}(\cos\widehat{x}, \sin\widehat{x}, \widehat{x}),$$

and ι_2 from the previous case. In this situation, the circle M will rotate along the spiral $\widehat{M}$. Checking the normal no-twist condition we came to the same conclusion that the path is uniquely determined by the initial angle θ_0.

Note that in all the cases above, the intrinsic rolling $t \mapsto (e^{i\varphi(t)}, \varphi(t))$ either uniquely induces a rolling, or the rolling is determined by an initial configuration of the normal tangent spaces, which corresponds to the initial

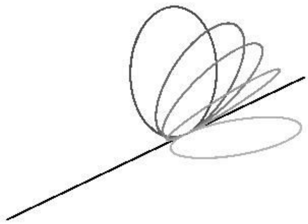

Fig. 2. Case 2: S^1 rolling on $\mathbb{R}$. Different tilting angles give different imbeddings, but equivalent rollings.

tilting angle θ_0. In fact it is also possible to find a choice of basis, consisting of normal parallel vector fields, so that the normal component of the rolling $p(t)$ is constant with respect to this basis. We show that this holds generally in Lemma 1 below.

Let $x : [0,\tau] \to M$ and $\widehat{x} : [0,\tau] \to \widehat{M}$ be two fixed curves. We denote by $\{e_j(t)\}_{j=1}^n$ a collection of parallel tangent vector fields along $x(t)$ forming an orthonormal basis for $T_{x(t)}M$ and by

$$\{\epsilon_\lambda(t)\}_{\lambda=1}^\nu$$

a collection of normal parallel vector fields along $x(t)$ forming an orthonormal basis for $T_{x(t)}M^\perp$. Such vector fields can be constructed by parallel transport and normal parallel transport along $x(t)$. Similarly, along $\widehat{x}(t)$, we define parallel frames $\{\hat{e}_i\}_{i=1}^n$ and $\{\hat{\epsilon}_\kappa\}_{\kappa=1}^\nu$. Recall that Latin indices $i, j, \ldots$ vary from 1 to n, while Greek ones $\kappa, \lambda, \ldots$ vary from 1 to ν.

The following lemma shows that the image of a parallel frame over M has constant coordinates in a parallel frame over $\widehat{M}$. This reflects the fact that rolling preserves parallel vector fields.

Lemma 1. *A curve $(q(t), p(t))$ in $Q \oplus P_{\iota,\hat{\iota}}$ in the fibers over $(x(t), \widehat{x}(t))$, satisfies* (II) *and* (III) *if and only if the matrices*

$$A(t) = (a_{ij}(t)) = (\langle \hat{e}_i, q(t)e_j\rangle), \quad B(t) = (b_{\kappa\lambda}(t)) = (\langle \hat{\epsilon}_\kappa(t), p(t)\epsilon_\lambda(t)\rangle)$$

are constant.

Proof. Let $(q(t), p(t))$ be an absolutely continuous curve. Then we have

$$\langle \hat{e}_i, \dot{\hat{e}}_j\rangle = \langle e_i, \dot{e}_j\rangle = 0$$

and

$$\dot{a}_{ij}(t) = \langle \dot{\hat{e}}_i, q(t)e_j\rangle + \left\langle \hat{e}_i, \frac{d}{dt}(q(t)e_j)\right\rangle$$

by the product rule. The vectors $q(t)^{-1}\hat{e}_i$ and $q(t)e_j$ are tangent, so we have

$$\langle q(t)^{-1}\hat{e}_i, \dot{e}_j\rangle = \langle \dot{\hat{e}}_i, q(t)e_j\rangle = 0$$

and

$$\begin{aligned}\dot{a}_{ij}(t) &= \langle \hat{e}_i, \dot{q}(t)e_j\rangle + \langle \hat{e}_i, q(t)\dot{e}_j\rangle + \langle \dot{\hat{e}}_i, q(t)e_j\rangle = \langle \hat{e}_i, \dot{q}(t)e_j\rangle + \langle q(t)^{-1}\hat{e}_i, \dot{e}_j\rangle \\ &= \left\langle \hat{e}_i, \frac{d}{dt}\big(q(t)e_j\big) - q(t)\dot{e}_j\right\rangle = \left\langle \hat{e}_i, \frac{D}{dt}q(t)e_j - q(t)\frac{D}{dt}e_j\right\rangle = 0.\end{aligned}$$

Thus (II) holds if and only if $\dot{a}_{ij}(t) = 0$. Similar result holds for the basis of the normal tangent bundle. □

The following theorem gives an answer to the first question raised at the beginning of Sec. 3.3.

Theorem 1. *Let $q : [0,\tau] \to Q$ be a given intrinsic rolling map without slipping or twisting with projection $\operatorname{pr}_{M\times\widehat{M}}\, q_0(t) = (x(t), \widehat{x}(t))$. Define the vector spaces*

$$V = \Big\{v(t) \textit{ is a parallel v. field along } x(t) \textit{ and } \langle v(t), \dot{x}(t)\rangle = 0 \textit{ for a.e. } t\Big\},$$

$$\widehat{V} = \Big\{\widehat{v}(t) \textit{ is a parallel v. field along } \widehat{x}(t) \textit{ and } \langle \widehat{v}(t), \dot{\widehat{x}}(t)\rangle = 0 \textit{ for a.e. } t\Big\},$$

with the inner product and orientation induced by the metric and orientation on M and $\widehat{M}$, respectively. Then $\dim V = \dim \widehat{V}$ and, if we denote this dimension by k, the following holds.

(a) *The map q is the unique intrinsic rolling of M on $\widehat{M}$ along $x(t)$ and $\widehat{x}(t)$ if and only if $k \le 1$.*
(b) *If $k \ge 2$, all the rollings along $x(t)$ and $\widehat{x}(t)$ differ from q by an element in $\mathrm{SO}(\widehat{V})$.*

Remark 6. Both the inner product and orientation are preserved under parallel transport. Hence, for any pair $v, w \in V$, the value of $\langle v(t), w(t)\rangle$ remains constant for any t. The metric on M therefore induces a well defined inner product on V. Similarly, we can say that a collection of vector fields is positively oriented if it has this property for one value of t (and consequently for all values of t). Similar considerations hold for $\widehat{V}$.

Proof. By Lemma 1, it is possible to find frames of parallel vector fields $\{e_i\}_{i=1}^n$ and $\{\hat{e}_i\}_{i=1}^n$ along $x(t)$ and $\widehat{x}(t)$, respectively, such that $q(t)e_i = \hat{e}_i$. Assume that the first k vector fields of each frame are orthogonal to $\dot{\widehat{x}}$. Note that $e_1, \dots, e_k$ is a basis for V, and $\hat{e}_1, \dots, \hat{e}_k$ is a basis for $\widehat{V}$.

Writing

$$\dot{\widehat{x}} = \sum_{i=1}^n \dot{\widehat{x}}_i(t)\hat{e}_i(t), \quad \dot{x} = \sum_{i=1}^n \dot{x}_i(t)e_i(t),$$

we get

$$\dot{\widehat{x}}_i(t) = \dot{x}_i(t), \quad \dot{\widehat{x}}_1(t) = \cdots = \dot{\widehat{x}}_k(t) = 0.$$

So, if $\widetilde{q}$ is any other rolling, then

$$A = (a_{ij}) = (\langle \hat{e}_i(t), \widetilde{q}(t)\, e_j(t)\rangle)$$

is clearly of the form

$$A = \begin{pmatrix} A' & 0 \\ 0 & \mathbf{1}_{n-k} \end{pmatrix}, \qquad A' \in \mathrm{SO}(k), \tag{7}$$

where $\mathbf{1}_{n-k}$ is the identity $\big((n-k)\times(n-k)\big)$-matrix. This will be unique if k is 0 or 1. If $k \geq 2$, there is more freedom, since it is not determined how an arbitrarily rolling $\widetilde{q}$ should map V into $\widehat{V}$.

The converse also holds, that is, for any matrix A on the form (7), there is a rolling corresponding to it. □

In particular, if the curve $x : [0,\tau] \to M$ is a geodesic, we have the following consequence of Theorem 1.

Corollary 1. *Assume that $x(t)$ is a geodesic in M. Then there exists an intrinsic rolling of M on $\widehat{M}$ along $(x(t), \widehat{x}(t))$ if and only if $\widehat{x}(t)$ is a geodesic with the same speed as $x(t)$. Moreover, if $n \geq 2$, and if $\widehat{V}$ is defined as in Theorem 1, then*

$$\dim \widehat{V} = n-1,$$

and all the rollings along $x(t)$ and $\widehat{x}(t)$ differ by an element in $\mathrm{SO}(\widehat{V})$.

Proof. By the no-slip and no-twist conditions, we have the equality

$$\frac{D}{dt}\dot{\widehat{x}}(t) = \frac{D}{dt} q(t)\dot{x}(t) = q(t)\frac{D}{dt}\dot{x}(t).$$

Thus, if $x(t)$ is a geodesic then $\widehat{x}(t)$ is also geodesic. In order to satisfy (I) we need to require that the speed of $\dot{\widehat{x}}(t)$ is the same as the speed of $\dot{x}(t)$. Conversely, the equality of speeds implies condition (I).

We start the construction of rolling map by choosing the vector field

$$e_1(t) = \frac{\dot{x}(t)}{\langle \dot{x}(t), \dot{x}(t)\rangle^{1/2}}$$

that is parallel along $x(t)$. Pick the remaining $n-1$ parallel vector fields so that they form an orthonormal basis together with $e_1(t)$ along the curve $x(t)$. We repeat the same construction for a parallel frame $\{\hat{e}_i(t)\}_{i=1}^n$ along $\widehat{x}(t)$. Define the intrinsic rolling $q(t)$ by

$$\begin{gathered} \langle \hat{e}_1(t), q(t)\, e_j(t)\rangle = \langle \hat{e}_j(t), q(t)\, e_1(t)\rangle = \delta_{1,j}, \\ A' = (\langle \hat{e}_{i+1}(t), q(t)\, e_{j+1}(t)\rangle)_{i,j=1}^{n-1}, \end{gathered} \tag{8}$$

where $A' \in \mathrm{SO}(n-1)$ will be a constant matrix. Conversely, we can construct a rolling by formulas (8) starting from $A' \in \mathrm{SO}(n-1)$. □

Concerning the problem of extending intrinsic rollings to extrinsic ones, the following theorem gives a complete answer to the question posed at the beginning of Sec. 3.3.

Theorem 2. *Let $q : [0, \tau] \to Q$ be an intrinsic rolling and let*

$$\iota : M \to \mathbb{R}^{n+\nu}, \quad \widehat{\iota} : \widehat{M} \to \mathbb{R}^{n+\nu}$$

be given imbeddings. Then, given an initial normal configuration

$$p_0 \in (P_{\iota,\widehat{\iota}})_{(x_0,\widehat{x}_0)}, \quad \textit{where } (x_0, \widehat{x}_0) = \mathrm{pr}_{M\times\widehat{M}}\, q(0),$$

there exists a unique rolling

$$(q, p) : [0, \tau] \to Q \oplus P_{\iota,\widehat{\iota}}$$

satisfying $p(0) = p_0$.

Proof. Let $\{\epsilon_\lambda(t)\}_{\lambda=1}^{\nu}$ and $\{\hat{\epsilon}_\kappa(t)\}_{\kappa=1}^{\nu}$ be normal parallel frames along $x(t)$ and $\widehat{x}(t)$, respectively. Let $B_0 \in \mathrm{SO}(\nu)$ be defined by

$$B_0 = (b_{\kappa\lambda}) = (\langle \widehat{\epsilon}_\kappa(0), p_0\, \epsilon_\lambda(0)\rangle) \,.$$

Then $p(t)$ must satisfy

$$b_{\kappa\lambda} = \langle \widehat{\epsilon}_\kappa(t), p(t)\, \epsilon_\lambda(t)\rangle,$$

by Lemma 1, and it is uniquely determined by this. □

Remark 7. Analogously to the spaces V and $\widehat{V}$ in Theorem 1, let us define the vector spaces

$$E = \{\epsilon(t) \text{ is a normal parallel vector field along } x(t)\} \,,$$

$$\widehat{E} = \{\widehat{\epsilon}(t) \text{ is a normal parallel vector field along } \widehat{x}(t)\} \,,$$

with inner product and orientation induced by $TM^\perp$ and $T\widehat{M}^\perp$ respectively, as mentioned in Remark 6. Both vector spaces have dimension ν. An extrinsic rolling (q, p) extending an intrinsic rolling q is determined up to a left action of $\mathrm{SO}(\widehat{E})$ or, equivalently, up to a right action of $\mathrm{SO}(E)$. Both $\mathrm{SO}(E)$ and $\mathrm{SO}(\widehat{E})$ are isomorphic to $\mathrm{SO}(\nu)$, but not canonically.

4. Distributions for rolling and intrinsic rolling

The aim of this section is to formulate the kinematic conditions of no-slipping and no-twisting in terms of a distribution. In this setting, a rolling will be an absolutely continuous curve almost everywhere tangent to this distribution.

4.1. **Local trivializations of Q.** Let $\pi : Q \oplus P_{\iota,\widehat{\iota}} \to M \times \widehat{M}$ denote the canonical projection. Consider a rolling $\gamma(t) = (q(t), p(t))$; we have

$$\pi \circ \gamma(t) = (x(t), \widehat{x}(t)) \,.$$

Given an arbitrary t_0 in the domain of $\gamma(t)$, let U and $\widehat{U}$ denote neighborhoods of $x(t_0)$ and $\widehat{x}(t_0)$ in M and $\widehat{M}$, respectively, such that both bundles TM and $TM^\perp$ trivialize being restricted to U. In the same way, we choose $\widehat{U}$ such that both $T\widehat{M}$ and $T\widehat{M}^\perp$ trivialize when they are restricted to $\widehat{U}$. This implies that the bundle

$$Q \oplus P_{\iota,\widehat{\iota}} \to M \times \widehat{M}$$

trivializes when it is restricted to $U \times \widehat{U}$. To see this, let

$$\{e_j\}_{j=1}^n, \quad \{\epsilon_\lambda\}_{\lambda=1}^\nu, \quad \{\hat{e}_i\}_{i=1}^n, \quad \{\hat{\epsilon}_\kappa\}_{\kappa=1}^\nu$$

denote positively oriented orthonormal bases of vector fields of $TM|_U$, $TM^\perp|_U$, $T\widehat{M}|_{\widehat{U}}$, and $T\widehat{M}^\perp|_{\widehat{U}}$, respectively. Then there is a trivialization

$$\begin{aligned} Q \oplus P_{\iota,\widehat{\iota}}|_{U\times\widehat{U}} &\xrightarrow{h} U \times \widehat{U} \times \mathrm{SO}(n) \times \mathrm{SO}(\nu), \\ (q,p) &\mapsto (x, \widehat{x}, A, B), \end{aligned} \tag{9}$$

given by the projections

$$x = \mathrm{pr}_U(q,p), \quad \widehat{x} = \mathrm{pr}_{\widehat{U}}(q,p),$$
$$A = (a_{ij})_{i,j=1}^n = (\langle qe_j, \hat{e}_i\rangle)_{i,j=1}^n \,, \quad B = (b_{\kappa\lambda})_{\kappa,\lambda=1}^\nu = (\langle p\epsilon_\lambda, \hat{\epsilon}_\kappa\rangle)_{\kappa,\lambda=1}^\nu \,.$$

The domain of γ can be chosen connected, containing t_0, and such that its image lies in $\pi^{-1}(U \times \widehat{U})$. Let us identify $\gamma(t)$ with its image under the trivialization given by $(x(t), \widehat{x}(t), A(t), B(t))$.

Each of requirements (I)–(III) can be written as restrictions to $\dot{\gamma}(t)$. We will show that all admissible values of $\dot{\gamma}(t)$ form a distribution; that is a smooth sub-bundle, of $T(Q \oplus P_{\iota,\widehat{\iota}})$. We will use the local trivializations to describe this distribution.

4.2. **Tangent space of** $\mathrm{SO}(n)$**.** Let U and $\widehat{U}$ be as in Sec. 4.1. Then we get in trivialization

$$T\pi^{-1}(U \times \widehat{U}) = TU \times T\widehat{U} \times T\,\mathrm{SO}(n) \times T\,\mathrm{SO}(\nu).$$

The decomposition requires that we present a detailed description of the tangent space of $\mathrm{SO}(n)$ in terms of left- and right-invariant vector fields.

We start by considering the imbedding of $\mathrm{SO}(n)$ in $\mathrm{GL}(n)$, the group of invertible real $(n \times n)$-matrices. Denote the matrix entries of a matrix A by (a_{ij}) and the transpose matrix by A^t. Then, differentiating the condition $A^tA = \mathbf{1}$, we obtain

$$T\,\mathrm{SO}(n) = \bigcap_{i\le j} \ker \omega_{ij}, \quad \omega_{ij} = \sum_{r=1}^n (a_{rj}\, da_{ri} + a_{ri}\, da_{rj}) \,.$$

It is clear that the tangent space at the identity 1 of $\mathrm{SO}(n)$ is spanned by

$$W_{ij}(1) := \frac{\partial}{\partial a_{ij}} - \frac{\partial}{\partial a_{ji}}, \quad 1 \leq i < j \leq n.$$

We denote $\mathfrak{so}(n) = \mathrm{span}\{W_{ij}(1)\}$ following the classical notation. We use the left translation of these vectors to define

$$W_{ij}(A) := A \cdot W_{ij}(1) = \sum_{r=1}^{n} \left(a_{ri} \frac{\partial}{\partial a_{rj}} - a_{rj} \frac{\partial}{\partial a_{ri}} \right) \tag{10}$$

as a global left-invariant basis of $T\,\mathrm{SO}(n)$. Note that the left and right actions in $T\,\mathrm{SO}(n)$ are described by

$$A \cdot \frac{\partial}{\partial a_{ij}} = \sum_{r=1}^{n} a_{ri} \frac{\partial}{\partial a_{rj}}, \quad \frac{\partial}{\partial a_{ij}} \cdot A = \sum_{s=1}^{n} a_{js} \frac{\partial}{\partial a_{is}}.$$

We have the following formula to switch from left to right translation:

$$\begin{aligned} A \cdot \frac{\partial}{\partial a_{ij}} = \sum_{r=1}^{n} a_{ri} \frac{\partial}{\partial a_{rj}} &= \sum_{l,r=1}^{n} a_{ri} \delta_{j,l} \frac{\partial}{\partial a_{rl}} \\ &= \sum_{l,r,s=1}^{n} a_{ri} a_{si} a_{sl} \frac{\partial}{\partial a_{rl}} = \sum_{r,s=1}^{n} a_{ri} a_{si} \left(\frac{\partial}{\partial a_{rs}} \cdot A \right), \end{aligned}$$

and the other way around,

$$\begin{aligned} \frac{\partial}{\partial a_{ij}} \cdot A = \sum_{s=1}^{n} a_{js} \frac{\partial}{\partial a_{is}} &= \sum_{l,s=1}^{n} a_{js} \delta_{i,l} \frac{\partial}{\partial a_{ls}} \\ &= \sum_{l,r,s=1}^{n} a_{js} a_{ir} a_{lr} \frac{\partial}{\partial a_{ls}} = \sum_{r,s=1}^{n} a_{js} a_{ir} \left(A \cdot \frac{\partial}{\partial a_{rs}} \right). \end{aligned}$$

Therefore, the right-invariant basis of $T\,\mathrm{SO}(n)$ can be written as

$$W_{ij}(1) \cdot A = \mathrm{Ad}(A^{-1}) W_{ij}(A) = \sum_{r<s} (a_{ir} a_{js} - a_{jr} a_{is}) W_{rs}(A).$$

If W_{ij} is defined by (10) and $i > j$, (so $W_{ij} = -W_{ji}$), then the bracket relations are given by

$$[W_{ij}, W_{kl}] = \delta_{j,k} W_{il} + \delta_{i,l} W_{jk} - \delta_{i,k} W_{jl} - \delta_{j,l} W_{ik}.$$

4.3. **Distributions.** Now we are ready to rewrite the kinematic conditions (I)–(III) as a distribution. Let $\gamma(t)$ be a rolling satisfying conditions (I)–(III). Consider it image under the trivializations. Then

$$\dot{\gamma}(t) = \dot{x}(t) + \dot{\hat{x}}(t) + \sum_{i,j=1}^{n} \dot{a}_{ij} \frac{\partial}{\partial a_{ij}} + \sum_{\kappa,\lambda=1}^{\nu} \dot{b}_{\kappa\lambda} \frac{\partial}{\partial b_{\kappa\lambda}}. \tag{11}$$

If we denote $\dot{x}(t)$ by $Z(t)$, then (I) holds if and only if

$$\dot{\hat{x}}(t) = q(t)Z(t).$$

We want to write the last two terms in (11) in the right-invariant basis of corresponding tangent spaces of $\mathrm{SO}(n)$ and $\mathrm{SO}(\nu)$ based on conditions (II) and (III). We start from (II) and note that

$$q(t)e_j = \sum_{i=1}^{n} a_{ij}(t)\hat{e}_i, \quad q^{-1}(t)\hat{e}_i = \sum_{j=1}^{n} a_{ij}(t)e_j$$

for orthonormal bases $\{e_j\}_{j=1}^n$ and $\{\widehat{e}_j\}_{j=1}^n$. Condition (II) holds if and only if

$$q\frac{D}{dt}e_j(x(t)) = \frac{D}{dt}qe_j(x(t))$$

for $j = 1, \ldots, n$, which yields

$$\begin{aligned}
0 &= \left\langle q\frac{D}{dt}e_j(x(t)) - \frac{D}{dt}qe_j(x(t)), \hat{e}_i \right\rangle \\
&= \left\langle \nabla_{Z(t)}e_j, q^{-1}\hat{e}_i \right\rangle - \left\langle \sum_{l=1}^{n} \dot{a}_{lj}\hat{e}_l, \hat{e}_i \right\rangle - \left\langle \sum_{l=1}^{n} a_{lj}\nabla_{qZ(t)}\hat{e}_l, \hat{e}_i \right\rangle \\
&= \sum_{l=1}^{n} a_{il}\left\langle \nabla_{Z(t)}e_j, e_l \right\rangle - \dot{a}_{ij} - \sum_{l=1}^{n} a_{lj}\left\langle \nabla_{qZ(t)}\hat{e}_l, \hat{e}_i \right\rangle
\end{aligned}$$

for every $i, j = 1, \ldots, n$. Hence the third term in (11) can be written as follows:

$$\begin{aligned}
\sum_{i,j=1}^{n} \dot{a}_{ij}\frac{\partial}{\partial a_{ij}} &= \sum_{i,j=1}^{n}\left(\sum_{l=1}^{n} a_{il}\left\langle \nabla_{Z(t)}e_j, e_l \right\rangle - \sum_{l=1}^{n} a_{lj}\left\langle \nabla_{qZ(t)}\hat{e}_l, \hat{e}_i \right\rangle\right)\frac{\partial}{\partial a_{ij}} \\
&= \sum_{j,l=1}^{n}\left\langle \nabla_{Z(t)}e_j, e_l \right\rangle A\cdot\frac{\partial}{\partial a_{lj}} - \sum_{i,l=1}^{n}\left\langle \nabla_{qZ(t)}\hat{e}_l, \hat{e}_i \right\rangle\frac{\partial}{\partial a_{il}}\cdot A \\
&= \sum_{i,j=1}^{n}\left\langle \nabla_{Z(t)}e_j, e_i \right\rangle A\cdot\frac{\partial}{\partial a_{ij}} - \sum_{i,j,r,s=1}^{n} a_{ir}a_{js}\left\langle \nabla_{qZ(t)}\hat{e}_j, \hat{e}_i \right\rangle A\cdot\frac{\partial}{\partial a_{rs}} \\
&= \sum_{i,j=1}^{n}\left(\left\langle \nabla_{Z(t)}e_j, e_i \right\rangle - \sum_{s=1}^{n} a_{sj}\left\langle \nabla_{qZ(t)}\hat{e}_s, \sum_{r=1}^{n} a_{ri}\hat{e}_r \right\rangle\right) A\cdot\frac{\partial}{\partial a_{ij}} \\
&= \sum_{i,j=1}^{n}\left(\left\langle \nabla_{Z(t)}e_j, e_i \right\rangle - \left\langle \nabla_{qZ(t)}qe_j, qe_i \right\rangle\right) A\cdot\frac{\partial}{\partial a_{ij}}. \qquad (12)
\end{aligned}$$

The coefficients in the basis $A \cdot \dfrac{\partial}{\partial_{ij}}$ in the sum (12) are skew symmetric, from the property of the Levi-Civita connection. Now we can write

$$\sum_{i,j=1}^{n} \dot{a}_{ij} \frac{\partial}{\partial a_{ij}} = \sum_{i<j} \left(\left\langle \nabla_{Z(t)} e_j, e_i \right\rangle - \left\langle \nabla_{qZ(t)} q e_j, q e_i \right\rangle \right) W_{ij}(A).$$

Writing this in a right-invariant basis, we obtain

$$\sum_{i,j=1}^{n} \dot{a}_{ij} \frac{\partial}{\partial a_{ij}} \\ = \sum_{i<j} \left(\left\langle \nabla_{Z(t)} q^{-1} \hat{e}_j, q^{-1} \hat{e}_i \right\rangle - \left\langle \nabla_{qZ(t)} \hat{e}_j, \hat{e}_i \right\rangle \right) \mathrm{Ad}(A^{-1}) W_{ij}(A).$$

Similarly, (III) holds if and only if

$$\sum_{\kappa,\lambda=1}^{\nu} \dot{b}_{\kappa\lambda} \frac{\partial}{\partial b_{\kappa\lambda}} = \sum_{\kappa<\lambda} \left(\left\langle \nabla^{\perp}_{Z(t)} \epsilon_\lambda, \epsilon_\kappa \right\rangle - \left\langle \nabla^{\perp}_{qZ(t)} p \epsilon_\lambda, p \epsilon_\kappa \right\rangle \right) W_{\kappa\lambda}(B) \\ = \sum_{\kappa<\lambda} \left(\left\langle \nabla^{\perp}_{Z(t)} p^{-1} \hat{\epsilon}_\lambda, p^{-1} \hat{\epsilon}_\kappa \right\rangle - \left\langle \nabla^{\perp}_{qZ(t)} \hat{\epsilon}_\lambda, \hat{\epsilon}_\kappa \right\rangle \right) \mathrm{Ad}(B^{-1}) W_{\kappa\lambda}(B).$$

Definition 4. If X is a vector field on M, then let us define the vector fields $\mathcal{V}(X)$ and $\mathcal{V}^{\perp}(X)$ on $Q \oplus P_{\iota,\widehat{\iota}}$ such that under any local trivialization h as in (9) and any $(q,p) \in \pi^{-1}(x)$, they satisfy

$$dh\left(\mathcal{V}(X)(q,p)\right) = \sum_{i<j} \left(\left\langle \nabla_{X(x)} e_j, e_i \right\rangle - \left\langle \nabla_{qX(x)} q e_j, q e_i \right\rangle \right) W_{ij}(A), \tag{13}$$

$$dh\left(\mathcal{V}^{\perp}(X)(q,p)\right) = \sum_{\kappa<\lambda} \left(\left\langle \nabla^{\perp}_{X(x)} \epsilon_\lambda, \epsilon_\kappa \right\rangle - \left\langle \nabla^{\perp}_{qX(x)} p \epsilon_\lambda, p \epsilon_\kappa \right\rangle \right) W_{\kappa\lambda}(B). \tag{14}$$

Note that if $Y(x) = X(x) = X_0 \in T_x M$, then

$$\mathcal{V}(Y)(q,p) = \mathcal{V}(X)(q,p) \quad \forall (q,p) \in (Q \oplus P_{\iota,\widehat{\iota}})_x.$$

Hence, we may define $\mathcal{V}(X_0)(q,p)$ whenever $X_0 \in T_x M$ and $(q,p) \in (Q \oplus P_{\iota,\widehat{\iota}})_x$. Also note that the map $X \mapsto \mathcal{V}(X)$ is linear. The same holds for $\mathcal{V}^{\perp}$.

Remark 8. At first glance, it may seem that all of the coefficients of $W_{ij}(A)$ and $W_{\kappa\lambda}(A)$ in (13) and (14) vanish by conditions (II) and (III). However, this is not valid. Even though, for any tangential vector field X

$$\frac{D}{dt} X(x(t)) = \nabla_{\dot{x}(t)} X(x(t)),$$

in general, $\nabla_{q\dot{x}(t)} q(t) e_j$ does not coincide with $\dfrac{D}{dt} q(t) e_j(x(t))$. To see this, note that

$$\frac{D}{dt}a_{sj}\hat{e}_s(\widehat{x}(t)) = \dot{a}_{sj}\hat{e}_s(\widehat{x}(t)) + a_{sj}\nabla_{\dot{\widehat{x}}(t)}\hat{e}_s(\widehat{x}(t))$$
$$= \dot{a}_{sj}\hat{e}_s(\widehat{x}(t)) + a_{sj}\nabla_{q\dot{x}(t)}\hat{e}_s(\widehat{x}(t)),$$

while

$$\nabla_{q\dot{x}(t)}a_{sj}\hat{e}_s(x(t)) = a_{sj}\nabla_{q\dot{x}(t)}\hat{e}_s(x(t)).$$

Similar relations hold for $\frac{D^\perp}{dt}$.

We sum up our considerations in this Section in the following result.

Proposition 2. *A curve $(q(t), p(t))$ in $Q \oplus P_{\iota,\widehat{\iota}}$ is a rolling if and only if it is a horizontal curve with respect to the distribution E, defined by*

$$E_{(q,p)} = \Big\{X_0 + qX_0 + \mathcal{V}(X_0)(q,p) + \mathcal{V}^\perp(X_0)(q,p) \;\Big|\; X_0 \in T_xM\Big\},$$

where $(q,p) \in (Q \oplus P_{\iota,\widehat{\iota}})_x$.

Using the same symbol to denote the restriction of $\mathcal{V}(X)$ to Q, we have the following assertion.

Proposition 3. *A curve $q(t)$ in Q is an intrinsic rolling if and only if it is a horizontal curve with respect to the distribution D, defined by*

$$D_q = \Big\{X_0 + qX_0 + \mathcal{V}(X_0)(q) \;\Big|\; X_0 \in T_xM\Big\}, \quad q \in Q_x.$$

5. Examples of rollings and their controllability

In this section, we show two examples of rolling configurations, namely, the sphere S^n rolling on $\mathbb{R}^n$ and the special Euclidean group SE(3) rolling on its Lie algebra $\mathfrak{se}(3)$. The first case is controllable; this follows from the fact that the distribution D in Proposition 3 is bracket generating and thus the Chow–Rashevskiĭ theorem holds (see [4, 12]). The second example is not controllable, which follows from the orbit theorem and a strong version of it for the case of analytic manifolds (see [1]).

For the second example, we study a particular intrinsic rolling, which we extend to an extrinsic rolling. This exemplifies the result obtained in Theorem 2.

5.1. A controllable example: S^n rolling on $\mathbb{R}^n$.

We want to illustrate the properties of the distribution D from Proposition 3, by proving that the unit sphere S^n in $\mathbb{R}^{n+1}$ rolling over $\mathbb{R}^n$ is a completely controllable system. This result was obtained in [17] by rewriting the kinematic equations as a left-invariant control system without drift evolving on $G = \mathbb{R}^n \times \mathrm{SO}(n+1)$, and then verifying the bracket generating condition at the identity of G. Left translation and the Chow–Rashevkiĭ theorem imply controllability of the system.

The aim of this subsection is to show the controllability of the system directly from the Chow–Rashevskiĭ theorem. It is important to stress that the following proof does not make use of any additional structure of the manifolds. The use of extra information about the geometry of the manifolds can lead to significant simplifications, as is the case in [17], but then the methods can be used only in particular situations.

Consider the unit sphere S^n as the submanifold of $\mathbb{R}^{n+1}$,

$$S^n = \left\{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; x_0^2 + \cdots + x_n^2 = 1 \right\},$$

with the induced metric.

For an arbitrary point $\tilde{x} = (\tilde{x}_0, \dots, \tilde{x}_n) \in S^n$, at least one of the coordinates $\tilde{x}_0, \dots, \tilde{x}_n$ does not vanish. Without lost of generality, we may assume that $\tilde{x}_n \neq 0$, and consider the neighborhood

$$U = \{(x_0, \dots, x_n) \in S^n \mid \pm x_n > 0\},$$

where the choice of $\pm$ depends on the sign of $\tilde{x}_n$. To simplify the notation, we define on U the functions

$$s_j(x) = \sum_{r=j}^{n} x_r^2, \quad j = 1, \dots, n.$$

The functions $s_j(x)$, $j = 1, \dots, n$, are always strictly positive on U, and we use them to define an orthonormal basis of TU. Define the vector fields e_j on U

$$e_j = \sqrt{\frac{s_j}{s_{j-1}}} \left(-\frac{\partial}{\partial x_{j-1}} + \frac{x_{j-1}}{s_j} \sum_{r=j}^{n} x_r \frac{\partial}{\partial x_r} \right), \quad j = 1, \dots, n. \tag{15}$$

These vector fields form an orthonormal basis of the tangent space over U and we denote by Γ_{ij}^k the Christoffel symbols with respect to the basis $\{e_i\}_{i=1}^n$. We set

$$\hat{e}_i = \frac{\partial}{\partial \widehat{x}_i}$$

to be the standard basis of $\mathbb{R}^n$.

Let us state two technical lemmas whose proofs can be obtained by direct calculations.

Lemma 2. *Let* $1 \leq i < j \leq n$. *Then*

$$\Gamma_{kj}^i = -\frac{x_{i-1}\delta_{k,j}}{\sqrt{s_{i-1}s_i}}$$

for all $k = 1, \dots, n$.

For convenience, we denote

$$\Gamma^k = \frac{x_{k-1}}{\sqrt{s_{k-1}s_k}} = -\Gamma_{aa}^k \quad \forall a = 1, \dots, n.$$

Remark 9. The properties of ∇ have the following consequences.

(i) The compatibility of ∇ with the metric and $\langle e_i, e_j\rangle = \delta_{i,j}$ imply that
$$\Gamma^i_{kj} = -\Gamma^j_{ki}.$$
In particular, $\Gamma^i_{ki} = 0$.

(ii) The symmetry of ∇ implies that if $l < k$, then
$$[e_k, e_l] = \nabla_{e_k} e_l - \nabla_{e_l} e_k = \sum_{i=1}^{n} \langle \nabla_{e_k} e_l - \nabla_{e_l} e_k, e_i\rangle\, e_i = \Gamma^l e_k.$$

Lemma 3. *For $k, l = 1, 2, \ldots, n$, we have*
$$e_k\left(\Gamma^l\right) = \begin{cases} 0, & k > l, \\ -\dfrac{1}{s_k}, & k = l, \\ -\Gamma^l\,\Gamma^k, & k < l. \end{cases}$$

Observe that
$$\nabla_{\hat{e}_k} \hat{e}_l = 0, \quad [\hat{e}_k, \hat{e}_l] = 0 \quad \forall k, l = 1, \ldots, n.$$

Consider the vector fields
$$X_k = e_k + q e_k + \mathcal{V}(e_k)$$
restricted to U, which span the distribution D introduced in Proposition 3. In this case, we have the explicit form
$$X_k(x, \hat{x}, A) = e_k(x) + \sum_{i=1}^{n} a_{ik} \hat{e}_i(\widehat{x}) - \sum_{i=1}^{k-1} \Gamma^i\, W_{ik}(A).$$

In order to determine $[X_k, X_l]$, we assume that $k > l$. Then
$$\begin{aligned}
[X_k, X_l] &= [e_k, e_l] - \sum_{i=1}^{k-1} \sum_{j=1}^{n} \Gamma^i\, W_{ik} a_{jl} \hat{e}_j + \sum_{j=1}^{l-1} \sum_{i=1}^{n} \Gamma^j\, W_{jl} a_{ik} \hat{e}_i \\
&\quad - \sum_{j=1}^{l-1} e_k(\Gamma^j) W_{jl} + \sum_{i=1}^{k-1} e_l(\Gamma^i) W_{ik} + \sum_{i=1}^{k-1} \sum_{j=1}^{l-1} \Gamma^i\,\Gamma^j [W_{ik}, W_{jl}] \\
&= \Gamma^l\, e_k - \sum_{i=1}^{k-1} \sum_{j=1}^{n} \Gamma^i \left(a_{ji}\delta_{k,l} - a_{jk}\delta_{i,l}\right) \hat{e}_j + \sum_{j=1}^{l-1} \sum_{i=1}^{n} \Gamma^j \left(a_{ij}\delta_{l,k} - a_{il}\delta_{j,k}\right) \hat{e}_i \\
&\quad - \frac{1}{s_l}\, W_{lk} - \sum_{i=l+1}^{k-1} \Gamma^i\,\Gamma^j\, W_{ik} + \sum_{i=1}^{k-1} \sum_{j=1}^{l-1} \Gamma^i\,\Gamma^j (-\delta_{i,l} W_{jk} + \delta_{i,j} W_{lk}) \\
&= \Gamma^l \left(e_k + \sum_{j=1}^{n} a_{jk} \hat{e}_j - \sum_{i=l+1}^{k-1} \Gamma^i\, W_{ik} - \sum_{j=1}^{l-1} \Gamma^j\, W_{jk} \right) - \frac{1}{s_l}\, W_{lk} + \sum_{j=1}^{l-1} (\Gamma^j)^2\, W_{lk} \\
&= \Gamma^l\, X_k - W_{lk}.
\end{aligned}$$

For $l < k$, we define the vector fields Y_{lk} by

$$Y_{lk} := [X_l, X_k] + \Gamma^l X_k = W_{lk}.$$

Finally, let

$$Z_1 = [Y_{12}, X_2] = \sum_{i=1}^{n} a_{i1}\hat{e}_i, \quad Z_k = [X_1, Y_{1k}] = \sum_{i=1}^{n} a_{ik}\hat{e}_i, \quad k = 2, \dots, n.$$

We conclude that the entire tangent space is spanned by the vector fields

$$\{X_k\}_{k=1}^n, \quad \{Y_{lk}\}_{1\le l<k\le n}, \quad \{Z_k\}_{k=1}^n.$$

Hence D is a regular bracket generating distribution of step 3, which implies that the system of rolling S^n over $\mathbb{R}^n$ is completely controllable.

5.2. **A non-controllable example:** SE(3) **rolling on** $\mathbb{R}^6$**.** We consider the case of SE(3) endowed with a left-invariant metric defined later, rolling over its tangent space $T_1\,\mathrm{SE}(3) = \mathfrak{se}(3)$ at the identity, with metric obtained by restricting the left-invariant metric on SE(3) to the identity. Our goal is to determine whether the system is controllable.

We give SE(3) coordinates as follows. For any $x \in \mathrm{SE}(3)$, there exist $C = (c_{ij}) \in \mathrm{SO}(3)$ and $r = (r_1, r_2, r_3) \in \mathbb{R}^3$ such that $x = (C, r)$ acts by the rule

$$x(y) = Cy + r \quad \forall y \in \mathbb{R}^3.$$

The tangent space of SE(3) at $x = (C, r)$ is spanned by the left-invariant vector fields

$$e_1 = Y_1 = \frac{1}{\sqrt{2}}\left(C\cdot\frac{\partial}{\partial c_{12}} - C\cdot\frac{\partial}{\partial c_{21}}\right) = \frac{1}{\sqrt{2}}\sum_{j=1}^{3}\left(c_{j1}\frac{\partial}{\partial c_{j2}} - c_{j2}\frac{\partial}{\partial c_{j1}}\right),$$

$$e_2 = Y_2 = \frac{1}{\sqrt{2}}\left(C\cdot\frac{\partial}{\partial c_{13}} - C\cdot\frac{\partial}{\partial c_{31}}\right) = \frac{1}{\sqrt{2}}\sum_{j=1}^{3}\left(c_{j1}\frac{\partial}{\partial c_{j3}} - c_{j3}\frac{\partial}{\partial c_{j1}}\right),$$

$$e_3 = Y_3 = \frac{1}{\sqrt{2}}\left(C\cdot\frac{\partial}{\partial c_{23}} - C\cdot\frac{\partial}{\partial c_{32}}\right) = \frac{1}{\sqrt{2}}\sum_{j=1}^{3}\left(c_{j2}\frac{\partial}{\partial c_{j3}} - c_{j3}\frac{\partial}{\partial c_{j2}}\right),$$

$$e_{k+3} = X_k = C\cdot\frac{\partial}{\partial r_k} = \sum_{j=1}^{3} c_{jk}\frac{\partial}{\partial r_j}, \quad k = 1, 2, 3.$$

Define a left-invariant metric on SE(3) by declaring the vectors $e_1, \dots, e_6$ to form an orthonormal basis. The mapping

$$\sum_{j=1}^{6} \widehat{x}_j e_j(1) \mapsto (\widehat{x}_1, \widehat{x}_2, \widehat{x}_3, \widehat{x}_4, \widehat{x}_5, \widehat{x}_6) \in \mathbb{R}^6$$

permits to identify $\mathfrak{se}(3)$ endowed with the induced metric with $\mathbb{R}^6$ endowed with the Euclidean metric. We write

$$\hat{e}_k = \frac{\partial}{\partial \widehat{x}_k}$$

on $\mathbb{R}^6$ and study the behavior of the intrinsic rollings of SE(3) on $\mathbb{R}^6$. Note that the configuration space Q is $\mathrm{SE}(3) \times \mathbb{R}^6 \times \mathrm{SO}(6)$, because both manifolds SE(3) and $\mathbb{R}^6$ are Lie groups, so their tangent bundles are trivial, and $\dim Q = 27$.

Let us denote by ∇ the Levi-Civita connection on SE(3) or $\mathbb{R}^6$ with respect to the corresponding Riemannian metrics defined above. The covariant derivatives $\nabla_{e_i} e_j$ are nonzero only in the following cases:

$$\nabla_{Y_1} Y_2 = -\frac{1}{2\sqrt{2}}\, Y_3, \quad \nabla_{Y_1} Y_3 = \frac{1}{2\sqrt{2}}\, Y_2, \quad \nabla_{Y_2} Y_3 = -\frac{1}{2\sqrt{2}}\, Y_1,$$
$$\nabla_{Y_1} X_k = \frac{1}{\sqrt{2}} \left(\delta_{2,k} X_1 - \delta_{1,k} X_2\right), \quad \nabla_{Y_2} X_k = \frac{1}{\sqrt{2}} \left(\delta_{3,k} X_1 - \delta_{1,k} X_3\right),$$
$$\nabla_{Y_3} X_k = \frac{1}{\sqrt{2}} \left(\delta_{3,k} X_2 - \delta_{2,k} X_3\right),$$

where $\delta_{i,j}$ denotes the Kronecker symbol. On the other hand, it is well known that

$$\nabla_{\hat{e}_i} \hat{e}_j = 0$$

for any i, j. Proposition 3 and Definition 4 show that the distribution D over Q is spanned by

$$\begin{aligned}
Z_1 &= Y_1 + qY_1 + \frac{1}{2\sqrt{2}} W_{23} + \frac{1}{\sqrt{2}} W_{45}, \\
Z_2 &= Y_2 + qY_2 - \frac{1}{2\sqrt{2}} W_{13} + \frac{1}{\sqrt{2}} W_{46}, \\
Z_3 &= Y_3 + qY_3 + \frac{1}{2\sqrt{2}} W_{12} + \frac{1}{\sqrt{2}} W_{56}, \\
K_1 &= X_1 + qX_1, \quad K_2 = X_2 + qX_2, \quad K_3 = X_3 + qX_3.
\end{aligned} \tag{16}$$

In order to determine the controllability of rolling SE(3) over $\mathbb{R}^6$, we employ the orbit theorem [6, 16]. In the case of D defined by the vector fields (16), straightforward calculations yield that the flag associated to D has the form

$$\begin{aligned}
D^2 &= D \oplus \operatorname{span}\{W_{12}, W_{13}, W_{23}\}, \\
D^3 &= D^2 \oplus \operatorname{span}\{qY_1, qY_2, qY_3\}, \\
D^4 &= D^3,
\end{aligned} \tag{17}$$

and so $\dim D^2 = 9$ and $\dim D^k = 12$ for all $k \geq 3$ and the step of D is 3.

Let $(x_0, \widehat{x}_0, A_0)$ be an arbitrary point in Q and let $\mathcal{O}_{(x_0, \widehat{x}_0, A_0)}$ denote the subset of all points in Q which are connected to $(x_0, \widehat{x}_0, A_0)$ by an intrinsic rolling. The orbit theorem asserts that at each point D^3 is contained in the tangent space of the orbits. However, since we know that D^3 has a local basis, we have the stronger result of

$$T_{(x,\widehat{x},A)}\mathcal{O}_{(x_0,\widehat{x}_0,A_0)} = D^3_{(x,\widehat{x},A)},$$

holding for all $(x,\widehat{x},A) \in \mathcal{O}_{(x_0,\widehat{x}_0,A_0)}$ (see [1, Chap. 5]).

It follows from (17) that $\mathcal{O}_{(x_0,\widehat{x}_0,A_0)}$ has dimension 12. Since $\mathcal{O}_{(x_0,\widehat{x}_0,A_0)}$ is not the entire Q, we conclude that the system is not controllable.

We end this section with a concrete example of an intrinsic rolling $q(t) = (x(t),\widehat{x}(t),A(t))$, where

$$x(0) = \mathrm{id}_{\mathbb{R}^3}, \quad \widehat{x}(0) = 0, \quad A(0) = \mathbf{1}.$$

Define the curve $x : [0,\tau] \to \mathrm{SE}(3)$ by

$$x(t)y = \begin{pmatrix} \cos\theta(t) & \sin\theta(t) & 0 \\ -\sin\theta(t) & \cos\theta(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \psi(t) \end{pmatrix}, \tag{18}$$

where $\theta(t)$ and $\psi(t)$ are absolutely continuous functions with

$$\theta(0) = \psi(0) = 0.$$

Then

$$\dot{x} = \sqrt{2}\,\dot{\theta}(t)Y_1 + \dot{\psi}(t)X_3$$

for almost every t, and the rolling has the form

$$\dot{q} = \sqrt{2}\,\dot{\theta}(t)Z_1 + \dot{\psi}(t)K_3.$$

This implies that

$$\dot{\widehat{x}}(t) = \sqrt{2}\,\dot{\theta}(t)\,qY_1 + \dot{\psi}(t)\,qX_3, \tag{19}$$

$$\dot{A}(t) = \dot{\theta}(t)\left(\frac{1}{2}W_{23}(A) + W_{45}(A)\right) \tag{20}$$

for almost every t. It follows from Eq. (19) that

$$\widehat{x}(t) = \left(\sqrt{2}\,\theta(t),0,0,0,0,\psi(t)\right)^t,$$

where ^{t} denotes transposition. By exponentiating Eq. (20) we obtain

$$A(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos\left(\frac{\theta(t)}{2}\right) & \sin\left(\frac{\theta(t)}{2}\right) & 0 & 0 & 0 \\ 0 & -\sin\left(\frac{\theta(t)}{2}\right) & \cos\left(\frac{\theta(t)}{2}\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta(t) & \sin\theta(t) & 0 \\ 0 & 0 & 0 & -\sin\theta(t) & \cos\theta(t) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

5.3. **Imbedding of** SE(n) **into Euclidean space.** Since it is less obvious how to extend an intrinsic rolling of SE(3) on $\mathfrak{se}(3)$ to an extrinsic rolling in ambient space, we describe an isometric imbedding of SE(n) into Euclidean space $\mathbb{R}^{(n+1)^2}$. Identify an element $\overline{C} \in \mathbb{R}^{(n+1)^2}$ with the matrix

$$\overline{C} = \begin{pmatrix} \overline{c}_{11} & \cdots & \overline{c}_{1,n+1} \\ \vdots & \ddots & \vdots \\ \overline{c}_{n+1,1} & \cdots & \overline{c}_{n+1,n+1} \end{pmatrix}.$$

Define the inner product on $\mathbb{R}^{(n+1)^2}$ by

$$\langle \overline{C}_1, \overline{C}_2 \rangle = \operatorname{trace}\left(\left(\overline{C}_1\right)^t \overline{C}_2 \right).$$

Note that since

$$\langle \overline{C}, \overline{C} \rangle = \sum_{i,j=1}^{n+1} |\overline{c}_{ij}|^2,$$

the metric $\langle \cdot, \cdot \rangle$ coincides with the Euclidean metric. From this we get that

$$\left\{ \frac{\partial}{\partial \overline{c}_{ij}} \right\}_{i,j=1}^{n+1}$$

is an orthonormal basis for the tangent bundle $T\mathbb{R}^{(n+1)^2}$ with respect to $\langle \cdot, \cdot \rangle$.

We define the imbedding of SE(3) into $\mathbb{R}^{(n+1)^2}$ by

$$\begin{aligned} \iota : \mathrm{SE}(n) &\to \mathbb{R}^{(n+1)^2}, \\ x = (C, r) &\mapsto \overline{C} = \begin{pmatrix} C & r \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

This mapping is in fact an isometry of SE(n) onto its image. To see this, note that the metrics coincide at the identity, and that the metric of $\mathbb{R}^{(n+1)^2}$, restricted to Image ι, is left-invariant under the action of SE(n). Hence, the metrics on SE(n) and Image ι coincide, and ι defines an isometric imbedding.

5.4. **Extrinsic rolling.** We will use the imbedding from Sec. 5.3 to construct an extrinsic rolling of SE(3) over $\mathfrak{se}(3)$ in $\mathbb{R}^{16}$. We use ∂_{ij} to denote $\partial/\partial \overline{c}_{ij}$. For the sake of clarity, we denote by M the image of SE(3) by ι.

Then the vector fields spanning TM are

$$\begin{aligned} e_1 = Y_1 &= \frac{1}{\sqrt{2}} \sum_{i=1}^{3} \left(\bar{c}_{i1} \partial_{i2} - \bar{c}_{i2} \partial_{i1} \right), \\ e_2 = Y_2 &= \frac{1}{\sqrt{2}} \sum_{i=1}^{3} \left(\bar{c}_{i1} \partial_{i3} - \bar{c}_{i3} \partial_{i1} \right), \\ e_3 = Y_3 &= \frac{1}{\sqrt{2}} \sum_{i=1}^{3} \left(\bar{c}_{i2} \partial_{i3} - \bar{c}_{i3} \partial_{i2} \right), \\ e_{3+k} = X_k &= \sum_{j=1}^{3} \bar{c}_{ik} \partial_{i4}, \quad k = 1, 2, 3, \end{aligned} \tag{21}$$

where we suppressed $d\iota$ in the notation. We introduce the following orthonormal basis of $TM^{\perp}$:

$$\begin{aligned} \Upsilon_1 &= \frac{1}{\sqrt{2}} \sum_{j=1}^{3} \left(\bar{c}_{j1} \partial_{j2} + \bar{c}_{j2} \partial_{j1} \right), \\ \Upsilon_2 &= \frac{1}{\sqrt{2}} \sum_{j=1}^{3} \left(\bar{c}_{j1} \partial_{j3} + \bar{c}_{j3} \partial_{j1} \right), \\ \Upsilon_3 &= \frac{1}{\sqrt{2}} \sum_{j=1}^{3} \left(\bar{c}_{j2} \partial_{j3} + \bar{c}_{j3} \partial_{j2} \right), \\ \Psi_\lambda &= \sum_{j=1}^{3} \bar{c}_{j\lambda} \partial_{j\lambda}, \quad \lambda = 1, 2, 3, \\ \Xi_\mu &= \partial_{4\mu}, \quad \mu = 1, 2, 3, 4. \end{aligned} \tag{22}$$

We denote by $\widehat{M}$ the image of $\mathbb{R}^6$ into $\mathbb{R}^{16}$ by the imbedding

$$(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3, \widehat{x}_4, \widehat{x}_5, \widehat{x}_6) \overset{\widehat{\iota}}{\mapsto} \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \widehat{x}_1 & \frac{1}{\sqrt{2}} \widehat{x}_2 & \widehat{x}_4 \\ -\frac{1}{\sqrt{2}} \widehat{x}_1 & 0 & \frac{1}{\sqrt{2}} \widehat{x}_3 & \widehat{x}_5 \\ -\frac{1}{\sqrt{2}} \widehat{x}_2 & -\frac{1}{\sqrt{2}} \widehat{x}_3 & 0 & \widehat{x}_6 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have the following orthonormal basis of $T\widehat{M}$:

$$\hat{e}_1 = \frac{1}{\sqrt{2}} (\partial_{12} - \partial_{21}), \quad \hat{e}_2 = \frac{1}{\sqrt{2}} (\partial_{13} - \partial_{31}), \quad \hat{e}_3 = \frac{1}{\sqrt{2}} (\partial_{23} - \partial_{32}),$$
$$\hat{e}_{3+k} = \partial_{k4}, \quad k = 1, 2, 3,$$

while the vector fields spanning $T\widehat{M}^{\perp}$ are

$$\hat{\epsilon}_1 = \frac{1}{\sqrt{2}}(\partial_{12} + \partial_{21}), \quad \hat{\epsilon}_2 = \frac{1}{\sqrt{2}}(\partial_{13} + \partial_{31}), \quad \hat{\epsilon}_3 = \frac{1}{\sqrt{2}}(\partial_{23} + \partial_{32}),$$
$$\hat{\epsilon}_{3+\kappa} = \partial_{\kappa\kappa}, \qquad \kappa = 1, 2, 3,$$
$$\hat{\epsilon}_{6+\kappa} = \partial_{4\kappa}, \qquad \kappa = 1, 2, 3, 4.$$

In order to extend an intrinsic rolling $q(t)$ with

$$\pi(q(t)) = (x(t), \widehat{x}(t)),$$

we find an orthonormal frame of normal parallel vector fields along $x(t)$ and $\widehat{x}(t)$. Along $\widehat{x}(t)$, we may use the restriction of

$$\{\hat{\epsilon}_\kappa\}_{\kappa=1}^{10}.$$

For the curve $x(t)$ the answer is more complicated.

First, we study the value of $\nabla^{\perp}$ for different choices of vector fields.

1. $\nabla^{\perp}_X \Xi_\mu = 0$ for any tangential vector field X and Ξ_μ as in Eq. (22);
2. $\nabla^{\perp}_{X_k} \Upsilon = 0$ for any normal vector field Υ and X_k as in Eq. (21);
3. Otherwise, the results are presented in the following table:

	Υ_1	Υ_2	Υ_3	Ψ_1	Ψ_2	Ψ_3
$\nabla^{\perp}_{Y_1}$	$\frac{1}{2}(\Psi_1 - \Psi_2)$	$-\frac{1}{2\sqrt{2}}\Upsilon_3$	$\frac{1}{2\sqrt{2}}\Upsilon_2$	$-\frac{1}{2}\Upsilon_1$	$\frac{1}{2}\Upsilon_1$	0
$\nabla^{\perp}_{Y_2}$	$-\frac{1}{2\sqrt{2}}\Upsilon_3$	$\frac{1}{2}(\Psi_1 - \Psi_3)$	$\frac{1}{2\sqrt{2}}\Upsilon_1$	$-\frac{1}{2}\Upsilon_2$	0	$\frac{1}{2}\Upsilon_2$
$\nabla^{\perp}_{Y_3}$	$-\frac{1}{2\sqrt{2}}\Upsilon_2$	$\frac{1}{2\sqrt{2}}\Upsilon_1$	$\frac{1}{2}(\Psi_2 - \Psi_3)$	0	$-\frac{1}{2}\Upsilon_3$	$\frac{1}{2}\Upsilon_3$

We use the relations above to construct an extrinsic rolling by making use of the curve (18).

Since

$$\dot{x}(t) = \sqrt{2}\,\dot{\theta}(t) Y_1(x(t)) + \dot{\psi}(t) X_3(x(t)),$$

the vector field

$$\Psi(t) = \sum_{\lambda=1}^{3} \left(\upsilon_\lambda(t)\Upsilon_\lambda(x(t)) + \upsilon_{3+\lambda}(t)\Psi_\lambda(x(t))\right)$$

is normal parallel along $x(t)$ if

$$\left(\dot{\upsilon}_1 - \frac{\dot{\theta}}{\sqrt{2}}(\upsilon_4 - \upsilon_5)\right)\Upsilon_1 + \left(\dot{\upsilon}_2 + \frac{\dot{\theta}}{2}\upsilon_3\right)\Upsilon_2 + \left(\dot{\upsilon}_3 - \frac{\dot{\theta}}{2}\upsilon_2\right)\Upsilon_3$$
$$+ \left(\dot{\upsilon}_4 + \frac{\dot{\theta}}{\sqrt{2}}\upsilon_1\right)\Psi_1 + \left(\dot{\upsilon}_5 - \frac{\dot{\theta}}{\sqrt{2}}\upsilon_1\right)\Psi_2 + \upsilon_6\Psi_3 = 0.$$

Hence we define a parallel orthonormal frame along $x(t)$ by

$$\epsilon_1(t) = \cos\theta\Upsilon_1(x(t)) - \frac{1}{\sqrt{2}}\sin\theta\Psi_1(x(t)) + \frac{1}{\sqrt{2}}\sin\theta\Psi_2(x(t)),$$

$$\epsilon_2(t) = \cos\left(\frac{\theta}{2}\right)\Upsilon_2(x(t)) + \sin\left(\frac{\theta}{2}\right)\Upsilon_3(x(t)),$$
$$\epsilon_3(t) = -\sin\left(\frac{\theta}{2}\right)\Upsilon_2(x(t)) + \cos\left(\frac{\theta}{2}\right)\Upsilon_3(x(t)),$$
$$\epsilon_4(t) = \frac{1}{\sqrt{2}}\sin\theta\Upsilon_1(x(t)) + \frac{\cos\theta+1}{2}\Psi_1(x(t)) + \frac{1-\cos\theta}{2}\Psi_2(x(t)),$$
$$\epsilon_5(t) = -\frac{1}{\sqrt{2}}\sin\theta\Upsilon_1(x(t)) + \frac{1-\cos\theta}{2}\Psi_1(x(t)) + \frac{1+\cos\theta}{2}\Psi_2(x(t)),$$
$$\epsilon_6(t) = \Psi_3(x(t)),$$
$$\epsilon_{6+\lambda}(t) = \Xi_\lambda(x(t)), \quad \lambda = 1,2,3,4.$$

Thus, $p(t)$ is represented by a constant matrix in the bases

$$\{\epsilon_\lambda(t)\}_{\lambda=1}^{10}, \quad \{\hat{\epsilon}_\kappa(t)\}_{\kappa=1}^{10}.$$

Let us choose $p(t)$ to be the identity in these bases, due to the given imbedding.

The curve $g(t) = (q(t), p(t))$ in SE(16) is given by

$$g(t)\overline{x} = \overline{A}\overline{x} + \overline{r}(t),$$

where the matrix $\overline{A}(t)$ is presented on p. 213 (here $\mathbf{0}_{m\times n}$ denotes the zero matrix of size $m \times n$ and $\mathbf{1}_6$ is the identity matrix of size 6×6) and

$$\overline{r}(t) = \left(-1,\ \frac{\theta}{\sqrt{2}},\ 0,\ 0,\ \frac{\theta}{\sqrt{2}},\ -1,\ 0,\ 0,\ 0,\ 0,\ -1,\ 0,\ 0,\ 0,\ 0,\ 0\right)^t.$$

References

1. A. Agrachev and Yu. Sachkov. Control theory from the geometric viewpoint. *Springer-Verlag* (2004).
2. R. Bryant and L. Hsu. Rigidity of integral curves of rank 2 distributions. *Invent. Math.* **114** (1993), No. 2, 435–461.
3. É. Cartan. Les systèmes de Pfaff, à cinq variables et les équations aux dérivées partielles du second ordre. *Ann. Sci. École Norm. Super.* **27** (1910), No. 3, 109–192.
4. W. L. Chow. Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung. *Math. Ann.* **117** (1939), 98–105.
5. M. Godoy Molina, E. Grong, I. Markina, and F. Silva Leite. Intrinsic rolling of manifolds. In: *Proc. CONTROLO'*2010, 8–10 *September* 2010, *University of Coimbra, Portugal* (2010).
6. R. Hermann. On the accessibility problem in control theory. In: *Int. Symp. Nonlinear Differential Equations and Nonlinear Mechanics. Academic Press, New York* (1963), pp. 325–332.
7. K. Hüper, M. Kleinsteuber, and F. Silva Leite. Rolling Stiefel manifolds. *Int. J. Systems Sci.* **39** (2008), No. 9, 881–887.

$$\overline{A}(t)=\left(\begin{array}{cccccccccc|c}
\cos^2\frac{\theta}{2} & -\frac{\sin\theta}{2} & 0 & 0 & \frac{\sin\theta}{2} & \frac{\cos\theta-1}{2} & 0 & 0 & 0 & 0 & \\
\frac{\sin\theta}{2} & \cos^2\frac{\theta}{2} & 0 & 0 & \sin^2\frac{\theta}{2} & \frac{\sin\theta}{2} & 0 & 0 & 0 & 0 & \\
0 & 0 & \cos\frac{\theta}{2} & 0 & 0 & 0 & \sin\frac{\theta}{2} & 0 & 0 & 0 & \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\
-\frac{\sin\theta}{2} & \sin^2\frac{\theta}{2} & 0 & 0 & \cos^2\frac{\theta}{2} & -\frac{\sin\theta}{2} & 0 & 0 & 0 & 0 & \mathbf{0}_{10\times 6}\\
\frac{\cos\theta-1}{2} & -\frac{\sin\theta}{2} & 0 & 0 & \frac{\sin\theta}{2} & \cos^2\frac{\theta}{2} & 0 & 0 & 0 & 0 & \\
0 & 0 & -\sin\frac{\theta}{2} & 0 & 0 & 0 & \cos\frac{\theta}{2} & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sin\frac{\theta}{2} & \cos\frac{\theta}{2} & \\
\hline
\multicolumn{10}{c|}{\mathbf{0}_{6\times 10}} & \mathbf{1}_6
\end{array}\right),$$

8. K. Hüper and F. Silva Leite. On the geometry of rolling and interpolation curves on S^n, SO_n, and Grassmann manifolds. *J. Dynam. Control Syst.* **13** (2007), No. 4, 467–502.
9. K. A. Krakowski and F. Silva Leite. Rolling maps in Riemannian manifolds. In: *Proc. CONTROLO'2010,* 8–10 *September* 2010*, University of Coimbra, Portugal* (2010).
10. B. O'Neill. Semi-Riemannian geometry. *Academic Press, Elsevier* (1983).
11. J. Nash. The imbedding problem for Riemannian manifolds. *Ann. Math.* **63** (1956), No. 2, 20–63.
12. P. K. Rashevskiĭ. About connecting two points of complete nonholonomic space by admissible curve. *Uch. Zap. Ped. Inst. K. Liebknecht* **2** (1938), 83–94.
13. E. D. Sontag. Mathematical Control Theory. *Springer-Verlag, New York* (1999), pp. 467–492.
14. R. W. Sharpe. Differential geometry. *Springer-Verlag, New York* (1997).
15. I. M. Singer and J. A. Thorpe. Lecture notes on elementary topology and geometry. *Springer-Verlag, New York–Heidelberg* (1976).
16. H. J. Sussmann. Orbits of families of vector fields and integrability of distributions. *Int. Symp. Trans. Amer. Math. Soc.* **180** (1973), 171–188.
17. J. A. Zimmerman. *Optimal control of the sphere S^n rolling on E^n. Math. Control Signals Systems* **17** (2005), No. 1, 14–37.

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