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SLAC-PUB--4320

DE87 012835

CONF-8610198--5

AN INTRODUCTION TO ACCELERATION MECHANISMS*

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*Work supported by Department of Energy contracts DE-AC03-76SF00515 (SLAC) and DE-AC02-76C0016 (BNL).

Invited talk presented at the Joint US-CERN School on Particle Accelerators:
Topical Course on Frontiers on Particle Beams
South Padre, Texas, October 23-29, 1986

(5) I - 31409 DR-0283-6

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1. Introduction

We wish to discuss the acceleration of charged particles by electromagnetic fields, i.e., by fields that are produced by the motion of other charged particles driven by some power source. It is convenient, from the start, to note that such electromagnetic fields may be separated into "near" and "far" fields. The distinction can easily be seen when we look at the form of the fields generated by a moving point charge. From Jackson[1] (p. 675, eq. 14.14):

$$\bar{E}(x,t) = e\left[\frac{\hat{n}-\bar{\beta}}{\gamma^2\left(1-\bar{\beta}\cdot\hat{n}\right)^3 R^2}\right] + \frac{e}{c}\left[\frac{\hat{n}_\times\left\{\left(\hat{n}-\bar{\beta}\right)\times\dot{\bar{\beta}}\right\}}{\left(1-\bar{\beta}\cdot\hat{n}\right)^3 R}\right]$$

where $\hat{n}$ is a unit vector from the particle to the observation point and R is the distance. $\bar{\beta},\dot{\bar{\beta}}$ and γ refer to the motion of the charge at time $[t-(R/c)]$, and e is the electric charge. One notes at once that the first part of this expression falls as $1/R^2$, just like any static electromagnetic field; it is the "near" field part. The second part falls only as $1/R$ and represents a propagating or radiating field; it is the "far" field.

Of course, we do not in general use fields generated by a single moving charge. It is thus often more convenient to look at the possible fields by examining Maxwell's equations and deducing the kinds of fields that can satisfy them. Taken from Jackson (p. 218, eq. 6.28) but expressed in MKS units:

$$\nabla\times\bar{E}(t) = -\frac{d\bar{B}(t)}{dt} \quad (1.1)$$

$$\nabla\cdot\bar{B}(t) = 0 \quad (1.2)$$

$$\nabla\times\bar{B}(t) = \mu\epsilon\,\frac{d\bar{E}(t)}{dt} \quad (1.3)$$

$$\nabla\cdot\bar{E}(t) = \frac{q}{\epsilon_o} \quad (1.4)$$

q is the space charge density, and μ and ϵ are the magnetic susceptibility and dielectric constants in the medium. ϵ_o is the dielectric constant in free space. Since any time-varying field can be represented as an integral of sinusoidally time-varying fields, (by taking a Fourier transform) we can write

$$\bar{E}(t) = \int \tilde{E}\,\exp\{-i\omega t\}d\omega \quad (1.5)$$

$$\bar{B}(t) = \int \tilde{B}\,\exp\{-i\omega t\}d\omega \quad (1.6)$$

$\hat{E}$ and $\hat{B}$ are complex and depend on the frequency ω. The E's and B's satisfy the following modified Maxwell's equations. We have now also made the assumption that there are no charges in the space in which these fields are present.

$$\nabla \times \hat{E} = i\omega \hat{B} \tag{1.7}$$

$$\nabla \cdot \hat{B} = 0 \tag{1.8}$$

$$\nabla \times \hat{B} = i\omega\mu\epsilon \hat{E} \tag{1.9}$$

$$\nabla \cdot \hat{E} = 0 \tag{1.10}$$

From vector algebra (see for instance, inside the front cover of Jackson) we can write

$$\nabla^2 \hat{E} = \nabla \cdot (\nabla \cdot \hat{E}) - \nabla \times (\nabla \times \hat{E}) \; .$$

From Eq. (1.10)

$$\nabla \times \hat{E} = 0 \; ,$$

from Eq. (1.7)

$$\nabla \times \hat{E} = i\omega \hat{B} \; ,$$

so

$$\nabla^2 \hat{E} = -i\omega \nabla \times \hat{B} \; ,$$

and from Eq. (1.9)

$$\nabla \times \hat{B} = -i\omega\mu\epsilon \hat{E}$$

which then gives us the wave equation:

$$\nabla^2 \hat{E} = -\mu\epsilon\omega^2 \hat{E} \; . \tag{1.11}$$

A similar calculation for B gives us

$$\nabla^2 \hat{B} = -\mu\epsilon\omega^2 \hat{B} \; . \tag{1.12}$$

All solutions to these equations can be expressed as sums of waves of the form

$$\begin{aligned} \hat{E} &= E_0 \exp\{i(\hat{k} \cdot \hat{r} - \omega t)\} \; , \\ \hat{B} &= B_0 \exp\{i(\hat{k} \cdot \hat{r} - \omega t)\} \; . \end{aligned} \tag{1.13}$$

If $\hat{k}$ is a real vector, i.e., if

$$\hat{k} \cdot \hat{r} = k_x \cdot x + k_y \cdot y + k_z \cdot z \; , \tag{1.14}$$

then these equations represent traveling waves moving in the direction of the vector $\hat{k}$ at a velocity equal to $\omega/|k| = 1/\sqrt{\mu\epsilon}$. Since $\mu_o \epsilon_o = 1/c^2$, by definition, where c equals the velocity of light, we see that the wave velocity is c/N, where the refracted index $N = \sqrt{\mu\epsilon/(\mu_0\epsilon_0)}$. Substituting Eq.(1.13) into Eqs. (1.8) and (1.10), we obtain

$$\tilde{E}_k \cdot \bar{k} = 0 , \tag{1.15}$$

$$\tilde{B}_k \cdot \bar{k} = 0 , \tag{1.16}$$

These show us that the waves that we are discussing are transversely polarized (again on the assumption that the vectors $\bar{k}$ are real). Finally, using Eq. (1.7), we can relate the B fields to the E fields, and obtain

$$\begin{aligned} \tilde{B}_k &= \frac{\bar{k}}{\omega} \times \tilde{E} , \\ |\tilde{B}| &= \frac{|\tilde{E}|}{c} , \end{aligned} \tag{1.17}$$

which shows us that the B fields and E fields are perpendicular to one another and that the magnitude of the E field is equal to the magnitude of the B field times c.

Thus we have found that solutions to Maxwell's equations in a space with no free charges include solutions that are plain, parallel, transversely polarized waves with a velocity of c/N, where N is the refractive index (see Figure 1).

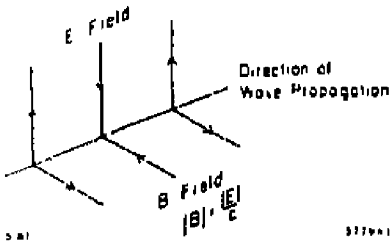

Figure 1

We will see later that these are not the only solutions. $\bar{k}$ need not be a real vector, and if complex, the waves represented by Eqs. (1.13) and (1.14) contain exponentials. These solutions fall in amplitude at distances which are far from any changes and thus cannot exist "*FAR*" from such sources. They are referred to as "*NEAR*" fields (see Sections 7-9). For the moment we will consider only "*FAR*" fields (see Figure 2).

Figure 2

Theorem 1. *General Acceleration Theorem.*

Consider a charge e moving in a field described by Maxwell's equations at a velocity $v = \beta c$. If

(a) $P_{rad} \approx 0$, *i.e.*, we neglect power radiated by the moving charge,

(b) $\beta \approx 1$, *i.e.*, the charge is moving near the velocity of light,

(c) $q = 0$, *i.e.*, the particle is moving in a region with no other free charges,

(d) $B_{static} = E_{static} = 0$, so that the charge moves on an approximately straight line,

(e) $N = 1$, *i.e.*, the particle moves in a medium with no refractive index,

(f) $(r/\lambda) \gg 1$, *i.e.*, the distance r to any source of field is large compared to the wavelength,

then, if there are no fields or static potential at $-\infty$ or $+\infty$:

$$\int_{-\infty}^{+\infty} \text{Acceleration} \approx 0 .$$

The proof is relatively trivial. Chose the axes so that the particle is moving on the z axis, then:

$$\int \text{Acceleration} = \int_{-\infty}^{+\infty} e\, E_z \exp\left(i(k_z \cdot z - \omega t)\right) . \tag{1.10}$$

For $\beta = 1$ we have

$$z = c(t - t_0) .$$

If θ is the angle between the vector direction of the wave $\vec{k}$ and the z axis then

$$k_z = |k| \cos\theta = \frac{\omega}{c} \cos\theta$$

and thus

$$\int \text{Acceleration} = e\, E_z \int_{-\infty}^{+\infty} \exp\left(i\omega(t - t_0) \cos\theta\right) .$$

If $\cos\theta \neq 1$, the RHS oscillates and the integral is zero.

If $\cos\theta_z = 1$, then the wave is propagating in the z direction and from the transverse polarization condition [Eq. (1.16)] $E_z = 0$. Thus in all cases, acceleration is zero. No playing with a focus or phase plates or holograms can change this.

In the following sections we will deal with acceleration that occurs when we relax one of the conditions listed above.

DISCLAIMER

2. Radiation Pressure

A) Nonrelativistic

In quantum terms we can understand that when a photon of electromagnetic energy interacts with (*i.e.*, Compton scatters off) an electron, then momentum is imparted to (*i.e.*, a force acts on) that electron. *Classically* we calculate the oscillatory motion given to the electron, the re-radiation of fields caused by this motion, and hence the resulting force.

For mathematical convenience, we consider the case of constant circularly polarized radiation [from Eqs. (1.13),(1.14)]:

$$
\begin{aligned}
E_x + iE_y &= E_0 \exp\{i(kz - \omega t)\} , \\
B_x + iB_y &= i \frac{E_0}{c} \exp\{i(kz - \omega t)\} .
\end{aligned}
\tag{2.1}
$$

Ignoring reradiation, the motion of the electron in the x, y plane is circular. The force is always inward (see Figure 3a):

$$
\begin{aligned}
|\vec{F}| &= m_e \ddot{x} = e E_0 , \\
&= m_e v \omega , \\
&= m_e \rho \omega^2 .
\end{aligned}
\tag{2.2}
$$

$$
\rho = \frac{E_0 e}{m_e \omega^2} . \tag{2.3}
$$

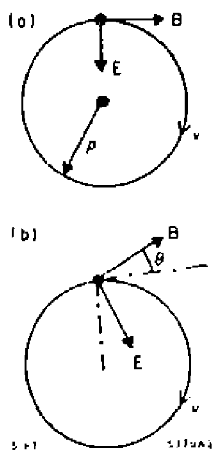

Figure 3

FOR EXAMPLE: *for the kind of field generated by a 1 cm-long focus of a 1.7 TW CO_2 laser, we might have:*

$$E_0 \approx 100 \text{ GV/m} ,$$

$$\lambda \approx 10 \, \mu\text{m} ,$$

$$e \approx 1.6 \times 10^{-19} \text{ C} ,$$

$$m_e \approx 9.1 \times 10^{-31} \text{ kg} ,$$

then:

$$\rho \approx .5 \, \mu\text{m} .$$

This circular motion will radiate a power P (see Jackson, p. 665):

$$P_{rad} = \frac{2}{3} \frac{e^2}{c^3} \quad \gamma^4 (\ddot{x})^2 \frac{1}{4\pi\epsilon_0} , \tag{2.4}$$

or, if nonrelativistic, and substituting

$$\ddot{x} = \frac{e E_0}{m_e} , \tag{2.5}$$

$$P_{rad} \approx \frac{2}{3} \frac{e^4 E_0^2}{m_e^2 c^3} \frac{1}{4\pi\epsilon_0} . \tag{2.6}$$

There are several ways of deriving the forward force on the electron:

(1) by considering the interference of incoming and radiated fields;

(2) by using energy and momentum conservation,

(3) by noting the consequences of a phase slip.

I will use the latter. By energy conservation a phase slip ϕ must exist between the electron motion and the incoming field (see Figure 3b). The power given to the electron from the field is,

$$P_{received} = E_0 e \, v \, \sin\phi , \tag{2.7}$$

where

$$v = \frac{e E_0}{m_e \omega} ,$$

and

$$P_{rec} = P_{rad} , \tag{2.8}$$

so using (2.6):

$$
\begin{aligned}
\frac{E_0^2 e^2}{m_e \omega} \sin\phi &= \frac{2}{3} \frac{e^4 E_0^2}{m_e^2 c^3} \frac{1}{4\pi \epsilon_0}, \\
\sin\phi &= \frac{2}{3} \left(\frac{e^2}{4\pi \epsilon_0 m_e c^2}\right) \frac{\omega}{c}
\end{aligned}
\tag{2.9}
$$

which, using the definition of the classical electron radius r_e,

$$
r_e = e^2 4\pi \epsilon_0 m_e c^2 , \tag{2.10}
$$

gives

$$
\sin\phi = \frac{2}{3} r_e \frac{\omega}{c} . \tag{2.11}
$$

FOR EXAMPLE: *for the above example*

$$
\omega = 1.88 \times 10^{14} \text{ sec}^{-1} ,
$$

$$
c = 3 \times 10^8 \text{ m/sec} ,
$$

$$
r_e = 2.8 \times 10^{-15} \text{ m} ,
$$

then

$$
\sin\phi \approx 1.2 \times 10^{-9} .
$$

which is a very small angle ϕ!

Referring again to Figure 3b, we see that there is now a finite force in the z direction

$$
\begin{aligned}
F_z &= e \bar{B} \times \bar{v} , \\
&= e \frac{E_0}{c} v \sin\phi . \\
v &= \frac{e E_0}{m_e \omega} ,
\end{aligned}
\tag{2.12}
$$

so

$$
\begin{aligned}
F_z &= \frac{e E_0}{c} \frac{e E_0}{m_e \omega} \frac{2}{3} r_e \frac{\omega}{c} , \\
F_z &= \frac{2}{3} (E_0 e)^2 \frac{r_e}{m_e c^2} ,
\end{aligned}
\tag{2.13}
$$

which is the expression for the nonrelativistic radiation pressure. It may be compared with the instantaneous radiation force $E_0\, e$:

$$f_F = \frac{F_s}{E_0\, e} = \frac{2}{3}\, E_0\, e\, \frac{r_e}{m_e c^2} . \tag{2.14}$$

which increases as the field E_0 increases, but even for our rather high power laser example, this fraction is very small.

$$\begin{aligned} E_0 &= 100\,\mathrm{GV/m} , \\ f_F &= 3.6 \times 10^{-10} . \end{aligned} \tag{2.15}$$

It is clear that radiation pressure is not a practical means of accelerating particles.

B) Relativistic

It might appear from Eq. (2.4) that if γ is large, the radiation might also become large $(P_{rad} \partial \gamma^4)$. But in this case we have to consider also the force on the moving electron from the B field of the wave

$$\begin{aligned} m_e \ddot{z} = F &= F_e + F_B , \\ &= e\, E_0 - e\, B_0\, v \\ &= e\, E_0 - e\, E_0\, \beta , \\ &= e\, E_0\, (1 - \beta) , \\ &\approx \frac{e\, E_0}{2\gamma^2} . \end{aligned} \tag{2.16}$$

So for $\gamma \gg 1$,

$$P \approx \frac{2}{3}\, \frac{e^2}{c^3}\, \gamma^4 \left(\frac{e\, E_0}{2\gamma^2}\right)^2 \frac{1}{4\pi\, \epsilon_0} , \tag{2.17}$$

and we see that radiated power, and thus force, has not increased.

Thus for all practical cases the first assumption of Theorem 1 is satisfied.

3. A Nonrelativistic Case (the Ponderomotive Force)

We have seen in Sec. 2 that a uniform electromagnetic wave induces only small forces on an electron. The same is not true for a time varying field.

Again we consider a circularly polarized wave [Eq. (2.1)]. Assuming the rate of change of amplitude is relatively small the motion is again approximately circular with [from Eq. (2.3)]:

$$\rho = \frac{E_0 e}{m_e \omega^2} \quad . \tag{3.1}$$

If we allow the magnitude of the wave E_0 to increase, then we have

$$\dot{\rho} = \frac{\dot{E}_0 e}{m_e \omega^2} \quad , \tag{3.2}$$

and the motion becomes a spiral (see Figure 4).

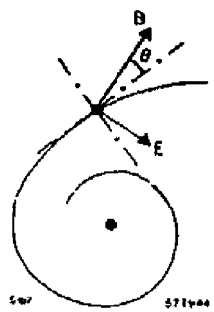

Figure 4

Now the stored energy in this orbital motion W is

$$\begin{aligned} W &= \frac{1}{2} m_e v^2 , \\ &= \frac{1}{2} m_e \omega^2 \rho^2 , \end{aligned} \tag{3.3}$$

and the rate of change of this energy will be

$$\dot{W} = m_e \omega^2 \rho \dot{\rho} \quad . \tag{3.4}$$

Using Eq.(3.2), we obtain

$$\dot{W} = \rho e \dot{E}_0 \quad , \tag{3.5}$$

which must equal the energy given to the electron. Again, as in Sec. 2, a phase ϕ_p must develop (see Figure 4) so that such an energy gain can take place:

$$\begin{aligned} \dot{W} &= E_0 e v \sin\phi_p , \\ &= E_0 e \omega \rho \sin\phi_p . \end{aligned} \tag{3.6}$$

Equating Eqs. (3.5) and (3.6) gives

$$\rho e \dot{E}_0 = E_0 e \omega \rho \sin\phi_p ,$$

so

$$\sin\phi_p = \frac{\dot{E}_0}{E_0} \cdot \frac{1}{\omega} . \tag{3.7}$$

Unlike the radiation pressure phase angle of Eq. (2.11), ϕ_p can be large. ϕ_p is of the order of the reciprocal of the number of cycles over which the amplitude charge occurs.

As in the radiation pressure case, this phase angle produces a finite force in the z direction:

$$\begin{aligned} F_z &= \vec{B} \times e\vec{v} , \\ &= \frac{E_0}{c} e v \sin\phi_p , \\ &= \frac{E_0 e}{c} \rho \omega \sin\phi_p , \end{aligned} \tag{3.8}$$

so with Eq. (3.7):

$$\begin{aligned} F_z &= \frac{E_0 e}{c} \rho \omega \frac{\dot{E}_0}{E_0} \frac{1}{\omega} , \\ F_z &= e \dot{E}_0 \frac{\rho}{c} , \end{aligned} \tag{3.9}$$

and the ratio of this to the radial force $E_0 e$:

$$f_F = \frac{F_z}{E_0 e} = e \dot{E} \frac{\rho}{c} \cdot \frac{1}{E_0 e}$$

using Eq. (3.1)

$$f_F = \frac{F_z}{E_0 e} = \frac{\dot{E}_0 e}{m \omega^2 c} . \tag{3.10}$$

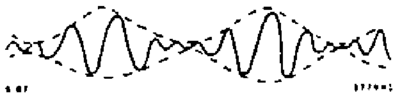

Figure 5

As an example of time varying fields, we consider the beat of two waves, each with amplitude E_0, at frequencies ω_1 and ω_2, where $\omega_1 \approx \omega_2$. The fields will be as shown in Figure 5. The resultant time varying amplitude will be:

$$E_0(t) \simeq 2E_0 \sin[(\omega_2 - \omega_1)\, t] \quad , \tag{3.11}$$

and the maximum rate of change of this amplitude:

$$\dot{E}_{0\,max} = 2E_0\,(\omega_2 - \omega_1) \quad . \tag{3.12}$$

The resulting maximum accelerating force [using Eq. (3.9)] is

$$F_{z\,max} = 2\,e\,E_0\,(\omega_2 - \omega_1)\,\frac{\rho}{c} \quad . \tag{3.13}$$

and the ratio of this force to the maximum radial force ($2e\,E_0$) is

$$\begin{aligned} f_F &= \frac{F_z}{2e\,E_0} = \frac{2e\,E_0}{2e\,E_0} \cdot (\omega_2 - \omega_1) \cdot \frac{\rho}{c} \,, \\ f_F &\simeq \frac{e\,E_0}{m_e\,c}\,\frac{\omega_2 - \omega_1}{\omega^2} \quad . \end{aligned} \tag{3.14}$$

As an example, we consider the beat from two lines of a CO_2 laser where

$$\frac{\omega_2 - \omega_1}{\omega} \approx 3\%$$

$$E_0 \simeq 100\ GV/m = 10^{11}\ V/m$$

$$\omega = 1.9 \times 10^{14}\ sec^{-1}$$

$$f_F = 3 \times 10^{-3}$$

$$F_z \simeq 1\ GV/m$$

which is not so negligible!

The acceleration, unfortunately does not continue indefinitely. The maximum velocity achieved

$$v_{max} = \int_0^t \frac{F_z}{m_e}\, dt \quad , \tag{3.15}$$

from Eq. (3.9)

$$F_z = \frac{2e\,\rho}{c}\,\frac{dE_0}{dt} \quad ,$$

from Eq. (2.3)

$$\rho = \frac{e\,E_0}{m_e\,\omega^2} \quad ,$$

so

$$\frac{v(t)}{c} = \int_0^t \frac{2e}{m_e\,c^2}\,\frac{e\,E_0}{m_e\,\omega^2}\,\frac{dE_0}{dt}\,dt \ ,$$

$$v(E_0) = \left(\frac{e}{m_e\,\omega\,c}\right)^2 2 \int_0^{E_0} E_0\, dE_0 \tag{3.16}$$

$$\frac{v_{max}}{c} = \frac{v_o}{c} + \left(\frac{e\,E_{0\,max}}{m_e\,\omega\,c}\right)^2 \quad . \tag{3.17}$$

For our example,

$$E_0 = 100\ \mathrm{GV/m}$$

$$\omega = 1.88 \times 10^{14}\ \mathrm{sec}^{-1}$$

$$m_e = 9.1 \times 10^{-31}\ \mathrm{kg}$$

$$e = 1.6 \times 10^{-19}\ \mathrm{C}$$

$$c = 3 \times 10^{8}\ \mathrm{m\ sec}^{-1}$$

$$\frac{v_{max}}{c} = \frac{v_{initial}}{c} + 0.1 \quad .$$

So that acceleration can even reach relativistic velocities with plausible laser power levels. However it is clear from Eq. (3.16) that integrating from any point with $E_0 = 0$ to any other point with E_0, the net acceleration is zero. This example suggests, although it does not prove, that condition (b) is not in fact required for the result of Theorem 1 to be correct.

4. Allow Free Charges (Plasma Beat Wave acceleration)

The Ponderamotive force (see Sec. 3) can be used to induce perturbations in an otherwise uniform distribution of charges, and the "*static*" fields generated by these perturbations can be used to accelerate particles. This idea was proposed by Willis for charges in a dense beam.[7] More realistically, Dawson proposed the perturbation of charges in a plasma.[5] In this case he explicitly proposed that beat waves be used (as discussed in Sec. 3) and that the resulting oscillations in electron plasma density be enhanced by selecting a plasma whose plasma frequency is equal to the beat frequency, *i.e.*,

$$\omega_1 - \omega_2 = \omega_p \quad . \tag{4.1}$$

In this case the amplitude of the plasma oscillations will increase until losses or nonlinearities overcome the amplification provided by the beat wave. The nonlinearities will clearly stop the process before the amplitude is such as to cause the plasma density, at its oscillating minima, to become zero.

To examine the magnitude of acceleration that we may expect, we return to Maxwell [Eqs. (1.1) to (1.4)]. If we assume that there is a solution in which the excess change density q is also periodic in t and z, *i.e.*, a plasma wave with

$$q = \rho e = \rho a \exp\{i(k_z z - \omega t)\} \, , \tag{4.2}$$

$$E = E_0 \exp\{i(k_z z - \omega t)\} \, . \tag{4.3}$$

Then substituting in Eqs. (1.1) to (1.4) for small a, one obtains a solution if

$$\omega_p = \left\{ \frac{\rho_o e^2}{\varepsilon_o m_e} \right\}^{1/2} \quad , \tag{4.4}$$

which frequency is defined as the "*plasma frequency.*"

The amplitude a cannot be greater than 1. If it were, the electron density would have to be negative. In practice, nonlinearities will limit a to $\approx$ 1/10.

From Eq. (1.4) we have

$$\nabla \cdot E = \frac{q}{\varepsilon_o} \quad .$$

That is,

$$\begin{aligned} \frac{dE}{dz} &= \frac{e}{\varepsilon_o} \rho \quad , \\ &= \frac{e}{\varepsilon_o} a \rho_o \exp\{i(k_z z - \omega t)\} \quad . \end{aligned} \tag{4.5}$$

Differentiating E from Eq. 4.3, we get

$$E_0 k_z = \frac{e}{\epsilon_0} a \rho_0 \quad .$$

Substituting for ρ_0 using Eq. (4.4), and for k_z using $k_z = \omega_p/c$:

$$E_0 = \frac{\omega_p m_e c a}{e} \quad ,$$

which gives the amplitude of longitudinal acceleration as a function of the wave density amplitude a. Clearly, higher fields are possible at higher plasma frequencies.

For an example consider the beat wave between two CO_2 laser frequencies 10% apart, then

$$\omega_p = 1.88 \times 10^{13} \quad (\lambda_p = 100\ \mu\text{m})$$

$$m_e = 9.1 \times 10^{-31} \quad \text{kg}$$

$$c = 3 \times 10^{8} \quad \text{m sec}^{-1}$$

$$e = 1.6 \times 10^{-19} \quad \text{C}$$

$$\rho_0 = 3.8 \times 10^{22}\ \text{m} \quad (3.8 \times 10^{16}\ \text{cm}^{-3}) \quad .$$

If

$$a = .1 \quad ,$$

then

$$E_0 = 3.2 \times 10^{9} \quad (3.2\ \text{GeV/m}) \quad .$$

Thus we see that with a quite reasonable plasma density, very high acceleration gradients are possible. Of course plasmas are complicated and the efficiency of driving the beat wave will not in general be 100%. Much study is being devoted to this mechanism.[4]

Note also that such a plasma wave can be excited by a bunch of low-energy particles passing through the plasma. The mechanism is then known as a "*plasma wake accelerator*."[5]

5. Allow External Magnetic Field B (Inverse Free Electron Laser)[6]

Before introducing the field, let us consider the forces on a relativistic particle traveling at a small angle θ to a plane parallel wave. For a time they will remain in phase. The forces on the particle are illustrated in Figure 6.

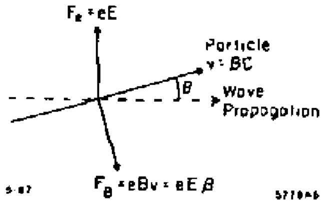

Figure 6

The electrostatic force

$$\vec{F}_e = e\,\vec{E}_\phi \quad (\text{perpendicular to wave direction}) \,, \tag{5.1}$$

the magnetic force

$$\vec{F}_B = e\,\vec{B}_\phi \times \vec{v} \quad , \tag{5.2}$$

but

$$|B| = \frac{|E|}{c} \quad ,$$

in vacuum, so

$$|F_B| = e\beta \; |E_\phi| \quad (\text{perpendicular to } \vec{v}) \quad . \tag{5.3}$$

In each case ϕ is the relative phase between the particle and the wave: $E_\phi = E \cos\theta$.

The accelerating field for small θ is

$$E_{acc} = E_\phi \sin\theta \approx E_\phi\,\theta \quad . \tag{5.4}$$

The deflecting field for small θ and $\beta \approx 1$ is

$$E_{deflection} = E_\phi\,(\beta - \cos\theta) \approx 0 \quad . \tag{5.5}$$

Thus we see that despite the polarization being nearly perpendicular to the velocity, the forces on the electron are almost entirely along the motion, i.e., accelerating or decelerating. No net acceleration will occur because as the phase slips, acceleration and deceleration will alternate. Let us estimate this rate of slip. Let z be the direction of wave propogation. Then:

$$v_z(wave) = c\ ,$$
$$v_z(particle) = c\beta\ \cos\theta\ \ .$$

Their phase ϕ will slip by 2π in a distance:

$$\Lambda = \frac{2\pi}{K}\ \ , \tag{5.6}$$

where

$$K \approx k\left(1 - \frac{1}{\beta\cos\theta}\right)\ \ , \tag{5.7}$$

$$K \approx k\left(\frac{\theta^2}{2} + \frac{1}{2\gamma^2}\right)\ \ . \tag{5.8}$$

If we assume $\theta^2/2 \gg 1/2\gamma^2$ then

$$\Lambda \approx \frac{2\lambda}{\theta^2}\ \ . \tag{5.9}$$

For a distance $\Lambda/2$ we can have acceleration, then it will reverse. Over a long distance there will be no net acceleration. How can we overcome this? Suppose after each distance $\Lambda/2$ we have a magnet that reverses the sign of θ (see Figure 7), acceleration will then continue indefinitely.

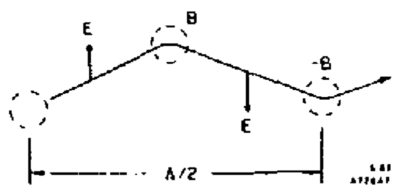

Figure 7

Since the acceleration is proportional to θ, we wish to maximize the amplitude of the zig-zag; with a fixed B, this is achieved by having a field everywhere but alternating its sign with an appropriate period. Such magnets are known as wigglers.

A more elegant (and easier to calculate) solution is to use a "*helical field*;" *i.e.*, a field that remains transverse but whose direction rotates about the axis. It is the field generated by a winding of two interleaved helical wires, carrying opposite currents. The motion of a charged particle in such a field will also be a helix with constant angle θ.

The helical field is given by

$$B_x + i\,B_y \;=\; B_0 \exp\{iKz\} \quad . \tag{5.10}$$

The force on the particle is then

$$\begin{aligned} F_x + i\,F_y &= i\,e\,v\,B_0\ \exp\{iKz\}\ , \\ &= \frac{dp_x}{dt} + i\,\frac{dp_y}{dt} \quad , \end{aligned} \tag{5.11}$$

so

$$\begin{aligned} p_x + i\,p_y &= \int i\,e\,v\,B_0 \exp\{iKz\}\ dt\ , \\ &= \frac{e\,v\,B_0}{K\,\frac{dz}{dt}} \exp\{iKz\} \quad , \end{aligned} \tag{5.12}$$

and

$$\begin{aligned} p_\perp &= \frac{e\,v\,B_0}{K\,\frac{dz}{dt}} \quad , \\ &\approx \frac{e\,B_0}{K} \quad \text{(for small } \theta\text{)} \quad . \end{aligned} \tag{5.13}$$

The transverse momentum is a constant, with its direction rotating about the axis (*i.e.*, a helix). The helix pitch angle θ is then:

$$\theta \;\approx\; \frac{p_\perp}{p_z} \;=\; \frac{e\,B_0}{K\,p_z} \quad . \tag{5.14}$$

Now, from Eqs. (5.6) and (5.9)

$$K \;\approx\; \frac{2\pi}{\lambda}\ \frac{\theta^2}{2} \quad .$$

Substituting in Eqs. (5.14) gives

$$\begin{aligned} \theta &\approx \left(\frac{e\,B\,\lambda}{\pi\,p_z}\right)^{1/3} , \\ &= \left(\frac{e\,B\,\lambda}{m_e\,c\,\pi\,\gamma}\right)^{1/3} , \end{aligned} \tag{5.15}$$

and from Eq. (5.4)

$$E_a \;=\; E_0\theta\cos\phi$$

so

$$E_a(max) \;=\; E_0\,\theta \quad .$$

For example, with a powerful CO_2 *laser one might have*

$$\lambda = 10\,\mu m\,,$$

$$B = 1\ Tesla$$

$$m_e = 9.1 \times 10^{-31}\ kgm\,,$$

$$c = 3 \times 10^{8}\ m\ sec^{-1}$$

$$E_0 = 10^{11}\ eV/m \quad (100\ GV/m)\,,$$

then for

$$\gamma = 100\ (50\ MeV)\,, \quad \theta = 1.2 \times 10^{-2}\,, \quad E_a = 1.2\ GV/m \quad ,$$
$$\gamma = 10^{5}\ (50\ GeV)\,, \quad \theta = 1.2 \times 10^{-3}\,, \quad E_a = .12\ GV/m \quad .$$

Thus we see that we can obtain good acceleration at low energies but it becomes less attractive as the energy goes up. In addition, C. Pellegrini[7] has pointed out that synchrotron radiation effectively limits the usefulness of the method above a few hundred GeV.

6. Allow Finite Refractive Index (Inverse Čerenkov Effect)

This mechanism was first discussed and subsequently demonstrated by R. Pantel.[9] As in Sec. 5, we can start by considering the interaction of a relativistic particle with a plane wave traveling in nearly the same direction. As in Eq. (5.4), the accelerating field is

$$E_{acc} = E_\phi \sin\theta \quad , \tag{6.1}$$

where θ is the angle between the particle and traveling plane wave and $E_\phi = E_0 \cos\theta$, where ϕ is the relative phase between the particle and that wave. The deflecting field will again be

$$E_{def} = E_\phi(\beta - \cos\theta) \quad , \tag{6.2}$$

which for $\beta \approx 1$ and θ small is negligible.

The phase ϕ will be given by

$$\phi = \phi_s + \omega\left(\frac{N z \cos\theta}{c} - t\right) \quad , \tag{6.3}$$

where N is the refractive index of the medium. If $N \neq 1$ then we can arrange to keep the phase ϕ constant by setting:

$$N \cos\theta = 1 \quad . \tag{6.4}$$

The accelerating field is then

$$E_{acc} = E_0 \sin\theta = \frac{E_0}{\sqrt{1 - \frac{1}{N^2}}} \quad . \tag{6.5}$$

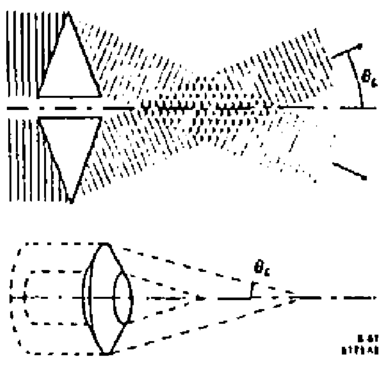

Figure 8

Clearly, this could be a very efficient and simple mechanism if N could be made reasonable large. Unfortunately, a large N implies a high density of gas and that will (1) break down at lower field and (2) cause Coulomb scattering of the beam. How serious these are would depend on the application.

A more "efficient" geometry than that of a plane wave is obtained at an "axicon" focus. This is the field obtained by adding waves, each at a fixed angle θ to the beam axis, but at all different azimuthal angles. It is obtained by passing a plane parallel wave through an "axicon" lens (see Figure 8). Such a field is also present in a circular waveguide excited in the TS_{on} mode. The fields are well known in this case, and are described by Bessel Functions of the first kind (see Jackson,[1] p. 367).

7. Acceleration Near a Planar e.m. Source (the Grating Accelerator)[9]

In our discussions so far we have restricted ourselves to propagating sinusoidal waves that are "far" waves by the definition given in the introduction. But it was noted, in that introduction, that there is a class of solutions to Maxwell's equations that are not sinusoidal propagating waves, and that exist only close to some source, *i.e.*, "near" waves.

Returning then to Maxwell's equations and their solutions discussed in Sec. 1 [Eq. (1.13)]:

$$\begin{aligned} \hat{E} &= \bar{E}_0 \exp\{i(\bar{k}\cdot\bar{r} - \omega t)\} \quad , \\ \hat{B} &= \bar{B}_0 \exp\{i(\bar{k}\cdot\bar{r} - \omega t)\} \quad , \end{aligned} \tag{7.1}$$

where $\bar{k}$ was a vector and [Eq. (1.14)]:

$$\bar{k}^2 = \frac{\omega^2}{c^2} = k_x^2 + k_y^2 + k_z^2 \quad . \tag{7.2}$$

If $\bar{k}$ is a real vector, then k_x, k_y, and k_z are all real and less than $(\omega/c)^2$, and Eq. (7.2) represents plane parallel propagating waves. These are "far waves". If the source is distributed over a plane surface, than the strength of such waves will remain independent of the distance y from that surface. If E_s represents the amplitudes at that surface then:

$$E(y) \approx E_s \quad , \tag{7.3}$$

However, another solution to Eq. (7.2) would be to allow $\bar{k}$ to be complex and thus allow one or more of k_x, k_y or k_z to be negative. If, for instance, k_y^2 is negative, where y is the direction away from the surface, then defining p to be real:

$$\begin{aligned} p^2 &= -k_y^2 \quad , \\ p &= -ik_y \quad , \end{aligned} \tag{7.4}$$

then Eq. (7.1) becomes

$$\hat{E} = E_0 \exp\{-py\} \exp\{i(k_z z + k_x x - \omega t)\} \quad . \tag{7.5}$$

In this case we see that the amplitude of the "wave" falls with the distance y from the surface

$$E(y) \approx \exp\{-py\} E_s \quad , \tag{7.6}$$

and such fields are "near" fields. Because they fall exponentially from the surface they are also referred to sometimes as "evanescent" waves.

For convenience let us consider waves traveling along the z direction, *i.e.*, $k_x = 0$:

$$\tilde{E} = \hat{E}_0 \exp\{-p_y\} \exp\{i(k_z z - \omega t)\} , \tag{7.7}$$

$$k_z^2 = \frac{\omega^2}{c^2} + p^2 . \tag{7.8}$$

Substituting into Eq. (1.10) gives

$$-E_y ip + E_z k_z = 0 \quad ,$$

and thus

$$E_z = \frac{ip}{k_z} E_y \quad . \tag{7.9}$$

So, unlike the far field case, we have a non-zero field in the direction of propagation. This field is 90° out of phase with the transverse field. The field pattern at a given time t is illustrated in Figure 9. As a function of time, the whole pattern advances along the z axis at a velocity

$$v_z = \frac{\omega}{k_z} = \frac{c}{\sqrt{1 + \frac{p^2 c^2}{\omega^2}}} \quad , \tag{7.10}$$

and we note that this wave velocity is less than the velocity of light. For this reason they are sometimes referred to as *"slow" waves.*

Because of this *"slowness"* one cannot accelerate relativistic particles along the direction of propagation of the surface wave. From this one derives:

<u>Theorem 2</u>. *Lawson's Theorem.*

> This states that for any one-sided system which is two-dimensional in character, no acceleration is possible. In our example we are referring to one-sided systems; if the fields are uniform in the x coordinate as above, it is two-dimensional, and thus there can be no acceleration.

The restriction is not so severe, however, since the uniformity in x can be broken even when the structure is itself uniform in x. An example of this is the grating accelerator.

Figure 9

GRATING ACCELERATOR

In a grating accelerator, slow waves are excited on the surface of a periodic structure in such a way that the direction of propagation of the waves are oriented diagonally across the surface from both sides. The two sets of waves generate a periodic wave pattern that is periodic in both z and x, *i.e.*, k_z and k_x are finite and real; k_y is imaginary leading to an exponential fall-off of field away from the grating surface. For acceleration in the z direction we require

$$k_z = \frac{\omega}{\beta c} \approx \frac{\omega}{c} \quad ,$$

since

$$\frac{\omega^2}{c^2} = k_x^2 + k_y^2 + k_z^2 \quad ,$$

thus

$$k_x^2 \approx -k_y^2 = p^2 \quad ,$$

and the field on a particle traveling at the velocity of light ($z = ct$) is

$$E_z = E_0 \exp\{-py\} \exp\{i(\phi_o + k_x x)\} \quad , \tag{7.11}$$

where

$$\phi_o = z_o - c\,t_o \quad .$$

It can be shown[9] that such a wave pattern together with higher space harmonics, form an Eigen solution over a suitable periodic structure with period $\lambda/2$; *i.e.*, the grating acts as a "cavity" that supports the accelerating mode without radiating away the stored energy. The extent of the field transversely (in direction x, in our example) can be limited by placing reflecting walls (see Figure 10). The fields can be excited either by introducing them at one end and allowing them to propagate along the z direction. Alternatively, they can be excited by incident radiation at an angle θ_z from the vertical where

$$k_z = \frac{\omega}{c} \cos\theta_z \quad , \tag{7.12}$$

(this angle assures the matching periodicity in the z direction). In addition, in order to couple the incoming radiation to the accelerating mode, the grating periodicity must have a small period $\approx \lambda$ component.

Figure 10

Let us now compare the magnitude of acceleration with the magnitude of surface fields. Returning to Maxwell [Eq. (1.4)]

$$\nabla \cdot \bar{E} = 0 \quad ,$$

which for

$$E = E_0 \exp\{-py + i(k_x x + k_z z - \omega t)\} \quad ,$$

gives

$$E_x k_x + E_z k_z + E_y i p = 0 \quad . \tag{7.13}$$

If we select the direction of polarization to have $E_x = 0$ (this maximizes acceleration) then

$$\frac{E_z}{E_y} = \frac{ip}{k_z} \approx i \frac{\lambda}{2\pi} p \quad ,$$

and

$$\frac{E_z(y)}{E_y(0)} = \frac{\lambda}{2\pi} p \exp\{-py\} \quad . \tag{7.14}$$

A large value of p gives a higher ratio of acceleration to deflecting fields but a high p also implies a more rapid exponential fall-off from the surface. For a fixed distance from the surface y the maximum acceleration is obtained when

$$p = \frac{1}{y} \quad ,$$

and then

$$\frac{E_{acc}}{E_{surface}} = \frac{1}{2\pi} \frac{\lambda}{y} = \frac{\lambda\!\!\!^{-}}{y} \quad . \tag{7.15}$$

Here we see, for the first time, the explicit requirement for near fields to be negligible:

$$y \gg \lambda \quad . \tag{7.16}$$

8. Acceleration Between Planar e.m. Sources (2D Linac)

This may not be a particularly practical case, but it can be understood without use of Bessel functions. It is also mathematically cute!

Consider two plane parallel "far" waves propagating in nearly the same direction , one at an angle θ above the beam direction and one at an angle $-\theta$ below the beam (see Figure 11). The magnitudes of the two waves are identical. The directions of polarization are vertical and opposite, so that on the axis the transverse electric fields exactly cancel:

$$
\begin{aligned}
k_x &= 0 \ \text{(for both waves)} , \\
k_y &= \frac{\omega}{c}\,\sin \pm\theta \approx \pm\frac{\omega\theta}{c} , \\
k_z &= \frac{\omega}{c}\,\cos\theta \approx \frac{\omega}{c} \ \text{(for both)} ,
\end{aligned}
\tag{8.1}
$$

$$
\begin{aligned}
E_{ox} &= 0 \ \text{(for both)} , \\
E_{oy} &= \pm E_0 \,\cos\theta , \\
E_{oz} &= E_0 \,\sin\theta \ \text{(for both)} ,
\end{aligned}
\tag{8.2}
$$

then

$$
\begin{aligned}
E_y &= E_0\,\cos\theta \left[\exp\left\{i\frac{\omega\theta}{c}\,y\right\} + \exp\left\{i\frac{\omega\theta}{c}\,y\right\}\right] \exp\left\{i\frac{\omega}{c}\,(\cos\theta\, z - ct)\right\} , \\
E_z &= E_0\,\sin\theta \left[\exp\left\{i\frac{\omega\theta}{c}\,y\right\} - \exp\left\{i\frac{\omega\theta}{c}\,y\right\}\right] \exp\left\{i\frac{\omega}{c}\,(\cos\theta\, z - ct)\right\} ;
\end{aligned}
\tag{8.3}
$$

i.e.,

$$
\begin{aligned}
E_y &= 2E_0\,\cos\theta\,\sin\left(\frac{\omega\theta}{c}\,y\right)\,\exp\left\{i\frac{\omega}{c}\,(\cos\theta\, z - ct)\right\} \\
E_z &= 2E_0\,\sin\theta\,\cos\left(\frac{\omega\theta}{c}\,y\right)\,\exp\left\{i\frac{\omega}{c}\,(\cos\theta\, z - ct)\right\} ,
\end{aligned}
\tag{8.4}
$$

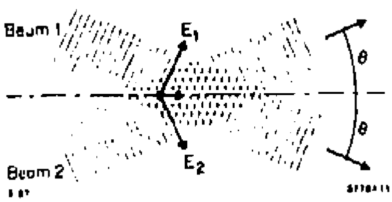

Figure 11

which for $\theta \to 0$ and $2E_0\theta \to A_0$

$$E_y \approx A_0 \frac{\omega}{c}\, y \,\exp\left\{i\frac{\omega}{c}\,(z-ct)\right\} \quad ,$$
$$E_z \approx A_0 \,\exp\left\{i\frac{\omega}{c}\,(z-ct)\right\} \quad . \tag{8.5}$$

Here we see that the accelerating field will remain in phase with the particle and need not go to zero (E_0 went to ∞, as $\theta \to 0$, but the observed fields can remain finite). We note moreover that the accelerating field is independent of transverse position x or y. However, the transverse field E_x rises linearly with y.

If the fields are generated at two surfaces at $\pm y$, then

$$\frac{E_{acc}(0)}{E_s(y)} = \frac{c}{\omega y} = \frac{1}{2\pi}\,\frac{\lambda}{y} = \frac{\lambdabar}{y} \quad . \tag{8.6}$$

We note that this is the same relation as found for the optimized planar grating accelerator [Eq. (7.15)]. Although we started with two far fields we have, by taking the limit $\theta \to 0$ obtained a "near" field solution.

9. Acceleration with Cylindrical Symmetry (Conventional Iris Loaded Linac Structure)

As in Sec. 8 we can again derive the near field solution by starting with a far field case and taking a limit. We start with the fields discussed in Sec. 6; *i.e.*, the inverse of Čerenkov fields (see Figure 8). These fields would be formed by taking the two interfering beams of Sec. 8 and rotating about the axis. Beams are approaching the axis at fixed angles θ to that axis, but from all azimuthal directions. The fields that result cannot be simply written down, since they involve infinite sums, but they can be written in terms of Bessel functions.

In general the fields in an axially symmetric case can be represented as sums of transverse electic (TE) and transverse Magnetic (TM) modes. Since only the TM modes contain accelerating fields, we will consider only these. With z along the cylindrical axis, ρ perpendicular to that axis and θ circumferential about it [from Jackson,[1] p. 367, eq. (8.117)]

$$E_z = E_0\, J_0(\rho\gamma) \cdot \exp\{i(k_z z - \omega t)\} ,$$

$$E_\rho = -E_0 \frac{ik_z}{\gamma} J_1(\rho\gamma) \cdot \exp\{i(k_z z - \omega t)\} ,$$

$$B_\phi = \frac{i\omega}{c\gamma} J_1(\rho\gamma) \cdot \exp\{i(k_z z - \omega t)\} .$$

(Note that we have exchanged E with B to obtain the TM case; the reference being for TE.)

In the above γ is not the relativistic parameter but:

$$\gamma^2 = \frac{\omega^2}{c^2} - k_z^2 ,$$

$$k_z = \frac{\omega}{c} \cos\theta , \tag{9.2}$$

thus

$$\gamma = \frac{\omega}{c} \sin\theta ,$$

where θ is the angle between the incoming plane parallel waves and the axis (as in Sec. 8). Thus

$$E_z = E_0\, J_0\left(\frac{\rho}{\lambda} \sin\theta\right) \cdot \exp\{i(k_z z - \omega t)\} ,$$

$$E_\rho = - E_0 \frac{i}{\sin\theta} J_1\left(\frac{\rho}{\lambda} \sin\theta\right) \cdot \exp\{i(k_z z - \omega t)\} , \tag{9.3}$$

$$c B_\phi = E_0 \frac{i}{\sin\theta} J_1\left(\frac{\rho}{\lambda} \sin\theta\right) \cdot \exp\{i(k_z z - \omega t)\} ,$$

where J_0 and J_1 are Bessel functions of the first kind.

As $\epsilon \to 0$ (Jackson,[1] p. 105, eq. 3.89)

$$J_0(\epsilon) \to 0 \quad ,$$
$$J_1(\epsilon) \to \frac{\epsilon}{2} \to 0 \quad . \tag{9.4}$$

So at $\rho = 0$ we obtain

$$E_z = E_0 \exp\{i(k_z z - \omega t)\} \ ,$$

$$E_\rho = B_\phi = 0 \ .$$

since $k_z = (\omega/c)\cos\theta$, continuous acceleration will only be obtained in the limit of $\theta \to 0$. In this limit, using Eq. (9.4):

$$E_z \to E_0 \exp\{i\phi_o\} \quad \text{(independent of } \rho) \ , \tag{9.6}$$

$$E_\rho \to -E_0\, i\, \frac{1}{\theta}\, \frac{\rho\theta}{2\lambda}\, \exp\{i\phi_o\} = -E_0\, i\, \frac{\rho}{\lambda}\, \exp\{i\phi_o\} \quad ,$$
$$c\, B_\phi \to E_0\, i\, \frac{1}{\theta}\, \frac{\rho\theta}{2\lambda}\, \exp\{i\phi_o\} = E_0\, i\, \frac{\rho}{\lambda}\, \exp\{i\phi_o\} \quad , \tag{9.7}$$

where $\phi_o = (\omega/c)(z_0 - ct_0)$ is the initial phase.

We note that as in Sec. 8 the limit is finite even though the terms $\to 0$. Again, as in Sec. 8, we can examine the accelerating field as the source is forced to be more distant.

If the source is at some large ρ, then the field at that source, $E_s \approx E_\rho(\rho) \gg E_z(\rho)$, and

$$\frac{E_{acc}}{E_s} \approx \frac{E_z(0)}{E_\rho(\rho)} = \frac{\lambda}{2\rho} \quad , \tag{9.8}$$

which, but for a factor of 2, is of the same form as Eqs. (8.6) and (7.15). So again we have a "*near*" field acceleration which falls, relative to source fields, as the wavelength divided by the distance to the source.

Returning to Eq. (9.6), we see that E_z is independent of ρ. It is a constant, despite the increasing radial field E_ρ. Since E_z is constant, it is clear that such a mode cannot exist in a simple circular waveguide. In fact, such an accelerating mode can only exist in structures that contain a dielectric (such as a dielectric loaded waveguide) or that are periodic (such as an iris loaded linac structure). See Figure 12.

Again looking at Eq. (9.6), we might expect that there is a focussing or defocussing force coming from the finite, but rising with ρ, radial field E_ρ. However, a radial force will also come

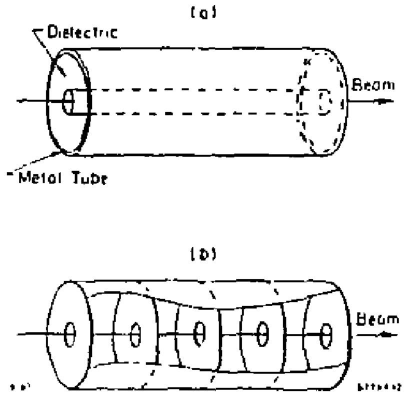

Figure 12

from the field B_ϕ and for a relativistic particle ($\beta \approx 1$)

$$F_\rho = e\,(E_\rho - B_\phi \beta c) \quad , \tag{9.9}$$

which from Eq. (9.6) is seen to give:

$$F_\rho \approx \frac{e\,E_z}{\gamma^2} \approx 0 \quad . \tag{9.10}$$

This conclusion is known as

Theorem 3. *Panofsky Wenzel Theorem.*

In any cylindrically symmetric system, wheth periodic, loaded or aperiodic, if we integrate from $-\infty$ to $+\infty$ (see Figure 13), any transverse deflection is proportional to $1/\gamma^2$ and is thus negligible for a sufficiently relativistic particle.

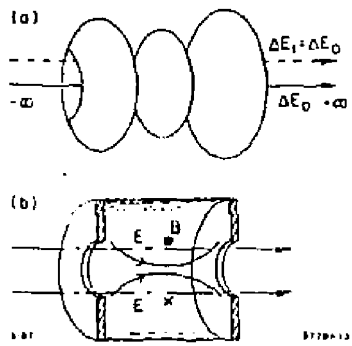

Figure 13

The generalization to any structure arises because, if integrated from $-\infty$ to $+\infty$ any mode that is not synchronous with the particle will vanish, and the only mode that is synchronous is that described by Eq. (9.6).

That the theorem is remarkable, is illustrated by Figure 13b. In a conventional accelerating cavity, it is clear that there are focussing E fields at the entry and defocussing fields at the exit. If the particle passes through as the phase ϕ changes sign, then these electric fields will give a net focussing force. By the theorem, these forces must be just cancelled by the magnetic fields which are zero on axis and rising on the radius. Since there is no very obvious connection between the circular B fields and the iris generated E fields, the cancellation is remarkable. It is nevertheless true!

Since the Panofsky Wenzel theorem is important, I will go through the basic derivation from Maxwell.

From Eq. (1.11)

$$\nabla^2 \bar{E} = -\frac{\omega^2}{c^2} \bar{E} \quad .$$

Writing this out for E_z in cylindrical coordinates (Jackson,[1] back cover):

$$\frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{dE_z}{d\rho} \right) + \frac{1}{\rho^2} \frac{d^2 E_z}{d\phi^2} + \frac{d^2 E_z}{dz^2} + \frac{\omega^2}{c^2} E_z = 0 \quad . \tag{9.11}$$

Given cylindrical symmetry

$$\frac{d^2 E_z}{d\phi^2} = 0 \quad . \tag{9.12}$$

For $\beta = 1$

$$\frac{d^2 E_z}{dz^2} = -k_z^2 E_z = -\omega^2 c^2 E_z \quad . \tag{9.13}$$

so

$$\frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{dE_z}{d\rho} \right) = 0 \quad , \tag{9.14}$$

and

$$\frac{dE_z}{d\rho} = \frac{const_1}{\rho} \quad , \tag{9.15}$$

$$E_z = const_2 + const_1 \, ln(\rho) \quad . \tag{9.16}$$

If $E_z \neq \infty$ at $\rho \to 0$, then $const_1 = 0$. Thus

$$E_z = const_2 \quad , \tag{9.17}$$

which is the same as Eq. (9.6) and is the precondition for Panosky Wenzel.

Now we go back to Maxwell Eq. (1.7)

$$\nabla \times \tilde{E} = i\omega\tilde{B} \quad .$$

Going again to the back cover of Jackson[1] and writing out the azimuthal cylindrical component:

$$\left(\frac{dE_\rho}{dz} - \frac{dE_z}{d\rho}\right) = i\omega\, B_\phi \quad . \tag{9.18}$$

From Eq. (9.17)

$$\frac{d\,E_z}{d\rho} = 0 \; ,$$

and for a wave propogating at c

$$\frac{dE_\rho}{dz} = i\,k_z\,E_\rho = i\,\frac{\omega}{c}\,E_\rho \; ,$$

so

$$i\omega\,B_\phi = i\,\frac{\omega}{c}\,E_\rho \quad . \tag{9.19}$$

Now the transverse force on a particle moving at $\beta = 1$ in the z direction:

$$F_\rho = e\,\left(E_\rho - c\,B_\phi\right) \quad ,$$

which from Eq. (9.19) gives

$$F_\rho = 0 \quad . \tag{9.20}$$

Wonderful!

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