

*Pacific
Journal of
Mathematics*

**AN INTRINSIC VOLUME FUNCTIONAL
ON ALMOST COMPLEX 6-MANIFOLDS
AND NEARLY KÄHLER GEOMETRY**

MISHA VERBITSKY

AN INTRINSIC VOLUME FUNCTIONAL ON ALMOST COMPLEX 6-MANIFOLDS AND NEARLY KÄHLER GEOMETRY

MISHA VERBITSKY

Let (M, I) be an almost complex 6-manifold. The obstruction to the integrability of almost complex structure $N: \Lambda^{0,1}(M) \rightarrow \Lambda^{2,0}(M)$ (the so-called Nijenhuis tensor) maps one 3-dimensional bundle to another 3-dimensional bundle. We say that Nijenhuis tensor is *nondegenerate* if it is an isomorphism. An almost complex manifold (M, I) is called *nearly Kähler* if it admits a Hermitian form ω such that $\nabla(\omega)$ is totally antisymmetric, ∇ being the Levi-Civita connection. We show that a nearly Kähler metric on a given almost complex 6-manifold with nondegenerate Nijenhuis tensor is unique (up to a constant). We interpret the nearly Kähler property in terms of G_2 -geometry and in terms of connections with totally antisymmetric torsion, obtaining a number of equivalent definitions.

We construct a natural diffeomorphism-invariant functional $I \rightarrow \int_M \text{Vol}_I$ on the space of almost complex structures on M , similar to the Hitchin functional, and compute its extrema in the following important case. Consider an almost complex structure I with nondegenerate Nijenhuis tensor, admitting a Hermitian connection with totally antisymmetric torsion. We show that the Hitchin-like functional $I \rightarrow \int_M \text{Vol}_I$ has an extremum in I if and only if (M, I) is nearly Kähler.

Introduction

Almost complex manifolds with nondegenerate Nijenhuis tensor. In geometry, two kinds of plane distributions often arise. There are integrable ones: complex structures, foliations, CR-structures. On the other hand, there are “maximally nonintegrable” distributions, such as the contact structures, where the obstruction

MSC2000: 53C15, 53C25.

Keywords: nearly Kähler, Gray manifold, Hitchin functional, Calabi–Yau, almost complex structure.

Verbitsky is an EPSRC advanced fellow supported by CRDF grant RM1-2354-MO02 and EPSRC grant GR/R77773/01.

to integrability is nowhere degenerate. Looking at almost complex structures in dimension 3, one finds that the obstruction to integrability, the so-called Nijenhuis tensor

$$N : \Lambda^2 T^{1,0}(M) \rightarrow T^{0,1}(M),$$

maps one 3-dimensional bundle to another 3-dimensional bundle. It is only natural to study the class of complex 3-manifolds such that N is nowhere degenerate.

Given such a manifold M , it is possible to construct a nowhere degenerate, positive volume form $\det N^* \otimes \overline{\det N^*}$ on M (for details, see (1-2)).

We study the extrema of this volume form, showing that these extrema correspond to an interesting geometric structure; see [Theorem 2.2](#).

In Hermitian geometry, one often encounters a special kind of almost complex Hermitian manifolds, called strictly nearly Kähler (NK-)manifolds, or Gray manifolds, after Alfred Gray; see [Definition 4.1](#). These manifolds can be characterized in terms of the G_2 -structure on their Riemannian cone, or in terms of a special set of equations reminiscent of Calabi–Yau equations; see [Section 4D](#).

We prove that a strictly nearly Kähler 3-manifold is uniquely determined by its almost complex structure; see [Corollary 3.3](#). Moreover, such manifolds are extrema of the volume functional associated with the Nijenhuis tensor; see [Theorem 2.2](#). This reminds of the construction of Hitchin’s functional on the space of all $SL(3, \mathbb{C})$ -structures on a manifold, having extrema on Calabi–Yau manifolds [[Hitchin 2000](#)].

This paper has the following structure. In [Section 1](#), we introduce the class of 3-manifolds with nowhere degenerate Nijenhuis tensor, and describe the basic structures associated with these manifolds. We give a sketch of a proof of the existence of a Hermitian connection with totally antisymmetric torsion, due to Friedrich and Ivanov, and show that such a Hermitian metric is uniquely determined by the almost complex structure, if the Nijenhuis tensor is nowhere degenerate.

In [Section 2](#), we introduce the nearly Kähler manifolds, giving several versions of their definition and listing some examples.

In [Section 3](#), we apply the results about connections with totally antisymmetric torsion to nearly Kähler geometry, showing that an almost complex structure determines the Hermitian structure on such a manifold uniquely, up to a constant multiplier.

In [Section 4](#), we give several additional versions of the definition of a nearly Kähler manifold, obtaining an explicit description of a Nijenhuis tensor in terms of an orthonormal frame. We also interpret the nearly Kähler structure on a manifold in terms of G_2 -geometry of its Riemannian cone. This is used to show that an NK-structure on a manifold M is uniquely determined by its metric, unless M is locally isometric to a 6-sphere; see [Proposition 4.7](#).

In Section 5, we study infinitesimal variations of an almost complex structure. We prove that NK-manifolds are extrema of an intrinsic volume functional described earlier. A partial converse result is also obtained. Given an almost complex manifold M with nowhere degenerate Nijenhuis tensor, admitting a Hermitian connection with totally antisymmetric torsion, M is an extremum of the intrinsic volume functional if and only if M is nearly Kähler.

1. Almost complex manifolds with nondegenerate Nijenhuis tensor

1A. Nijenhuis tensor on 6-manifolds. Let (M, I) be an almost complex manifold. The Nijenhuis tensor maps two $(1, 0)$ -vector fields to the $(0, 1)$ -part of their commutator. This map is C^∞ -linear, and vanishes, as the Newlander–Nirenberg theorem implies, precisely when I is integrable. We write the Nijenhuis tensor as

$$N : \Lambda^2 T^{1,0}(M) \rightarrow T^{0,1}(M).$$

The dual map

$$(1-1) \quad N^* : \Lambda^{0,1}(M) \rightarrow \Lambda^{2,0}(M)$$

is also called the Nijenhuis tensor. Cartan’s formula implies that N^* acts on $\Lambda^1(M)$ as the $(2, -1)$ -part of the de Rham differential.

When one studies the distributions, one is usually interested in integrable ones (such as $T^{1,0}(M) \subset TM \otimes \mathbb{C}$ for complex or CR-manifolds) or ones where the obstruction to integrability is nowhere degenerate (such as a contact distribution).

For the Nijenhuis tensor in complex dimension > 3 , nondegeneracy does not make much sense, because the space $\text{Hom}(\Lambda^{0,1}(M), \Lambda^{2,0}(M))$ becomes quite complicated. However, for $n = 3$, both sides of (1-1) are 3-dimensional, and we can define the nondegeneracy as follows.

Definition 1.1. Let (M, I) be an almost complex manifold of real dimension 6, and $N : \Lambda^2 T^{1,0}(M) \rightarrow T^{0,1}(M)$ the Nijenhuis tensor. We say that N is *nondegenerate* if N is an isomorphism everywhere. Then (M, I) is called an *almost complex 6-manifold with nowhere degenerate Nijenhuis tensor*.

Remark 1.2. Such manifolds were investigated by R. Bryant. His results were presented at a conference [Bryant 2000], but never published. The present author unfortunately did not attend the conference and was not aware of his work.

The first thing one notices is that the determinant $\det N^*$ gives a section

$$\det N^* \in \Lambda^{3,0}(M)^{\otimes 2} \otimes \Lambda^{3,0}(M)^*.$$

Taking

$$(1-2) \quad \det N^* \otimes \overline{\det N^*} \in \Lambda^{3,0}(M) \otimes \Lambda^{0,3}(M) = \Lambda^6(M),$$

we obtain a nowhere degenerate real volume form Vol_I on M . This form is called *the canonical volume form associated with the Nijenhuis tensor*. This gives a functional $\Psi : I \rightarrow \int_M \text{Vol}_I$ on the space of almost complex structures. One of the purposes of this paper is to investigate the critical points of the functional Ψ , in the spirit of Hitchin's work [2000; 2001].

1B. Connections with totally antisymmetric torsion. Let (M, g) be a Riemannian manifold, $\nabla : TM \rightarrow TM \otimes \Lambda^1 M$ a connection, and $T \subset \Lambda^2 M \otimes TM$ its torsion. Identifying TM and $\Lambda^1 M$ via g , we may consider T as an element in $\Lambda^2 M \otimes \Lambda^1 M$, that is, a 3-form on TM . If T is totally skew-symmetric as a 3-form on TM , we say that ∇ is a *connection with totally skew-symmetric* (or *totally antisymmetric*) torsion. If, in addition, M is Hermitian, and ∇ preserves the Hermitian structure, we say that ∇ is a *Hermitian connection with totally antisymmetric torsion*.

Connections with totally skew-symmetric torsion are extremely useful in physics and differential geometry. An important example of such a connection is provided by a theorem of Bismut [1989].

Theorem 1.3. *Let (M, I) be a complex manifold, and g a Hermitian metric. Then M admits a unique connection with totally skew-symmetric torsion preserving I and g . \square*

Connections with totally skew-symmetric torsion were studied at great length by Friedrich, Ivanov and others; see for example [Friedrich and Ivanov 2002; Friedrich 2003; Agricola and Friedrich 2004]. Bismut's theorem requires the base manifold to be complex. Motivated by string theory, Friedrich and Ivanov generalized Bismut's theorem to nonintegrable almost complex manifolds [Friedrich and Ivanov 2002]. For completeness, we sketch a proof of their theorem below.

Theorem 1.4. *Let (M, I, ω) be an almost complex Hermitian manifold, and*

$$N : \Lambda^2 T^{1,0}(M) \rightarrow T^{0,1}(M)$$

the Nijenhuis tensor. Consider the 3-linear form

$$(1-3) \quad \begin{aligned} &\rho : T^{1,0}(M) \times T^{1,0}(M) \times T^{1,0}(M) \rightarrow \mathbb{C}, \\ &\rho(x, y, z) := \omega(N(x, y), z) \end{aligned}$$

Then M admits a connection ∇ with totally skew-symmetric torsion preserving (ω, I) if and only if ρ is skew-symmetric. Moreover, such a connection is unique.

Sketch of a proof. Theorem 1.4 is proven essentially in the same way as one proves Bismut's theorem and the existence and uniqueness of a Levi-Civita connection. Let (M, I, g) be a Hermitian manifold, and ∇_0 a Hermitian connection. Then all Hermitian connections can be obtained by taking $\nabla(A) := \nabla_0 + A$, where A is

a 1-form with coefficients in the algebra $\mathfrak{u}(TM)$ of all skew-Hermitian endomorphisms. The torsion T_A of $\nabla(A)$ is written as

$$T_A = T_0 + \text{Alt}_{12}(A),$$

where T_0 is a torsion of ∇_0 , and Alt_{12} denotes the antisymmetrization of

$$\Lambda^1(M) \otimes \mathfrak{u}(TM) \subset \Lambda^1(M) \otimes \Lambda^1(M) \otimes TM$$

over the first two indices. We identify $\mathfrak{u}(TM)$ with $\Lambda^{1,1}(M)$ in a standard way. Then [Theorem 1.4](#) can be reinterpreted as a statement about linear-algebraic properties of the operator

$$(1-4) \quad \text{Alt}_{12} : \Lambda^1(M) \otimes \Lambda^{1,1}(M) \rightarrow (\Lambda^2(M) \otimes \Lambda^1(M))^{(2,1)+(1,2)}$$

(where the superscript $(\dots)^{(2,1)+(1,2)}$ means taking $(2, 1) + (1, 2)$ -part with respect to the Hodge decomposition), as follows.

By definition, the Nijenhuis tensor N is a section of $\Lambda^{2,0} \otimes T^{0,1}$. Identifying $T^{0,1}$ with $\Lambda^{1,0}$ via g , we can consider N as an element of $\Lambda^{2,0} \otimes \Lambda^{1,0}$. By Cartan’s formula, N is equal to the $(3, 0)$ -part of the torsion. Therefore, the existence of a connection with totally skew-symmetric torsion implies that [\(1-3\)](#) is skew-symmetric.

Conversely, assume that [\(1-3\)](#) is skew-symmetric. Since [\(1-4\)](#) maps $\Lambda^1(M) \otimes \Lambda^{1,1}(M)$ to $(2, 1) \oplus (1, 2)$ -tensors, the $(3, 0)$ and $(0, 3)$ -parts of torsion stay skew-symmetric if we modify the connection by adding $A \in \Lambda^1 \otimes \mathfrak{u}(TM)$. Denote by T_1 the $(2, 1) \oplus (1, 2)$ -part of the torsion T_0 . To prove [Theorem 1.4](#), we need to find $A \in \Lambda^1(M) \otimes \Lambda^{1,1}(M)$ such that $T_1 - \text{Alt}_{12}(A)$ is totally skew-symmetric.

The map $\text{Alt}_{12} : \Lambda^1(M) \otimes \Lambda^2(M) \rightarrow \Lambda^2(M) \otimes \Lambda^1(M)$ is an isomorphism, as a dimension count implies (this map has no kernel, which is easy to see). Therefore, [\(1-4\)](#) is injective. Using the dimension count again, we find that cokernel of [\(1-4\)](#) projects isomorphically into

$$\Lambda^{2,1}(M) \oplus \Lambda^{1,2}(M) \subset \Lambda^2(M) \otimes \Lambda^1(M).$$

Therefore, for any T_1 in $(2, 1) \oplus (1, 2)$ -part of $\Lambda^2(M) \otimes \Lambda^1(M)$ there exists $A \in \Lambda^1(M) \otimes \Lambda^{1,1}(M)$ and $B \in \Lambda^{2,1}(M) \oplus \Lambda^{1,2}(M)$ such that $T_1 = \text{Alt}_{12}(A) + B$. \square

1C. Connections with antisymmetric torsion on almost complex 6-manifolds.

Let (M, I) be an almost complex manifold, N its Nijenhuis tensor. To obtain all Hermitian connections with totally skew-symmetric torsion on (M, I) , one needs to find all metrics g for which the tensor $\omega(N(x, y), z)$ is skew-symmetric. As [Theorem 1.4](#) implies, these metrics are precisely those for which such a connection exists.

We also prove the following proposition.

Proposition 1.5. *Let (M, I) be an almost complex 6-manifold with Nijenhuis tensor which is nondegenerate in a dense subset of M , and g a Hermitian metric admitting a connection with totally antisymmetric torsion. Then g is uniquely determined by I , up to conformal equivalence. Moreover, the Riemannian metric g determines I uniquely, unless (M, g) is locally isometric to a 6-sphere.*

Proof. This is [Proposition 3.1](#) and [Proposition 4.7](#). □

1D. Correspondence with the results of R. Bryant. Since the first version of this paper was written, the previously unpublished results of R. Bryant appeared in a fundamental and important preprint [2006]. There is a significant overlap with our research, though the presentation and terminology are different. The property (1-3) (which is equivalent to the existence of Hermitian connection with totally antisymmetric curvature) is called “Nijenhuis tensor of real type” in [Bryant 2006]. The main focus of that preprint is the so-called “quasiintegrable almost complex manifold”: manifolds with Nijenhuis tensor of real type, which is at every point of M either nondegenerate (of constant signature) or zero. Examples of such structures are found. In particular, all twistor spaces of Kähler surfaces with sign-definite holomorphic bisectional curvature are shown to be quasiintegrable. A variant of [Theorem 2.2](#) is also proven. It is shown that nearly Kähler manifolds are critical points of the functional Vol_I [Bryant 2006, Proposition 8].

2. Nearly Kähler manifolds: an introduction

Nearly Kähler manifolds (also known as K -spaces or almost Tachibana spaces) were defined and studied by Alfred Gray in [1965; 1970; 1971; 1976] in a general context of intrinsic torsion of $U(n)$ -structures and weak holonomies. An almost complex Hermitian manifold (M, I) is called *nearly Kähler* if $\nabla_X(I)X = 0$, for any vector fields X (∇ denotes the Levi-Civita connection). In other words, the tensor $\nabla\omega$ must be totally skew-symmetric, for ω the Hermitian form on M . If $\nabla_X(\omega) \neq 0$ for any nonzero vector field X , M is called *strictly nearly Kähler*.

In this section, we give an overview of known results and “folk theorems” of nearly Kähler geometry. Most of this theory was known (in a different context) since 1980s, when the study of Killing spinors was initiated [Baum et al. 1991].

2A. Splitting theorems for nearly Kähler manifolds. As V. F. Kirichenko proved, nearly Kähler manifolds admit a connection with totally antisymmetric, parallel torsion [Kiričenko 1977]. This observation was used to prove a splitting theorem for nearly Kähler manifolds: any nearly Kähler manifold is locally a Riemannian product of a Kähler manifold and a strictly nearly Kähler one [Gray 1976; Nagy 2002b].

A powerful classification theorem for Riemannian manifolds admitting an orthogonal connection with irreducible connection and parallel torsion was obtained by R. Cleyton and A. Swann [2004]. They proved that any such manifold either is locally homogeneous, or has vanishing torsion, or has weak holonomy G_2 (in dimension 7) or $SU(3)$ (in dimension 6).

Using Kirichenko theorem, this result can be used to obtain a classification of nearly Kähler manifolds. P.-A. Nagy has shown [2002a] that any strictly nearly Kähler manifold is locally a product of locally homogeneous manifolds, strictly nearly Kähler 6-manifolds, and twistor spaces of quaternionic Kähler manifolds of positive Ricci curvature, equipped with the Eells–Salamon metric.

These days the term “nearly Kähler” usually denotes strictly nearly Kähler 6-manifolds. In sequel we shall follow this usage, often omitting “strictly” and “6-dimensional”.

In dimension 6, a manifold is (strictly) nearly Kähler if and only if it admits a Killing spinor [Grunewald 1990]. Therefore, such a manifold is Einstein, with positive Einstein constant.

As one can easily show (see Theorem 4.2), strictly nearly Kähler 6-manifolds can be defined as 6-manifolds with structure group $SU(3)$ and fundamental forms $\omega \in \Lambda_{\mathbb{R}}^{1,1}(M)$, $\Omega \in \Lambda^{3,0}(M)$, satisfying $d\omega = 3\lambda \operatorname{Re} \Omega$ and $d \operatorname{Im} \Omega = -2\lambda\omega^2$. An excellent introduction to nearly Kähler geometry is found in [Moroianu et al. 2005].

The most puzzling aspect of nearly Kähler geometry is a complete lack of nonhomogeneous examples. With the exception of four homogeneous cases described below (Section 2C), no other compact examples of strictly nearly Kähler 6-manifolds are known to exist.

2B. Nearly Kähler manifolds in G_2 -geometry and physics. Nearly Kähler manifolds have many uses in geometry and physics. Along with Calabi–Yau manifolds, nearly Kähler manifolds appear as target spaces for supersymmetric sigma-models, solving equations of type II string theory. These manifolds are the only 6-manifolds admitting a Killing spinor. This implies that a Riemannian cone $C(M)$ of a nearly Kähler manifold has a parallel spinor.

Let (M, g) be a Riemannian manifold. Recall that the *Riemannian cone* of (M, g) is a product $M \times \mathbb{R}^{>0}$, with a metric $gt^2 \oplus \lambda \cdot dt^2$, where t is a unit parameter on $\mathbb{R}^{>0}$, and λ a constant. It is well known that M admits a real Killing spinor if and only if $C(M)$ admits a parallel spinor (for appropriate choice of λ). Then, $C(M)$ has restricted holonomy, for any nearly Kähler 6-manifold. It is easy to check that in fact $C(M)$ has holonomy G_2 . This explains a tremendous importance that nearly Kähler manifolds play in G_2 -geometry.

We give a brief introduction of G_2 -geometry, following [Hitchin 2000] and [Joyce 2000]. Let V^7 be a 7-dimensional real vector space. The group $GL(7, \mathbb{R})$

acts on $\Lambda^3(V^7)$ with two open orbits. For ν in one of these orbits, its stabilizer $St(\nu) \subset GL(7, \mathbb{R})$ is 14-dimensional, as a dimension count insures. It is easy to check that $St(\nu)$ is a real form of a Lie group G_2 . For one of these orbits, $St(\nu)$ is a compact form of G_2 , for another one it is noncompact. A 3-form $\nu \in \Lambda^3(V^7)$ is called *stable* if its stabilizer is a compact form of G_2 .

A 7-manifold X equipped with a 3-form ρ is called a G_2 -manifold if ρ is stable everywhere in X . In this case, the structure group of X is reduced to G_2 . Also, X is equipped with a natural Riemannian structure:

$$(2-1) \quad x, y \rightarrow \int_X (\rho \lrcorner x) \wedge (\rho \lrcorner x) \wedge \rho \quad (x, y \in TM).$$

A G_2 -manifold is called *parallel* if $\nabla \rho = 0$, where ∇ is the Levi-Civita connection associated with this Riemannian structure.

Isolated singularities of G_2 -manifolds are of paramount importance in physics; see [Acharya and Gukov 2004; Atiyah and Witten 2002]. A simplest example of an isolated singular point is a conical singularity.

A metric space X with marked points x_1, \dots, x_n is called a *space with isolated singularities*, if $X \setminus \{x_1, \dots, x_n\}$ is a Riemannian manifold. Consider a space (X, x) with a single singular point. The singularity $x \in X$ is called *conical* if X is equipped with a flow acting on X by homotheties and contracting X to x . In this case, $X \setminus x$ is isomorphic to a Riemannian cone of a Riemannian manifold M .

It is easy to check that the cone $C(M)$ of a nearly Kähler manifold is equipped with a parallel G_2 -structure, and, conversely, every conical singularity of a parallel G_2 -manifold is obtained as $C(M)$, for some nearly Kähler manifold M [Hitchin 2001; Ivanov et al. 2006]. For completeness' sake, we give a sketch of a proof of this result in Proposition 4.5.

The idea of this correspondence is quite clear. Let $X = C(M)$ be a parallel G_2 -manifold, and ω_C its 3-form. Unless X is flat, we may assume that X has holonomy which is equal to G_2 and not its proper subgroup. Indeed, if holonomy of X is less than G_2 , by Berger's classification of irreducible holonomies, X is represented (as a Riemannian manifold) as a product of manifolds of smaller dimension. However, the singular point of the metric completion \bar{X} is isolated, and this precludes such a decomposition, unless \bar{X} is smooth. In the latter case, X is flat.

Since the holonomy of X is (strictly) G_2 , the 3-form can be reconstructed from the Riemannian structure uniquely. After rescaling, we may assume that the Riemannian structure on $X = C(M)$ is homogeneous of weight 2, with respect to the action of $\mathbb{R}^{>0}$ on $C(M)$. Then ω_C is homogeneous of weight 3. Homogeneous G_2 -structures on $C(M)$ correspond naturally to $SU(3)$ -structures on M . We write ω_C as $t^2 \pi^* \omega \wedge dt + t^3 \pi^* \rho$, where ρ, ω are forms on M , and $\pi : C(M) \rightarrow M$ is the standard projection. From a local coordinate expression of a G_2 -form, we find

that ω is a Hermitian form corresponding to an almost complex structure I , and $\rho = \text{Re } \Omega$ for a nowhere degenerate $(3, 0)$ -form Ω on (M, I) .

The converse is proven by the same computation: given an $SU(3)$ -manifold (M, I, ω, Ω) , we write a 3-form

$$(2-2) \quad \omega_C := t^2 \pi^* \omega \wedge dt + t^3 \pi^* \rho,$$

on $C(M)$, and show that it is a G_2 -structure, using a coordinate expression for a G_2 -form.

As Fernandez and Gray proved in [1982], a G_2 -manifold (X, ω_C) is parallel if and only if ω_C is harmonic. For the form (2-2), $d\omega_C = 0$ is translated into $d\omega = 3\rho$.

Since $*\rho = I\rho$ and $*\omega = \omega^2$, the condition $d^*\omega_C = 0$ becomes $dI\rho = -2\omega^2$. After an appropriate rescaling, we find that this is precisely the condition defining the nearly Kähler structure Theorem 4.2. Therefore, $C(M)$ is a G_2 -manifold if and only if M is nearly Kähler (see Section 4C for a more detailed argument).

The correspondence between conical singularities of G_2 -manifolds and nearly Kähler geometry can be used further to study the locally conformally parallel G_2 -manifolds (see also [Ivanov et al. 2006]). A *locally conformally parallel G_2 -manifold* is a 7-manifold M with a covering \tilde{M} equipped with a parallel G_2 -structure, with the deck transform acting on \tilde{M} by homotheties. Since homotheties preserve the Levi-Civita connection $\tilde{\nabla}$ on \tilde{M} , $\tilde{\nabla}$ descends to a torsion-free connection on M , which is no longer orthogonal, but preserves the conformal class of a metric. Such a connection is called a *Weyl connection*, and a conformal manifold of dimension > 2 equipped with a torsion-free connection preserving the conformal class is called a *Weyl manifold*. The Weyl manifolds are a subject of much study in conformal geometry; see for example [Dragomir and Ornea 1998] and the reference therein.

The key theorem of Weyl geometry is proven by P. Gauduchon [1984]. He has shown that any compact Weyl manifold is equipped with a privileged metric in its conformal class. This metric (called a *Gauduchon metric* now) is defined as follows.

Let $(M, [g], \nabla)$ be a compact Weyl manifold, where $[g]$ is a conformal class, and $g \in [g]$ any metric within this conformal class. Since $\nabla[g] = 0$, we have $\nabla(g) = g \otimes \theta$, where θ is a 1-form, called a *Lee form*. A metric g is called *Gauduchon* if θ satisfies $d^*\theta = 0$. A Gauduchon metric is unique (up to a complex multiplier).

Let now $(M, \nabla, [g])$ be a Weyl manifold with a Ricci-flat connection ∇ . In [1995], Gauduchon has shown that the Lee form θ of the Gauduchon metric on M is parallel with respect to the Levi-Civita connection associated with this metric.

Applying this argument to a compact locally conformally parallel G_2 -manifold M , we obtain that the Lee form is parallel. From this one infers that the parallel

G_2 -covering \tilde{M} of M is a cone over some Riemannian manifold S ; see for example [Verbitskiĭ 2004, Proposition 11.1], also see [Kamishima and Ornea 2005] and [Gini et al. 2005]. Using the argument stated above, we find that this manifold is in fact nearly Kähler. Therefore, S is Einstein, with positive Ricci curvature. Since M is compact, S is complete, and by Myers theorem, S is actually compact; see [Verbitskiĭ 2004, Remark 10.7]. Now, the argument which proves Theorem 12.1 of [Verbitskiĭ 2004] can be used to show that $\dim H^1(M, \mathbb{Q}) = 1$, and $M = C(S)/\mathbb{Z}$. This gives the following structure theorem, which is proven independently in [Ivanov et al. 2006].

Theorem 2.1. *Let M be a compact locally conformally parallel G_2 -manifold. Then $M = C(S)/\mathbb{Z}$, where S is a nearly Kähler manifold, and the \mathbb{Z} -action on $C(S) \cong S \times \mathbb{R}^{>0}$ is generated by a map $(x, t) \mapsto (\varphi(x), qt)$, where $|q| > 1$ is a real number, and $\varphi : S \rightarrow S$ an automorphism of nearly Kähler structure. \square*

2C. Examples of nearly Kähler manifolds. Just as the conical singularities of parallel G_2 -manifolds correspond to nearly Kähler manifolds, the conical singularities of $Spin(7)$ -manifolds correspond to the so-called “nearly parallel” G_2 -manifolds (see [Ivanov 2004]). A G_2 -manifold (M, ω) is called *nearly parallel* if $d\omega = c * \omega$, where c is some constant. The analogy between nearly Kähler 6-manifolds and nearly parallel G_2 -manifolds is almost perfect. These manifolds admit a connection with totally antisymmetric torsion and have weak holonomy $SU(3)$ and G_2 respectively. N. Hitchin realized nearly Kähler 6-manifolds and nearly parallel G_2 -manifolds as extrema of a certain functional, called *Hitchin functional* by physicists (see [Hitchin 2001]).

However, examples of nearly parallel G_2 -manifolds are found in profusion (every 3-Sasakian manifold is nearly parallel G_2), and compact nearly Kähler manifolds are rare.

Only four compact examples are known (see the list below); all of them homogeneous. In [Butruille 2005] it was shown that any homogeneous nearly Kähler 6-manifold belongs to this list.

- (1) The 6-dimensional sphere S^6 . Since the cone $C(S^6)$ is flat, S^6 is a nearly Kähler manifold, as shown in Section 2B. The almost complex structure on S^6 is reconstructed from the octonion action, and the metric is standard.
- (2) $S^3 \times S^3$, with the complex structure mapping ξ_i to ξ'_i , ξ'_i to $-\xi_i$, where ξ_i, ξ'_i , $i = 1, 2, 3$ is a basis of left invariant 1-forms on the first and the second component.
- (3) Given a selfdual Einstein Riemannian 4-manifold M with positive Einstein constant, one defines its *twistor space* $\text{Tw}(M)$ as a total space of a bundle of unit

spheres in $\Lambda^2_-(M)$ of antiselfdual 2-forms. Then $\text{Tw}(M)$ has a natural Kähler-Einstein structure (I_+, g) , obtained by interpreting unit vectors in $\Lambda^2_-(M)$ as complex structure operators on TM . Changing the sign of I_+ on TM , we obtain an almost complex structure I_- which is also compatible with the metric g [Eells and Salamon 1985]. A straightforward computation insures that $(\text{Tw}(M), I_-, g)$ is nearly Kähler [Muškarov 1987].

As N. Hitchin proved [1981], there are only two compact selfdual Einstein 4-manifolds: S^4 and $\mathbb{C}P^2$. The corresponding twistor spaces are $\mathbb{C}P^3$ and the flag space $F(1, 2)$. The almost complex structure operator I_- induces a nearly Kähler structure on these two symmetric spaces.

2D. Nearly Kähler manifolds are extrema of volume on almost complex manifolds with nowhere degenerate Nijenhuis tensor. Let (M, I, ω) be a nearly Kähler manifold, and $N^* : \Lambda^{0,1}(M) \rightarrow \Lambda^{2,0}(M)$ the Nijenhuis tensor. By Cartan’s formula, N^* is the $(2, -1)$ -part of the de Rham differential (with respect to the Hodge decomposition). In Theorem 4.2, it is shown that $d\omega$ is a real part of a nowhere degenerate $(3, 0)$ -form Ω . Therefore, the 3-form

$$\omega(N(x, y), z) = d\omega(x, y, z) = \text{Re } \Omega(x, y, z)$$

is nowhere degenerate on $T^{1,0}(M)$. So the Nijenhuis tensor N is nowhere degenerate.

The main result of this paper is the following theorem, which is analogous to [Hitchin 2001].

Theorem 2.2. *Let (M, I) be a compact almost complex 6-manifold with nowhere degenerate Nijenhuis tensor admitting a Hermitian connection with totally antisymmetric torsion. Consider the functional $I \rightarrow \int_M \text{Vol}_I$ on the space of such manifolds constructed in Section 1A. Then this functional has a critical point at I if and only if (M, I) admits a nearly Kähler metric.*

Proof. This follows from Proposition 5.6 and Theorem 4.2. □

Remark 2.3. As follows from Corollary 3.3, the nearly Kähler metric on (M, I) is uniquely determined by the almost complex structure.

3. Almost complex structures and connections with totally antisymmetric torsion

Let (M, I) be a 6-dimensional almost complex manifold, and

$$N^* : \Lambda^{0,1}(M) \rightarrow \Lambda^{2,0}(M)$$

its Nijenhuis tensor. Given a point $x \in M$, the operator $N^*|_{\Lambda^{0,1}_x(M)}$ can a priori take any value within $\text{Hom}(\Lambda^{0,1}(M), \Lambda^{2,0}(M))$. For $N^*|_{\Lambda^{0,1}_x(M)}$ generic, the stabilizer

$St(N_x^*)$ of N_x^* within $GL(T_x M)$ is 2-dimensional. If we fix a complex parameter, the eigenspaces of N_x^* (taken in appropriate sense) define a frame in TM . Thus, a geometry of a “very generic” 6-dimensional almost complex manifold is rather trivial.

However, for a N_x^* inside a 10-dimensional subspace

$$W_0 \subset \text{Hom}(\Lambda^{0,1}(M), \Lambda^{2,0}(M))$$

(see [Remark 3.2](#)), the stabilizer $St(N_x^*)$ contains $SU(3)$, and the geometry of (M, I) becomes more interesting.

Proposition 3.1. *Let (M, I) be an almost complex 6-manifold with Nijenhuis tensor which is nondegenerate in a dense set. Assume that (M, I) admits a Hermitian structure ω and a Hermitian connection with totally antisymmetric torsion. Then ω is uniquely determined by I , up to conformal equivalence.*

Proof. Consider the map

$$(3-1) \quad C := \text{Id} \otimes N^* : \Lambda^{1,1}(M) \rightarrow \Lambda^{1,0}(M) \otimes \Lambda^{2,0}(M)$$

obtained by acting with the Nijenhuis tensor $N^* : \Lambda^{0,1}(M) \rightarrow \Lambda^{2,0}(M)$ on the second tensor multiplier of $\Lambda^{1,1}(M) \cong \Lambda^{1,0}(M) \otimes \Lambda^{0,1}(M)$. Then C maps ω to a 3-form $x, y, z \mapsto \omega(N(x, y), z)$. As [Theorem 1.4](#) implies, (M, I, ω) admits a Hermitian connection with totally antisymmetric torsion if and only if $C(\omega)$ lies inside a 1-dimensional space

$$\Lambda^{3,0}(M) \subset \Lambda^{1,0}(M) \otimes \Lambda^{2,0}(M).$$

However, C is an isomorphism in a dense subset of M ; hence, all ω which satisfy the conditions of [Theorem 1.4](#) are proportional. □

Remark 3.2. The same argument can be used to prove that an almost complex manifold admits a Hermitian connection with totally antisymmetric torsion if and only if $C^{-1}(\Lambda^{3,0}(M))$ contains a Hermitian form. This is the space W_0 alluded to in the beginning of this section.

[Proposition 3.1](#) leads to the following corollary.

Corollary 3.3. *Let (M, I) be an almost complex 6-manifold. Then (M, I) admits at most one strictly nearly Kähler metric, up to a constant multiplier.*

Proof. Let ω_1 and ω_2 be nearly Kähler metrics on (M, I) . Since (M, I, ω_i) is strictly nearly Kähler, the 3-form $C(\omega_i) \in \Lambda^{3,0}(M)$ is nowhere degenerate; see (3-1). Therefore, (M, I) has nowhere degenerate Nijenhuis tensor. Then, by [Proposition 3.1](#), ω_i are proportional: $\omega_1 = f\omega_2$. However, $d\omega_i^2 = 0$ on any nearly Kähler 3-manifold; see for example [Theorem 4.2 \(ii\)](#). Then $2fd\omega_2^2 = 0$. This implies $df = 0$, because the map $\eta \mapsto \eta \wedge \omega_2^2$ is an isomorphism on $\Lambda^1(M)$. □

Remark 3.4. The converse is also true: unless (M, g) is locally isometric to a 6-sphere, the Riemannian metric g determines the nearly Kähler almost complex structure I uniquely; see [Proposition 4.7](#).

4. Nearly Kähler geometry and Hermitian connections with totally antisymmetric torsion

4A. Hermitian structure on $\Lambda^{3,0}(M)$ and nearly Kähler manifolds. Let (M, I) be an almost complex 6-manifold, and $\Omega \in \Lambda^{3,0}(M)$ a nondegenerate $(3, 0)$ -form. Then $\Omega \wedge \overline{\Omega}$ is a positive volume form on M . This gives a $\text{Vol}(M)$ -valued Hermitian structure on $\Lambda^{3,0}(M)$. If M is in addition Hermitian, then M is equipped with a natural volume form Vol_h associated with the metric, and the map

$$\Omega \mapsto \frac{\Omega \wedge \overline{\Omega}}{\text{Vol}_h}$$

can be considered as a Hermitian metric on $\Lambda^{3,0}(M)$. This metric agrees with the usual Riemann-Hodge pairing known from algebraic geometry, when I is integrable. The following definition is a restatement of the classical one; see [Section 2](#).

Definition 4.1. Let (M, I, ω) be an almost complex Hermitian manifold, and ∇ the Levi-Civita connection. Then (M, I, ω) is called *nearly Kähler* if the tensor $\nabla\omega$ is totally antisymmetric, that is, $\nabla\omega \subset \Lambda^3(M)$.

The following theorem is a main result of this section.

Theorem 4.2. *Let (M, I, ω) be an almost complex Hermitian 6-manifold equipped with a $(3, 0)$ -form Ω . Assume that Ω satisfies $3\lambda \text{Re } \Omega = d\omega$, and $|\Omega|_\omega = 1$, where λ is a constant, and $|\cdot|_\omega$ is the Hermitian metric on $\Lambda^{3,0}(M)$ constructed above. Then the following conditions are equivalent.*

- (i) M admits a Hermitian connection with totally antisymmetric torsion.
- (ii) $d\Omega = -2\sqrt{-1} \lambda \omega^2$.
- (iii) (M, I, ω) is nearly Kähler, and $d\omega = \nabla\omega$.

The equivalence of (ii) and (iii) is known; see for example the second part of the proof of [Theorem 6](#) in [\[Hitchin 2001\]](#).

The existence of Hermitian connections with totally antisymmetric torsion on nearly Kähler manifolds is also well known (see [Section 2](#)). This connection is written as $\nabla_{NK} = \nabla + T$, where ∇ is the Levi-Civita connection on M , and T the operator obtained from the 3-form $3\lambda \text{Im } \Omega$ by raising one of the indices. The torsion of ∇_{NK} is totally antisymmetric by construction (it is equal T). Also by construction, we find that $T(\omega) = -3\lambda \text{Re } \Omega$, hence $\nabla_{NK}(\omega) = 0$. Therefore, ∇_{NK} is a Hermitian connection with totally antisymmetric torsion. This takes care of the implication (iii) \Rightarrow (i).

To prove [Theorem 4.2](#), it remains to prove that (i) implies (ii); we do that in [Section 4B](#). For completeness' sake, we sketch the proof of the implication (ii) \Rightarrow (iii) in [Section 4D](#).

Remark 4.3. As [Corollary 3.3](#) shows, a non-Kähler nearly Kähler metric on M is uniquely determined by the almost complex structure I .

4B. Connections with totally antisymmetric torsion and Nijenhuis tensor.

Lemma 4.4. *In the assumptions of [Theorem 4.2](#), (i) implies (ii).*

Proof. Step 1: We show that $d\Omega \in \Lambda^{2,2}(M)$. Were (M, I) integrable, the differential d would have only (0,1)- and (1,0)-part with respect to the Hodge decomposition: $d = d^{1,0} + d^{0,1}$. For a general almost complex manifold, d splits into 4 parts:

$$d = d^{2,-1} + d^{1,0} + d^{0,1} + d^{-1,2}.$$

This follows immediately from the Leibniz rule. However,

$$(4-1) \quad 0 = d^2\omega = d(\Omega + \overline{\Omega}) = d\Omega + d\overline{\Omega}.$$

Since $\Lambda^{p,q}(M)$ vanishes for p or $q > 3$, we also have

$$(4-2) \quad d\Omega + d\overline{\Omega} = d^{0,1}\Omega + d^{-1,2}\Omega + d^{2,-1}\overline{\Omega} + d^{1,0}\overline{\Omega}.$$

The four terms on the right hand side of (4-2) have Hodge types (3, 1), (2, 2), (2, 2) and (1, 3). Since their sum vanishes by (4-1), we obtain

$$d^{0,1}\Omega = 0, \quad d^{1,0}\overline{\Omega} = 0, \quad d^{2,-1}\overline{\Omega} = -d^{-1,2}\Omega.$$

Then (4-2) gives

$$(4-3) \quad d\Omega = -d^{2,-1}\overline{\Omega} = d^{-1,2}\Omega.$$

Step 2:

$$(4-4) \quad d^{2,-1}|_{\Lambda^{1,1}(M)} = \wedge \circ \text{Id} \otimes N^*,$$

where $N^* : \Lambda^{0,1}(M) \rightarrow \Lambda^{2,0}(M)$ is the Nijenhuis tensor,

$$\text{Id} \otimes N^* : \Lambda^{1,1}(M) \rightarrow \Lambda^{2,0}(M) \otimes \Lambda^{1,0}(M)$$

acts as N^* on the second multiplier of $\Lambda^{1,1}(M) \cong \Lambda^{1,0}(M) \otimes \Lambda^{0,1}(M)$, and \wedge denotes the exterior product. (4-4) is immediately implied by the Cartan's formula for the de Rham differential.

Step 3: From the existence of Hermitian connection with totally antisymmetric torsion we obtain that the form

$$\omega(N(x, y), z) : T^{1,0}M \times T^{1,0}M \times T^{1,0}M \rightarrow \mathbb{C}$$

is totally antisymmetric; see [Theorem 1.4](#). From (4-4) it follows that

$$(4-5) \quad \omega(N(x, y), z) = d\omega = 3\lambda \operatorname{Re} \Omega.$$

Consider an orthonormal frame dz_1, dz_2, dz_3 in $\Lambda^{1,0}(M)$, satisfying

$$\Omega = dz_1 \wedge dz_2 \wedge dz_3$$

(such a frame exists because $|\Omega|_\omega = 1$). Then (4-5) gives

$$(4-6) \quad N^*(d\bar{z}_i) = \lambda d\check{z}_i,$$

where $d\check{z}_1 = dz_2 \wedge dz_3, d\check{z}_2 = -dz_1 \wedge dz_3, d\check{z}_3 = dz_1 \wedge dz_2$.

Step 4: Using Cartan’s formula as in [Step 2](#), we express $d^{-1,2}\Omega$ through the Nijenhuis tensor. Then (4-3) can be used to write $d\Omega = d^{-1,2}\Omega$ in terms of N^* . Finally, (4-6) allows us to write $d^{-1,2}\Omega$ in coordinates, yielding $d\Omega = -2\sqrt{-1}\lambda\omega^2$. \square

4C. G_2 -structures on cones of Hermitian 6-manifolds.

Proposition 4.5. *Let (M, I, ω) be an almost complex Hermitian manifold, $\Omega \in \Lambda^{3,0}(M)$ a $(3, 0)$ -form which satisfies $d\omega = 3\lambda \operatorname{Re} \Omega$, for some real constant, and $|\Omega|_\omega = 1$. Assume, in addition, that $d\Omega = -2\sqrt{-1}\lambda\omega^2$. Consider the cone $C(M) = M \times \mathbb{R}^{>0}$, equipped with a 3-form $\rho = 3t^2\omega \wedge dt + t^3d\omega$, where t is the unit parameter on the $\mathbb{R}^{>0}$ -component. Then $(C(M), \rho)$ is a parallel, G_2 -manifold; see [Section 2B](#). Moreover, any parallel G_2 -structure ρ' on $C(M)$ is obtained in this way if ρ' is homogeneous of weight 3 with respect to the natural action of $\mathbb{R}^{>0}$ on $C(M)$.*

Proof. As Fernandez and Gray has shown in [1982], to show that a G_2 -structure ρ is parallel it suffices to prove that $d\rho = d^*\rho = 0$. Clearly, $d\rho = 0$, because

$$d\rho = 3t^2d\omega \wedge dt + 3t^2dt \wedge d\omega = 0.$$

On the other hand, $*(\omega \wedge dt) = \frac{1}{2}t^2\omega^2$, and $*d\omega = -3dt \wedge I(d\omega)$, where $*$ is taken with respect to the cone metric on $C(M)$. This is clear, because (ω, Ω) defines an $SU(3)$ -structure on M , and $d\omega = 3\lambda \operatorname{Re} \Omega$. Then

$$(4-7) \quad *\rho = \frac{3}{2}t^4\omega^2 - 3t^3dt \wedge I(d\omega).$$

Since $d\Omega = -2\sqrt{-1}\lambda\omega^2$ and $3\lambda d \operatorname{Re} \Omega = d^2\omega = 0$, we obtain $d \operatorname{Im} \Omega = -2\lambda\omega^2$. This gives $dI(d\omega) = -2\omega^2$, because $\lambda I(d\omega) = \operatorname{Im} \Omega$. Then (4-7) implies

$$d(*\rho) = 6t^3dt \wedge \omega^2 + 3t^3dt \wedge dI(d\omega) = 6t^3dt \wedge \omega^2 - 6t^3dt \wedge \omega^2 = 0.$$

We proved that $C(M)$ is a parallel G_2 -manifold. The converse statement is straightforward. \square

In [Section 2B](#), it is shown that the holonomy of $C(M)$ is strictly G_2 , unless it is flat (in the latter case, M is locally isometric to a sphere). Therefore, [Proposition 4.5](#) implies the following corollary.

Corollary 4.6. *In the assumptions of [Proposition 4.5](#), the almost complex structure is uniquely determined by the metric, unless M is locally isometric to a 6-sphere. \square*

4D. Near Kählerness obtained from G_2 -geometry. Now we can conclude the proof of [Theorem 4.2](#).

Proof of (iii) from (ii). Let M be a 6-manifold satisfying the assumptions of [Theorem 4.2 \(ii\)](#). Consider the cone $C(M)$ equipped with a parallel G_2 -structure ρ as in [Proposition 4.5](#). Let g_0 be a cone metric on $C(M)$. From the argument used to prove [Proposition 4.5](#), it is clear that g_0 is a metric induced by the 3-form ρ as in (2-1).

Consider the map $C(M) \xrightarrow{\tau} M \times \mathbb{R}$ induced by $(m, t) \mapsto (m, \log t)$, and let $g_1 = \tau^*g_\pi$ be induced by the product metric g_π on $M \times \mathbb{R}$. Denote by ∇_0, ∇_1 the corresponding Levi-Civita connections. We know that $\nabla_0(\rho) = 0$, and we need to show that

$$(4-8) \quad \nabla_1(\omega) = d\omega.$$

The metrics g_0, g_1 are proportional: $g_1 = g_0e^{-t}$. This allows one to relate the Levi-Civita connections ∇_1 and ∇_0 (see for example [\[Ornea and Piccinni 1998\]](#)) by

$$\nabla_1 = \nabla_0 + \frac{1}{2}A,$$

where $A : TM \rightarrow \text{End}(\Lambda^1(M))$ is an $\text{End}(\Lambda^1(M))$ -valued 1-form mapping $X \in TM$ to

$$(4-9) \quad (\theta, X) \text{Id} - X \otimes \theta + X^\sharp \otimes \theta^\sharp,$$

with θ the 1-form defined by $\nabla_0(g_1) = g_1 \otimes \theta$, $X \otimes \theta$ the tensor product of X and θ considered as an endomorphism of $\Lambda^1(M)$, and $X^\sharp \otimes \theta^\sharp$ the dual endomorphism.

From (4-9) and $\nabla_0(\rho) = 0$ we obtain

$$(4-10) \quad (\nabla_1)_X(\rho) = (X, \theta)\rho - (\rho \lrcorner X) \wedge \theta + (\rho \lrcorner \theta^\sharp) \wedge X^\sharp.$$

Since $\theta = \frac{dt}{t}$, we have $\nabla_1(\theta) = 0$, and ∇_1 preserves the decomposition $\Lambda^*(C(M)) \cong \Lambda^*(M) \oplus dt \wedge \Lambda^*(M)$. Restricting ourselves to the $dt \wedge \Lambda^*(M)$ -summand of this decomposition and applying (4-10), we find

$$(\nabla_1)_X(t^3\omega \wedge \theta) = t^3(d\omega \lrcorner X) \wedge \theta$$

for any X orthogonal to dt . Since g_1 is a product metric on $C(M) \cong M \times \mathbb{R}$, this leads to $\nabla\omega = d\omega$, where ∇ is the Levi-Civita connection on M . This implies

(4-8). So we deduced [Theorem 4.2 \(iii\)](#) from [\(ii\)](#) and the proof of [Theorem 4.2](#) is finished. □

Using [Corollary 4.6](#), we also obtain the following useful proposition.

Proposition 4.7. *Let (M, I, g) be a nearly Kähler manifold. Then the almost complex structure is uniquely determined by the Riemannian structure, unless M is locally isometric to a 6-sphere.*

5. Almost complex structures on 6-manifolds and their infinitesimal variations

5A. Hitchin functional and the volume functional. Let (M, I) be an almost complex 6-manifold with nowhere degenerate Nijenhuis tensor N , and $\text{Vol}_I = \det N^* \otimes \det N^*$ the corresponding volume form as in [\(1-2\)](#). In this section we study the extrema of the functional

$$I \xrightarrow{\Psi} \int_M \text{Vol}_I.$$

A similar functional was studied by N. Hitchin for 6- and 7-manifolds equipped with a stable 3-form [[Hitchin 2001](#)]. Since then, this functional has acquired a pivotal role in string theory and M-theory, under the name ‘‘Hitchin functional’’.

Our first step is to describe the variation of Ψ . We denote by \mathfrak{M} the space of all almost complex structures with nowhere degenerate Nijenhuis tensor on M .

Let (M, I, ω) be an almost complex manifold with nowhere degenerate Nijenhuis tensor

$$N \in \text{Hom}(\Lambda^2 T^{1,0}(M), T^{0,1}(M)),$$

$\delta \in T_I \mathfrak{M}$ an infinitesimal variation of I , and

$$N_\delta \in \text{Hom}(\Lambda^2 T^{1,0}(M), T^{0,1}(M))$$

the corresponding variation of the Nijenhuis tensor. Consider the form

$$\rho := \omega(N(x, y), z)$$

associated with the Hermitian structure on M as in [Theorem 1.4](#). After rescaling ω , we assume that

$$(5-1) \quad |\rho|_\omega = 1.$$

Since the Nijenhuis tensor is nowhere degenerate, ρ is also nowhere degenerate. Therefore, ρ can be used to identify $T^{0,1}(M)$ and $\Lambda^2 T^{1,0}(M)$, and we may consider N_δ as an endomorphism of $\Lambda^{0,1}(M)$. Notice that this identification maps N to the identity automorphism of $\Lambda^{0,1}(M)$.

Claim 5.1. *With these assumptions,*

$$\frac{d\Psi}{dI}(\delta) = 2 \operatorname{Re} \int_M \operatorname{Tr} N_\delta \operatorname{Vol}_I .$$

Proof. It is well known that

$$\frac{d(\det A)}{dt} = \det A \operatorname{Tr} \left(A^{-1} \frac{dA}{dt} \right)$$

for any matrix A . Applying that to the map

$$N^* \otimes \bar{N}^* : \Lambda^{1,0}(M) \otimes \Lambda^{0,1}(M) \rightarrow \Lambda^{0,2}(M) \otimes \Lambda^{2,0}(M),$$

we obtain that

$$(5-2) \quad \frac{d(\det(N^* \otimes \bar{N}^*))}{dI}(\delta) = \operatorname{Tr} \left(\frac{N_\delta^* \otimes \bar{N}^* + N^* \otimes \bar{N}_\delta^*}{N^* \otimes \bar{N}^*} \right) \cdot \det(N^* \otimes \bar{N}^*).$$

However, after we identify $\Lambda^{1,0}(M)$ and $\Lambda^{0,2}(M)$ as above, N becomes an identity, and (5-2) gives

$$\frac{d(\det(N^* \otimes \bar{N}^*))}{dI}(\delta) = 2 \operatorname{Re} \operatorname{Tr} N_\delta \operatorname{Vol}_I . \quad \square$$

Remark 5.2. We find that the extrema of the functional $\Psi(M, I) = \int_M \operatorname{Vol}_I$ are precisely those almost complex structures for which $\operatorname{Re} \operatorname{Tr} N_\delta = 0$ for any infinitesimal variation δ of I .

5B. Variations of almost complex structures and the Nijenhuis tensor. It is convenient, following Kodaira and Spencer, to consider infinitesimal variations of almost complex structures as tensors $\delta \in \Lambda^{0,1}(M) \otimes T^{1,0}(M)$. Indeed, a complex structure on a vector space V , $\dim_{\mathbb{R}} V = 2d$, can be considered as a point of the Grassmannian of d -dimensional planes in $V \otimes \mathbb{C}$. The tangent space to a Grassmannian at a point $W \subset V \otimes \mathbb{C}$ is given by $\operatorname{Hom}(W, V \otimes \mathbb{C}/W)$.

Consider the $(0, 1)$ -part $\nabla^{0,1}$ of the Levi-Civita connection

$$\nabla^{0,1} \delta \in \Lambda^{0,1}(M) \otimes T^{1,0}(M) \otimes \Lambda^{0,1}(M),$$

and let

$$\bar{\partial} : \Lambda^{0,1}(M) \otimes T^{1,0}(M) \rightarrow \Lambda^{0,2}(M) \otimes T^{1,0}(M)$$

denote the composition of $\nabla^{0,1}$ with the exterior multiplication map

$$\Lambda^{0,1}(M) \otimes T^{1,0}(M) \otimes \Lambda^{0,1}(M) \rightarrow \Lambda^{0,2}(M) \otimes T^{1,0}(M).$$

The following claim is well known.

Claim 5.3. *Let (M, I) be an almost complex manifold, and*

$$\delta \in \Lambda^{0,1}(M) \otimes T^{1,0}(M)$$

an infinitesimal variation of almost complex structure. Denote by $N_\delta \subset \Lambda^{2,0}(M) \otimes T^{0,1}(M)$ the corresponding infinitesimal variation of the Nijenhuis tensor (see [Section 5A](#)). Then $\bar{N}_\delta = \bar{\partial}\delta$, where $\bar{\partial}$ is the differential operator defined above.

Proof. The proof follows from a direct computation; see for example [[Kodaira and Spencer 1958](#)]. □

[Claim 5.3](#) can be used to study the deformation properties of the functional $I \xrightarrow{\Psi} \int_M \text{Vol}_I$ constructed above; see [Section 5A](#). Indeed, from [Remark 5.2](#) it follows that Ψ has an extremum at I if and only if $\text{Re Tr } N_\delta = 0$ for any $\delta \in \Lambda^{0,1}(M) \otimes T^{1,0}(M)$. Using the identification $T^{1,0}(M) \cong \Lambda^{2,0}(M)$, provided by the nondegenerate $(3, 0)$ -form as above, we can consider δ as a $(2, 1)$ -form on M . Then

$$\bar{\partial}\delta \in \Lambda^{0,2}(M) \otimes \Lambda^{2,0}(M) = \Lambda^{2,2}(M)$$

is the $(2, 2)$ -part of $d\delta$. Under these identifications, and using $|\rho|_\omega = 1$ from [\(5-1\)](#), we can express $\text{Tr } \bar{N}_\delta$ as

$$(5-3) \quad \text{Tr } \bar{N}_\delta = \frac{\bar{\partial}\delta \wedge \omega}{\text{Vol}_I},$$

where $\bar{\partial}$ is a $(0, 1)$ -part of the de Rham differential. This gives the following claim.

Claim 5.4. *Let (M, I, ω) be an almost complex Hermitian 6-manifold with nowhere degenerate Nijenhuis tensor. Assume that the corresponding 3-form ρ satisfies $|\rho|_\omega = 1$; see [\(5-1\)](#). Consider the functional $\Psi(I) = \int_M \text{Vol}_I$ on the space of such almost complex structures. Then*

$$\frac{d\Psi}{dI}(\delta) = 2 \text{Re} \int_M \bar{\partial}\delta \wedge \omega,$$

where $\delta \in \Lambda^{0,1}(M) \otimes T^{1,0}(M)$ is an infinitesimal deformation of an almost complex structure I , considered as a $(2, 1)$ -form on M .

Proof. The claim is implied immediately by [\(5-3\)](#) and [Claim 5.1](#). □

Comparing [Claim 5.4](#) with [Remark 5.2](#), we find the following corollary.

Corollary 5.5. *In assumptions of [Claim 5.4](#), I is an extremum of Ψ if and only if*

$$(5-4) \quad \text{Re} \int_M \bar{\partial}\delta \wedge \omega = 0$$

for any $\delta \in \Lambda^{2,1}(M)$. □

Integrating by parts, we find that (5-4) is equivalent to

$$\operatorname{Re} \int_M \delta \wedge \bar{\partial} \omega = 0$$

and to $\bar{\partial} \omega = 0$. This gives the following proposition.

Proposition 5.6. *Let (M, I, ω) be an almost complex Hermitian 6-manifold with nowhere degenerate Nijenhuis tensor. Consider the functional $\Psi(I) = \int_M \operatorname{Vol}_I$ on the space of such almost complex structures on M . Then I is an extremum of Ψ if and only if $d\omega$ lies in $\Lambda^{3,0}(M) \oplus \Lambda^{0,3}(M)$. \square*

Now, Proposition 5.6 together with Theorem 4.2 implies Theorem 2.2. Notice that by Corollary 3.3, the nearly Kähler Hermitian structure on (M, I) is (up to a constant multiplier) uniquely determined by I .

Acknowledgments

I am grateful to Robert Bryant, Nigel Hitchin, Paul-Andi Nagy and Uwe Semmelmann for valuable advices and consultations. P.-A. Nagy also suggested adding Proposition 4.7. Much gratitude to the referee for useful suggestions and the invaluable help of finding a multitude of minor errors.

References

- [Acharya and Gukov 2004] B. S. Acharya and S. Gukov, “M theory and singularities of exceptional holonomy manifolds”, *Phys. Rep.* **392**:3 (2004), 121–189. [MR 2004k:53072](#) [arXiv hep-th/0409191](#)
- [Agricola and Friedrich 2004] I. Agricola and T. Friedrich, “On the holonomy of connections with skew-symmetric torsion”, *Math. Ann.* **328**:4 (2004), 711–748. [MR 2005f:53072](#) [Zbl 1055.53031](#) [arXiv math.DG/0305069](#)
- [Atiyah and Witten 2002] M. Atiyah and E. Witten, “M-theory dynamics on a manifold of G_2 holonomy”, *Adv. Theor. Math. Phys.* **6**:1 (2002), 1–106. [MR 2004f:53046](#) [Zbl 1033.81065](#) [arXiv hep-th/0107177](#)
- [Baum et al. 1991] H. Baum, T. Friedrich, R. Grunewald, and I. Kath, *Twistors and Killing spinors on Riemannian manifolds*, Teubner-Texte zur Mathematik **124**, B. G. Teubner Verlagsgesellschaft, Stuttgart, 1991. [MR 94a:53077](#) [Zbl 0734.53003](#)
- [Bismut 1989] J.-M. Bismut, “A local index theorem for non-Kähler manifolds”, *Math. Ann.* **284**:4 (1989), 681–699. [MR 91i:58140](#) [Zbl 0666.58042](#)
- [Bryant 2000] R. Bryant, “Pseudoholomorphic Bundles over Almost Hermitian 6-Manifolds”, 2000. Talk at the International Congress on Differential Geometry in memory of Alfred Gray (Bilbao, 2000).
- [Bryant 2006] R. L. Bryant, “On the geometry of almost complex 6-manifolds”, *Asian J. Math.* **10**:3 (2006), 561–605. [MR 2007g:53029](#) [Zbl 1114.53026](#) [arXiv math.DG/0508428](#)
- [Butruille 2005] J.-B. Butruille, “Classification des variétés approximativement kähleriennes homogènes”, *Ann. Global Anal. Geom.* **27**:3 (2005), 201–225. [MR 2006f:53060](#) [Zbl 1079.53044](#) [arXiv math.DG/0401152](#)

- [Cleyton and Swann 2004] R. Cleyton and A. Swann, “Einstein metrics via intrinsic or parallel torsion”, *Math. Z.* **247**:3 (2004), 513–528. [MR 2005i:53054](#) [Zbl 1069.53041](#) [arXiv math.DG/0211446](#)
- [Dragomir and Ornea 1998] S. Dragomir and L. Ornea, *Locally conformal Kähler geometry*, Progress in Mathematics **155**, Birkhäuser, Boston, 1998. [MR 99a:53081](#) [Zbl 0887.53001](#)
- [Eells and Salamon 1985] J. Eells and S. Salamon, “Twistorial construction of harmonic maps of surfaces into four-manifolds”, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **12**:4 (1985), 589–640 (1986). [MR 87i:58042](#) [Zbl 0627.58019](#)
- [Fernández and Gray 1982] M. Fernández and A. Gray, “Riemannian manifolds with structure group G_2 ”, *Ann. Mat. Pura Appl.* (4) **132** (1982), 19–45 (1983). [MR 84e:53056](#) [Zbl 0524.53023](#)
- [Friedrich 2003] T. Friedrich, “On types of non-integrable geometries”, pp. 99–113 in *Proceedings of the 22nd Winter School “Geometry and Physics”* (Srní, Czech Republic, 2002), 2003. [MR 2004e:53039](#) [Zbl 1079.53041](#) [arXiv math.DG/0205149](#)
- [Friedrich and Ivanov 2002] T. Friedrich and S. Ivanov, “Parallel spinors and connections with skew-symmetric torsion in string theory”, *Asian J. Math.* **6**:2 (2002), 303–335. [MR 2003m:53070](#) [Zbl pre01925761](#) [arXiv math.DG/0102142](#)
- [Gauduchon 1984] P. Gauduchon, “La 1-forme de torsion d’une variété hermitienne compacte”, *Math. Ann.* **267**:4 (1984), 495–518. [MR 87a:53101](#) [Zbl 0523.53059](#)
- [Gauduchon 1995] P. Gauduchon, “Structures de Weyl–Einstein, espaces de twisteurs et variétés de type $S^1 \times S^3$ ”, *J. Reine Angew. Math.* **469** (1995), 1–50. [MR 97d:53048](#) [Zbl 0858.53039](#)
- [Gini et al. 2005] R. Gini, L. Ornea, and M. Parton, “Locally conformal Kähler reduction”, *J. Reine Angew. Math.* **581** (2005), 1–21. [MR MR2132669 \(2006c:53077\)](#) [Zbl 1080.53066](#) [arXiv math.DG/0208208](#)
- [Gray 1965] A. Gray, “Minimal varieties and almost Hermitian submanifolds”, *Michigan Math. J.* **12** (1965), 273–287. [MR 32 #1658](#) [Zbl 0132.16702](#)
- [Gray 1970] A. Gray, “Nearly Kähler manifolds”, *J. Differential Geometry* **4** (1970), 283–309. [MR 42 #2404](#) [Zbl 0201.54401](#)
- [Gray 1971] A. Gray, “Weak holonomy groups”, *Math. Z.* **123** (1971), 290–300. [MR 45 #2614](#) [Zbl 0222.53043](#)
- [Gray 1976] A. Gray, “The structure of nearly Kähler manifolds”, *Math. Ann.* **223**:3 (1976), 233–248. [MR 54 #6010](#) [Zbl 0345.53019](#)
- [Grunewald 1990] R. Grunewald, “Six-dimensional Riemannian manifolds with a real Killing spinor”, *Ann. Global Anal. Geom.* **8**:1 (1990), 43–59. [MR 92a:58146](#) [Zbl 0704.53050](#)
- [Hitchin 1981] N. Hitchin, “Kählerian twistor spaces”, *Proc. London Math. Soc.* (3) **43**:1 (1981), 133–150. [MR 84b:32014](#) [Zbl 0474.14024](#)
- [Hitchin 2000] N. Hitchin, “The geometry of three-forms in six dimensions”, *J. Differential Geom.* **55**:3 (2000), 547–576. [MR 2002m:53070](#) [Zbl 1036.53042](#) [arXiv math.DG/0010054](#)
- [Hitchin 2001] N. Hitchin, “Stable forms and special metrics”, pp. 70–89 in *Global differential geometry: the mathematical legacy of Alfred Gray* (Bilbao, 2000), edited by M. Fernández and J. A. Wolf, Contemp. Math. **288**, Amer. Math. Soc., Providence, RI, 2001. [MR 2003f:53065](#) [Zbl 1004.53034](#) [arXiv math.DG/0107101](#)
- [Ivanov 2004] S. Ivanov, “Connections with torsion, parallel spinors and geometry of Spin(7) manifolds”, *Math. Res. Lett.* **11**:2-3 (2004), 171–186. [MR 2005e:53071](#) [Zbl 1073.53065](#) [arXiv math.DG/0111216](#)
- [Ivanov et al. 2006] S. Ivanov, M. Parton, and P. Piccinni, “Locally conformal parallel G_2 and Spin(7) manifolds”, *Math. Res. Lett.* **13**:2-3 (2006), 167–177. [MR 2007e:53057](#) [Zbl 1118.53028](#) [arXiv math/0509038](#)

- [Joyce 2000] D. D. Joyce, *Compact manifolds with special holonomy*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000. [MR 2001k:53093](#) [Zbl 1027.53052](#)
- [Kamishima and Ornea 2005] Y. Kamishima and L. Ornea, “Geometric flow on compact locally conformally Kähler manifolds”, *Tohoku Math. J. (2)* **57**:2 (2005), 201–221. [MR 2006g:53112](#) [Zbl 1083.53068](#) [arXiv math.DG/0105040](#)
- [Kiričenko 1977] V. F. Kiričenko, “ K -spaces of maximal rank”, *Mat. Zametki* **22**:4 (1977), 465–476. English translation: *Math. Notes* **22**: 4 (1978) 751–757. [MR 57 #13756](#) [Zbl 0414.53018](#)
- [Kodaira and Spencer 1958] K. Kodaira and D. C. Spencer, “On deformations of complex analytic structures. I, II”, *Ann. of Math. (2)* **67** (1958), 328–466. [MR 22 #3009](#) [Zbl 0128.16901](#)
- [Moroianu et al. 2005] A. Moroianu, P.-A. Nagy, and U. Semmelmann, “Unit Killing vector fields on nearly Kähler manifolds”, *Internat. J. Math.* **16**:3 (2005), 281–301. [MR 2006e:53056](#) [Zbl 1077.53038](#) [arXiv math.DG/0406492](#)
- [Muškarov 1987] O. Muškarov, “Structures presque hermitiennes sur des espaces twistoriels et leurs types”, *C. R. Acad. Sci. Paris Sér. I Math.* **305**:7 (1987), 307–309. [MR 89b:53068](#) [Zbl 0626.53028](#)
- [Nagy 2002a] P.-A. Nagy, “Nearly Kähler geometry and Riemannian foliations”, *Asian J. Math.* **6**:3 (2002), 481–504. [MR 2003m:53043](#) [Zbl 1041.53021](#) [arXiv math.DG/0203038](#)
- [Nagy 2002b] P.-A. Nagy, “On nearly-Kähler geometry”, *Ann. Global Anal. Geom.* **22**:2 (2002), 167–178. [MR 2003g:53073](#) [Zbl 1020.53030](#) [arXiv math.DG/0110065](#)
- [Ornea and Piccinni 1998] L. Ornea and P. Piccinni, “Weyl structures on quaternionic manifolds”, pp. 231–236 in *Quaternionic structures in mathematics and physics* (Trieste, 1994), edited by G. Gentili et al., SISSA, Trieste, 1998. [MR 99k:53098](#) [Zbl 0946.53019](#) [arXiv math.DG/0105041](#)
- [Verbitskiĭ 2004] M. S. Verbitskiĭ, “Theorems on the vanishing of cohomology for locally conformally hyper-Kähler manifolds”, *Tr. Mat. Inst. Steklova* **246** (2004), 64–91. [MR 2005h:53071](#) [Zbl 1101.53027](#) [arXiv math.DG/0302219](#)

Received September 13, 2007. Revised November 1, 2007.

MISHA VERBITSKY
INSTITUTE OF THEORETICAL AND EXPERIMENTAL PHYSICS
B. CHEREMUSHKINSKAYA, 25
MOSCOW, 117259
RUSSIA
verbit@mccme.ru