AN INTRODUCTION TO DIFFERENTIAL GEOMETRY WITH APPLICATIONS TO ELASTICITY

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Contents

PREFACE

This book is based on lectures delivered over the years by the author at the Université Pierre et Marie Curie, Paris, at the University of Stuttgart, and at City University of Hong Kong. Its two-fold aim is to give thorough introductions to the basic theorems of differential geometry and to elasticity theory in curvilinear coordinates.

The treatment is essentially self-contained and proofs are complete. The prerequisites essentially consist in a working knowledge of basic notions of analysis and functional analysis, such as differential calculus, integration theory and Sobolev spaces, and some familiarity with ordinary and partial differential equations.

In particular, no *a priori* knowledge of differential geometry or of elasticity theory is assumed.

In the first chapter, we review the basic notions, such as the metric tensor and covariant derivatives, arising when a three-dimensional open set is equipped with curvilinear coordinates. We then prove that the vanishing of the Riemann curvature tensor is sufficient for the existence of isometric immersions from a simply-connected open subset of $\mathbb{R}^n$ equipped with a Riemannian metric into a Euclidean space of the same dimension. We also prove the corresponding uniqueness theorem, also called rigidity theorem.

In the second chapter, we study basic notions about surfaces, such as their two fundamental forms, the Gaussian curvature and covariant derivatives. We then prove the fundamental theorem of surface theory, which asserts that the Gauß and Codazzi-Mainardi equations constitute sufficient conditions for two matrix fields defined in a simply-connected open subset of $\mathbb{R}^2$ to be the two fundamental forms of a surface in a three-dimensional Euclidean space. We also prove the corresponding rigidity theorem.

In addition to such "classical" theorems, which constitute special cases of the fundamental theorem of Riemannian geometry, we also include in both chapters recent results which have not yet appeared in book form, such as the continuity of a surface as a function of its fundamental forms.

The third chapter, which heavily relies on Chapter 1, begins by a detailed derivation of the equations of nonlinear and linearized three-dimensional elasticity in terms of arbitrary curvilinear coordinates. This derivation is then followed by a detailed mathematical treatment of the existence, uniqueness, and regularity of solutions to the equations of linearized three-dimensional elasticity in curvilinear coordinates. This treatment includes in particular a direct proof of the three-dimensional Korn inequality in curvilinear coordinates.

The fourth and last chapter, which heavily relies on Chapter 2, begins by a detailed description of the nonlinear and linear equations proposed by W.T. Koiter for modeling thin elastic shells. These equations are "two-dimensional", in the sense that they are expressed in terms of two curvilinear coordinates used for defining the middle surface of the shell. The existence, uniqueness, and regularity of solutions to the linear Koiter equations is then established, thanks this time to a fundamental "Korn inequality on a surface" and to an "infinitesimal rigid displacement lemma on a surface". This chapter also includes a brief introduction to other two-dimensional shell equations.

Interestingly, notions that pertain to differential geometry *per se*, such as covariant derivatives of tensor fields, are also introduced in Chapters 3 and 4, where they appear most naturally in the derivation of the basic boundary value problems of three-dimensional elasticity and shell theory.

Occasionally, portions of the material covered here are adapted from excerpts from my book "Mathematical Elasticity, Volume III: Theory of Shells", published in 2000 by North-Holland, Amsterdam; in this respect, I am indebted to Arjen Sevenster for his kind permission to rely on such excerpts. Otherwise, the bulk of this work was substantially supported by two grants from the Research Grants Council of Hong Kong Special Administrative Region, China [Project No. 9040869, CityU 100803 and Project No. 9040966, CityU 100604].

Last but not least, I am greatly indebted to Roger Fosdick for his kind suggestion some years ago to write such a book, for his permanent support since then, and for his many valuable suggestions after he carefully read the entire manuscript.

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Chapter 1

THREE-DIMENSIONAL DIFFERENTIAL GEOMETRY

INTRODUCTION

Let Ω be an open subset of $\mathbb{R}^3$, let $\mathbf{E}^3$ denote a three-dimensional Euclidean space, and let $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$ be a smooth injective immersion. We begin by reviewing (Sections 1.1 to 1.3) basic definitions and properties arising when the three-dimensional open subset $\boldsymbol{\Theta}(\Omega)$ of $\mathbf{E}^3$ is equipped with the coordinates of the points of Ω as its *curvilinear coordinates*.

Of fundamental importance is the *metric tensor* of the set $\boldsymbol{\Theta}(\Omega)$, whose covariant and contravariant components $g_{ij} = g_{ji} : \Omega \to \mathbb{R}$ and $g^{ij} = g^{ji} : \Omega \to \mathbb{R}$ are given by (Latin indices or exponents take their values in $\{1, 2, 3\}$):

$$g_{ij} = \boldsymbol{g}_i \cdot \boldsymbol{g}_j \text{ and } g^{ij} = \boldsymbol{g}^i \cdot \boldsymbol{g}^j, \text{ where } \boldsymbol{g}_i = \partial_i \boldsymbol{\Theta} \text{ and } \boldsymbol{g}^j \cdot \boldsymbol{g}_i = \delta_i^j.$$

The vector fields $\boldsymbol{g}_i : \Omega \to \mathbb{R}^3$ and $\boldsymbol{g}^j : \Omega \to \mathbb{R}^3$ respectively form the *covariant*, and *contravariant, bases* in the set $\boldsymbol{\Theta}(\Omega)$.

It is shown in particular how *volumes*, *areas*, and *lengths*, in the set $\boldsymbol{\Theta}(\Omega)$ are computed in terms of its curvilinear coordinates, by means of the functions g_{ij} and g^{ij} (Theorem 1.3-1).

We next introduce in Section 1.4 the fundamental notion of *covariant derivatives $v_{i\|j}$ of a vector field $v_i\boldsymbol{g}^i : \Omega \to \mathbb{R}^3$* defined by means of its covariant components v_i over the contravariant bases $\boldsymbol{g}^i$. Covariant derivatives constitute a generalization of the usual partial derivatives of vector fields defined by means of their Cartesian components. As illustrated by the equations of nonlinear and linearized elasticity studied in Chapter 3, covariant derivatives naturally appear when a system of partial differential equations with a vector field as the unknown (the displacement field in elasticity) is expressed in terms of curvilinear coordinates.

It is a basic fact that the symmetric and positive-definite matrix field (g_{ij}) defined on Ω in this fashion cannot be arbitrary. More specifically (Theorem 1.5-1), its components and some of their partial derivatives must satisfy *necessary conditions* that take the form of the following relations (meant to hold for all $i, j, k, q \in \{1, 2, 3\}$): Let the functions Γ_{ijq} and Γ^p_{ij} be defined by

$$\Gamma_{ijq} = \frac{1}{2}(\partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij}) \text{ and } \Gamma^p_{ij} = g^{pq}\Gamma_{ijq}, \text{ where } (g^{pq}) = (g_{ij})^{-1}.$$

Then, necessarily,

$$\partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma^p_{ij}\Gamma_{kqp} - \Gamma^p_{ik}\Gamma_{jqp} = 0 \text{ in } \Omega.$$

The functions Γ_{ijq} and Γ^p_{ij} are the *Christoffel symbols of the first*, and *second, kind* and the functions

$$R_{qijk} = \partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma^p_{ij}\Gamma_{kqp} - \Gamma^p_{ik}\Gamma_{jqp}$$

are the covariant components of the *Riemann curvature tensor* of the set $\boldsymbol{\Theta}(\Omega)$.

We then focus our attention on the reciprocal questions:

Given an open subset Ω of $\mathbb{R}^3$ and a smooth enough symmetric and positive-definite matrix field (g_{ij}) defined on Ω, when is it the metric tensor field of an open set $\boldsymbol{\Theta}(\Omega) \subset \mathbf{E}^3$, i.e., when does there exist an immersion $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$ such that $g_{ij} = \partial_i \boldsymbol{\Theta} \cdot \partial_j \boldsymbol{\Theta}$ in Ω?

If such an immersion exists, to what extent is it unique?

As shown in Theorems 1.6-1 and 1.7-1, the answers turn out to be remarkably simple to state (but not so simple to prove, especially the first one!): *Under the assumption that Ω is simply-connected, the necessary conditions*

$$R_{qijk} = 0 \text{ in } \Omega$$

are also sufficient for the existence of such an immersion $\boldsymbol{\Theta}$.

Besides, *if Ω is connected, this immersion is unique up to isometries of* $\mathbf{E}^3$. This means that, if $\widetilde{\boldsymbol{\Theta}} : \Omega \to \mathbf{E}^3$ is any other smooth immersion satisfying

$$g_{ij} = \partial_i \widetilde{\boldsymbol{\Theta}} \cdot \partial_j \widetilde{\boldsymbol{\Theta}} \text{ in } \Omega,$$

there then exist a vector $\boldsymbol{c} \in \mathbf{E}^3$ and an orthogonal matrix Q of order three such that

$$\boldsymbol{\Theta}(x) = \boldsymbol{c} + \mathbf{Q}\widetilde{\boldsymbol{\Theta}}(x) \text{ for all } x \in \Omega.$$

Together, the above existence and uniqueness theorems constitute an important special case of the *fundamental theorem of Riemannian geometry* and as such, constitute the core of Chapter 1.

We conclude this chapter by showing (Theorem 1.8-5) that *the equivalence class of* $\boldsymbol{\Theta}$*, defined in this fashion* modulo *isometries of* $\mathbf{E}^3$*, depends continuously on the matrix field* (g_{ij}) *with respect to appropriate Fréchet topologies.*

1.1 CURVILINEAR COORDINATES

To begin with, we list some notations and conventions that will be consistently used throughout.

All spaces, matrices, etc., considered here are *real*.

Latin indices and exponents range in the set $\{1, 2, 3\}$, save when otherwise indicated, e.g., when they are used for indexing sequences, and the summation convention with respect to repeated indices or exponents is systematically used in conjunction with this rule. For instance, the relation

$$\boldsymbol{g}_i(x) = g_{ij}(x)\boldsymbol{g}^j(x)$$

means that

$$\boldsymbol{g}_i(x) = \sum_{j=1}^{3} g_{ij}(x)\boldsymbol{g}^j(x) \text{ for } i = 1, 2, 3.$$

Kronecker's symbols are designated by δ_i^j, δ_{ij}, or δ^{ij} according to the context.

Let $\mathbf{E}^3$ denote a *three-dimensional Euclidean space*, let $\boldsymbol{a} \cdot \boldsymbol{b}$ and $\boldsymbol{a} \wedge \boldsymbol{b}$ denote the Euclidean inner product and exterior product of $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{E}^3$, and let $|\boldsymbol{a}| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}}$ denote the Euclidean norm of $\boldsymbol{a} \in \mathbf{E}^3$. The space $\mathbf{E}^3$ is endowed with an *orthonormal basis* consisting of three vectors $\widehat{\boldsymbol{e}}^i = \widehat{\boldsymbol{e}}_i$. Let $\widehat{x}_i$ denote the *Cartesian coordinates* of a point $\widehat{x} \in \mathbf{E}^3$ and let $\widehat{\partial}_i := \partial/\partial\widehat{x}_i$.

In addition, let there be given a *three-dimensional vector space* in which three vectors $\boldsymbol{e}^i = \boldsymbol{e}_i$ form a basis. *This space will be identified with* $\mathbb{R}^3$. Let x_i denote the coordinates of a point $x \in \mathbb{R}^3$ and let $\partial_i := \partial/\partial x_i$, $\partial_{ij} := \partial^2/\partial x_i \partial x_j$, and $\partial_{ijk} := \partial^3/\partial x_i \partial x_j \partial x_k$.

Let there be given an *open* subset $\widehat{\Omega}$ of $\mathbf{E}^3$ and assume that there exist an *open* subset Ω of $\mathbb{R}^3$ and an *injective* mapping $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$ such that $\boldsymbol{\Theta}(\Omega) = \widehat{\Omega}$. Then each point $\widehat{x} \in \widehat{\Omega}$ can be unambiguously written as

$$\widehat{x} = \boldsymbol{\Theta}(x),\ x \in \Omega,$$

and the three coordinates x_i of x are called the **curvilinear coordinates** of $\widehat{x}$ (Figure 1.1-1). Naturally, *there are infinitely many ways of defining curvilinear coordinates in a given open set* $\widehat{\Omega}$, depending on how the open set Ω and the mapping $\boldsymbol{\Theta}$ are chosen!

Examples of curvilinear coordinates include the well-known *cylindrical* and *spherical coordinates* (Figure 1.1-2).

In a different, but equally important, approach, an open subset Ω of $\mathbb{R}^3$ together with a mapping $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$ are instead *a priori* given.

If $\boldsymbol{\Theta} \in \mathcal{C}^0(\Omega; \mathbf{E}^3)$ and $\boldsymbol{\Theta}$ is injective, the set $\widehat{\Omega} := \boldsymbol{\Theta}(\Omega)$ is *open* by the *invariance of domain theorem* (for a proof, see, e.g., Nirenberg [1974, Corollary 2, p. 17] or Zeidler [1986, Section 16.4]), and curvilinear coordinates inside $\widehat{\Omega}$ are unambiguously defined in this case.

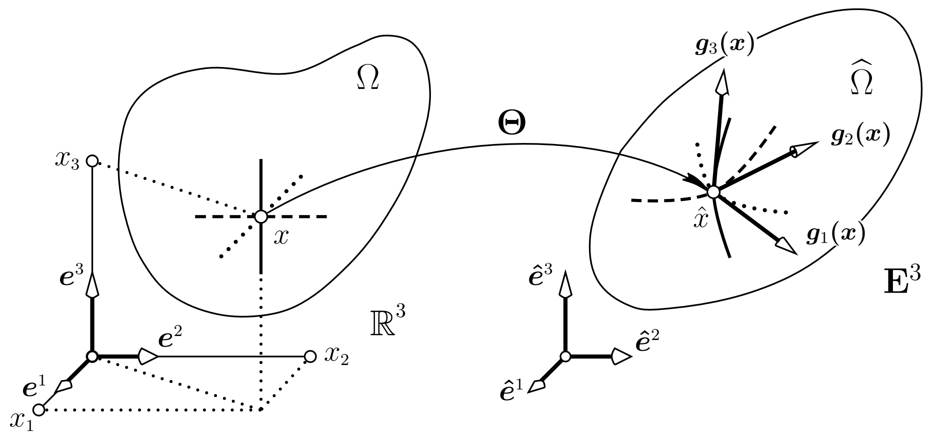

Figure 1.1-1: *Curvilinear coordinates and covariant bases in an open set* $\widehat{\Omega} \subset \mathbf{E}^3$. The three coordinates x_1, x_2, x_3 of $x \in \Omega$ are the curvilinear coordinates of $\widehat{x} = \boldsymbol{\Theta}(x) \in \widehat{\Omega}$. If the three vectors $\boldsymbol{g}_i(x) = \partial_i \boldsymbol{\Theta}(x)$ are linearly independent, they form the covariant basis at $\widehat{x} = \boldsymbol{\Theta}(x)$ and they are tangent to the coordinate lines passing through $\widehat{x}$.

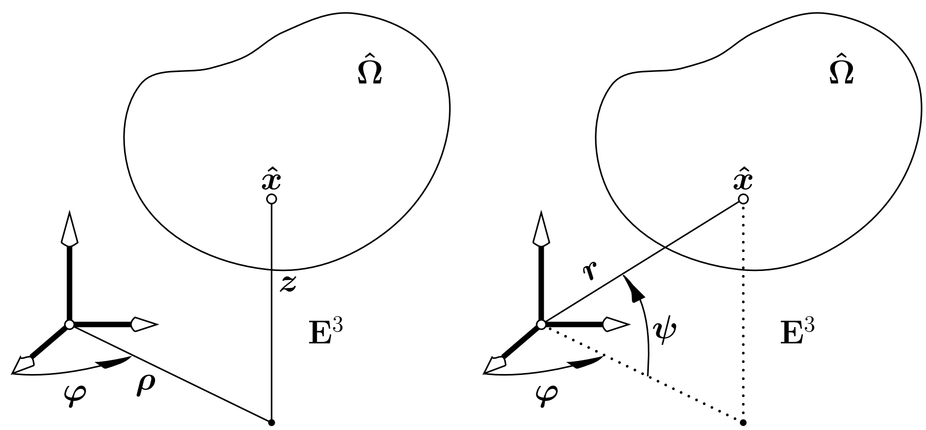

Figure 1.1-2: *Two familiar examples of curvilinear coordinates.* Let the mapping $\boldsymbol{\Theta}$ be defined by

$$\boldsymbol{\Theta} : (\varphi, \rho, z) \in \Omega \to (\rho \cos\varphi, \rho \sin\varphi, z) \in \mathbf{E}^3.$$

Then (φ, ρ, z) are the cylindrical coordinates of $\widehat{x} = \boldsymbol{\Theta}(\varphi, \rho, z)$. Note that $(\varphi + 2k\pi, \rho, z)$ or $(\varphi + \pi + 2k\pi, -\rho, z)$, $k \in \mathbb{Z}$, are also cylindrical coordinates of the same point $\widehat{x}$ and that φ is not defined if $\widehat{x}$ is the origin of $\mathbf{E}^3$.

Let the mapping $\boldsymbol{\Theta}$ be defined by

$$\boldsymbol{\Theta} : (\varphi, \psi, r) \in \Omega \to (r \cos\psi \cos\varphi, r \cos\psi \sin\varphi, r \sin\psi) \in \mathbf{E}^3.$$

Then (φ, ψ, r) are the spherical coordinates of $\widehat{x} = \boldsymbol{\Theta}(\varphi, \psi, r)$. Note that $(\varphi + 2k\pi, \psi + 2\ell\pi, r)$ or $(\varphi + 2k\pi, \psi + \pi + 2\ell\pi, -r)$ are also spherical coordinates of the same point $\widehat{x}$ and that φ and ψ are not defined if $\widehat{x}$ is the origin of $\mathbf{E}^3$.

In both cases, the covariant basis at $\widehat{x}$ and the coordinate lines are represented with self-explanatory notations.

If $\boldsymbol{\Theta} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ and the three vectors $\partial_i \boldsymbol{\Theta}(x)$ are linearly independent at all $x \in \Omega$, the set $\widehat{\Omega}$ is again *open* (for a proof, see, e.g., Schwartz [1992] or Zeidler [1986, Section 16.4]), but curvilinear coordinates may be defined only locally in this case: Given $x \in \Omega$, all that can be asserted (by the local inversion theorem) is the existence of an open neighborhood V of x in Ω such that the restriction of $\boldsymbol{\Theta}$ to V is a $\mathcal{C}^1$-diffeomorphism, hence an injection, of V onto $\boldsymbol{\Theta}(V)$.

1.2 METRIC TENSOR

Let Ω be an open subset of $\mathbb{R}^3$ and let

$$\boldsymbol{\Theta} = \Theta_i \widehat{\boldsymbol{e}}^i : \Omega \to \mathbf{E}^3$$

be a mapping that is *differentiable at a point* $x \in \Omega$. If $\boldsymbol{\delta x}$ is such that $(x + \boldsymbol{\delta x}) \in \Omega$, then

$$\boldsymbol{\Theta}(x + \boldsymbol{\delta} x) = \boldsymbol{\Theta}(x) + \boldsymbol{\nabla\Theta}(x)\boldsymbol{\delta x} + o(\boldsymbol{\delta x}),$$

where the 3×3 matrix $\boldsymbol{\nabla\Theta}(x)$ and the column vector $\boldsymbol{\delta x}$ are defined by

$$\boldsymbol{\nabla\Theta}(x) := \begin{pmatrix} \partial_1\Theta_1 & \partial_2\Theta_1 & \partial_3\Theta_1 \\ \partial_1\Theta_2 & \partial_2\Theta_2 & \partial_3\Theta_2 \\ \partial_1\Theta_3 & \partial_2\Theta_3 & \partial_3\Theta_3 \end{pmatrix}(x) \text{ and } \boldsymbol{\delta x} = \begin{pmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{pmatrix}.$$

Let the three vectors $\boldsymbol{g}_i(x) \in \mathbb{R}^3$ be defined by

$$\boldsymbol{g}_i(x) := \partial_i \boldsymbol{\Theta}(x) = \begin{pmatrix} \partial_i\Theta_1 \\ \partial_i\Theta_2 \\ \partial_i\Theta_3 \end{pmatrix}(x),$$

i.e., $\boldsymbol{g}_i(x)$ *is the* i-th *column vector of the matrix* $\boldsymbol{\nabla\Theta}(x)$. Then the expansion of $\boldsymbol{\Theta}$ about x may be also written as

$$\boldsymbol{\Theta}(x + \boldsymbol{\delta x}) = \boldsymbol{\Theta}(x) + \delta x^i \boldsymbol{g}_i(x) + o(\boldsymbol{\delta x}).$$

If in particular $\boldsymbol{\delta x}$ is of the form $\boldsymbol{\delta x} = \delta t \boldsymbol{e}_i$, where $\delta t \in \mathbb{R}$ and $\boldsymbol{e}_i$ is one of the basis vectors in $\mathbb{R}^3$, this relation reduces to

$$\boldsymbol{\Theta}(x + \delta t \boldsymbol{e}_i) = \boldsymbol{\Theta}(x) + \delta t \boldsymbol{g}_i(x) + o(\delta t).$$

A mapping $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$ is an **immersion at** $x \in \Omega$ if it is differentiable at x and the matrix $\boldsymbol{\nabla\Theta}(x)$ is invertible or, equivalently, if the three vectors $\boldsymbol{g}_i(x) = \partial_i \boldsymbol{\Theta}(x)$ are linearly independent.

Assume from now on in this section that *the mapping* $\boldsymbol{\Theta}$ *is an immersion at* x. Then *the three vectors* $\boldsymbol{g}_i(x)$ constitute the **covariant basis** at the point $\widehat{x} = \boldsymbol{\Theta}(x)$.

In this case, the last relation thus shows that *each vector* $\boldsymbol{g}_i(x)$ *is tangent to the* i-th **coordinate line** *passing through* $\widehat{x} = \boldsymbol{\Theta}(x)$, defined as the image by $\boldsymbol{\Theta}$ of the points of Ω that lie on the line parallel to $\boldsymbol{e}_i$ passing through x

(there exist t_0 and t_1 with $t_0 < 0 < t_1$ such that the i-th coordinate line is given by $t \in \,]t_0, t_1[\, \to \boldsymbol{f}_i(t) := \boldsymbol{\Theta}(x + t\boldsymbol{e}_i)$ in a neighborhood of $\widehat{x}$; hence $\boldsymbol{f}_i'(0) = \partial_i \boldsymbol{\Theta}(x) = \boldsymbol{g}_i(x)$); see Figures 1.1-1 and 1.1-2.

Returning to a general increment $\boldsymbol{\delta x} = \delta x^i \boldsymbol{e}_i$, we also infer from the expansion of $\boldsymbol{\Theta}$ about x that (recall that we use the summation convention):

$$\begin{aligned}|\boldsymbol{\Theta}(x + \boldsymbol{\delta x}) - \boldsymbol{\Theta}(x)|^2 &= \boldsymbol{\delta x}^T \boldsymbol{\nabla\Theta}(x)^T \boldsymbol{\nabla\Theta}(x) \boldsymbol{\delta x} + o\big(|\boldsymbol{\delta x}|^2\big)\\ &= \delta x^i \boldsymbol{g}_i(x) \cdot \boldsymbol{g}_j(x) \delta x^j + o\big(|\boldsymbol{\delta x}|^2\big).\end{aligned}$$

Note that, here and subsequently, we use standard notations from matrix algebra. For instance, $\boldsymbol{\delta x}^T$ stands for the transpose of the column vector $\boldsymbol{\delta x}$ and $\boldsymbol{\nabla\Theta}(x)^T$ designates the transpose of the matrix $\boldsymbol{\nabla\Theta}(x)$, the element at the i-th row and j-th column of a matrix $\mathbf{A}$ is noted $(\mathbf{A})_{ij}$, etc.

In other words, the principal part with respect to $\boldsymbol{\delta x}$ of the length between the points $\boldsymbol{\Theta}(x + \boldsymbol{\delta x})$ and $\boldsymbol{\Theta}(x)$ is $\{\delta x^i \boldsymbol{g}_i(x) \cdot \boldsymbol{g}_j(x) \delta x^j\}^{1/2}$. This observation suggests to define a matrix $(g_{ij}(x))$ of order three, by letting

$$g_{ij}(x) := \boldsymbol{g}_i(x) \cdot \boldsymbol{g}_j(x) = (\boldsymbol{\nabla\Theta}(x)^T \boldsymbol{\nabla\Theta}(x))_{ij}.$$

The elements $g_{ij}(x)$ of this symmetric matrix are called the **covariant components of the metric tensor** at $\widehat{x} = \boldsymbol{\Theta}(x)$.

Note that *the matrix* $\boldsymbol{\nabla\Theta}(x)$ *is invertible* and that *the matrix* $(g_{ij}(x))$ *is positive definite*, since the vectors $\boldsymbol{g}_i(x)$ are assumed to be linearly independent.

The three vectors $\boldsymbol{g}_i(x)$ being linearly independent, *the nine relations*

$$\boldsymbol{g}^i(x) \cdot \boldsymbol{g}_j(x) = \delta^i_j$$

unambiguously define three linearly independent vectors $\boldsymbol{g}^i(x)$. To see this, let *a priori* $\boldsymbol{g}^i(x) = X^{ik}(x)\boldsymbol{g}_k(x)$ in the relations $\boldsymbol{g}^i(x) \cdot \boldsymbol{g}_j(x) = \delta^i_j$. This gives $X^{ik}(x) g_{kj}(x) = \delta^i_j$; consequently, $X^{ik}(x) = g^{ik}(x)$, where

$$(g^{ij}(x)) := (g_{ij}(x))^{-1}.$$

Hence $\boldsymbol{g}^i(x) = g^{ik}(x)\boldsymbol{g}_k(x)$. These relations in turn imply that

$$\begin{aligned}\boldsymbol{g}^i(x) \cdot \boldsymbol{g}^j(x) &= \big(g^{ik}(x)\boldsymbol{g}_k(x)\big) \cdot \big(g^{j\ell}(x)\boldsymbol{g}_\ell(x)\big)\\ &= g^{ik}(x) g^{j\ell}(x) g_{k\ell}(x) \;=\; g^{ik}(x)\delta^j_k \;=\; g^{ij}(x),\end{aligned}$$

and thus the vectors $\boldsymbol{g}^i(x)$ are *linearly independent* since the matrix $(g^{ij}(x))$ is positive definite. We would likewise establish that $\boldsymbol{g}_i(x) = g_{ij}(x)\boldsymbol{g}^j(x)$.

The three vectors $\boldsymbol{g}^i(x)$ form the **contravariant basis** at the point $\widehat{x} = \boldsymbol{\Theta}(x)$ and the elements $g^{ij}(x)$ of the symmetric positive definite matrix $(g^{ij}(x))$ are the **contravariant components of the metric tensor** at $\widehat{x} = \boldsymbol{\Theta}(x)$.

Let us record for convenience the fundamental relations that exist between the vectors of the covariant and contravariant bases and the covariant and contravariant components of the metric tensor at a point $x \in \Omega$ where the mapping $\boldsymbol{\Theta}$ is an immersion:

$$\begin{aligned}g_{ij}(x) = \boldsymbol{g}_i(x) \cdot \boldsymbol{g}_j(x) \quad &\text{and} \quad g^{ij}(x) = \boldsymbol{g}^i(x) \cdot \boldsymbol{g}^j(x),\\ \boldsymbol{g}_i(x) = g_{ij}(x)\boldsymbol{g}^j(x) \quad &\text{and} \quad \boldsymbol{g}^i(x) = g^{ij}(x)\boldsymbol{g}_j(x).\end{aligned}$$

A mapping $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$ is an **immersion** if it is an immersion at each point in Ω, i.e., if $\boldsymbol{\Theta}$ is differentiable in Ω and the three vectors $\boldsymbol{g}_i(x) = \partial_i \boldsymbol{\Theta}(x)$ are linearly independent at each $x \in \Omega$.

If $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$ is an immersion, the vector fields $\boldsymbol{g}_i : \Omega \to \mathbb{R}^3$ and $\boldsymbol{g}^i : \Omega \to \mathbb{R}^3$ respectively form the **covariant**, and **contravariant bases**.

To conclude this section, we briefly explain in what sense the components of the "metric tensor" may be *"covariant"* or *"contravariant"*.

Let Ω and $\widetilde{\Omega}$ be two domains in $\mathbb{R}^3$ and let $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$ and $\widetilde{\boldsymbol{\Theta}} : \widetilde{\Omega} \to \mathbf{E}^3$ be two $\mathcal{C}^1$-diffeomorphisms such that $\boldsymbol{\Theta}(\Omega) = \widetilde{\boldsymbol{\Theta}}(\widetilde{\Omega})$ and such that the vectors $\boldsymbol{g}_i(x) := \partial_i \boldsymbol{\Theta}(x)$ and $\widetilde{\boldsymbol{g}}_i(\widetilde{x}) = \widetilde{\partial}_i \widetilde{\boldsymbol{\Theta}}(\widetilde{x})$ of the covariant bases at the *same* point $\boldsymbol{\Theta}(x) = \widetilde{\boldsymbol{\Theta}}(\widetilde{x}) \in \mathbf{E}^3$ are linearly independent. Let $\boldsymbol{g}^i(x)$ and $\widetilde{\boldsymbol{g}}^i(\widetilde{x})$ be the vectors of the corresponding contravariant bases at the same point $\widehat{x}$. A simple computation then shows that

$$\boldsymbol{g}_i(x) = \frac{\partial \chi^j}{\partial x_i}(x)\widetilde{\boldsymbol{g}}_j(\widetilde{x}) \text{ and } \boldsymbol{g}^i(x) = \frac{\partial \widetilde{\chi}^i}{\partial \widetilde{x}_j}(\widetilde{x})\widetilde{\boldsymbol{g}}^j(\widetilde{x}),$$

where $\boldsymbol{\chi} = (\chi^j) := \widetilde{\boldsymbol{\Theta}}^{-1} \circ \boldsymbol{\Theta} \in \mathcal{C}^1(\Omega; \widetilde{\Omega})$ (hence $\widetilde{x} = \boldsymbol{\chi}(x)$) and $\widetilde{\boldsymbol{\chi}} = (\widetilde{\chi}^i) := \boldsymbol{\chi}^{-1} \in \mathcal{C}^1(\widetilde{\Omega}; \Omega)$.

Let $g_{ij}(x)$ and $\widetilde{g}_{ij}(\widetilde{x})$ be the covariant components, and let $g^{ij}(x)$ and $\widetilde{g}^{ij}(\widetilde{x})$ be the contravariant components, of the metric tensor at the *same* point $\boldsymbol{\Theta}(x) = \widetilde{\boldsymbol{\Theta}}(\widetilde{x}) \in \mathbf{E}^3$. Then a simple computation shows that

$$g_{ij}(x) = \frac{\partial \chi^k}{\partial x_i}(x)\frac{\partial \chi^\ell}{\partial x_j}(x)\widetilde{g}_{k\ell}(\widetilde{x}) \text{ and } g^{ij}(x) = \frac{\partial \widetilde{\chi}^i}{\partial \widetilde{x}_k}(\widetilde{x})\frac{\partial \widetilde{\chi}^j}{\partial \widetilde{x}_\ell}(\widetilde{x})\widetilde{g}^{k\ell}(\widetilde{x}).$$

These formulas explain why the components $g_{ij}(x)$ and $g^{ij}(x)$ are respectively called "covariant" and "contravariant": *Each* index in $g_{ij}(x)$ "varies like" that of the corresponding vector of the *covariant* basis *under a change of curvilinear coordinates*, while *each* exponent in $g^{ij}(x)$ "varies like" that of the corresponding vector of the *contravariant* basis.

Remark. What is exactly the "second-order tensor" hidden behind its covariant components $g_{ij}(x)$ or its contravariant exponents $g^{ij}(x)$ is beautifully explained in the gentle introduction to tensors given by Antman [1995, Chapter 11, Sections 1 to 3]; it is also shown in *ibid.* that the same "tensor" also has "mixed" components $g^i_j(x)$, which turn out to be simply the Kronecker symbols δ^i_j. □

In fact, analogous justifications apply as well to the components of all the other "tensors" that will be introduced later on. Thus, for instance, the covariant components $v_i(x)$ and $\widetilde{v}_i(x)$, and the contravariant components $v^i(x)$ and $\widetilde{v}^i(x)$ (both with self-explanatory notations), of a vector at the *same* point $\boldsymbol{\Theta}(x) = \widetilde{\boldsymbol{\Theta}}(\widetilde{x})$ satisfy (cf. Section 1.4)

$$v_i(x)\boldsymbol{g}^i(x) = \widetilde{v}_i(\widetilde{x})\widetilde{\boldsymbol{g}}^i(\widetilde{x}) = v^i(x)\boldsymbol{g}_i(x) = \widetilde{v}^i(\widetilde{x})\widetilde{\boldsymbol{g}}_i(\widetilde{x}).$$

It is then easily verified that

$$v_i(x) = \frac{\partial \chi^j}{\partial x_i}(x)\widetilde{v}_j(\widetilde{x}) \text{ and } v^i(x) = \frac{\partial \widetilde{\chi}^i}{\partial \widetilde{x}_j}(\widetilde{x})\widetilde{v}^j(\widetilde{x}).$$

In other words, the components $v_i(x)$ "vary like" the vectors $\boldsymbol{g}_i(x)$ of the *covariant* basis *under a change of curvilinear coordinates*, while the components $v^i(x)$ of a vector "vary like" the vectors $\boldsymbol{g}^i(x)$ of the *contravariant* basis. This is why they are respectively called *"covariant"* and *"contravariant"*. A vector is an example of a "first-order" tensor.

Likewise, it is easily checked that each exponent in the "contravariant" components $A^{ijk\ell}(x)$ of the three-dimensional elasticity tensor in curvilinear coordinates introduced in Section 3.4 again "varies like" that of the corresponding vector of the *contravariant* basis under a change of curvilinear coordinates.

Remark. See again Antman [1995, Chapter 11, Sections 1 to 3] to decipher the *"fourth-order tensor"* hidden behind such contravariant components $A^{ijk\ell}(x)$. □

Note, however, that we shall no longer provide such commentaries in the sequel. We leave it instead to the reader to verify in each instance that any index or exponent appearing in a component of a "tensor" indeed behaves according to its nature.

The reader interested by such questions will find exhaustive treatments of tensor analysis, particularly as regards its relevance to elasticity, in Boothby [1975], Marsden & Hughes [1983, Chapter 1], or Simmonds [1994].

1.3 VOLUMES, AREAS, AND LENGTHS IN CURVILINEAR COORDINATES

We now review fundamental formulas showing how *volume, area*, and *length elements* at a point $\widehat{x} = \boldsymbol{\Theta}(x)$ in the set $\widehat{\Omega} = \boldsymbol{\Theta}(\Omega)$ can be expressed in terms of the matrices $\boldsymbol{\nabla\Theta}(x)$, $(g_{ij}(x))$, and matrix $(g^{ij}(x))$.

These formulas thus highlight the crucial rôle played by the matrix $(g_{ij}(x))$ for computing "metric" notions at the point $\widehat{x} = \boldsymbol{\Theta}(x)$. Indeed, the "metric tensor" well deserves its name!

A **domain** in $\mathbb{R}^d, d \geq 2$, is a bounded, open, and connected subset D of $\mathbb{R}^d$ with a Lipschitz-continuous boundary, the set D being locally on one side of its boundary. All relevant details needed here about domains are found in Nečas [1967] or Adams [1975].

Given a domain $D \subset \mathbb{R}^3$ with boundary Γ, we let $\mathrm{d}x$ denote the *volume element* in D, $\mathrm{d}\Gamma$ denote the *area element* along Γ, and $\boldsymbol{n} = n_i\widehat{\boldsymbol{e}}^i$ denote the *unit* ($|\boldsymbol{n}| = 1$) *outer normal vector* along Γ ($\mathrm{d}\Gamma$ is well defined and $\boldsymbol{n}$ is defined $\mathrm{d}\Gamma$-almost everywhere since Γ is assumed to be Lipschitz-continuous).

Note also that the assumptions made on the mapping $\boldsymbol{\Theta}$ in the next theorem guarantee that, if D is a domain in $\mathbb{R}^3$ such that $\overline{D} \subset \Omega$, then $\{\widehat{D}\}^- \subset \widehat{\Omega}$,

$\{\boldsymbol{\Theta}(D)\}^- = \boldsymbol{\Theta}(\overline{D})$, and the boundaries $\partial\widehat{D}$ of $\widehat{D}$ and ∂D of D are related by $\partial\widehat{D} = \boldsymbol{\Theta}(\partial D)$ (see, e.g., Ciarlet [1988, Theorem 1.2-8 and Example 1.7]).

If $\mathbf{A}$ is a square matrix, $\mathbf{Cof}\,\mathbf{A}$ denotes the *cofactor matrix* of $\mathbf{A}$. Thus $\mathbf{Cof}\,\mathbf{A} = (\det\mathbf{A})\mathbf{A}^{-T}$ if $\mathbf{A}$ is invertible.

Theorem 1.3-1. *Let Ω be an open subset of $\mathbb{R}^3$, let $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$ be an injective and smooth enough immersion, and let $\widehat{\Omega} = \boldsymbol{\Theta}(\Omega)$.*

(a) *The volume element $\mathrm{d}\widehat{x}$ at $\widehat{x} = \boldsymbol{\Theta}(x) \in \widehat{\Omega}$ is given in terms of the volume element $\mathrm{d}x$ at $x \in \Omega$ by*

$$\mathrm{d}\widehat{x} = |\det\boldsymbol{\nabla}\boldsymbol{\Theta}(x)|\,\mathrm{d}x = \sqrt{g(x)}\,\mathrm{d}x, \text{ where } g(x) := \det(g_{ij}(x)).$$

(b) *Let D be a domain in $\mathbb{R}^3$ such that $\overline{D} \subset \Omega$. The area element $\mathrm{d}\widehat{\Gamma}(\widehat{x})$ at $\widehat{x} = \boldsymbol{\Theta}(x) \in \partial\widehat{D}$ is given in terms of the area element $\mathrm{d}\Gamma(x)$ at $x \in \partial D$ by*

$$\mathrm{d}\widehat{\Gamma}(\widehat{x}) = |\,\mathbf{Cof}\,\boldsymbol{\nabla}\boldsymbol{\Theta}(x)\boldsymbol{n}(x)|\,\mathrm{d}\Gamma(x) = \sqrt{g(x)}\sqrt{n_i(x)g^{ij}(x)n_j(x)}\,\mathrm{d}\Gamma(x),$$

where $\boldsymbol{n}(x) := n_i(x)\boldsymbol{e}^i$ denotes the unit outer normal vector at $x \in \partial D$.

(c) *The length element $\mathrm{d}\widehat{\ell}(\widehat{x})$ at $\widehat{x} = \boldsymbol{\Theta}(x) \in \widehat{\Omega}$ is given by*

$$\mathrm{d}\widehat{\ell}(\widehat{x}) = \left\{\boldsymbol{\delta x}^T\boldsymbol{\nabla}\boldsymbol{\Theta}(x)^T\boldsymbol{\nabla}\boldsymbol{\Theta}(x)\boldsymbol{\delta x}\right\}^{1/2} = \left\{\delta x^i g_{ij}(x)\delta x^j\right\}^{1/2},$$

where $\boldsymbol{\delta x} = \delta x^i\boldsymbol{e}_i$.

Proof. The relation $\mathrm{d}\widehat{x} = |\det\boldsymbol{\nabla}\boldsymbol{\Theta}(x)|\,\mathrm{d}x$ between the volume elements is well known. The second relation in (a) follows from the relation $g(x) = |\det\boldsymbol{\nabla}\boldsymbol{\Theta}(x)|^2$, which itself follows from the relation $(g_{ij}(x)) = \boldsymbol{\nabla}\boldsymbol{\Theta}(x)^T\boldsymbol{\nabla}\boldsymbol{\Theta}(x)$.

Indications about the proof of the relation between the area elements $\mathrm{d}\widehat{\Gamma}(\widehat{x})$ and $\mathrm{d}\Gamma(x)$ given in (b) are found in Ciarlet [1988, Theorem 1.7-1] (in this formula, $\boldsymbol{n}(x) = n_i(x)\boldsymbol{e}^i$ is identified with the column vector in $\mathbb{R}^3$ with $n_i(x)$ as its components). Using the relations $\mathbf{Cof}(\mathbf{A}^T) = (\mathbf{Cof}\,\mathbf{A})^T$ and $\mathbf{Cof}(\mathbf{AB}) = (\mathbf{Cof}\,\mathbf{A})(\mathbf{Cof}\,\mathbf{B})$, we next have:

$$\begin{aligned}|\,\mathbf{Cof}\,\boldsymbol{\nabla}\boldsymbol{\Theta}(x)\boldsymbol{n}(x)|^2 &= \boldsymbol{n}(x)^T\,\mathbf{Cof}\left(\boldsymbol{\nabla}\boldsymbol{\Theta}(x)^T\boldsymbol{\nabla}\boldsymbol{\Theta}(x)\right)\boldsymbol{n}(x)\\ &= g(x)n_i(x)g^{ij}(x)n_j(x).\end{aligned}$$

Either expression of the length element given in (c) recalls that $\mathrm{d}\widehat{\ell}(\widehat{x})$ is by definition the principal part with respect to $\boldsymbol{\delta x} = \delta x^i\boldsymbol{e}_i$ of the length $|\boldsymbol{\Theta}(x+\boldsymbol{\delta x}) - \boldsymbol{\Theta}(x)|$, whose expression precisely led to the introduction of the matrix $(g_{ij}(x))$ in Section 1.2. ☐

The relations found in Theorem 1.3-1 are used in particular for computing *volumes, areas,* and *lengths* inside $\widehat{\Omega}$ by means of integrals inside Ω, i.e., in terms of the *curvilinear coordinates* used in the open set $\widehat{\Omega}$ (Figure 1.3-1):

Let D be a domain in $\mathbb{R}^3$ such that $\overline{D} \subset \Omega$, let $\widehat{D} := \boldsymbol{\Theta}(D)$, and let $\widehat{f} \in L^1(\widehat{D})$ be given. Then

$$\int_{\widehat{D}} \widehat{f}(\widehat{x})\,\mathrm{d}\widehat{x} = \int_D (\widehat{f}\circ\boldsymbol{\Theta})(x)\sqrt{g(x)}\,\mathrm{d}x.$$

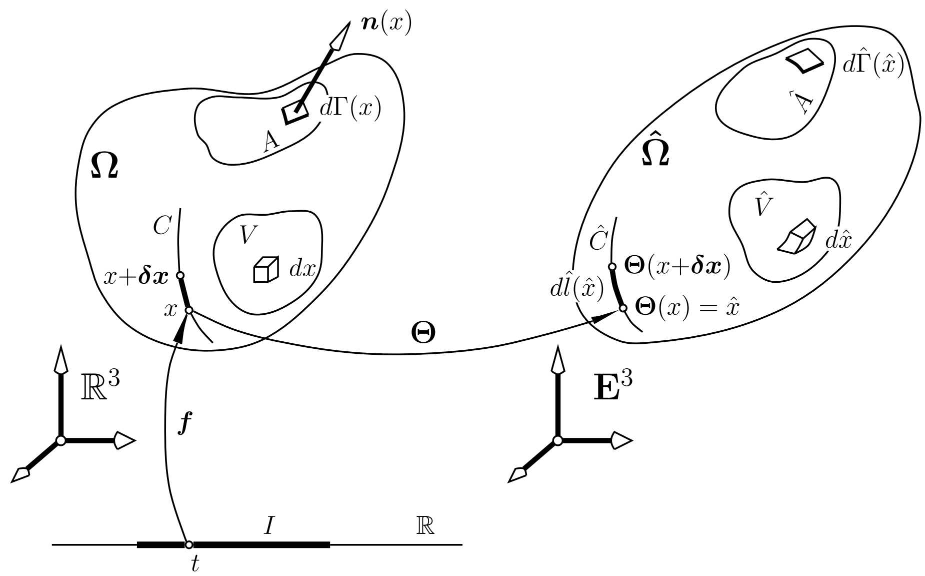

Figure 1.3-1: *Volume, area, and length elements in curvilinear coordinates.* The elements $\mathrm{d}\widehat{x}, \mathrm{d}\widehat{\Gamma}(\widehat{x})$, and $\mathrm{d}\widehat{\ell}(\widehat{x})$ at $\widehat{x} = \boldsymbol{\Theta}(x) \in \widehat{\Omega}$ are expressed in terms of $\mathrm{d}x$, $\mathrm{d}\Gamma(x)$, and $\boldsymbol{\delta x}$ at $x \in \Omega$ by means of the covariant and contravariant components of the metric tensor; cf. Theorem 1.3-1. Given a domain D such that $\overline{D} \subset \Omega$ and a $\mathrm{d}\Gamma$-measurable subset Σ of ∂D, the corresponding relations are used for computing the volume of $\widehat{D} = \boldsymbol{\Theta}(D) \subset \widehat{\Omega}$, the area of $\widehat{\Sigma} = \boldsymbol{\Theta}(\Sigma) \subset \partial\widehat{D}$, and the length of a curve $\widehat{C} = \boldsymbol{\Theta}(C) \subset \widehat{\Omega}$, where $C = \boldsymbol{f}(I)$ and I is a compact interval of $\mathbb{R}$.

In particular, the *volume* of $\widehat{D}$ is given by

$$\mathit{vol}\,\widehat{D} := \int_{\widehat{D}} \mathrm{d}\widehat{x} = \int_D \sqrt{g(x)}\,\mathrm{d}x.$$

Next, let $\Gamma := \partial D$, let Σ be a $\mathrm{d}\Gamma$-measurable subset of Γ, let $\widehat{\Sigma} := \boldsymbol{\Theta}(\Sigma) \subset \partial\widehat{D}$, and let $\widehat{h} \in L^1(\widehat{\Sigma})$ be given. Then

$$\int_{\widehat{\Sigma}} \widehat{h}(\widehat{x})\,\mathrm{d}\widehat{\Gamma}(\widehat{x}) = \int_{\Sigma} (\widehat{h} \circ \boldsymbol{\Theta})(x)\sqrt{g(x)}\sqrt{n_i(x)g^{ij}(x)n_j(x)}\,\mathrm{d}\Gamma(x).$$

In particular, the *area* of $\widehat{\Sigma}$ is given by

$$\mathit{area}\,\widehat{\Sigma} := \int_{\widehat{\Sigma}} \mathrm{d}\widehat{\Gamma}(\widehat{x}) = \int_{\Sigma} \sqrt{g(x)}\sqrt{n_i(x)g^{ij}(x)n_j(x)}\,\mathrm{d}\Gamma(x).$$

Finally, consider a curve $C = \boldsymbol{f}(I)$ in Ω, where I is a compact interval of $\mathbb{R}$ and $\boldsymbol{f} = f^i\boldsymbol{e}_i : I \to \Omega$ is a smooth enough injective mapping. Then the *length* of the curve $\widehat{C} := \boldsymbol{\Theta}(C) \subset \widehat{\Omega}$ is given by

$$\mathit{length}\,\widehat{C} := \int_I \Big|\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\Theta} \circ \boldsymbol{f})(t)\Big|\,\mathrm{d}t = \int_I \sqrt{g_{ij}(\boldsymbol{f}(t))\frac{\mathrm{d}f^i}{\mathrm{d}t}(t)\frac{\mathrm{d}f^j}{\mathrm{d}t}(t)}\,\mathrm{d}t.$$

This relation shows in particular that *the lengths of curves inside the open set $\boldsymbol{\Theta}(\Omega)$ are precisely those induced by the Euclidean metric of the space* $\mathbf{E}^3$. For this reason, the set $\boldsymbol{\Theta}(\Omega)$ is said to be **isometrically imbedded** in $\mathbf{E}^3$.

1.4 COVARIANT DERIVATIVES OF A VECTOR FIELD

Suppose that a *vector field* is defined in an open subset $\widehat{\Omega}$ of $\mathbf{E}^3$ by means of its *Cartesian components* $\widehat{v}_i : \widehat{\Omega} \to \mathbb{R}$, i.e., this field is defined by its values $\widehat{v}_i(\widehat{x})\widehat{\boldsymbol{e}}^i$ at each $\widehat{x} \in \widehat{\Omega}$, where the vectors $\widehat{\boldsymbol{e}}^i$ constitute the orthonormal basis of $\mathbf{E}^3$; see Figure 1.4-1.

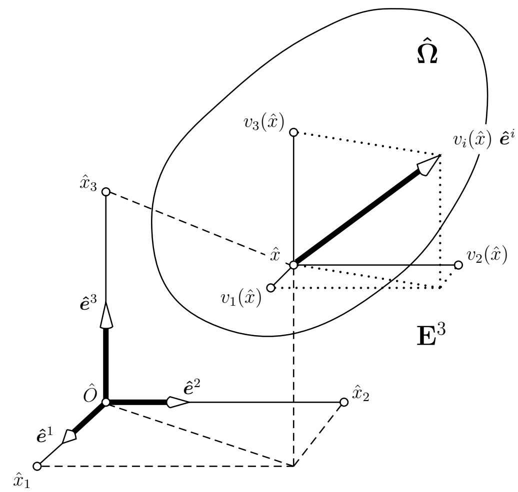

Figure 1.4-1: *A vector field in Cartesian coordinates.* At each point $\widehat{x} \in \widehat{\Omega}$, the vector $\widehat{v}_i(\widehat{x})\widehat{\boldsymbol{e}}^i$ is defined by its Cartesian components $\widehat{v}_i(\widehat{x})$ over an orthonormal basis of $\mathbf{E}^3$ formed by three vectors $\widehat{\boldsymbol{e}}^i$.

An example of a vector field in Cartesian coordinates is provided by the displacement field of an elastic body with $\{\widehat{\Omega}\}^-$ as its reference configuration; cf. Section 3.1.

Suppose now that the open set $\widehat{\Omega}$ is equipped with *curvilinear coordinates* from an open subset Ω of $\mathbb{R}^3$, by means of an injective mapping $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$ satisfying $\boldsymbol{\Theta}(\Omega) = \widehat{\Omega}$.

How does one define appropriate components of the same vector field, but this time in terms of these curvilinear coordinates? It turns out that the proper way to do so consists in defining three functions $v_i : \overline{\Omega} \to \mathbb{R}$ by requiring that (Figure 1.4-2)

$$v_i(x)\boldsymbol{g}^i(x) := \widehat{v}_i(\widehat{x})\widehat{\boldsymbol{e}}^i \text{ for all } \widehat{x} = \boldsymbol{\Theta}(x),\ x \in \Omega,$$

where the three vectors $\boldsymbol{g}^i(x)$ form the *contravariant basis* at $\widehat{x} = \boldsymbol{\Theta}(x)$ (Section 1.2). Using the relations $\boldsymbol{g}^i(x) \cdot \boldsymbol{g}_j(x) = \delta^i_j$ and $\widehat{\boldsymbol{e}}^i \cdot \widehat{\boldsymbol{e}}_j = \delta^i_j$, we immediately find how the old and new components are related, viz.,

$$v_j(x) = v_i(x)\boldsymbol{g}^i(x) \cdot \boldsymbol{g}_j(x) = \widehat{v}_i(\widehat{x})\widehat{\boldsymbol{e}}^i \cdot \boldsymbol{g}_j(x),$$
$$\widehat{v}_i(\widehat{x}) = \widehat{v}_j(\widehat{x})\widehat{\boldsymbol{e}}^j \cdot \widehat{\boldsymbol{e}}_i = v_j(x)\boldsymbol{g}^j(x) \cdot \widehat{\boldsymbol{e}}_i.$$

The three components $v_i(x)$ are called the **covariant components of the vector** $v_i(x)\boldsymbol{g}^i(x)$ *at* $\widehat{x}$, and the three functions $v_i : \Omega \to \mathbb{R}$ defined in this fashion are called the **covariant components of the vector field** $v_i\boldsymbol{g}^i : \Omega \to \mathbf{E}^3$.

Suppose next that we wish to compute a partial derivative $\widehat{\partial}_j\widehat{v}_i(\widehat{x})$ at a point $\widehat{x} = \boldsymbol{\Theta}(x) \in \widehat{\Omega}$ in terms of the partial derivatives $\partial_\ell v_k(x)$ and of the values $v_q(x)$ (which are also expected to appear by virtue of the chain rule). Such a task is required for example if we wish to write a system of partial differential equations whose unknown is a vector field (such as the equations of nonlinear or linearized elasticity) in terms of *ad hoc* curvilinear coordinates.

As we now show, carrying out such a transformation naturally leads to a fundamental notion, that of *covariant derivatives of a vector field*.

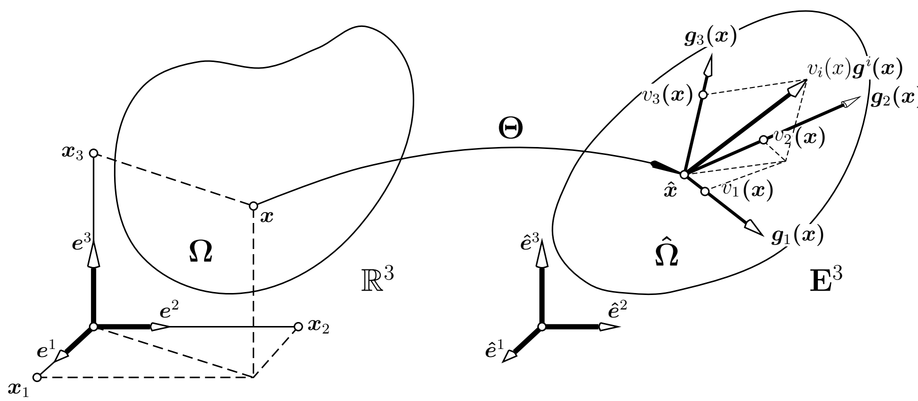

Figure 1.4-2: *A vector field in curvilinear coordinates.* Let there be given a vector field in Cartesian coordinates defined at each $\widehat{x} \in \widehat{\Omega}$ by its Cartesian components $\widehat{v}_i(\widehat{x})$ over the vectors $\widehat{\boldsymbol{e}}^i$ (Figure 1.4-1). In curvilinear coordinates, the same vector field is defined at each $x \in \Omega$ by its covariant components $v_i(x)$ over the contravariant basis vectors $\boldsymbol{g}^i(x)$ in such a way that $v_i(x)\boldsymbol{g}^i(x) = \widehat{v}_i(\widehat{x})\boldsymbol{e}^i$, $\widehat{x} = \boldsymbol{\Theta}(x)$.

An example of a vector field in curvilinear coordinates is provided by the displacement field of an elastic body with $\{\widehat{\Omega}\}^- = \boldsymbol{\Theta}(\overline{\Omega})$ as its reference configuration; cf. Section 3.2.

Theorem 1.4-1. *Let* Ω *be an open subset of* $\mathbb{R}^3$ *and let* $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$ *be an injective immersion that is also a* $\mathcal{C}^2$*-diffeomorphism of* Ω *onto* $\widehat{\Omega} := \boldsymbol{\Theta}(\Omega)$. *Given a vector field* $\widehat{v}_i\widehat{\boldsymbol{e}}^i : \widehat{\Omega} \to \mathbb{R}^3$ *in Cartesian coordinates with components* $\widehat{v}_i \in \mathcal{C}^1(\widehat{\Omega})$, *let* $v_i\boldsymbol{g}^i : \Omega \to \mathbb{R}^3$ *be the same field in curvilinear coordinates, i.e., that defined by*

$$\widehat{v}_i(\widehat{x})\widehat{\boldsymbol{e}}^i = v_i(x)\boldsymbol{g}^i(x) \text{ for all } \widehat{x} = \boldsymbol{\Theta}(x),\ x \in \Omega.$$

Then $v_i \in \mathcal{C}^1(\Omega)$ *and for all* $x \in \Omega$,

$$\widehat{\partial}_j \widehat{v}_i(\widehat{x}) = \big(v_{k\|\ell}[\boldsymbol{g}^k]_i[\boldsymbol{g}^\ell]_j\big)(x),\ \widehat{x} = \boldsymbol{\Theta}(x),$$

where

$$v_{i\|j} := \partial_j v_i - \Gamma^p_{ij} v_p \text{ and } \Gamma^p_{ij} := \boldsymbol{g}^p \cdot \partial_i \boldsymbol{g}_j,$$

and

$$[\boldsymbol{g}^i(x)]_k := \boldsymbol{g}^i(x) \cdot \widehat{\boldsymbol{e}}_k$$

denotes the i*-th component of* $\boldsymbol{g}^i(x)$ *over the basis* $\{\widehat{\boldsymbol{e}}_1, \widehat{\boldsymbol{e}}_2, \widehat{\boldsymbol{e}}_3\}$.

Proof. The following convention holds throughout this proof: The simultaneous appearance of $\widehat{x}$ and x in an equality means that they are related by $\widehat{x} = \boldsymbol{\Theta}(x)$ and that the equality in question holds for all $x \in \Omega$.

(i) *Another expression of* $[\boldsymbol{g}^i(x)]_k := \boldsymbol{g}^i(x) \cdot \widehat{\boldsymbol{e}}_k$.

Let $\boldsymbol{\Theta}(x) = \Theta^k(x)\widehat{\boldsymbol{e}}_k$ and $\widehat{\boldsymbol{\Theta}}(\widehat{x}) = \widehat{\Theta}^i(\widehat{x})\boldsymbol{e}_i$, where $\widehat{\boldsymbol{\Theta}} : \widehat{\Omega} \to \mathbf{E}^3$ denotes the inverse mapping of $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$. Since $\widehat{\boldsymbol{\Theta}}(\boldsymbol{\Theta}(x)) = x$ for all $x \in \Omega$, the chain rule shows that the matrices $\boldsymbol{\nabla\Theta}(x) := (\partial_j \Theta^k(x))$ (the row index is k) and $\widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{\Theta}}(\widehat{x}) := (\widehat{\partial}_k \widehat{\Theta}^i(\widehat{x}))$ (the row index is i) satisfy

$$\widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{\Theta}}(\widehat{x})\boldsymbol{\nabla\Theta}(x) = \mathbf{I},$$

or equivalently,

$$\widehat{\partial}_k \widehat{\Theta}^i(\widehat{x}) \partial_j \Theta^k(x) = \begin{pmatrix} \widehat{\partial}_1 \widehat{\Theta}^i(\widehat{x}) & \partial_2 \widehat{\Theta}^i(\widehat{x}) & \partial_3 \widehat{\Theta}^i(\widehat{x}) \end{pmatrix} \begin{pmatrix} \partial_j \Theta^1(x) \\ \partial_j \Theta^2(x) \\ \partial_j \Theta^3(x) \end{pmatrix} = \delta^i_j.$$

The components of the above column vector being precisely those of the vector $\boldsymbol{g}_j(x)$, the components of the above row vector must be those of the vector $\boldsymbol{g}^i(x)$ since $\boldsymbol{g}^i(x)$ is uniquely defined for each exponent i by the three relations $\boldsymbol{g}^i(x) \cdot \boldsymbol{g}_j(x) = \delta^i_j$, $j = 1, 2, 3$. Hence the k-th component of $\boldsymbol{g}^i(x)$ over the basis $\{\widehat{\boldsymbol{e}}_1, \widehat{\boldsymbol{e}}_2, \widehat{\boldsymbol{e}}_3\}$ can be also expressed in terms of the inverse mapping $\widehat{\boldsymbol{\Theta}}$, as:

$$[\boldsymbol{g}^i(x)]_k = \widehat{\partial}_k \widehat{\Theta}^i(\widehat{x}).$$

(ii) *The functions* $\Gamma^q_{\ell k} := \boldsymbol{g}^q \cdot \partial_\ell \boldsymbol{g}_k \in \mathcal{C}^0(\Omega)$.

We next compute the derivatives $\partial_\ell \boldsymbol{g}^q(x)$ (the fields $\boldsymbol{g}^q = g^{qr}\boldsymbol{g}_r$ are of class $\mathcal{C}^1$ on Ω since $\boldsymbol{\Theta}$ is assumed to be of class $\mathcal{C}^2$). These derivatives will be needed in (iii) for expressing the derivatives $\widehat{\partial}_j \widehat{u}_i(\widehat{x})$ as functions of x (recall that $\widehat{u}_i(\widehat{x}) = u_k(x)[\boldsymbol{g}^k(x)]_i$). Recalling that the vectors $\boldsymbol{g}^k(x)$ form a basis, we may write *a priori*

$$\partial_\ell \boldsymbol{g}^q(x) = -\Gamma^q_{\ell k}(x)\boldsymbol{g}^k(x),$$

thereby unambiguously defining functions $\Gamma^q_{\ell k} : \Omega \to \mathbb{R}$. To find their expressions in terms of the mappings $\boldsymbol{\Theta}$ and $\widehat{\boldsymbol{\Theta}}$, we observe that

$$\Gamma^q_{\ell k}(x) = \Gamma^q_{\ell m}(x)\delta^m_k = \Gamma^q_{\ell m}(x)\boldsymbol{g}^m(x) \cdot \boldsymbol{g}_k(x) = -\partial_\ell \boldsymbol{g}^q(x) \cdot \boldsymbol{g}_k(x).$$

Hence, noting that $\partial_\ell(\boldsymbol{g}^q(x) \cdot \boldsymbol{g}_k(x)) = 0$ and $[\boldsymbol{g}^q(x)]_p = \widehat{\partial}_p \widehat{\Theta}^q(\widehat{x})$, we obtain

$$\Gamma^q_{\ell k}(x) = \boldsymbol{g}^q(x) \cdot \partial_\ell \boldsymbol{g}_k(x) = \widehat{\partial}_p \widehat{\Theta}^q(\widehat{x})\partial_{\ell k}\Theta^p(x) = \Gamma^q_{k\ell}(x).$$

Since $\boldsymbol{\Theta} \in \mathcal{C}^2(\Omega; \mathbf{E}^3)$ and $\widehat{\boldsymbol{\Theta}} \in \mathcal{C}^1(\widehat{\Omega}; \mathbb{R}^3)$ by assumption, the last relations show that $\Gamma^q_{\ell k} \in \mathcal{C}^0(\Omega)$.

(iii) *The partial derivatives* $\widehat{\partial}_i \widehat{v}_i(\widehat{x})$ *of the Cartesian components of the vector field* $\widehat{v}_i \widehat{\boldsymbol{e}}^i \in \mathcal{C}^1(\widehat{\Omega}; \mathbb{R}^3)$ *are given at each* $\widehat{x} = \boldsymbol{\Theta}(x) \in \widehat{\Omega}$ *by*

$$\widehat{\partial}_j \widehat{v}_i(\widehat{x}) = v_{k\|\ell}(x)[\boldsymbol{g}^k(x)]_i[\boldsymbol{g}^\ell(x)]_j,$$

where

$$v_{k\|\ell}(x) := \partial_\ell v_k(x) - \Gamma^q_{\ell k}(x)v_q(x),$$

and $[\boldsymbol{g}^k(x)]_i$ *and* $\Gamma^q_{\ell k}(x)$ *are defined as in* (i) *and* (ii).

We compute the partial derivatives $\widehat{\partial}_j \widehat{v}_i(\widehat{x})$ as functions of x by means of the relation $\widehat{v}_i(\widehat{x}) = v_k(x)[\boldsymbol{g}^k(x)]_i$. To this end, we first note that a differentiable function $w : \Omega \to \mathbb{R}$ satisfies

$$\widehat{\partial}_j w(\widehat{\boldsymbol{\Theta}}(\widehat{x})) = \partial_\ell w(x)\widehat{\partial}_j \widehat{\Theta}^\ell(\widehat{x}) = \partial_\ell w(x)[\boldsymbol{g}^\ell(x)]_j,$$

by the chain rule and by (i). In particular then,

$$\begin{aligned}\widehat{\partial}_j \widehat{v}_i(\widehat{x}) &= \widehat{\partial}_j v_k(\widehat{\boldsymbol{\Theta}}(\widehat{x}))[\boldsymbol{g}^k(x)]_i + v_q(x)\widehat{\partial}_j[\boldsymbol{g}^q(\widehat{\boldsymbol{\Theta}}(\widehat{x}))]_i \\ &= \partial_\ell v_k(x)[\boldsymbol{g}^\ell(x)]_j[\boldsymbol{g}^k(x)]_i + v_q(x)\big(\partial_\ell[\boldsymbol{g}^q(x)]_i\big)[\boldsymbol{g}^\ell(x)]_j \\ &= \left(\partial_\ell v_k(x) - \Gamma^q_{\ell k}(x)v_q(x)\right)[\boldsymbol{g}^k(x)]_i[\boldsymbol{g}^\ell(x)]_j,\end{aligned}$$

since $\partial_\ell \boldsymbol{g}^q(x) = -\Gamma^q_{\ell k}(x)\boldsymbol{g}^k(x)$ by (ii). □

The functions

$$v_{i\|j} = \partial_j v_i - \Gamma^p_{ij} v_p$$

defined in Theorem 1.4-1 are called the **first-order covariant derivatives of the vector field** $v_i \boldsymbol{g}^i : \Omega \to \mathbb{R}^3$.

The functions

$$\Gamma^p_{ij} = \boldsymbol{g}^p \cdot \partial_i \boldsymbol{g}_j$$

are called the **Christoffel symbols of the second kind** (the Christoffel symbols of the first kind are introduced in the next section).

The following result summarizes properties of covariant derivatives and Christoffel symbols that are constantly used.

Theorem 1.4-2. *Let the assumptions on the mapping* $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$ *be as in* Theorem 1.4-1, *and let there be given a vector field* $v_i \boldsymbol{g}^i : \Omega \to \mathbb{R}^3$ *with covariant components* $v_i \in \mathcal{C}^1(\Omega)$.

(a) *The first-order covariant derivatives* $v_{i\|j} \in \mathcal{C}^0(\Omega)$ *of the vector field* $v_i \boldsymbol{g}^i : \Omega \to \mathbb{R}^3$*, which are defined by*

$$v_{i\|j} := \partial_j v_i - \Gamma^p_{ij} v_p, \text{ where } \Gamma^p_{ij} := \boldsymbol{g}^p \cdot \partial_i \boldsymbol{g}_j,$$

can be also defined by the relations

$$\partial_j (v_i \boldsymbol{g}^i) = v_{i\|j} \boldsymbol{g}^i \iff v_{i\|j} = \big\{\partial_j (v_k \boldsymbol{g}^k)\big\} \cdot \boldsymbol{g}_i.$$

(b) *The Christoffel symbols* $\Gamma^p_{ij} := \boldsymbol{g}^p \cdot \partial_i \boldsymbol{g}_j = \Gamma^p_{ji} \in \mathcal{C}^0(\Omega)$ *satisfy the relations*

$$\partial_i \boldsymbol{g}^p = -\Gamma^p_{ij} \boldsymbol{g}^j \text{ and } \partial_j \boldsymbol{g}_q = \Gamma^i_{jq} \boldsymbol{g}_i.$$

Proof. It remains to verify that the covariant derivatives $v_{i\|j}$, defined in Theorem 1.4-1 by

$$v_{i\|j} = \partial_j v_i - \Gamma^p_{ij} v_p,$$

may be equivalently defined by the relations

$$\partial_j (v_i \boldsymbol{g}^i) = v_{i\|j} \boldsymbol{g}^i.$$

These relations unambiguously define the functions $v_{i\|j} = \{\partial_j (v_k \boldsymbol{g}^k)\} \cdot \boldsymbol{g}_i$ since the vectors $\boldsymbol{g}^i$ are linearly independent at all points of Ω by assumption. To this end, we simply note that, by definition, the Christoffel symbols satisfy $\partial_i \boldsymbol{g}^p = -\Gamma^p_{ij} \boldsymbol{g}^j$ (cf. part (ii) of the proof of Theorem 1.4-1); hence

$$\partial_j (v_i \boldsymbol{g}^i) = (\partial_j v_i) \boldsymbol{g}^i + v_i \partial_j \boldsymbol{g}^i = (\partial_j v_i) \boldsymbol{g}^i - v_i \Gamma^i_{jk} \boldsymbol{g}^k = v_{i\|j} \boldsymbol{g}^i.$$

To establish the other relations $\partial_j \boldsymbol{g}_q = \Gamma^i_{jq} \boldsymbol{g}_i$, we note that

$$0 = \partial_j (\boldsymbol{g}^p \cdot \boldsymbol{g}_q) = -\Gamma^p_{ji} \boldsymbol{g}^i \cdot \boldsymbol{g}_q + \boldsymbol{g}^p \cdot \partial_j \boldsymbol{g}_q = -\Gamma^p_{qj} + \boldsymbol{g}^p \cdot \partial_j \boldsymbol{g}_q.$$

Hence

$$\partial_j \boldsymbol{g}_q = (\partial_j \boldsymbol{g}_q \cdot \boldsymbol{g}^p) \boldsymbol{g}_p = \Gamma^p_{qj} \boldsymbol{g}_p.$$

□

Remark. The Christoffel symbols Γ^p_{ij} can be also defined solely in terms of the components of the metric tensor; see the proof of Theorem 1.5-1. □

If the affine space $\mathbf{E}^3$ is identified with $\mathbb{R}^3$ and $\boldsymbol{\Theta}(x) = x$ for all $x \in \Omega$, the relation $\partial_j (v_i \boldsymbol{g}^i)(x) = (v_{i\|j} \boldsymbol{g}^i)(x)$ reduces to $\widehat{\partial}_j (\widehat{v}_i(\widehat{x}) \widehat{\boldsymbol{e}}^i) = (\widehat{\partial}_j \widehat{v}_i(\widehat{x})) \widehat{\boldsymbol{e}}^i$. In this sense, *a covariant derivative of the first order constitutes a generalization of a partial derivative of the first order in Cartesian coordinates.*

1.5 NECESSARY CONDITIONS SATISFIED BY THE METRIC TENSOR; THE RIEMANN CURVATURE TENSOR

It is remarkable that *the components* $g_{ij} = g_{ji} : \Omega \to \mathbb{R}$ *of the metric tensor of an open set* $\boldsymbol{\Theta}(\Omega) \subset \mathbf{E}^3$ (Section 1.2), defined by a smooth enough immersion $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$, *cannot be arbitrary functions.*

As shown in the next theorem, they must satisfy relations that take the form:

$$\partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma^p_{ij} \Gamma_{kqp} - \Gamma^p_{ik} \Gamma_{jqp} = 0 \text{ in } \Omega,$$

where the functions Γ_{ijq} and Γ^p_{ij} have simple expressions in terms of the functions g_{ij} and of some of their partial derivatives (as shown in the next proof, it so happens that the functions Γ^p_{ij} as defined in Theorem 1.5-1 coincide with the Christoffel symbols introduced in the previous section; this explains why they are denoted by the same symbol). Note that, according to the rule governing Latin indices and exponents, these relations are meant to hold for all $i, j, k, q \in \{1, 2, 3\}$.

Theorem 1.5-1. *Let* Ω *be an open set in* $\mathbb{R}^3$, *let* $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ *be an immersion, and let*

$$g_{ij} := \partial_i \boldsymbol{\Theta} \cdot \partial_j \boldsymbol{\Theta}$$

denote the covariant components of the metric tensor of the set $\boldsymbol{\Theta}(\Omega)$. *Let the functions* $\Gamma_{ijq} \in \mathcal{C}^1(\Omega)$ *and* $\Gamma^p_{ij} \in \mathcal{C}^1(\Omega)$ *be defined by*

$$\Gamma_{ijq} := \frac{1}{2}(\partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij}),$$
$$\Gamma^p_{ij} := g^{pq} \Gamma_{ijq} \text{ where } (g^{pq}) := (g_{ij})^{-1}.$$

Then, necessarily,

$$\partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma^p_{ij} \Gamma_{kqp} - \Gamma^p_{ik} \Gamma_{jqp} = 0 \text{ in } \Omega.$$

Proof. Let $\boldsymbol{g}_i = \partial_i \boldsymbol{\Theta}$. It is then immediately verified that the functions Γ_{ijq} are also given by

$$\Gamma_{ijq} = \partial_i \boldsymbol{g}_j \cdot \boldsymbol{g}_q.$$

For each $x \in \Omega$, let the three vectors $\boldsymbol{g}^j(x)$ be defined by the relations $\boldsymbol{g}^j(x) \cdot \boldsymbol{g}_i(x) = \delta^j_i$. Since we also have $\boldsymbol{g}^j = g^{ij} \boldsymbol{g}_i$, the last relations imply that $\Gamma^p_{ij} = \partial_i \boldsymbol{g}_j \cdot \boldsymbol{g}^p$. Therefore,

$$\partial_i \boldsymbol{g}_j = \Gamma^p_{ij} \boldsymbol{g}_p$$

since $\partial_i \boldsymbol{g}_j = (\partial_i \boldsymbol{g}_j \cdot \boldsymbol{g}^p) \boldsymbol{g}_p$. Differentiating the same relations yields

$$\partial_k \Gamma_{ijq} = \partial_{ik} \boldsymbol{g}_j \cdot \boldsymbol{g}_q + \partial_i \boldsymbol{g}_j \cdot \partial_k \boldsymbol{g}_q,$$

so that the above relations together give

$$\partial_i \boldsymbol{g}_j \cdot \partial_k \boldsymbol{g}_q = \Gamma^p_{ij} \boldsymbol{g}_p \cdot \partial_k \boldsymbol{g}_q = \Gamma^p_{ij} \Gamma_{kqp}.$$

Consequently,

$$\partial_{ik}\boldsymbol{g}_j \cdot \boldsymbol{g}_q = \partial_k \Gamma_{ijq} - \Gamma^p_{ij}\Gamma_{kqp}.$$

Since $\partial_{ik}\boldsymbol{g}_j = \partial_{ij}\boldsymbol{g}_k$, we also have

$$\partial_{ik}\boldsymbol{g}_j \cdot \boldsymbol{g}_q = \partial_j \Gamma_{ikq} - \Gamma^p_{ik}\Gamma_{jqp},$$

and thus the required necessary conditions immediately follow. □

Remark. The vectors $\boldsymbol{g}_i$ and $\boldsymbol{g}^j$ introduced above form the covariant and contravariant bases and the functions g^{ij} are the contravariant components of the metric tensor (Section 1.2). □

As shown in the above proof, the necessary conditions $R_{qijk} = 0$ thus simply constitute a re-writing of the relations $\partial_{ik}\boldsymbol{g}_j = \partial_{ki}\boldsymbol{g}_j$ in the form of the equivalent relations $\partial_{ik}\boldsymbol{g}_j \cdot \boldsymbol{g}_q = \partial_{ki}\boldsymbol{g}_j \cdot \boldsymbol{g}_q$.

The functions

$$\Gamma_{ijq} = \frac{1}{2}(\partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij}) = \partial_i \boldsymbol{g}_j \cdot \boldsymbol{g}_q = \Gamma_{jiq}$$

and

$$\Gamma^p_{ij} = g^{pq}\Gamma_{ijq} = \partial_i \boldsymbol{g}_j \cdot \boldsymbol{g}^p = \Gamma^p_{ji}$$

are the **Christoffel symbols of the first**, and **second, kinds**. We saw in Section 1.4 that the Christoffel symbols of the second kind also naturally appear in a different context (that of covariant differentiation).

Finally, the functions

$$R_{qijk} := \partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma^p_{ij}\Gamma_{kqp} - \Gamma^p_{ik}\Gamma_{jqp}$$

are the **covariant components of the Riemann curvature tensor** of the set $\boldsymbol{\Theta}(\Omega)$. The relations $R_{qijk} = 0$ found in Theorem 1.4-1 thus express that the *Riemann curvature tensor of the set* $\boldsymbol{\Theta}(\Omega)$ (equipped with the metric tensor with covariann components g_{ij}) *vanishes.*

1.6 EXISTENCE OF AN IMMERSION DEFINED ON AN OPEN SET IN $\mathbb{R}^3$ WITH A PRESCRIBED METRIC TENSOR

Let $\mathbb{M}^3, \mathbb{S}^3$, and $\mathbb{S}^3_>$ denote the sets of all square matrices of order three, of all symmetric matrices of order three, and of all symmetric positive definite matrices of order three.

As in Section 1.2, the matrix representing the Fréchet derivative at $x \in \Omega$ of a differentiable mapping $\boldsymbol{\Theta} = (\Theta_\ell) : \Omega \to \mathbf{E}^3$ is denoted

$$\boldsymbol{\nabla\Theta}(x) := (\partial_j \Theta_\ell(x)) \in \mathbb{M}^3,$$

where ℓ is the row index and j the column index (equivalently, $\boldsymbol{\nabla\Theta}(x)$ is the matrix of order three whose j-th column vector is $\partial_j\boldsymbol{\Theta}$).

So far, we have considered that we are given an open set $\Omega \subset \mathbb{R}^3$ and a smooth enough immersion $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$, thus allowing us to define a matrix field

$$\mathbf{C} = (g_{ij}) = \boldsymbol{\nabla\Theta}^T\boldsymbol{\nabla\Theta} : \Omega \to \mathbb{S}^3_>,$$

where $g_{ij} : \Omega \to \mathbb{R}$ are the covariant components of the *metric tensor* of the open set $\boldsymbol{\Theta}(\Omega) \subset \mathbf{E}^3$.

We now turn to the *reciprocal questions*:

Given an open subset Ω of $\mathbb{R}^3$ and a smooth enough matrix field $\mathbf{C} = (g_{ij}) : \Omega \to \mathbb{S}^3_>$, when is $\mathbf{C}$ the metric tensor field of an open set $\boldsymbol{\Theta}(\Omega) \subset \mathbf{E}^3$? Equivalently, *when does there exist an immersion* $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$ *such that*

$$\mathbf{C} = \boldsymbol{\nabla\Theta}^T\boldsymbol{\nabla\Theta} \text{ in } \Omega,$$

or equivalently, such that

$$g_{ij} = \partial_i\boldsymbol{\Theta} \cdot \partial_j\boldsymbol{\Theta} \text{ in } \Omega?$$

If such an immersion exists, to what extent is it unique?

The answers are remarkably simple: *If* Ω *is simply-connected, the necessary conditions*

$$\partial_j\Gamma_{ikq} - \partial_k\Gamma_{ijq} + \Gamma^p_{ij}\Gamma_{kqp} - \Gamma^p_{ik}\Gamma_{jqp} = 0 \text{ in } \Omega$$

found in Theorem 1.7-1 *are also sufficient for the existence of such an immersion. If* Ω *is connected, this immersion is unique up to isometries in* $\mathbf{E}^3$.

Whether the immersion found in this fashion is *globally injective* is a different issue, which accordingly should be resolved by different means.

This result comprises two essentially distinct parts, a *global existence result* (Theorem 1.6-1) and a *uniqueness result* (Theorem 1.7-1). Note that these two results are established under *different assumptions* on the set Ω and on the smoothness of the field (g_{ij}).

In order to put these results in a wider perspective, let us make a brief incursion into *Riemannian Geometry.* For detailed treatments, see classic texts such as Choquet-Bruhat, de Witt-Morette & Dillard-Bleick [1977], Marsden & Hughes [1983], Berger [2003], or Gallot, Hulin & Lafontaine [2004].

Considered as a three-dimensional manifold, an open set $\Omega \subset \mathbb{R}^3$ equipped with an immersion $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$ becomes an example of a *Riemannian manifold* $(\Omega; (g_{ij}))$, i.e., a *manifold*, the set Ω, equipped with a *Riemannian metric*, the symmetric positive-definite matrix field $(g_{ij}) : \Omega \to \mathbb{S}^3_>$ defined in this case by $g_{ij} := \partial_i\boldsymbol{\Theta} \cdot \partial_j\boldsymbol{\Theta}$ in Ω. More generally, a **Riemannian metric on a manifold** is a twice covariant, symmetric, positive-definite tensor field acting on vectors in the tangent spaces to the manifold (these tangent spaces coincide with $\mathbb{R}^3$ in the present instance).

This particular Riemannian manifold $(\Omega; (g_{ij}))$ possesses the remarkable property that *its metric is the same as that of the surrounding space* $\mathbf{E}^3$. More specifically, $(\Omega; (g_{ij}))$ is **isometrically immersed** in the Euclidean space $\mathbf{E}^3$, in the sense that there exists an immersion $\mathbf{\Theta} : \Omega \to \mathbf{E}^3$ that satisfies the relations $g_{ij} = \partial_i \mathbf{\Theta} \cdot \partial_j \mathbf{\Theta}$. Equivalently, the length of any curve in the Riemannian manifold $(\Omega; (g_{ij}))$ is the same as the length of its image by $\mathbf{\Theta}$ in the Euclidean space $\mathbf{E}^3$ (see Theorem 1.3-1).

The first question above can thus be rephrased as follows: *Given an open subset* Ω *of* $\mathbb{R}^3$ *and a positive-definite matrix field* $(g_{ij}) : \Omega \to \mathbb{S}^3_>$*, when is the Riemannian manifold* $(\Omega; (g_{ij}))$ **flat**, in the sense that it can be locally isometrically immersed in a Euclidean space of the *same* dimension (three)?

The answer to this question can then be rephrased as follows (compare with the statement of Theorem 1.6-1 below): *Let* Ω *be a simply-connected open subset of* $\mathbb{R}^3$*. Then a Riemannian manifold* $(\Omega; (g_{ij}))$ *with a Riemannian metric* (g_{ij}) *of class* $\mathcal{C}^2$ *in* Ω *is flat if and only if its Riemannian curvature tensor vanishes in* Ω*.* Recast as such, this result becomes a special case of the **fundamental theorem on flat Riemannian manifolds**, which holds for a general finite-dimensional Riemannian manifold.

The answer to the second question, viz., the issue of uniqueness, can be rephrased as follows (compare with the statement of Theorem 1.7-1 in the next section): *Let* Ω *be a connected open subset of* $\mathbb{R}^3$*. Then the isometric immersions of a flat Riemannian manifold* $(\Omega; (g_{ij}))$ *into a Euclidean space* $\mathbf{E}^3$ *are unique up to isometries of* $\mathbf{E}^3$. Recast as such, this result likewise becomes a special case of the so-called **rigidity theorem**; cf. Section 1.7.

Recast as such, these two theorems together constitute a special case (that where the dimensions of the manifold and of the Euclidean space are both equal to three) of the **fundamental theorem of Riemannian Geometry**. This theorem addresses the same *existence* and *uniqueness* questions in the more general setting where Ω is replaced by a p-dimensional manifold and $\mathbf{E}^3$ is replaced by a $(p+q)$-dimensional Euclidean space (the "fundamental theorem of surface theory", established in Sections 2.8 and 2.9, constitutes another important special case). When the p-dimensional manifold is an open subset of $\mathbb{R}^{p+q}$, an outline of a self-contained proof is given in Szopos [2005].

Another fascinating question (which will not be addressed here) is the following: Given again an open subset Ω of $\mathbb{R}^3$ equipped with a symmetric, positive-definite matrix field $(g_{ij}) : \Omega \to \mathbb{S}^3$, assume this time that *the Riemannian manifold* $(\Omega; (g_{ij}))$ *is no longer flat*, i.e., its Riemannian curvature tensor no longer vanishes in Ω. *Can such a Riemannian manifold still be isometrically immersed, but this time in a higher-dimensional Euclidean space*? Equivalently, does there exist a Euclidean space $\mathbf{E}^d$ with $d > 3$ and does there exist an immersion $\mathbf{\Theta} : \Omega \to \mathbf{E}^d$ such that $g_{ij} = \partial_i \mathbf{\Theta} \cdot \partial_j \mathbf{\Theta}$ in Ω?

The answer is yes, according to the following beautiful **Nash theorem**, so named after Nash [1954]: *Any* p*-dimensional Riemannian manifold equipped with a continuous metric can be isometrically immersed in a Euclidean space of dimension* $2p$ *with an immersion of class* $\mathcal{C}^1$*; it can also be isometrically immersed in a Euclidean space of dimension* $(2p+1)$ *with a globally injective immersion of class* $\mathcal{C}^1$.

Let us now humbly return to the question of existence raised at the beginning of this section, i.e., when the manifold is an open set in $\mathbb{R}^3$.

Theorem 1.6-1. *Let Ω be a connected and simply-connected open set in $\mathbb{R}^3$ and let $\mathbf{C} = (g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ be a matrix field that satisfies*

$$R_{qijk} := \partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma^p_{ij} \Gamma_{kqp} - \Gamma^p_{ik} \Gamma_{jqp} = 0 \text{ in } \Omega,$$

where

$$\Gamma_{ijq} := \frac{1}{2}(\partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij}),$$
$$\Gamma^p_{ij} := g^{pq} \Gamma_{ijq} \text{ with } (g^{pq}) := (g_{ij})^{-1}.$$

Then there exists an immersion $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ such that

$$\mathbf{C} = \boldsymbol{\nabla}\boldsymbol{\Theta}^T \boldsymbol{\nabla}\boldsymbol{\Theta} \text{ in } \Omega.$$

Proof. The proof relies on a simple, yet crucial, observation. When a smooth enough immersion $\boldsymbol{\Theta} = (\Theta_\ell) : \Omega \to \mathbf{E}^3$ is *a priori* given (as it was so far), its components Θ_ℓ satisfy the relations $\partial_{ij}\Theta_\ell = \Gamma^p_{ij} \partial_p \Theta_\ell$, which are nothing but another way of writing the relations $\partial_i \boldsymbol{g}_j = \Gamma^p_{ij} \boldsymbol{g}_p$ (see the proof of Theorem 1.5-1). This observation thus suggests to begin by solving (see part (ii)) the system of partial differential equations

$$\partial_i F_{\ell j} = \Gamma^p_{ij} F_{\ell p} \text{ in } \Omega,$$

whose solutions $F_{\ell j} : \Omega \to \mathbb{R}$ then constitute natural candidates for the partial derivatives $\partial_j \Theta_\ell$ of the unknown immersion $\boldsymbol{\Theta} = (\Theta_\ell) : \Omega \to \mathbf{E}^3$ (see part (iii)).

To begin with, we establish in (i) relations that will in turn allow us to re-write the sufficient conditions

$$\partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma^p_{ij} \Gamma_{kqp} - \Gamma^p_{ik} \Gamma_{jqp} = 0 \text{ in } \Omega$$

in a slightly different form, more appropriate for the existence result of part (ii). Note that the positive definiteness of the symmetric matrices (g_{ij}) is not needed for this purpose.

(i) *Let Ω be an open subset of $\mathbb{R}^3$ and let there be given a field $(g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3)$ of symmetric invertible matrices. The functions $\Gamma_{ijq}, \Gamma^p_{ij}$, and g^{pq} being defined by*

$$\Gamma_{ijq} := \frac{1}{2}(\partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij}), \quad \Gamma^p_{ij} := g^{pq} \Gamma_{ijq}, \quad (g^{pq}) := (g_{ij})^{-1},$$

define the functions

$$R_{qijk} := \partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma^p_{ij} \Gamma_{kqp} - \Gamma^p_{ik} \Gamma_{jqp},$$
$$R^p_{\cdot ijk} := \partial_j \Gamma^p_{ik} - \partial_k \Gamma^p_{ij} + \Gamma^\ell_{ik} \Gamma^p_{j\ell} - \Gamma^\ell_{ij} \Gamma^p_{k\ell}.$$

Then

$$R^p_{\cdot ijk} = g^{pq} R_{qijk} \text{ and } R_{qijk} = g_{pq} R^p_{\cdot ijk}.$$

Using the relations

$$\Gamma_{jq\ell} + \Gamma_{\ell jq} = \partial_j g_{q\ell} \text{ and } \Gamma_{ikq} = g_{q\ell}\Gamma_{ik}^{\ell},$$

which themselves follow from the definitions of the functions Γ_{ijq} and Γ_{ij}^p, and noting that

$$(g^{pq}\partial_j g_{q\ell} + g_{q\ell}\partial_j g^{pq}) = \partial_j(g^{pq}g_{q\ell}) = 0,$$

we obtain

$$\begin{aligned} g^{pq}(\partial_j\Gamma_{ikq} - \Gamma_{ik}^r\Gamma_{jqr}) &= \partial_j\Gamma_{ik}^p - \Gamma_{ikq}\partial_j g^{pq} - \Gamma_{ik}^{\ell} g^{pq}(\partial_j g_{q\ell} - \Gamma_{\ell jq}) \\ &= \partial_j\Gamma_{ik}^p + \Gamma_{ik}^{\ell}\Gamma_{j\ell}^p - \Gamma_{ik}^{\ell}(g^{pq}\partial_j g_{q\ell} + g_{q\ell}\partial_j g^{pq}) \\ &= \partial_j\Gamma_{ik}^p + \Gamma_{ik}^{\ell}\Gamma_{j\ell}^p. \end{aligned}$$

Likewise,

$$g^{pq}(\partial_k\Gamma_{ijq} - \Gamma_{ij}^r\Gamma_{kqr}) = \partial_k\Gamma_{ij}^p - \Gamma_{ij}^{\ell}\Gamma_{k\ell}^p,$$

and thus the relations $R_{\cdot ijk}^p = g^{pq}R_{qijk}$ are established. The relations $R_{qijk} = g_{pq}R_{\cdot ijk}^p$ are clearly equivalent to these ones.

We next establish the existence of solutions to the system

$$\partial_i F_{\ell j} = \Gamma_{ij}^p F_{\ell p} \text{ in } \Omega.$$

(ii) *Let Ω be a connected and simply-connected open subset of $\mathbb{R}^3$ and let there be given functions $\Gamma_{ij}^p = \Gamma_{ji}^p \in \mathcal{C}^1(\Omega)$ satisfying the relations*

$$\partial_j\Gamma_{ik}^p - \partial_k\Gamma_{ij}^p + \Gamma_{ik}^{\ell}\Gamma_{j\ell}^p - \Gamma_{ij}^{\ell}\Gamma_{k\ell}^p = 0 \text{ in } \Omega,$$

which are equivalent to the relations

$$\partial_j\Gamma_{ikq} - \partial_k\Gamma_{ijq} + \Gamma_{ij}^p\Gamma_{kqp} - \Gamma_{ik}^p\Gamma_{jqp} = 0 \text{ in } \Omega,$$

by part (i).

Let a point $x^0 \in \Omega$ and a matrix $(F_{\ell j}^0) \in \mathbb{M}^3$ be given. Then there exists one, and only one, field $(F_{\ell j}) \in \mathcal{C}^2(\Omega; \mathbb{M}^3)$ that satisfies

$$\begin{aligned} \partial_i F_{\ell j}(x) &= \Gamma_{ij}^p(x)F_{\ell p}(x),\ x \in \Omega, \\ F_{\ell j}(x^0) &= F_{\ell j}^0. \end{aligned}$$

Let x^1 be an arbitrary point in the set Ω, *distinct from* x^0. Since Ω is *connected*, there exists a *path* $\boldsymbol{\gamma} = (\gamma^i) \in \mathcal{C}^1([0,1]; \mathbb{R}^3)$ *joining* x^0 *to* x^1 *in* Ω; this means that

$$\boldsymbol{\gamma}(0) = x^0,\ \boldsymbol{\gamma}(1) = x^1, \text{ and } \boldsymbol{\gamma}(t) \in \Omega \text{ for all } 0 \le t \le 1.$$

Assume that a matrix field $(F_{\ell j}) \in \mathcal{C}^1(\Omega; \mathbb{M}^3)$ satisfies $\partial_i F_{\ell j}(x) = \Gamma_{ij}^p(x)F_{\ell p}(x)$, $x \in \Omega$. Then, *for each integer* $\ell \in \{1,2,3\}$, the three functions $\zeta_j \in \mathcal{C}^1([0,1])$ defined by (for simplicity, the dependence on ℓ is dropped)

$$\zeta_j(t) := F_{\ell j}(\boldsymbol{\gamma}(t)),\ 0 \le t \le 1,$$

satisfy the following *Cauchy problem for a linear system of three ordinary differential equations with respect to three unknowns*:

$$\frac{\mathrm{d}\zeta_j}{\mathrm{d}t}(t) = \Gamma^p_{ij}(\boldsymbol{\gamma}(t))\frac{\mathrm{d}\gamma^i}{\mathrm{d}t}(t)\zeta_p(t),\ 0 \le t \le 1,$$
$$\zeta_j(0) = \zeta_j^0,$$

where the *initial values* ζ_j^0 are given by

$$\zeta_j^0 := F^0_{\ell j}.$$

Note in passing that the three Cauchy problems obtained by letting $\ell = 1, 2$, or 3 only differ by their initial values ζ_j^0.

It is well known that a Cauchy problem of the form (with self-explanatory notations)

$$\frac{\mathrm{d}\boldsymbol{\zeta}}{\mathrm{d}t}(t) = \mathbf{A}(t)\boldsymbol{\zeta}(t),\ 0 \le t \le 1,$$
$$\boldsymbol{\zeta}(0) = \boldsymbol{\zeta}^0,$$

has *one* and *only one* solution $\boldsymbol{\zeta} \in \mathcal{C}^1([0,1];\mathbb{R}^3)$ if $\mathbf{A} \in \mathcal{C}^0([0,1];\mathbb{M}^3)$ (see, e.g., Schwartz [1992, Theorem 4.3.1, p. 388]). Hence each one of the three Cauchy problems has one and only one solution.

Incidentally, this result already shows that, *if it exists, the unknown field* $(F_{\ell j})$ *is unique.*

In order that the three values $\zeta_j(1)$ found by solving the above Cauchy problem for a given integer $\ell \in \{1,2,3\}$ be acceptable candidates for the three unknown values $F_{\ell j}(x^1)$, they must be of course *independent of the path chosen for joining* x^0 *to* x^1.

So, let $\boldsymbol{\gamma}_0 \in \mathcal{C}^1([0,1];\mathbb{R}^3)$ and $\boldsymbol{\gamma}_1 \in \mathcal{C}^1([0,1];\mathbb{R}^3)$ be two paths joining x^0 to x^1 in Ω. The open set Ω being *simply-connected*, there exists a *homotopy* $\mathbf{G} = (G^i) : [0,1] \times [0,1] \to \mathbb{R}^3$ *joining* $\boldsymbol{\gamma}_0$ *to* $\boldsymbol{\gamma}_1$ *in* Ω, i.e., such that

$$\mathbf{G}(\cdot,0) = \boldsymbol{\gamma}_0,\ \mathbf{G}(\cdot,1) = \boldsymbol{\gamma}_1,\ \mathbf{G}(t,\lambda) \in \Omega \text{ for all } 0 \le t \le 1,\ 0 \le \lambda \le 1,$$
$$\mathbf{G}(0,\lambda) = x^0 \text{ and } \mathbf{G}(1,\lambda) = x^1 \text{ for all } 0 \le \lambda \le 1,$$

and smooth enough in the sense that

$$\mathbf{G} \in \mathcal{C}^1([0,1] \times [0,1];\mathbb{R}^3) \text{ and } \frac{\partial}{\partial t}\Big(\frac{\partial \mathbf{G}}{\partial \lambda}\Big) = \frac{\partial}{\partial \lambda}\Big(\frac{\partial \mathbf{G}}{\partial t}\Big) \in \mathcal{C}^0([0,1] \times [0,1];\mathbb{R}^3).$$

Let $\boldsymbol{\zeta}(\cdot,\lambda) = (\zeta_j(\cdot,\lambda)) \in \mathcal{C}^1([0,1];\mathbb{R}^3)$ denote for each $0 \le \lambda \le 1$ the solution of the Cauchy problem corresponding to the path $\mathbf{G}(\cdot,\lambda)$ joining x^0 to x^1. We thus have

$$\frac{\partial \zeta_j}{\partial t}(t,\lambda) = \Gamma^p_{ij}(\mathbf{G}(t,\lambda))\frac{\partial G^i}{\partial t}(t,\lambda)\zeta_p(t,\lambda) \text{ for all } 0 \le t \le 1,\ 0 \le \lambda \le 1,$$
$$\zeta_j(0,\lambda) = \zeta_j^0 \text{ for all } 0 \le \lambda \le 1.$$

Our objective is to show that

$$\frac{\partial \zeta_j}{\partial \lambda}(1,\lambda) = 0 \text{ for all } 0 \leq \lambda \leq 1,$$

as this relation will imply that $\zeta_j(1,0) = \zeta_j(1,1)$, as desired. For this purpose, a direct differentiation shows that, for all $0 \leq t \leq 1$, $0 \leq \lambda \leq 1$,

$$\frac{\partial}{\partial \lambda}\Big(\frac{\partial \zeta_j}{\partial t}\Big) = \{\Gamma^q_{ij}\Gamma^p_{kq} + \partial_k \Gamma^p_{ij}\}\zeta_p \frac{\partial G^k}{\partial \lambda}\frac{\partial G^i}{\partial t} + \Gamma^p_{ij}\zeta_p \frac{\partial}{\partial \lambda}\Big(\frac{\partial G^i}{\partial t}\Big) + \sigma_q \Gamma^q_{ij}\frac{\partial G^i}{\partial t},$$

where

$$\sigma_j := \frac{\partial \zeta_j}{\partial \lambda} - \Gamma^p_{kj}\zeta_p \frac{\partial G^k}{\partial \lambda},$$

on the one hand (in the relations above and below, $\Gamma^q_{ij}, \partial_k \Gamma^p_{ij}$, etc., stand for $\Gamma^q_{ij}(\mathbf{G}(\cdot,\cdot)), \partial_k \Gamma^p_{ij}(\mathbf{G}(\cdot,\cdot))$, etc.).

On the other hand, a direct differentiation of the equation defining the functions σ_j shows that, for all $0 \leq t \leq 1, 0 \leq \lambda \leq 1$,

$$\frac{\partial}{\partial t}\Big(\frac{\partial \zeta_j}{\partial \lambda}\Big) = \frac{\partial \sigma_j}{\partial t} + \Big\{\partial_i \Gamma^p_{kj}\frac{\partial G^i}{\partial t}\zeta_p + \Gamma^q_{kj}\frac{\partial \zeta_q}{\partial t}\Big\}\frac{\partial G^k}{\partial \lambda} + \Gamma^p_{ij}\zeta_p \frac{\partial}{\partial t}\Big(\frac{\partial G^i}{\partial \lambda}\Big).$$

But $\dfrac{\partial \zeta_j}{\partial t} = \Gamma^p_{ij}\dfrac{\partial G^i}{\partial t}\zeta_p$, so that we also have

$$\frac{\partial}{\partial t}\Big(\frac{\partial \zeta_j}{\partial \lambda}\Big) = \frac{\partial \sigma_j}{\partial t} + \{\partial_i \Gamma^p_{kj} + \Gamma^q_{kj}\Gamma^p_{iq}\}\zeta_p \frac{\partial G^i}{\partial t}\frac{\partial G^k}{\partial \lambda} + \Gamma^p_{ij}\zeta_p \frac{\partial}{\partial t}\Big(\frac{\partial G^i}{\partial \lambda}\Big).$$

Hence, subtracting the above relations and noting that $\dfrac{\partial}{\partial \lambda}\Big(\dfrac{\partial \zeta_j}{\partial t}\Big) = \dfrac{\partial}{\partial t}\Big(\dfrac{\partial \zeta_j}{\partial \lambda}\Big)$ and $\dfrac{\partial}{\partial \lambda}\Big(\dfrac{\partial G^i}{\partial t}\Big) = \dfrac{\partial}{\partial t}\Big(\dfrac{\partial G^i}{\partial \lambda}\Big)$ by assumption, we infer that

$$\frac{\partial \sigma_j}{\partial t} + \{\partial_i \Gamma^p_{kj} - \partial_k \Gamma^p_{ij} + \Gamma^q_{kj}\Gamma^p_{iq} - \Gamma^q_{ij}\Gamma^p_{kq}\}\zeta_p \frac{\partial G^k}{\partial \lambda}\frac{\partial G^i}{\partial t} - \Gamma^q_{ij}\frac{\partial G^i}{\partial t}\sigma_q = 0.$$

The assumed symmetries $\Gamma^p_{ij} = \Gamma^p_{ji}$ combined with the assumed relations $\partial_j \Gamma^p_{ik} - \partial_k \Gamma^p_{ij} + \Gamma^\ell_{ik}\Gamma^p_{j\ell} - \Gamma^\ell_{ij}\Gamma^p_{k\ell} = 0$ in Ω show that

$$\partial_i \Gamma^p_{kj} - \partial_k \Gamma^p_{ij} + \Gamma^q_{kj}\Gamma^p_{iq} - \Gamma^q_{ij}\Gamma^p_{kq} = 0,$$

on the one hand. On the other hand,

$$\sigma_j(0,\lambda) = \frac{\partial \zeta_j}{\partial \lambda}(0,\lambda) - \Gamma^p_{kj}(\mathbf{G}(0,\lambda))\zeta_p(0,\lambda)\frac{\partial G^k}{\partial \lambda}(0,\lambda) = 0,$$

since $\zeta^0_j(0,\lambda) = \zeta^0_j$ and $\mathbf{G}(0,\lambda) = x^0$ for all $0 \leq \lambda \leq 1$. Therefore, for any *fixed* value of the parameter $\lambda \in [0,1]$, each function $\sigma_j(\cdot,\lambda)$ satisfies a *Cauchy problem for an ordinary differential equation*, viz.,

$$\frac{\mathrm{d}\sigma_j}{\mathrm{d}t}(t,\lambda) = \Gamma^q_{ij}(\mathbf{G}(t,\lambda))\frac{\partial G^i}{\partial t}(t,\lambda)\sigma_q(t,\lambda),\ 0 \leq t \leq 1,$$
$$\sigma_j(0,\lambda) = 0.$$

But the solution of such a Cauchy problem is unique; hence $\sigma_j(t,\lambda) = 0$ for all $0 \le t \le 1$. In particular then,

$$\begin{aligned}\sigma_j(1,\lambda) &= \frac{\partial \zeta_j}{\partial \pi}(1,\lambda) - \Gamma^p_{kj}(\mathbf{G}(1,\lambda))\zeta_p(1,\lambda)\frac{\partial G^k}{\partial \pi}(1,\lambda)\\ &= 0 \text{ for all } 0 \le \lambda \le 1,\end{aligned}$$

and thus $\dfrac{\partial \zeta_j}{\partial \lambda}(1,\lambda) = 0$ for all $0 \le \lambda \le 1$, since $\mathbf{G}(1,\lambda) = x^1$ for all $0 \le \lambda \le 1$.

For each integer ℓ, we may thus unambiguously define a vector field $(F_{\ell j}) : \Omega \to \mathbb{R}^3$ by letting

$$F_{\ell j}(x^1) := \zeta_j(1) \text{ for any } x^1 \in \Omega,$$

where $\boldsymbol{\gamma} \in \mathcal{C}^1([0,1];\mathbb{R}^3)$ is any path joining x^0 to x^1 in Ω and the vector field $(\zeta_j) \in \mathcal{C}^1([0,1])$ is the solution to the Cauchy problem

$$\begin{aligned}&\frac{\mathrm{d}\zeta_j}{\mathrm{d}t}(t) = \Gamma^p_{ij}(\boldsymbol{\gamma}(t))\frac{\mathrm{d}\gamma^i}{\mathrm{d}t}(t)\zeta_p(t),\ 0 \le t \le 1,\\ &\zeta_j(0) = \zeta^0_j,\end{aligned}$$

corresponding to such a path.

To establish that such a vector field is indeed the ℓ-th row-vector field of the unknown matrix field we are seeking, we need to show that $(F_{\ell j})^3_{j=1} \in \mathcal{C}^1(\Omega;\mathbb{R}^3)$ and that this field does satisfy the partial differential equations $\partial_i F_{\ell j} = \Gamma^p_{ij} F_{\ell p}$ in Ω corresponding to the fixed integer ℓ used in the above Cauchy problem.

Let x be an arbitrary point in Ω and let the integer $i \in \{1,2,3\}$ be *fixed* in what follows. Then there exist $x^1 \in \Omega$, a path $\boldsymbol{\gamma} \in \mathcal{C}^1([0,1];\mathbb{R}^3)$ joining x^0 to x^1, $\tau \in]0,1[$, and an open interval $I \subset [0,1]$ containing τ such that

$$\boldsymbol{\gamma}(t) = x + (t-\tau)\boldsymbol{e}_i \text{ for } t \in I,$$

where $\boldsymbol{e}_i$ is the i-th basis vector in $\mathbb{R}^3$. Since each function ζ_j is continuously differentiable in $[0,1]$ and satisfies $\dfrac{\mathrm{d}\zeta_j}{\mathrm{d}t}(t) = \Gamma^p_{ij}(\boldsymbol{\gamma}(t))\dfrac{\mathrm{d}\gamma^i}{\mathrm{d}t}(t)\zeta_p(t)$ for all $0 \le t \le 1$, we have

$$\begin{aligned}\zeta_j(t) &= \zeta_j(\tau) + (t-\tau)\frac{\mathrm{d}\zeta_j}{\mathrm{d}t}(\tau) + o(t-\tau)\\ &= \zeta_j(\tau) + (t-\tau)\Gamma^p_{ij}(\boldsymbol{\gamma}(\tau))\zeta_p(\tau) + o(t-\tau)\end{aligned}$$

for all $t \in I$. Equivalently,

$$F_{\ell j}(x + (t-\tau)\boldsymbol{e}_i) = F_{\ell j}(x) + (t-\tau)\Gamma^p_{ij}(x)F_{\ell p}(x) + o(t-x).$$

This relation shows that each function $F_{\ell j}$ possesses partial derivatives in the set Ω, given at each $x \in \Omega$ by

$$\partial_i F_{\ell p}(x) = \Gamma^p_{ij}(x)F_{\ell p}(x).$$

Consequently, the matrix field $(F_{\ell j})$ is of class $\mathcal{C}^1$ in Ω (its partial derivatives are continuous in Ω) and it satisfies the partial differential equations $\partial_i F_{\ell j} = \Gamma_{ij}^p F_{\ell p}$ in Ω, as desired. Differentiating these equations shows that the matrix field $(F_{\ell j})$ is in fact of class $\mathcal{C}^2$ in Ω.

In order to conclude the proof of the theorem, it remains to adequately choose the initial values $F_{\ell j}^0$ at x^0 in step (ii).

(iii) *Let Ω be a connected and simply-connected open subset of $\mathbb{R}^3$ and let $(g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ be a matrix field satisfying*

$$\partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma_{ij}^p \Gamma_{kqp} - \Gamma_{ik}^p \Gamma_{jqp} = 0 \text{ in } \Omega,$$

the functions $\Gamma_{ijq}, \Gamma_{ij}^p$, and g^{pq} being defined by

$$\Gamma_{ijq} := \frac{1}{2}(\partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij}), \quad \Gamma_{ij}^p := g^{pq}\Gamma_{ijq}, \quad (g^{pq}) := (g_{ij})^{-1}.$$

Given an arbitrary point $x^0 \in \Omega$, let $(F_{\ell j}^0) \in \mathbb{S}^3_>$ denote the square root of the matrix $(g_{ij}^0) := (g_{ij}(x^0)) \in \mathbb{S}^3_>$.

Let $(F_{\ell j}) \in \mathcal{C}^2(\Omega; \mathbb{M}^3)$ denote the solution to the corresponding system

$$\begin{aligned} \partial_i F_{\ell j}(x) &= \Gamma_{ij}^p(x) F_{\ell p}(x),\ x \in \Omega, \\ F_{\ell j}(x^0) &= F_{\ell j}^0, \end{aligned}$$

which exists and is unique by parts (i) *and* (ii). *Then there exists an immersion $\boldsymbol{\Theta} = (\Theta_\ell) \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ such that*

$$\partial_j \Theta_\ell = F_{\ell j} \text{ and } g_{ij} = \partial_i \boldsymbol{\Theta} \cdot \partial_j \boldsymbol{\Theta} \text{ in } \Omega.$$

To begin with, we show that *the three vector fields defined by*

$$\boldsymbol{g}_j := (F_{\ell j})_{\ell=1}^3 \in \mathcal{C}^2(\Omega; \mathbb{R}^3)$$

satisfy

$$\boldsymbol{g}_i \cdot \boldsymbol{g}_j = g_{ij} \text{ in } \Omega.$$

To this end, we note that, by construction, these fields satisfy

$$\begin{aligned} \partial_i \boldsymbol{g}_j &= \Gamma_{ij}^p \boldsymbol{g}_p \text{ in } \Omega, \\ \boldsymbol{g}_j(x^0) &= \boldsymbol{g}_j^0, \end{aligned}$$

where $\boldsymbol{g}_j^0$ is the j-th column vector of the matrix $(F_{\ell j}^0) \in \mathbb{S}^3_>$. Hence the matrix field $(\boldsymbol{g}_i \cdot \boldsymbol{g}_j) \in \mathcal{C}^2(\Omega; \mathbb{M}^3)$ satisfies

$$\begin{aligned} \partial_k(\boldsymbol{g}_i \cdot \boldsymbol{g}_j) &= \Gamma_{kj}^m(\boldsymbol{g}_m \cdot \boldsymbol{g}_i) + \Gamma_{ki}^m(\boldsymbol{g}_m \cdot \boldsymbol{g}_j) \text{ in } \Omega, \\ (\boldsymbol{g}_i \cdot \boldsymbol{g}_j)(x^0) &= g_{ij}^0. \end{aligned}$$

The definitions of the functions Γ_{ijq} and Γ_{ij}^p imply that

$$\partial_k g_{ij} = \Gamma_{ikj} + \Gamma_{jki} \text{ and } \Gamma_{ijq} = g_{pq}\Gamma_{ij}^p.$$

Hence the matrix field $(g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ satisfies

$$\partial_k g_{ij} = \Gamma^m_{kj} g_{mi} + \Gamma^m_{ki} g_{mj} \text{ in } \Omega,$$
$$g_{ij}(x^0) = g^0_{ij}.$$

Viewed as a system of partial differential equations, together with initial values at x^0, with respect to the matrix field $(g_{ij}) : \Omega \to \mathbb{M}^3$, the above system can have *at most one solution* in the space $\mathcal{C}^2(\Omega; \mathbb{M}^3)$.

To see this, let $x^1 \in \Omega$ be distinct from x^0 and let $\boldsymbol{\gamma} \in \mathcal{C}^1([0,1]; \mathbb{R}^3)$ be any path joining x^0 to x^1 in Ω, as in part (ii). Then the nine functions $g_{ij}(\boldsymbol{\gamma}(t))$, $0 \leq t \leq 1$, satisfy a Cauchy problem for a linear system of nine ordinary differential equations and this system has *at most one* solution.

An inspection of the two above systems therefore shows that their solutions are identical, i.e., that $\boldsymbol{g}_i \cdot \boldsymbol{g}_j = g_{ij}$.

It remains to show that *there exists an immersion* $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ *such that*

$$\partial_i \boldsymbol{\Theta} = \boldsymbol{g}_i \text{ in } \Omega,$$

where $\boldsymbol{g}_i := (F_{\ell j})^3_{\ell=1}$.

Since the functions Γ^p_{ij} satisfy $\Gamma^p_{ij} = \Gamma^p_{ji}$, any solution $(F_{\ell j}) \in \mathcal{C}^2(\Omega; \mathbb{M}^3)$ of the system

$$\partial_i F_{\ell j}(x) = \Gamma^p_{ij}(x) F_{\ell p}(x),\ x \in \Omega,$$
$$F_{\ell j}(x^0) = F^0_{\ell j}$$

satisfies

$$\partial_i F_{\ell j} = \partial_j F_{\ell i} \text{ in } \Omega.$$

The open set Ω being *simply-connected, Poincaré's lemma* (for a proof, see, e.g., Flanders [1989], Schwartz [1992, Vol. 2, Theorem 59 and Corollary 1, p. 234–235], or Spivak [1999]) shows that, *for each integer* ℓ, there exists a function $\Theta_\ell \in \mathcal{C}^3(\Omega)$ such that

$$\partial_i \Theta_\ell = F_{\ell i} \text{ in } \Omega,$$

or, equivalently, such that the mapping $\boldsymbol{\Theta} := (\Theta_\ell) \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ satisfies

$$\partial_i \boldsymbol{\Theta} = \boldsymbol{g}_i \text{ in } \Omega.$$

That $\boldsymbol{\Theta}$ is an immersion follows from the assumed invertibility of the matrices (g_{ij}). The proof is thus complete. □

Remarks. (1) The assumptions

$$\partial_j \Gamma^p_{ik} - \partial_k \Gamma^p_{ij} + \Gamma^\ell_{ik} \Gamma^p_{j\ell} - \Gamma^\ell_{ij} \Gamma^p_{k\ell} = 0 \text{ in } \Omega,$$

made in part (ii) on the functions $\Gamma^p_{ij} = \Gamma^p_{ji}$ are thus *sufficient* conditions for the equations $\partial_i F_{\ell j} = \Gamma^p_{ij} F_{\ell p}$ in Ω to have solutions. Conversely, a simple computation shows that they are also *necessary* conditions, simply expressing that, if these equations have a solution, then necessarily $\partial_{ik}F_{\ell j} = \partial_{ki}F_{\ell j}$ in Ω.

It is no surprise that these necessary conditions are of the same nature as those of Theorem 1.5-1, viz., $\partial_{ik}\boldsymbol{g}_j = \partial_{ij}\boldsymbol{g}_k$ in Ω.

(2) The assumed positive definiteness of the matrices (g_{ij}) is used only in part (iii), for defining *ad hoc* initial vectors $\boldsymbol{g}_i^0$. $\square$

The definitions of the functions Γ_{ij}^p and Γ_{ijq} imply that the functions

$$R_{qijk} := \partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma_{ij}^p \Gamma_{kqp} - \Gamma_{ik}^p \Gamma_{jqp}$$

satisfy, for all i, j, k, p,

$$R_{qijk} = R_{jkqi} = -R_{qikj},$$
$$R_{qijk} = 0 \text{ if } j = k \text{ or } q = i.$$

These relations in turn imply that *the eighty-one sufficient conditions*

$$R_{qijk} = 0 \text{ in } \Omega \text{ for all } i, j, k, q \in \{1, 2, 3\},$$

are satisfied if and only if the six relations

$$R_{1212} = R_{1213} = R_{1223} = R_{1313} = R_{1323} = R_{2323} = 0 \text{ in } \Omega$$

are satisfied (as is immediately verified, there are other sets of six relations that will suffice as well, again owing to the relations satisfied by the functions R_{qijk} for all i, j, k, q).

To conclude, we briefly review various *extensions of the fundamental existence result* of Theorem 1.6-1. First, a quick look at its proof reveals that it holds *verbatim* in *any dimension* $d \geq 2$, i.e., with $\mathbb{R}^3$ replaced by $\mathbb{R}^d$ and $\mathbf{E}^3$ by a d-dimensional Euclidean space $\mathbf{E}^d$. This extension only demands that Latin indices and exponents now range in the set $\{1, 2, \ldots, d\}$ and that the sets of matrices $\mathbb{M}^3, \mathbb{S}^3$, and $\mathbb{S}^3_>$ be replaced by their d-dimensional counterparts $\mathbb{M}^d, \mathbb{S}^d$, and $\mathbb{S}^d_>$.

The *regularity assumptions* on the components g_{ij} of the symmetric positive definite matrix field $\mathbf{C} = (g_{ij})$ made in Theorem 1.6-1, viz., that $g_{ij} \in \mathcal{C}^2(\Omega)$, can be significantly weakened. More specifically, C. Mardare [2003] has shown that the existence theorem still holds if $g_{ij} \in \mathcal{C}^1(\Omega)$, with a resulting mapping $\boldsymbol{\Theta}$ in the space $\mathcal{C}^2(\Omega; \mathbf{E}^d)$. Then S. Mardare [2004] has shown that the existence theorem still holds if $g_{ij} \in W^{2,\infty}_{\text{loc}}(\Omega)$, with a resulting mapping $\boldsymbol{\Theta}$ in the space $W^{2,\infty}_{\text{loc}}(\Omega; \mathbf{E}^d)$. As expected, the sufficient conditions $R_{qijk} = 0$ in Ω of Theorem 1.6-1 are then assumed to hold only in the sense of distributions, viz., as

$$\int_\Omega \{-\Gamma_{ikq}\partial_j\varphi + \Gamma_{ijq}\partial_k\varphi + \Gamma_{ij}^p\Gamma_{kqp}\varphi - \Gamma_{ik}^p\Gamma_{jqp}\varphi\}\,\mathrm{d}x = 0$$

for all $\varphi \in \mathcal{D}(\Omega)$.

The existence result has also been extended "*up to the boundary of the set* Ω" by Ciarlet & C. Mardare [2004a]. More specifically, assume that the set Ω

satisfies the "*geodesic property*" (in effect, a mild smoothness assumption on the boundary $\partial\Omega$, satisfied in particular if $\partial\Omega$ is Lipschitz-continuous) and that the functions g_{ij} and their partial derivatives of order ≤ 2 can be extended by continuity to the closure $\overline{\Omega}$, the symmetric matrix field extended in this fashion remaining positive-definite over the set $\overline{\Omega}$. Then the immersion $\boldsymbol{\Theta}$ and its partial derivatives of order ≤ 3 can be also extended by continuity to $\overline{\Omega}$.

Ciarlet & C. Mardare [2004a] have also shown that, if in addition the geodesic distance is equivalent to the Euclidean distance on Ω (a property stronger than the "geodesic property", but again satisfied if the boundary $\partial\Omega$ is Lipschitz-continuous), then a matrix field $(g_{ij}) \in \mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n_>)$ with a Riemann curvature tensor vanishing in Ω can be extended to a matrix field $(\widetilde{g}_{ij}) \in \mathcal{C}^2(\widetilde{\Omega}; \mathbb{S}^n_>)$ defined on a connected open set $\widetilde{\Omega}$ containing $\overline{\Omega}$ and whose Riemann curvature tensor still vanishes in $\widetilde{\Omega}$. This result relies on the existence of continuous extensions to $\overline{\Omega}$ of the immersion $\boldsymbol{\Theta}$ and its partial derivatives of order ≤ 3 and on a deep extension theorem of Whitney [1934].

1.7 UNIQUENESS UP TO ISOMETRIES OF IMMERSIONS WITH THE SAME METRIC TENSOR

In Section 1.6, we have established the *existence* of an immersion $\boldsymbol{\Theta} : \Omega \subset \mathbb{R}^3 \to \mathbf{E}^3$ giving rise to a set $\boldsymbol{\Theta}(\Omega)$ with a prescribed metric tensor, provided the given metric tensor field satisfies *ad hoc* sufficient conditions. We now turn to the question of *uniqueness* of such immersions.

This uniqueness result is the object of the next theorem, aptly called a **rigidity theorem** in view of its geometrical interpretation: It asserts that, if two immersions $\widetilde{\boldsymbol{\Theta}} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ and $\boldsymbol{\Theta} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ share the same metric tensor field, then the set $\boldsymbol{\Theta}(\Omega)$ is obtained by subjecting the set $\widetilde{\boldsymbol{\Theta}}(\Omega)$ either to a *rotation* (represented by an orthogonal matrix $\mathbf{Q}$ with $\det \mathbf{Q} = 1$), or to a *symmetry with respect to a plane followed by a rotation* (together represented by an orthogonal matrix $\mathbf{Q}$ with $\det \mathbf{Q} = -1$), then by subjecting the rotated set to a *translation* (represented by a vector $\boldsymbol{c}$).

The terminology "rigidity theorem" reflects that such a geometric transformation indeed corresponds to the idea of a "*rigid transformation*" of the set $\boldsymbol{\Theta}(\Omega)$ (provided a symmetry is included in this definition).

Let $\mathbb{O}^3$ denote the set of all orthogonal matrices of order three.

Theorem 1.7-1. *Let Ω be a connected open subset of $\mathbb{R}^3$ and let $\boldsymbol{\Theta} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ and $\widetilde{\boldsymbol{\Theta}} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ be two immersions such that their associated metric tensors satisfy*

$$\boldsymbol{\nabla}\boldsymbol{\Theta}^T \boldsymbol{\nabla}\boldsymbol{\Theta} = \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^T \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}} \text{ in } \Omega.$$

Then there exist a vector $\boldsymbol{c} \in \mathbf{E}^3$ and an orthogonal matrix $\mathbf{Q} \in \mathbb{O}^3$ such that

$$\boldsymbol{\Theta}(x) = \boldsymbol{c} + \mathbf{Q}\widetilde{\boldsymbol{\Theta}}(x) \text{ for all } x \in \Omega.$$

Proof. For convenience, *the three-dimensional vector space* $\mathbb{R}^3$ *is identified throughout this proof with the Euclidean space* $\mathbf{E}^3$. In particular then, $\mathbb{R}^3$ inherits the inner product and norm of $\mathbf{E}^3$. The *spectral norm* of a matrix $\mathbf{A} \in \mathbb{M}^3$ is denoted

$$|\mathbf{A}| := \sup\{|\mathbf{A}\boldsymbol{b}|;\, \boldsymbol{b} \in \mathbb{R}^3,\, |\boldsymbol{b}| = 1\}.$$

To begin with, we consider the *special case* where $\widetilde{\mathbf{\Theta}} : \Omega \to \mathbf{E}^3 = \mathbb{R}^3$ is the *identity mapping*. The issue of uniqueness reduces in this case to finding $\mathbf{\Theta} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ such that

$$\boldsymbol{\nabla}\mathbf{\Theta}(x)^T \boldsymbol{\nabla}\mathbf{\Theta}(x) = \mathbf{I} \text{ for all } x \in \Omega.$$

Parts (i) to (iii) are devoted to solving these equations.

(i) We first establish that *a mapping* $\mathbf{\Theta} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ *that satisfies*

$$\boldsymbol{\nabla}\mathbf{\Theta}(x)^T \boldsymbol{\nabla}\mathbf{\Theta}(x) = \mathbf{I} \text{ for all } x \in \Omega$$

is locally an isometry: Given any point $x^0 \in \Omega$*, there exists an open neighborhood* V *of* x^0 *contained in* Ω *such that*

$$|\mathbf{\Theta}(y) - \mathbf{\Theta}(x)| = |y - x| \text{ for all } x, y \in V.$$

Let B be an open ball centered at x^0 and contained in Ω. Since the set B is convex, the *mean-value theorem* (for a proof, see, e.g., Schwartz [1992]) can be applied. It shows that

$$|\mathbf{\Theta}(y) - \mathbf{\Theta}(x)| \leq \sup_{z \in]x,y[} |\boldsymbol{\nabla}\mathbf{\Theta}(z)||y - x| \text{ for all } x, y \in B.$$

Since the spectral norm of an orthogonal matrix is one, we thus have

$$|\mathbf{\Theta}(y) - \mathbf{\Theta}(x)| \leq |y - x| \text{ for all } x, y \in B.$$

Since the matrix $\boldsymbol{\nabla}\mathbf{\Theta}(x^0)$ is invertible, the *local inversion theorem* (for a proof, see, e.g., Schwartz [1992]) shows that there exist an open neighborhood V of x^0 contained in Ω and an open neighborhood $\widehat{V}$ of $\mathbf{\Theta}(x^0)$ in $\mathbf{E}^3$ such that the restriction of $\mathbf{\Theta}$ to V is a $\mathcal{C}^1$-diffeomorphism from V onto $\widehat{V}$. Besides, there is no loss of generality in assuming that V is contained in B and that $\widehat{V}$ is convex (to see this, apply the local inversion theorem first to the restriction of $\mathbf{\Theta}$ to B, thus producing a "first" neighborhood V' of x^0 contained in B, then to the restriction of the inverse mapping obtained in this fashion to an open ball V centered at $\mathbf{\Theta}(x^0)$ and contained in $\mathbf{\Theta}(V')$).

Let $\mathbf{\Theta}^{-1} : \widehat{V} \to V$ denote the inverse mapping of $\mathbf{\Theta} : V \to \widehat{V}$. The chain rule applied to the relation $\mathbf{\Theta}^{-1}(\mathbf{\Theta}(x)) = x$ for all $x \in V$ then shows that

$$\widehat{\boldsymbol{\nabla}}\mathbf{\Theta}^{-1}(\widehat{x}) = \boldsymbol{\nabla}\mathbf{\Theta}(x)^{-1} \text{ for all } \widehat{x} = \mathbf{\Theta}(x), x \in V.$$

The matrix $\widehat{\boldsymbol{\nabla}}\boldsymbol{\Theta}^{-1}(\widehat{x})$ being thus orthogonal for all $\widehat{x} \in \widehat{V}$, the mean-value theorem applied in the convex set $\widehat{V}$ shows that

$$|\boldsymbol{\Theta}^{-1}(\widehat{y}) - \boldsymbol{\Theta}^{-1}(\widehat{x})| \leq |\widehat{y} - \widehat{x}| \text{ for all } \widehat{x}, \widehat{y} \in \widehat{V},$$

or equivalently, that

$$|y - x| \leq |\boldsymbol{\Theta}(y) - \boldsymbol{\Theta}(x)| \text{ for all } x, y \in V.$$

The restriction of the mapping $\boldsymbol{\Theta}$ to the open neighborhood V of x^0 is thus an isometry.

(ii) We next establish that, *if a mapping* $\boldsymbol{\Theta} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ *is locally an isometry*, in the sense that, given any $x^0 \in \Omega$, there exists an open neighborhood V of x^0 contained in Ω such that $|\boldsymbol{\Theta}(y) - \boldsymbol{\Theta}(x)| = |y - x|$ for all $x, y \in V$, *then its derivative is locally constant*, in the sense that

$$\boldsymbol{\nabla}\boldsymbol{\Theta}(x) = \boldsymbol{\nabla}\boldsymbol{\Theta}(x^0) \text{ for all } x \in V.$$

The set V being that found in (i), let the differentiable function $F : V \times V \to \mathbb{R}$ be defined for all $x = (x_p) \in V$ and all $y = (y_p) \in V$ by

$$F(x, y) := (\Theta_\ell(y) - \Theta_\ell(x))(\Theta_\ell(y) - \Theta_\ell(x)) - (y_\ell - x_\ell)(y_\ell - x_\ell).$$

Then $F(x, y) = 0$ for all $x, y \in V$ by (i). Hence

$$G_i(x, y) := \frac{1}{2}\frac{\partial F}{\partial y_i}(x, y) = \frac{\partial \Theta_\ell}{\partial y_i}(y)(\Theta_\ell(y) - \Theta_\ell(x)) - \delta_{i\ell}(y_\ell - x_\ell) = 0$$

for all $x, y \in V$. For a fixed $y \in V$, each function $G_i(\cdot, y) : V \to \mathbb{R}$ is differentiable and its derivative vanishes. Consequently,

$$\frac{\partial G_i}{\partial x_i}(x, y) = -\frac{\partial \Theta_\ell}{\partial y_i}(y)\frac{\partial \Theta_\ell}{\partial x_j}(x) + \delta_{ij} = 0 \text{ for all } x, y \in V,$$

or equivalently, in matrix form,

$$\boldsymbol{\nabla}\boldsymbol{\Theta}(y)^T \boldsymbol{\nabla}\boldsymbol{\Theta}(x) = \mathbf{I} \text{ for all } x, y \in V.$$

Letting $y = x^0$ in this relation shows that

$$\boldsymbol{\nabla}\boldsymbol{\Theta}(x) = \boldsymbol{\nabla}\boldsymbol{\Theta}(x^0) \text{ for all } x \in V.$$

(iii) By (ii), the mapping $\boldsymbol{\nabla}\boldsymbol{\Theta} : \Omega \to \mathbb{M}^3$ is differentiable and its derivative vanishes in Ω. Therefore *the mapping* $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$ *is twice differentiable and its second Fréchet derivative vanishes in* Ω. The open set Ω being *connected*, a classical result from differential calculus (see, e.g., Schwartz [1992, Theorem 3.7.10]) shows that the mapping $\boldsymbol{\Theta}$ is *affine* in Ω, i.e., that there exists a vector

$\boldsymbol{c} \in \mathbf{E}^3$ and a matrix $\mathbf{Q} \in \mathbb{M}^3$ such that (the notation $\boldsymbol{ox}$ designates the column vector with components x_i)

$$\boldsymbol{\Theta}(x) = \boldsymbol{c} + \mathbf{Q}\boldsymbol{ox} \text{ for all } x \in \Omega.$$

Since $\mathbf{Q} = \boldsymbol{\nabla\Theta}(x^0)$ and $\boldsymbol{\nabla\Theta}(x^0)^T\boldsymbol{\nabla\Theta}(x^0) = \mathbf{I}$ by assumption, the matrix $\mathbf{Q}$ is *orthogonal*.

(iv) We now consider the general equations $g_{ij} = \widetilde{g}_{ij}$ in Ω, noting that they also read

$$\boldsymbol{\nabla\Theta}(x)^T\boldsymbol{\nabla\Theta}(x) = \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}(x)^T\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}(x) \text{ for all } x \in \Omega.$$

Given any point $x^0 \in \Omega$, let the neighborhoods V of x^0 and $\widehat{V}$ of $\boldsymbol{\Theta}(x^0)$ and the mapping $\boldsymbol{\Theta}^{-1} : \widehat{V} \to V$ be defined as in part (i) (by assumption, the mapping $\boldsymbol{\Theta}$ is an immersion; hence the matrix $\boldsymbol{\nabla\Theta}(x^0)$ is invertible).

Consider the composite mapping

$$\widehat{\boldsymbol{\Phi}} := \widetilde{\boldsymbol{\Theta}} \circ \boldsymbol{\Theta}^{-1} : \widehat{V} \to \mathbf{E}^3.$$

Clearly, $\widehat{\boldsymbol{\Phi}} \in \mathcal{C}^1(\widehat{V}; \mathbf{E}^3)$ and

$$\begin{aligned}\widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{\Phi}}(\widehat{x}) &= \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}(x)\widehat{\boldsymbol{\nabla}}\boldsymbol{\Theta}^{-1}(\widehat{x}) \\ &= \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}(x)\boldsymbol{\nabla\Theta}(x)^{-1} \text{ for all } \widehat{x} = \boldsymbol{\Theta}(x), x \in V.\end{aligned}$$

Hence the assumed relations

$$\boldsymbol{\nabla\Theta}(x)^T\boldsymbol{\nabla\Theta}(x) = \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}(x)^T\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}(x) \text{ for all } x \in \Omega$$

imply that

$$\widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{\Phi}}(\widehat{x})^T\widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{\Phi}}(\widehat{x}) = \mathbf{I} \text{ for all } x \in V.$$

By parts (i) to (iii), there thus exist a vector $\boldsymbol{c} \in \mathbb{R}^3$ and a matrix $\mathbf{Q} \in \mathbb{O}^3$ such that

$$\widehat{\boldsymbol{\Phi}}(\widehat{x}) = \widetilde{\boldsymbol{\Theta}}(x) = \boldsymbol{c} + \mathbf{Q}\boldsymbol{\Theta}(x) \text{ for all } \widehat{x} = \boldsymbol{\Theta}(x), x \in V,$$

hence such that

$$\boldsymbol{\Xi}(x) := \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}(x)\boldsymbol{\nabla\Theta}(x)^{-1} = \mathbf{Q} \text{ for all } x \in V.$$

The *continuous* mapping $\boldsymbol{\Xi} : V \to \mathbb{M}^3$ defined in this fashion is thus *locally constant* in Ω. As in part (iii), we conclude from the assumed *connectedness* of Ω that the mapping $\boldsymbol{\Xi}$ is *constant in* Ω. Thus the proof is complete. □

An **isometry of** $\mathbf{E}^3$ is a mapping $\mathbf{J} : \mathbf{E}^3 \to \mathbf{E}^3$ of the form $\mathbf{J}(x) = \boldsymbol{c} + \mathbf{Q}\,\boldsymbol{ox}$ for all $x \in \mathbf{E}^3$, with $\boldsymbol{c} \in \mathbf{E}^3$ and $\mathbf{Q} \in \mathbb{O}^3$ (an analogous definition holds *verbatim* in any Euclidean space of dimension $d \geq 2$). Clearly, *an isometry preserves distances* in the sense that

$$|\mathbf{J}(y) - \mathbf{J}(x)| = |y - x| \text{ for all } x, y \in \Omega.$$

Remarkably, the converse is also true, according to the classical **Mazur-Ulam theorem**, which asserts the following: *Let* Ω *be a connected subset in* $\mathbb{R}^d$*, and let* $\boldsymbol{\Theta} : \Omega \to \mathbb{R}^d$ *be a mapping that satisfies*

$$|\boldsymbol{\Theta}(y) - \boldsymbol{\Theta}(x)| = |y - x| \text{ for all } x, y \in \Omega.$$

Then $\boldsymbol{\Theta}$ *is an isometry of* $\mathbb{R}^d$*.*

Parts (ii) and (iii) of the above proof thus provide a proof of this theorem under the additional assumption that the mapping $\boldsymbol{\Theta}$ is of class $\mathcal{C}^1$ (the extension from $\mathbb{R}^3$ to $\mathbb{R}^d$ is trivial).

In Theorem 1.7-1, the special case where $\boldsymbol{\Theta}$ is the identity mapping of $\mathbb{R}^3$ identified with $\mathbf{E}^3$ is the classical **Liouville theorem**. This theorem thus asserts that *if a mapping* $\boldsymbol{\Theta} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ *is such that* $\boldsymbol{\nabla}\boldsymbol{\Theta}(x) \in \mathbb{O}^3$ *for all* $x \in \Omega$*, where* Ω *is an open connected subset of* $\mathbb{R}^3$*, then* $\boldsymbol{\Theta}$ *is an isometry.*

Two mappings $\boldsymbol{\Theta} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ and $\widetilde{\boldsymbol{\Theta}} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ are said to be **isometrically equivalent** if there exist $\boldsymbol{c} \in \mathbf{E}^3$ and $\mathbf{Q} \in \mathbb{O}^3$ such that $\boldsymbol{\Theta} = \boldsymbol{c} + \mathbf{Q}\widetilde{\boldsymbol{\Theta}}$ in Ω, i.e., such that $\boldsymbol{\Theta} = \mathbf{J} \circ \widetilde{\boldsymbol{\Theta}}$, where $\mathbf{J}$ is an *isometry* of $\mathbf{E}^3$. Theorem 1.7-1 thus asserts that *two immersions* $\boldsymbol{\Theta} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ *and* $\widetilde{\boldsymbol{\Theta}} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ *share the same metric tensor field over an open connected subset* Ω *of* $\mathbb{R}^3$ *if and only if they are isometrically equivalent.*

Remark. In terms of covariant components g_{ij} of metric tensors, parts (i) to (iii) of the above proof provide the solution to the equations $g_{ij} = \delta_{ij}$ in Ω, while part (iv) provides the solution to the equations $g_{ij} = \partial_i\widetilde{\boldsymbol{\Theta}} \cdot \partial_j\widetilde{\boldsymbol{\Theta}}$ in Ω, where $\widetilde{\boldsymbol{\Theta}} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ is a given immersion. □

While the immersions $\boldsymbol{\Theta}$ found in Theorem 1.6-1 are thus only defined up to isometries in $\mathbf{E}^3$, they become *uniquely determined* if they are required to satisfy *ad hoc* additional conditions, according to the following corollary to Theorems 1.6-1 and 1.7-1.

Theorem 1.7-2. *Let the assumptions on the set* Ω *and on the matrix field* $\mathbf{C}$ *be as in* Theorem 1.6-1*, let a point* $x_0 \in \Omega$ *be given, and let* $\mathbf{F}_0 \in \mathbb{M}^3$ *be any matrix that satisfies*

$$\mathbf{F}_0^T\mathbf{F}_0 = \mathbf{C}(x_0).$$

Then there exists one and only one immersion $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ *that satisfies*

$$\boldsymbol{\nabla}\boldsymbol{\Theta}(x)^T\boldsymbol{\nabla}\boldsymbol{\Theta}(x) = \mathbf{C}(x) \text{ for all } x \in \Omega,$$
$$\boldsymbol{\Theta}(x_0) = \mathbf{0} \text{ and } \boldsymbol{\nabla}\boldsymbol{\Theta}(x_0) = \mathbf{F}_0.$$

Proof. Given any immersion $\boldsymbol{\Phi} \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ that satisfies $\boldsymbol{\nabla}\boldsymbol{\Phi}(x)^T\boldsymbol{\nabla}\boldsymbol{\Phi}(x) = \mathbf{C}(x)$ for all $x \in \Omega$ (such immersions exist by Theorem 1.6-1), let the mapping $\boldsymbol{\Theta} : \Omega \to \mathbb{R}^3$ be defined by

$$\boldsymbol{\Theta}(x) := \mathbf{F}_0\boldsymbol{\nabla}\boldsymbol{\Phi}(x_0)^{-1}(\boldsymbol{\Phi}(x) - \boldsymbol{\Phi}(x_0)) \text{ for all } x \in \Omega.$$

Then it is immediately verified that this mapping $\boldsymbol{\Theta}$ satisfies the announced properties.

Besides, it is uniquely determined. To see this, let $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ and $\boldsymbol{\Phi} \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ be two immersions that satisfy

$$\boldsymbol{\nabla}\boldsymbol{\Theta}(x)^T \boldsymbol{\nabla}\boldsymbol{\Theta}(x) = \boldsymbol{\nabla}\boldsymbol{\Phi}(x)^T \boldsymbol{\nabla}\boldsymbol{\Phi}(x) \text{ for all } x \in \Omega.$$

Hence there exist (by Theorem 1.7-1) $\boldsymbol{c} \in \mathbb{R}^3$ and $\mathbf{Q} \in \mathbb{O}^3$ such that $\boldsymbol{\Phi}(x) = \boldsymbol{c} + \mathbf{Q}\boldsymbol{\Theta}(x)$ for all $x \in \Omega$, so that $\boldsymbol{\nabla}\boldsymbol{\Phi}(x) = \mathbf{Q}\boldsymbol{\nabla}\boldsymbol{\Theta}(x)$ for all $x \in \Omega$. The relation $\boldsymbol{\nabla}\boldsymbol{\Theta}(x_0) = \boldsymbol{\nabla}\boldsymbol{\Phi}(x_0)$ then implies that $\mathbf{Q} = \mathbf{I}$ and the relation $\boldsymbol{\Theta}(x_0) = \boldsymbol{\Phi}(x_0)$ in turn implies that $\boldsymbol{c} = \mathbf{0}$. □

Remark. One possible choice for the matrix $\mathbf{F}_0$ is the square root of the symmetric positive-definite matrix $\mathbf{C}(x_0)$. □

Theorem 1.7-1 constitutes the "classical" rigidity theorem, in that both immersions $\boldsymbol{\Theta}$ and $\widetilde{\boldsymbol{\Theta}}$ are assumed to be in the space $\mathcal{C}^1(\Omega; \mathbf{E}^3)$. The next theorem is an extension, due to Ciarlet & C. Mardare [2003], that covers the case where *one of the mappings belongs to the Sobolev space* $H^1(\Omega; \mathbf{E}^3)$.

The way the result in part (i) of the next proof is derived is due to Friesecke, James & Müller [2002]; the result of part (i) itself goes back to Reshetnyak [1967].

Let $\mathbb{O}^3_+$ denote the set of all *proper orthogonal matrices of order three*, i.e., of all orthogonal matrices $\mathbf{Q} \in \mathbb{O}^3$ with $\det \mathbf{Q} = 1$.

Theorem 1.7-3. *Let Ω be a connected open subset of $\mathbb{R}^3$, let $\boldsymbol{\Theta} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ be a mapping that satisfies*

$$\det \boldsymbol{\nabla}\boldsymbol{\Theta} > 0 \text{ in } \Omega,$$

and let $\widetilde{\boldsymbol{\Theta}} \in H^1(\Omega; \mathbf{E}^3)$ be a mapping that satisfies

$$\det \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}} > 0 \text{ a.e. in } \Omega \text{ and } \boldsymbol{\nabla}\boldsymbol{\Theta}^T \boldsymbol{\nabla}\boldsymbol{\Theta} = \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^T \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}} \text{ a.e. in } \Omega.$$

Then there exist a vector $\boldsymbol{c} \in \mathbf{E}^3$ and a matrix $\mathbf{Q} \in \mathbb{O}^3_+$ such that

$$\widetilde{\boldsymbol{\Theta}}(x) = \boldsymbol{c} + \mathbf{Q}\boldsymbol{\Theta}(x) \text{ for almost all } x \in \Omega.$$

Proof. The Euclidean space $\mathbf{E}^3$ is identified with the space $\mathbb{R}^3$ throughout the proof.

(i) *To begin with, consider the special case where* $\boldsymbol{\Theta}(x) = x$ for all $x \in \Omega$. In other words, we are given a mapping $\widetilde{\boldsymbol{\Theta}} \in \mathbf{H}^1(\Omega)$ that satisfies $\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}(x) \in \mathbb{O}^3_+$ for almost all $x \in \Omega$. Hence

$$\mathbf{Cof}\,\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}(x) = (\det \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}(x))\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}(x)^{-T} = \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}(x)^{-T} \text{ for almost all } x \in \Omega,$$

on the one hand. Since, on the other hand,

$$\operatorname{div} \mathbf{Cof}\,\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}} = \mathbf{0} \text{ in } (\mathcal{D}'(B))^3$$

in any open ball B such that $\overline{B} \subset \Omega$ (to see this, combine the density of $\mathcal{C}^2(\overline{B})$ in $H^1(B)$ with the classical Piola identity in the space $\mathcal{C}^2(\overline{B})$; for a proof of this identity, see, e.g., Ciarlet [1988, Theorem 1.7.1]), we conclude that

$$\Delta\widetilde{\mathbf{\Theta}} = \operatorname{div} \mathbf{Cof}\, \boldsymbol{\nabla}\widetilde{\mathbf{\Theta}} = \mathbf{0} \text{ in } (\mathcal{D}'(B))^3.$$

Hence $\widetilde{\mathbf{\Theta}} = (\widetilde{\Theta}_j) \in (\mathcal{C}^\infty(\Omega))^3$. For such mappings, the identity

$$\Delta(\partial_i\widetilde{\Theta}_j\partial_i\widetilde{\Theta}_j) = 2\partial_i\widetilde{\Theta}_j\partial_i(\Delta\widetilde{\Theta}_j) + 2\partial_{ik}\widetilde{\Theta}_j\partial_{ik}\widetilde{\Theta}_j,$$

together with the relations $\Delta\widetilde{\Theta}_j = 0$ and $\partial_i\widetilde{\Theta}_j\partial_i\widetilde{\Theta}_j = 3$ in Ω, shows that $\partial_{ik}\widetilde{\Theta}_j = 0$ in Ω. The assumed connectedness of Ω then implies that there exist a vector $\boldsymbol{c} \in \mathbf{E}^3$ and a matrix $\mathbf{Q} \in \mathbb{O}^3_+$ (by assumption, $\boldsymbol{\nabla}\widetilde{\mathbf{\Theta}}(x) \in \mathbb{O}^3_+$ for almost all $x \in \Omega$) such that

$$\widetilde{\mathbf{\Theta}}(x) = \boldsymbol{c} + \mathbf{Q}\,\boldsymbol{ox} \text{ for almost all } x \in \Omega.$$

(ii) *Consider next the general case.* Let $x_0 \in \Omega$ be given. Since $\mathbf{\Theta}$ is an immersion, the local inversion theorem can be applied; there thus exist bounded open neighborhoods U of x_0 and $\widehat{U}$ of $\mathbf{\Theta}(x_0)$ satisfying $\overline{U} \subset \Omega$ and $\{\widehat{U}\}^- \subset \mathbf{\Theta}(\Omega)$, such that the restriction $\mathbf{\Theta}_U$ of $\mathbf{\Theta}$ to U can be extended to a $\boldsymbol{\mathcal{C}}^1$-diffeomorphism from $\overline{U}$ onto $\{\widehat{U}\}^-$.

Let $\mathbf{\Theta}_U^{-1} : \widehat{U} \to U$ denote the inverse mapping of $\mathbf{\Theta}_U$, which therefore satisfies $\widehat{\boldsymbol{\nabla}}\mathbf{\Theta}_U^{-1}(\widehat{x}) = \boldsymbol{\nabla}\mathbf{\Theta}(x)^{-1}$ for all $\widehat{x} = \mathbf{\Theta}(x) \in \widehat{U}$ (the notation $\widehat{\boldsymbol{\nabla}}$ indicates that differentiation is carried out with respect to the variable $\widehat{x} \in \widehat{U}$). Define the composite mapping

$$\widehat{\mathbf{\Phi}} := \widetilde{\mathbf{\Theta}} \cdot \mathbf{\Theta}_U^{-1} : \widehat{U} \to \mathbb{R}^3.$$

Since $\widetilde{\mathbf{\Theta}} \in \mathbf{H}^1(U)$ and $\mathbf{\Theta}_U^{-1}$ can be extended to a $\mathcal{C}^1$-diffeomorphism from $\{\widehat{U}\}^-$ onto $\overline{U}$, it follows that $\widehat{\mathbf{\Phi}} \in H^1(\widehat{U};\mathbb{R}^3)$ and that

$$\widehat{\boldsymbol{\nabla}}\widehat{\mathbf{\Phi}}(\widehat{x}) = \boldsymbol{\nabla}\widetilde{\mathbf{\Theta}}(x)\widehat{\boldsymbol{\nabla}}\mathbf{\Theta}_U^{-1}(\widehat{x}) = \boldsymbol{\nabla}\widetilde{\mathbf{\Theta}}(x)\boldsymbol{\nabla}\mathbf{\Theta}(x)^{-1}$$

for almost all $\widehat{x} = \mathbf{\Theta}(x) \in \widehat{U}$ (see, e.g., Adams [1975, Chapter 3]). Hence the assumptions $\det \boldsymbol{\nabla}\mathbf{\Theta} > 0$ in Ω, $\det \boldsymbol{\nabla}\widetilde{\mathbf{\Theta}} > 0$ a.e. in Ω, and $\boldsymbol{\nabla}\mathbf{\Theta}^T\boldsymbol{\nabla}\mathbf{\Theta} = \boldsymbol{\nabla}\widetilde{\mathbf{\Theta}}^T\boldsymbol{\nabla}\widetilde{\mathbf{\Theta}}$ a.e. in Ω, together imply that $\widehat{\boldsymbol{\nabla}}\widehat{\mathbf{\Phi}}(\widehat{x}) \in \mathbb{O}^3_+$ for almost all $\widehat{x} \in \widehat{U}$. By (i), there thus exist $\boldsymbol{c} \in \mathbf{E}^3$ and $\mathbf{Q} \in \mathbb{O}^3_+$ such that

$$\widehat{\mathbf{\Phi}}(\widehat{x}) = \widetilde{\mathbf{\Theta}}(x) = \boldsymbol{c} + \mathbf{Q}\,\boldsymbol{o}\widehat{\boldsymbol{x}} \text{ for almost all } \widehat{x} = \mathbf{\Theta}(x) \in \widehat{U},$$

or equivalently, such that

$$\mathbf{\Xi}(x) := \boldsymbol{\nabla}\widetilde{\mathbf{\Theta}}(x)\boldsymbol{\nabla}\mathbf{\Theta}(x)^{-1} = \mathbf{Q} \text{ for almost all } x \in U.$$

Since the point $x_0 \in \Omega$ is arbitrary, this relation shows that $\mathbf{\Xi} \in \mathbf{L}^1_{\text{loc}}(\Omega)$. By a classical result from distribution theory (cf. Schwartz [1966, Section 2.6]), we conclude from the assumed connectedness of Ω that $\mathbf{\Xi}(x) = \mathbf{Q}$ for almost all $x \in \Omega$, and consequently that

$$\widetilde{\mathbf{\Theta}}(x) = \boldsymbol{c} + \mathbf{Q}\mathbf{\Theta}(x) \text{ for almost all } x \in \Omega.$$

$\square$

Remarks. (1) The existence of $\widetilde{\mathbf{\Theta}} \in H^1(\Omega; \mathbf{E}^3)$ satisfying the assumptions of Theorem 1.7-3 thus implies that $\mathbf{\Theta} \in H^1(\Omega; \mathbf{E}^3)$ and $\widetilde{\mathbf{\Theta}} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$.

(2) If $\widetilde{\mathbf{\Theta}} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$, the assumptions $\det \boldsymbol{\nabla}\mathbf{\Theta} > 0$ in Ω and $\det \boldsymbol{\nabla}\widetilde{\mathbf{\Theta}} > 0$ in Ω are no longer necessary; but then it can only be concluded that $\mathbf{Q} \in \mathbb{O}^3$: This is the *classical rigidity theorem* (Theorem 1.7-1), of which *Liouville's theorem* is the special case corresponding to $\mathbf{\Theta}(x) = x$ for all $x \in \Omega$.

(3) The result established in part (i) of the above proof asserts that, given a connected open subset Ω of $\mathbb{R}^3$, if a mapping $\mathbf{\Theta} \in H^1(\Omega; \mathbf{E}^3)$ is such that $\boldsymbol{\nabla}\mathbf{\Theta}(x) \in \mathbb{O}^3_+$ for almost all $x \in \Omega$, then there exist $\boldsymbol{c} \in \mathbf{E}^3$ and $\mathbf{Q} \in \mathbb{O}^3_+$ such that $\mathbf{\Theta}(x) = \boldsymbol{c} + \mathbf{Q}\boldsymbol{ox}$ for almost all $x \in \Omega$. This result thus constitutes a *generalization of Liouville's theorem*.

(4) By contrast, if the mapping $\widetilde{\mathbf{\Theta}}$ is assumed to be instead in the space $H^1(\Omega; \mathbf{E}^3)$ (as in Theorem 1.7-3), an assumption about the sign of $\det \boldsymbol{\nabla}\widetilde{\mathbf{\Theta}}$ becomes necessary. To see this, let for instance Ω be an open ball centered at the origin in $\mathbb{R}^3$, let $\mathbf{\Theta}(x) = x$, and let $\widetilde{\mathbf{\Theta}}(x) = x$ if $x_1 \geq 0$ and $\widetilde{\mathbf{\Theta}}(x) = (-x_1, x_2, x_3)$ if $x_1 < 0$. Then $\widetilde{\mathbf{\Theta}} \in H^1(\Omega; \mathbf{E}^3)$ and $\boldsymbol{\nabla}\widetilde{\mathbf{\Theta}} \in \mathbb{O}^3$ a.e. in Ω; yet there does not exist any orthogonal matrix such that $\widetilde{\mathbf{\Theta}}(x) = \mathbf{Q}\,\boldsymbol{ox}$ for all $x \in \Omega$, since $\widetilde{\mathbf{\Theta}}(\Omega) \subset \{x \in \mathbb{R}^3;\, x_1 \geq 0\}$ (this counter-example was kindly communicated to the author by Sorin Mardare).

(5) Surprisingly, the assumption $\det \boldsymbol{\nabla}\mathbf{\Theta} > 0$ in Ω *cannot* be replaced by the weaker assumption $\det \boldsymbol{\nabla}\mathbf{\Theta} > 0$ a.e. in Ω. To see this, let for instance Ω be an open ball centered at the origin in $\mathbb{R}^3$, let $\mathbf{\Theta}(x) = (x_1 x_2^2, x_2, x_3)$ and let $\widetilde{\mathbf{\Theta}}(x) = \mathbf{\Theta}(x)$ if $x_2 \geq 0$ and $\widetilde{\mathbf{\Theta}}(x) = (-x_1 x_2^2, -x_2, x_3)$ if $x_2 < 0$ (this counter-example was kindly communicated to the author by Hervé Le Dret).

(6) If a mapping $\mathbf{\Theta} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ satisfies $\det \boldsymbol{\nabla}\mathbf{\Theta} > 0$ in Ω, then $\mathbf{\Theta}$ is an immersion. Conversely, if Ω is a connected open set and $\mathbf{\Theta} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ is an immersion, then either $\det \boldsymbol{\nabla}\mathbf{\Theta} > 0$ in Ω or $\det \boldsymbol{\nabla}\mathbf{\Theta} < 0$ in Ω. The assumption that $\det \boldsymbol{\nabla}\mathbf{\Theta} > 0$ in Ω made in Theorem 1.7-3 is simply intended to fix ideas (a similar result clearly holds under the other assumption).

(7) A little further ado shows that the conclusion of Theorem 1.7-3 is still valid if $\widetilde{\mathbf{\Theta}} \in H^1(\Omega; \mathbf{E}^3)$ is replaced by the weaker assumption $\widetilde{\mathbf{\Theta}} \in H^1_{\text{loc}}(\Omega; \mathbf{E}^3)$. $\square$

Like the existence results of Section 1.6, the uniqueness theorems of this section hold *verbatim* in any dimension $d \geq 2$, with $\mathbb{R}^3$ replaced by $\mathbb{R}^d$ and $\mathbf{E}^d$ by a d-dimensional Euclidean space.

1.8 CONTINUITY OF AN IMMERSION AS A FUNCTION OF ITS METRIC TENSOR

Let Ω be a connected and simply-connected open subset of $\mathbb{R}^3$. Together, Theorems 1.6-1 and 1.7-1 establish the existence of a mapping $\mathcal{F}$ that associates with any matrix field $\mathbf{C} = (g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ satisfying

$$R_{qijk} := \partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma^p_{ij} \Gamma_{kqp} - \Gamma^p_{ik} \Gamma_{jqp} = 0 \text{ in } \Omega,$$

where the functions Γ_{ijq} and Γ^p_{ij} are defined in terms of the functions g_{ij} as in Theorem 1.6-1, a well-defined element $\mathcal{F}(\mathbf{C})$ in the quotient set $\mathcal{C}^3(\Omega; \mathbf{E}^3)/\mathcal{R}$, where $(\boldsymbol{\Theta}, \widetilde{\boldsymbol{\Theta}}) \in \mathcal{R}$ means that there exist a vector $\boldsymbol{a} \in \mathbf{E}^3$ and a matrix $\mathbf{Q} \in \mathbb{O}^3$ such that $\boldsymbol{\Theta}(x) = \boldsymbol{a} + \mathbf{Q}\widetilde{\boldsymbol{\Theta}}(x)$ for all $x \in \Omega$.

A natural question thus arises as to whether there exist natural topologies on the space $\mathcal{C}^2(\Omega; \mathbb{S}^3)$ and on the quotient set $\mathcal{C}^3(\Omega; \mathbf{E}^3)/\mathcal{R}$ such that the mapping $\mathcal{F}$ defined in this fashion is *continuous*.

Equivalently, *is an immersion a continuous function of its metric tensor?*

The object of this section, which is based on Ciarlet & Laurent [2003], is to provide an affirmative answer to this question (see Theorem 1.8-5).

Note that such a question is not only clearly relevant to differential geometry *per se*, but it also naturally arises in *nonlinear three-dimensional elasticity*. As we shall see more specifically in Section 3.1, a smooth enough immersion $\boldsymbol{\Theta} = (\Theta_i) : \Omega \to \mathbf{E}^3$ may be thought of in this context as a *deformation* of the set Ω viewed as a *reference configuration* of a *nonlinearly elastic body* (although such an immersion should then be in addition injective and orientation-preserving in order to qualify for this definition; for details, see, e.g., Ciarlet [1988, Section 1.4] or Antman [1995, Section 12.1]). In this context, the associated matrix

$$\mathbf{C}(x) = (g_{ij}(x)) = \boldsymbol{\nabla}\boldsymbol{\Theta}(x)^T \boldsymbol{\nabla}\boldsymbol{\Theta}(x),$$

is called the (right) *Cauchy-Green tensor at* x and the matrix

$$\boldsymbol{\nabla}\boldsymbol{\Theta}(x) = (\partial_j \Theta_i(x)) \in \mathbb{M}^3,$$

representing the Fréchet derivative of the mapping $\boldsymbol{\Theta}$ at x, is called the *deformation gradient at* x.

The Cauchy-Green tensor field $\mathbf{C} = \boldsymbol{\nabla}\boldsymbol{\Theta}^T \boldsymbol{\nabla}\boldsymbol{\Theta} : \Omega \to \mathbb{S}^3_>$ associated with a deformation $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$ plays a major role in the theory of nonlinear three-dimensional elasticity, since the response function, or the stored energy function, of a frame-indifferent elastic, or hyperelastic, material necessarily depends on the deformation gradient through the Cauchy-Green tensor (for a detailed description see, e.g., Ciarlet [1988, Chapters 3 and 4]). As already suggested by Antman [1976], the Cauchy-Green tensor field of the *unknown deformed configuration* could thus also be regarded as the "*primary*" *unknown* rather than the deformation itself as is customary.

To begin with, we list some specific notations that will be used in this section for addressing the question raised above. Given a matrix $\mathbf{A} \in \mathbb{M}^3$, we let $\rho(\mathbf{A})$ denote its spectral radius (i.e., the largest modulus of the eigenvalues of $\mathbf{A}$) and we let

$$|\mathbf{A}| := \sup_{\begin{cases} \boldsymbol{b}\in\mathbb{R}^3 \\ \boldsymbol{b}\neq\mathbf{0}\end{cases}} \frac{|\mathbf{A}\boldsymbol{b}|}{|\boldsymbol{b}|} = \{\rho(\mathbf{A}^T\mathbf{A})\}^{1/2}$$

denote its spectral norm.

Let Ω be an open subset of $\mathbb{R}^3$. The notation $K \Subset \Omega$ means that K is a compact subset of Ω. If $g \in \mathcal{C}^\ell(\Omega;\mathbb{R}), \ell \geq 0$, and $K \Subset \Omega$, we define the *semi-norms*

$$|g|_{\ell,K} = \sup_{\begin{cases} x\in K \\ |\alpha|=\ell\end{cases}} |\partial^\alpha g(x)| \quad \text{and} \quad \|g\|_{\ell,K} = \sup_{\begin{cases} x\in K \\ |\alpha|\leq\ell\end{cases}} |\partial^\alpha g(x)|,$$

where ∂^α stands for the standard multi-index notation for partial derivatives. If $\mathbf{\Theta} \in \mathcal{C}^\ell(\Omega;\mathbf{E}^3)$ or $\mathbf{A} \in \mathcal{C}^\ell(\Omega;\mathbb{M}^3)$, $\ell \geq 0$, and $K \Subset \Omega$, we likewise set

$$|\mathbf{\Theta}|_{\ell,K} = \sup_{\begin{cases} x\in K \\ |\alpha|=\ell\end{cases}} |\partial^\alpha \mathbf{\Theta}(x)| \quad \text{and} \quad \|\mathbf{\Theta}\|_{\ell,K} = \sup_{\begin{cases} x\in K \\ |\alpha|\leq\ell\end{cases}} |\partial^\alpha \mathbf{\Theta}(x)|,$$

$$|\mathbf{A}|_{\ell,K} = \sup_{\begin{cases} x\in K \\ |\alpha|=\ell\end{cases}} |\partial^\alpha \mathbf{A}(x)| \quad \text{and} \quad \|\mathbf{A}\|_{\ell,K} = \sup_{\begin{cases} x\in K \\ |\alpha|\leq\ell\end{cases}} |\partial^\alpha \mathbf{A}(x)|,$$

where $|\cdot|$ denotes either the Euclidean vector norm or the matrix spectral norm.

The next sequential continuity results (Theorems 1.8-1, 1.8-2, and 1.8-3) constitute key steps toward establishing the continuity of the mapping $\mathcal{F}$ (see Theorem 1.8-5). Note that the functions R^n_{qijk} occurring in their statements are meant to be constructed from the functions g^n_{ij} in the same way that the functions R_{qijk} are constructed from the functions g_{ij}. To begin with, we establish the sequential continuity of the mapping $\mathcal{F}$ at $\mathbf{C} = \mathbf{I}$.

Theorem 1.8-1. *Let Ω be a connected and simply-connected open subset of $\mathbb{R}^3$. Let $\mathbf{C}^n = (g^n_{ij}) \in \mathcal{C}^2(\Omega;\mathbb{S}^3_>)$, $n \geq 0$, be matrix fields satisfying $R^n_{qijk} = 0$ in Ω, $n \geq 0$, such that*

$$\lim_{n\to\infty} \|\mathbf{C}^n - \mathbf{I}\|_{2,K} = 0 \text{ for all } K \Subset \Omega.$$

Then there exist immersions $\mathbf{\Theta}^n \in \mathcal{C}^3(\Omega;\mathbf{E}^3)$ satisfying $(\boldsymbol{\nabla}\mathbf{\Theta}^n)^T\boldsymbol{\nabla}\mathbf{\Theta}^n = \mathbf{C}^n$ in Ω, $n \geq 0$, such that

$$\lim_{n\to\infty} \|\mathbf{\Theta}^n - \boldsymbol{id}\|_{3,K} = 0 \text{ for all } K \Subset \Omega$$

where $\boldsymbol{id}$ denotes the identity mapping of the set Ω, the space $\mathbb{R}^3$ being identified here with $\mathbf{E}^3$ (in other words, $\boldsymbol{id}(x) = x$ for all $x \in \Omega$).

Proof. The proof is broken into four parts, numbered (i) to (iv). The first part is a preliminary result about matrices (for convenience, it is stated here for matrices of order three, but it holds as well for matrices of arbitrary order).

(i) *Let matrices* $\mathbf{A}^n \in \mathbb{M}^3$, $n \geq 0$, *satisfy*

$$\lim_{n\to\infty} (\mathbf{A}^n)^T \mathbf{A}^n = \mathbf{I}.$$

Then there exist matrices $\mathbf{Q}^n \in \mathbb{O}^3$, $n \geq 0$, *that satisfy*

$$\lim_{n\to\infty} \mathbf{Q}^n \mathbf{A}^n = \mathbf{I}.$$

Since the set $\mathbb{O}^3$ is compact, there exist matrices $\mathbf{Q}^n \in \mathbb{O}^3$, $n \geq 0$, such that

$$|\mathbf{Q}^n \mathbf{A}^n - \mathbf{I}| = \inf_{\mathbf{R}\in\mathbb{O}^3} |\mathbf{R}\mathbf{A}^n - \mathbf{I}|.$$

We assert that the matrices $\mathbf{Q}^n$ defined in this fashion satisfy $\lim_{n\to\infty} \mathbf{Q}^n\mathbf{A}^n = \mathbf{I}$. For otherwise, there would exist a subsequence $(\mathbf{Q}^p)_{p\geq 0}$ of the sequence $(\mathbf{Q}^n)_{n\geq 0}$ and $\delta > 0$ such that

$$|\mathbf{Q}^p \mathbf{A}^p - \mathbf{I}| = \inf_{\mathbf{R}\in\mathbb{O}^3} |\mathbf{R}\mathbf{A}^p - \mathbf{I}| \geq \delta \text{ for all } p \geq 0.$$

Since

$$\lim_{p\to\infty} |\mathbf{A}^p| = \lim_{p\to\infty} \sqrt{\rho((\mathbf{A}^p)^T\mathbf{A}^p)} = \sqrt{\rho(\mathbf{I})} = 1,$$

the sequence $(\mathbf{A}^p)_{p\geq 0}$ is bounded. Therefore there exists a further subsequence $(\mathbf{A}^q)_{q\geq 0}$ that converges to a matrix $\mathbf{S}$, which is orthogonal since

$$\mathbf{S}^T\mathbf{S} = \lim_{q\to\infty} (\mathbf{A}^q)^T \mathbf{A}^q = \mathbf{I}.$$

But then

$$\lim_{q\to\infty} \mathbf{S}^T \mathbf{A}^q = \mathbf{S}^T\mathbf{S} = \mathbf{I},$$

which contradicts $\inf_{\mathbf{R}\in\mathbb{O}^3} |\mathbf{R}\mathbf{A}^q - \mathbf{I}| \geq \delta$ for all $q \geq 0$. This proves (i).

In the remainder of this proof, the matrix fields $\mathbf{C}^n, n \geq 0$, are meant to be those appearing in the statement of Theorem 1.8-1 and the notation $\boldsymbol{id}$ stands for the identity mapping of the set Ω.

(ii) *Let mappings* $\boldsymbol{\Theta}^n \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$, $n \geq 0$, *satisfy* $(\boldsymbol{\nabla}\boldsymbol{\Theta}^n)^T \boldsymbol{\nabla}\boldsymbol{\Theta}^n = \mathbf{C}^n$ *in* Ω (such mappings exist by Theorem 1.6-1). *Then*

$$\lim_{n\to\infty} |\boldsymbol{\Theta}^n - \boldsymbol{id}|_{\ell,K} = \lim_{n\to\infty} |\boldsymbol{\Theta}^n|_{\ell,K} = 0 \textit{ for all } K \Subset \Omega \textit{ and for } \ell = 2, 3.$$

As usual, given any immersion $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$, let $\boldsymbol{g}_i = \partial_i \boldsymbol{\Theta}$, let $g_{ij} = \boldsymbol{g}_i \cdot \boldsymbol{g}_j$, and let the vectors $\boldsymbol{g}^q$ be defined by the relations $\boldsymbol{g}_i \cdot \boldsymbol{g}^q = \delta_i^q$. It is then immediately verified that

$$\partial_{ij}\boldsymbol{\Theta} = \partial_i \boldsymbol{g}_j = (\partial_i \boldsymbol{g}_j \cdot \boldsymbol{g}_q)\boldsymbol{g}^q = \frac{1}{2}(\partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij})\boldsymbol{g}^q.$$

Applying this relation to the mappings $\mathbf{\Theta}^n$ thus gives

$$\partial_{ij}\mathbf{\Theta}^n = \frac{1}{2}(\partial_j g^n_{iq} + \partial_i g^n_{jq} - \partial_q g^n_{ij})(\boldsymbol{g}^q)^n,\ n \geq 0,$$

where the vectors $(\boldsymbol{g}^q)^n$ are defined by means of the relations $\partial_i\mathbf{\Theta}^n \cdot (\boldsymbol{g}^q)^n = \delta^q_i$. Let K denote an arbitrary compact subset of Ω. On the one hand,

$$\lim_{n\to\infty} |\partial_j g^n_{iq} + \partial_i g^n_{jq} - \partial_q g^n_{ij}|_{0,K} = 0,$$

since $\lim_{n\to\infty} |g^n_{ij}|_{1,K} = \lim_{n\to\infty} |g^n_{ij} - \delta_{ij}|_{1,K} = 0$ by assumption. On the other hand, the norms $|(\boldsymbol{g}^q)^n|_{0,K}$ are bounded independently of $n \geq 0$; to see this, observe that $(\boldsymbol{g}^q)^n$ is the q-th column vector of the matrix $(\boldsymbol{\nabla}\mathbf{\Theta}^n)^{-1}$, then that

$$\begin{aligned}|(\boldsymbol{\nabla}\mathbf{\Theta}^n)^{-1}|_{0,K} &= |\{\rho((\boldsymbol{\nabla}\mathbf{\Theta}^n)^{-T}(\boldsymbol{\nabla}\mathbf{\Theta}^n)^{-1})\}^{1/2}|_{0,K} \\ &= |\{\rho((g^n_{ij})^{-1})\}^{1/2}|_{0,K} \leq \{|(g^n_{ij})^{-1}|_{0,K}\}^{1/2},\end{aligned}$$

and, finally, that

$$\lim_{n\to\infty} |(g^n_{ij}) - \mathbf{I}|_{0,K} = 0 \Longrightarrow \lim_{n\to\infty} |(g^n_{ij})^{-1} - \mathbf{I}|_{0,K} = 0.$$

Consequently,

$$\lim_{n\to\infty} |\mathbf{\Theta}^n - \boldsymbol{id}|_{2,K} = \lim_{n\to\infty} |\mathbf{\Theta}^n|_{2,K} = 0 \text{ for all } K \Subset \Omega.$$

Differentiating the relations $\partial_i\boldsymbol{g}_j \cdot \boldsymbol{g}_q = \frac{1}{2}(\partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij})$ yields

$$\begin{aligned}\partial_{ijp}\mathbf{\Theta} &= \partial_{ip}\boldsymbol{g}_j = (\partial_{ip}\boldsymbol{g}_j \cdot \boldsymbol{g}_q)\boldsymbol{g}^q \\ &= \Big(\frac{1}{2}(\partial_{jp}g_{iq} + \partial_{ip}g_{jq} - \partial_{pq}g_{ij}) - \partial_i\boldsymbol{g}_j \cdot \partial_p\boldsymbol{g}_q\Big)\boldsymbol{g}^q.\end{aligned}$$

Observing that $\lim_{n\to\infty} |g^n_{ij}|_{\ell,K} = \lim_{n\to\infty} |g^n_{ij} - \delta_{ij}|_{\ell,K} = 0$ for $\ell = 1, 2$ by assumption and recalling that the norms $|(\boldsymbol{g}^q)^n|_{0,K}$ are bounded independently of $n \geq 0$, we likewise conclude that

$$\lim_{n\to\infty} |\mathbf{\Theta}^n - \boldsymbol{id}|_{3,K} = \lim_{n\to\infty} |\mathbf{\Theta}^n|_{3,K} = 0 \text{ for all } K \Subset \Omega.$$

(iii) *There exist mappings* $\widetilde{\mathbf{\Theta}}^n \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ *that satisfy* $(\boldsymbol{\nabla}\widetilde{\mathbf{\Theta}}^n)^T\boldsymbol{\nabla}\widetilde{\mathbf{\Theta}}^n = \mathbf{C}^n$ *in* Ω, $n \geq 0$, *and*

$$\lim_{n\to\infty} |\widetilde{\mathbf{\Theta}}^n - \boldsymbol{id}|_{1,K} = 0 \text{ for all } K \Subset \Omega.$$

Let $\boldsymbol{\psi}^n \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ be mappings that satisfy $(\boldsymbol{\nabla}\boldsymbol{\psi}^n)^T\boldsymbol{\nabla}\boldsymbol{\psi}^n = \mathbf{C}^n$ in Ω, $n \geq 0$ (such mappings exist by Theorem 1.6-1), and let x_0 denote a point in the set Ω. Since $\lim_{n\to\infty} \boldsymbol{\nabla}\boldsymbol{\psi}^n(x_0)^T\boldsymbol{\nabla}\boldsymbol{\psi}^n(x_0) = \mathbf{I}$ by assumption, part (i) implies that there exist orthogonal matrices $\mathbf{Q}^n(x_0)$, $n \geq 0$, such that

$$\lim_{n\to\infty} \mathbf{Q}^n(x_0)\boldsymbol{\nabla}\boldsymbol{\psi}^n(x_0) = \mathbf{I}.$$

Then the mappings $\widetilde{\mathbf{\Theta}}^n \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$, $n \geq 0$, defined by

$$\widetilde{\mathbf{\Theta}}^n(x) := \mathbf{Q}^n(x_0)\boldsymbol{\psi}^n(x),\ x \in \Omega,$$

satisfy

$$(\boldsymbol{\nabla}\widetilde{\mathbf{\Theta}}^n)^T \boldsymbol{\nabla}\widetilde{\mathbf{\Theta}}^n = \mathbf{C}^n \text{ in } \Omega,$$

so that their gradients $\boldsymbol{\nabla}\widetilde{\mathbf{\Theta}}^n \in \mathcal{C}^2(\Omega; \mathbb{M}^3)$ satisfy

$$\lim_{n\to\infty} |\partial_i \boldsymbol{\nabla}\widetilde{\mathbf{\Theta}}^n|_{0,K} = \lim_{n\to\infty} |\widetilde{\mathbf{\Theta}}^n|_{2,K} = 0 \text{ for all } K \Subset \Omega,$$

by part (ii). In addition,

$$\lim_{n\to\infty} \boldsymbol{\nabla}\widetilde{\mathbf{\Theta}}^n(x_0) = \lim_{n\to\infty} \mathbf{Q}^n \boldsymbol{\nabla}\boldsymbol{\psi}^n(x_0) = \mathbf{I}.$$

Hence a classical theorem about the differentiability of the limit of a sequence of mappings that are continuously differentiable on a connected open set and that take their values in a Banach space (see, e.g., Schwartz [1992, Theorem 3.5.12]) shows that *the mappings* $\boldsymbol{\nabla}\widetilde{\mathbf{\Theta}}^n$ *uniformly converge on every compact subset of* Ω *toward a limit* $\mathbf{R} \in \mathcal{C}^1(\Omega; \mathbb{M}^3)$ *that satisfies*

$$\partial_i \mathbf{R}(x) = \lim_{n\to\infty} \partial_i \boldsymbol{\nabla}\widetilde{\mathbf{\Theta}}^n(x) = \mathbf{0} \text{ for all } x \in \Omega.$$

This shows that $\mathbf{R}$ is a constant mapping since Ω is connected. Consequently, $\mathbf{R} = \mathbf{I}$ since in particular $\mathbf{R}(x_0) = \lim_{n\to\infty} \boldsymbol{\nabla}\widetilde{\mathbf{\Theta}}^n(x_0) = \mathbf{I}$. We have therefore established that

$$\lim_{n\to\infty} |\widetilde{\mathbf{\Theta}}^n - \boldsymbol{id}|_{1,K} = \lim_{n\to\infty} |\boldsymbol{\nabla}\widetilde{\mathbf{\Theta}}^n - \mathbf{I}|_{0,K} = 0 \text{ for all } K \Subset \Omega.$$

(iv) *There exist mappings* $\mathbf{\Theta}^n \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ *satisfying* $(\boldsymbol{\nabla}\mathbf{\Theta}^n)^T \boldsymbol{\nabla}\mathbf{\Theta}^n = \mathbf{C}^n$ *in* Ω, $n \geq 0$, and

$$\lim_{n\to\infty} |\mathbf{\Theta}^n - \boldsymbol{id}|_{\ell,K} = 0 \text{ for all } K \Subset \Omega \text{ and for } \ell = 0, 1.$$

The mappings

$$\mathbf{\Theta}^n := \left(\widetilde{\mathbf{\Theta}}^n - \{\widetilde{\mathbf{\Theta}}^n(x_0) - x_0\}\right) \in \mathcal{C}^3(\Omega; \mathbf{E}^3),\ n \geq 0,$$

clearly satisfy

$$\begin{aligned}
(\boldsymbol{\nabla}\mathbf{\Theta}^n)^T \boldsymbol{\nabla}\mathbf{\Theta}^n &= \mathbf{C}^n \text{ in } \Omega,\ n \geq 0,\\
\lim_{n\to\infty} |\mathbf{\Theta}^n - \boldsymbol{id}|_{1,K} = \lim_{n\to\infty} |\boldsymbol{\nabla}\mathbf{\Theta}^n - \mathbf{I}|_{0,K} &= 0 \text{ for all } K \Subset \Omega,\\
\mathbf{\Theta}^n(x_0) &= x_0,\ n \geq 0.
\end{aligned}$$

Again applying the theorem about the differentiability of the limit of a sequence of mappings used in part (iii), we conclude from the last two relations that the mappings $\boldsymbol{\Theta}^n$ uniformly converge on every compact subset of Ω toward a limit $\boldsymbol{\Theta} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ that satisfies

$$\boldsymbol{\nabla\Theta}(x) = \lim_{n\to\infty} \boldsymbol{\nabla\Theta}^n(x) = \mathbf{I} \text{ for all } x \in \Omega.$$

This shows that $(\boldsymbol{\Theta} - \boldsymbol{id})$ is a constant mapping since Ω is connected. Consequently, $\boldsymbol{\Theta} = \boldsymbol{id}$ since in particular $\boldsymbol{\Theta}(x_0) = \lim_{n\to\infty} \boldsymbol{\Theta}^n(x_0) = x_0$. We have thus established that

$$\lim_{n\to\infty} |\boldsymbol{\Theta}^n - \boldsymbol{id}|_{0,K} = 0 \text{ for all } K \Subset \Omega.$$

This completes the proof of Theorem 1.8-1. $\square$

We next establish the sequential continuity of the mapping $\mathcal{F}$ at those matrix fields $\mathbf{C} \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ that can be written as $\mathbf{C} = \boldsymbol{\nabla\Theta}^T\boldsymbol{\nabla\Theta}$ with an *injective* mapping $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$.

Theorem 1.8-2. *Let Ω be a connected and simply-connected open subset of $\mathbb{R}^3$. Let $\mathbf{C} = (g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ and $\mathbf{C}^n = (g^n_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$, $n \geq 0$, be matrix fields satisfying respectively $R_{qijk} = 0$ in Ω and $R^n_{qijk} = 0$ in Ω, $n \geq 0$, such that*

$$\lim_{n\to\infty} \|\mathbf{C}^n - \mathbf{C}\|_{2,K} = 0 \text{ for all } K \Subset \Omega.$$

Assume that there exists an injective immersion $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ such that $\boldsymbol{\nabla\Theta}^T\boldsymbol{\nabla\Theta} = \mathbf{C}$ in Ω. Then there exist immersions $\boldsymbol{\Theta}^n \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ satisfying $(\boldsymbol{\nabla\Theta}^n)^T\boldsymbol{\nabla\Theta}^n = \mathbf{C}^n$ in Ω, $n \geq 0$, such that

$$\lim_{n\to\infty} \|\boldsymbol{\Theta}^n - \boldsymbol{\Theta}\|_{3,K} = 0 \text{ for all } K \Subset \Omega.$$

Proof. The assumptions made on the mapping $\boldsymbol{\Theta} : \Omega \subset \mathbb{R}^3 \to \mathbf{E}^3$ imply that the set $\widehat{\Omega} := \boldsymbol{\Theta}(\Omega) \subset \mathbf{E}^3$ is open, connected, and simply-connected, and that the inverse mapping $\widehat{\boldsymbol{\Theta}} : \widehat{\Omega} \subset \mathbf{E}^3 \to \mathbb{R}^3$ belongs to the space $\mathcal{C}^3(\widehat{\Omega}; \mathbb{R}^3)$. Define the matrix fields $(\widehat{g}^n_{ij}) \in \mathcal{C}^2(\widehat{\Omega}; \mathbb{S}^3_>)$, $n \geq 0$, by letting

$$(\widehat{g}^n_{ij}(\widehat{x})) := \boldsymbol{\nabla\Theta}(x)^{-T}(g^n_{ij}(x))\boldsymbol{\nabla\Theta}(x)^{-1} \text{ for all } \widehat{x} = \boldsymbol{\Theta}(x) \in \widehat{\Omega}.$$

Given any compact subset $\widehat{K}$ of $\widehat{\Omega}$, let $K := \widehat{\boldsymbol{\Theta}}(\widehat{K})$. Since $\lim_{n\to\infty} \|g^n_{ij} - g_{ij}\|_{2,K} = 0$ because K is a compact subset of Ω, the definition of the functions $\widehat{g}^n_{ij} : \widehat{\Omega} \to \mathbb{R}$ and the chain rule together imply that

$$\lim_{n\to\infty} \|\widehat{g}^n_{ij} - \delta_{ij}\|_{2,\widehat{K}} = 0.$$

Given $\widehat{x} = (\widehat{x}_i) \in \widehat{\Omega}$, let $\widehat{\partial}_i = \partial/\partial\widehat{x}_i$. Let $\widehat{R}^n_{qijk}$ denote the functions constructed from the functions $\widehat{g}^n_{ij}$ in the same way that the functions R_{qijk} are constructed from the functions g_{ij}. Since it is easily verified that these functions satisfy $\widehat{R}^n_{qijk} = 0$ in $\widehat{\Omega}$, Theorem 1.8-1 applied over the set $\widehat{\Omega}$ shows that there exist mappings $\widehat{\boldsymbol{\Theta}}^n \in \mathcal{C}^3(\widehat{\Omega}; \mathbf{E}^3)$ satisfying

$$\widehat{\partial}_i \widehat{\boldsymbol{\Theta}}^n \cdot \widehat{\partial}_j \widehat{\boldsymbol{\Theta}}^n = \widehat{g}^n_{ij} \text{ in } \widehat{\Omega},\ n \geq 0,$$

such that

$$\lim_{n\to\infty} \|\widehat{\boldsymbol{\Theta}}^n - \widehat{\boldsymbol{id}}\|_{3,\widehat{K}} = 0 \text{ for all } \widehat{K} \Subset \widehat{\Omega},$$

where $\widehat{\boldsymbol{id}}$ denotes the identity mapping of the set $\widehat{\Omega}$, the space $\mathbf{E}^3$ being identified here with $\mathbb{R}^3$. Define the mappings $\boldsymbol{\Theta}^n \in \mathcal{C}^3(\Omega; \mathbb{S}^3_>)$, $n \geq 0$, by letting

$$\boldsymbol{\Theta}^n(x) = \widehat{\boldsymbol{\Theta}}^n(\widehat{x}) \text{ for all } x = \widehat{\boldsymbol{\Theta}}(\widehat{x}) \in \Omega.$$

Given any compact subset K of Ω, let $\widehat{K} := \boldsymbol{\Theta}(K)$. Since $\lim_{n\to\infty} \|\widehat{\boldsymbol{\Theta}}^n - \widehat{\boldsymbol{id}}\|_{3,\widehat{K}} = 0$, the definition of the mappings $\boldsymbol{\Theta}^n$ and the chain rule together imply that

$$\lim_{n\to\infty} \|\boldsymbol{\Theta}^n - \boldsymbol{\Theta}\|_{3,K} = 0,$$

on the one hand. Since, on the other hand, $(\boldsymbol{\nabla}\boldsymbol{\Theta}^n)^T \boldsymbol{\nabla}\boldsymbol{\Theta}^n = \mathbf{C}^n$ in Ω, the proof is complete. ☐

We are now in a position to establish the sequential continuity of the mapping $\mathcal{F}$ at *any* matrix field $\mathbf{C} \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ that can be written as $\mathbf{C} = \boldsymbol{\nabla}\boldsymbol{\Theta}^T \boldsymbol{\nabla}\boldsymbol{\Theta}$ with $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; E^3)$.

Theorem 1.8-3. *Let Ω be a connected and simply-connected open subset of $\mathbb{R}^3$. Let $\mathbf{C} = (g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ and $\mathbf{C}^n = (g^n_{ij}) \in \mathcal{C}^2(\Omega, \mathbb{S}^3_>)$, $n \geq 0$, be matrix fields respectively satisfying $R_{qijk} = 0$ in Ω and $R^n_{qijk} = 0$ in Ω, $n \geq 0$, such that*

$$\lim_{n\to\infty} \|\mathbf{C}^n - \mathbf{C}\|_{2,K} = 0 \text{ for all } K \Subset \Omega.$$

Let $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ be any immersion that satisfies $\boldsymbol{\nabla}\boldsymbol{\Theta}^T \boldsymbol{\nabla}\boldsymbol{\Theta} = \mathbf{C}$ in Ω (such immersions exist by Theorem 1.6-1). *Then there exist immersions $\boldsymbol{\Theta}^n \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ satisfying $(\boldsymbol{\nabla}\boldsymbol{\Theta}^n)^T \boldsymbol{\nabla}\boldsymbol{\Theta}^n = \mathbf{C}^n$ in Ω, $n \geq 0$, such that*

$$\lim_{n\to\infty} \|\boldsymbol{\Theta}^n - \boldsymbol{\Theta}\|_{3,K} = 0 \text{ for all } K \Subset \Omega.$$

Proof. The proof is broken into four parts. In what follows, $\mathbf{C}$ and $\mathbf{C}^n$ designate matrix fields possessing the properties listed in the statement of the theorem.

(i) *Let $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ be any mapping that satisfies $\boldsymbol{\nabla}\boldsymbol{\Theta}^T \boldsymbol{\nabla}\boldsymbol{\Theta} = \mathbf{C}$ in Ω. Then there exist a countable number of open balls $B_r \subset \Omega$, $r \geq 1$, such that $\Omega = \bigcup_{r=1}^\infty B_r$ and such that, for each $r \geq 1$, the set $\bigcup_{s=1}^r B_s$ is connected and the restriction of $\boldsymbol{\Theta}$ to B_r is injective.*

Given any $x \in \Omega$, there exists an open ball $V_x \subset \Omega$ such that the restriction of $\boldsymbol{\Theta}$ to V_x is injective. Since $\Omega = \bigcup_{x\in\Omega} V_x$ can also be written as a countable union of compact subsets of Ω, there already exist countably many such open balls, denoted V_r, $r \geq 1$, such that $\Omega = \bigcup_{r=1}^{\infty} V_r$.

Let $r_1 := 1$, $B_1 := V_{r_1}$, and $r_2 := 2$. If the set $B_{r_1} \cup V_{r_2}$ is connected, let $B_2 := V_{r_2}$ and $r_3 := 3$. Otherwise, there exists a path γ_1 in Ω joining the centers of V_{r_1} and V_{r_2} since Ω is connected. Then there exists a finite set $I_1 = \{r_1(1), r_1(2), \cdots, r_1(N_1)\}$ of integers, with $N_1 \geq 1$ and $2 < r_1(1) < r_1(2) < \cdots < r_1(N_1)$, such that

$$\gamma_1 \subset V_{r_1} \cup V_{r_2} \cup \Big(\bigcup_{r\in I_1} V_r \Big).$$

Furthermore there exists a permutation σ_1 of $\{1, 2, \dots, N_1\}$ such that the sets $V_{r_1} \cup (\bigcup_{s=1}^{r} V_{\sigma_1(s)})$, $1 \leq r \leq N_1$, and $V_{r_1} \cup (\bigcup_{s=1}^{N_1} V_{\sigma_1(s)}) \cup V_{r_2}$ are connected. Let

$$B_r := V_{\sigma_1(r-1)},\ 2 \leq r \leq N_1 + 1, \quad B_{N_1+2} := V_{r_2},$$
$$r_3 := \min\big\{ i \in \{\sigma_1(1), \dots, \sigma_1(N_1)\};\ i \geq 3 \big\}.$$

If the set $(\bigcup_{r=1}^{N_1+2} B_r) \cup V_{r_3}$ is connected, let $B_{N_1+3} := V_{r_3}$. Otherwise, apply the same argument as above to a path γ_2 in Ω joining the centers of V_{r_2} and V_{r_3}, and so forth.

The iterative procedure thus produces a countable number of open balls B_r, $r \geq 1$, that possess the announced properties. In particular, $\Omega = \bigcup_{r=1}^{\infty} B_r$ since, by construction, the integer r_i appearing at the i-th stage satisfies $r_i \geq i$.

(ii) By Theorem 1.8-2, there exist mappings $\boldsymbol{\Theta}_1^n \in \mathcal{C}^3(B_1; \mathbf{E}^3)$ and $\widetilde{\boldsymbol{\Theta}}_2^n \in \mathcal{C}^3(B_2; \mathbf{E}^3)$, $n \geq 0$, that satisfy

$$(\boldsymbol{\nabla}\boldsymbol{\Theta}_1^n)^T \boldsymbol{\nabla}\boldsymbol{\Theta}_1^n = \mathbf{C}^n \text{ in } B_1 \quad \text{and} \quad \lim_{n\to\infty} \|\boldsymbol{\Theta}_1^n - \boldsymbol{\Theta}\|_{3,K} = 0 \text{ for all } K \Subset B_1,$$
$$(\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}_2^n)^T \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}_2^n = \mathbf{C}^n \text{ in } B_2 \quad \text{and} \quad \lim_{n\to\infty} \|\widetilde{\boldsymbol{\Theta}}_2^n - \boldsymbol{\Theta}\|_{3,K} = 0 \text{ for all } K \Subset B_2,$$

and by Theorem 1.7-1, there exist vectors $\boldsymbol{c}^n \in \mathbf{E}^3$ and matrices $\mathbf{Q}^n \in \mathbb{O}^3$, $n \geq 0$, such that

$$\widetilde{\boldsymbol{\Theta}}_2^n(x) = \boldsymbol{c}^n + \mathbf{Q}^n \boldsymbol{\Theta}_1^n(x) \text{ for all } x \in B_1 \cap B_2.$$

Then we assert that

$$\lim_{n\to\infty} \boldsymbol{c}^n = \mathbf{0} \text{ and } \lim_{n\to\infty} \mathbf{Q}^n = \mathbf{I}.$$

Let $(\mathbf{Q}^p)_{p\geq 0}$ be a subsequence of the sequence $(\mathbf{Q}^n)_{n\geq 0}$ that converges to a (necessarily orthogonal) matrix $\mathbf{Q}$ and let x_1 denote a point in the set $B_1 \cap B_2$. Since $\boldsymbol{c}^p = \widetilde{\boldsymbol{\Theta}}_2^p(x_1) - \mathbf{Q}^p \boldsymbol{\Theta}_1(x_1)$ and $\lim_{n\to\infty} \widetilde{\boldsymbol{\Theta}}_2^p(x_1) = \lim_{n\to\infty} \boldsymbol{\Theta}_1^p(x_1) = \boldsymbol{\Theta}(x_1)$, the subsequence $(\boldsymbol{c}^p)_{p\geq 0}$ also converges. Let $\boldsymbol{c} := \lim_{p\to\infty} \boldsymbol{c}^p$. Thus

$$\begin{aligned}\boldsymbol{\Theta}(x) &= \lim_{p\to\infty} \widetilde{\boldsymbol{\Theta}}_2^p(x) \\ &= \lim_{p\to\infty} (\boldsymbol{c}^p + \mathbf{Q}^p \boldsymbol{\Theta}_1^p(x)) = \boldsymbol{c} + \mathbf{Q}\boldsymbol{\Theta}(x) \text{ for all } x \in B_1 \cap B_2,\end{aligned}$$

on the one hand. On the other hand, the differentiability of the mapping $\boldsymbol{\Theta}$ implies that

$$\boldsymbol{\Theta}(x) = \boldsymbol{\Theta}(x_1) + \boldsymbol{\nabla}\boldsymbol{\Theta}(x_1)(x - x_1) + o(|x - x_1|) \text{ for all } x \in B_1 \cap B_2.$$

Note that $\boldsymbol{\nabla}\boldsymbol{\Theta}(x_1)$ is an invertible matrix, since $\boldsymbol{\nabla}\boldsymbol{\Theta}(x_1)^T\boldsymbol{\nabla}\boldsymbol{\Theta}(x_1) = (g_{ij}(x_1))$.

Let $\boldsymbol{b} := \boldsymbol{\Theta}(x_1)$ and $\mathbf{A} := \boldsymbol{\nabla}\boldsymbol{\Theta}(x_1)$. Together, the last two relations imply that

$$\boldsymbol{b} + \mathbf{A}(x - x_1) = \boldsymbol{c} + \mathbf{Q}\boldsymbol{b} + \mathbf{Q}\mathbf{A}(x - x_1) + o(|x - x_1|),$$

and hence (letting $x = x_1$ shows that $\boldsymbol{b} = \boldsymbol{c} + \mathbf{Q}\boldsymbol{b}$) that

$$\mathbf{A}(x - x_1) = \mathbf{Q}\mathbf{A}(x - x_1) + o(|x - x_1|) \text{ for all } x \in B_1 \cap B_2.$$

The invertibility of $\mathbf{A}$ thus implies that $\mathbf{Q} = \mathbf{I}$ and therefore that $\boldsymbol{c} = \boldsymbol{b} - \mathbf{Q}\boldsymbol{b} = \mathbf{0}$. The uniqueness of these limits shows that the whole sequences $(\mathbf{Q}^n)_{n\geq 0}$ and $(\boldsymbol{c}^n)_{n\geq 0}$ converge.

(iii) *Let the mappings* $\boldsymbol{\Theta}_2^n \in \mathcal{C}^3(B_1 \cup B_2; \mathbf{E}^3)$, $n \geq 0$, *be defined by*

$$\boldsymbol{\Theta}_2^n(x) := \boldsymbol{\Theta}_1^n(x) \text{ for all } x \in B_1,$$
$$\boldsymbol{\Theta}_2^n(x) := (\mathbf{Q}^n)^T(\widetilde{\boldsymbol{\Theta}}_2^n(x) - \boldsymbol{c}^n) \text{ for all } x \in B_2.$$

Then

$$(\boldsymbol{\nabla}\boldsymbol{\Theta}_2^n)^T\boldsymbol{\nabla}\boldsymbol{\Theta}_2^n = \mathbf{C}^n \text{ in } B_1 \cup B_2$$

(as is clear), *and*

$$\lim_{n\to\infty} \|\boldsymbol{\Theta}_2^n - \boldsymbol{\Theta}\|_{3,K} = 0 \text{ for all } K \Subset B_1 \cup B_2.$$

The plane containing the intersection of the boundaries of the open balls B_1 and B_2 is the common boundary of two closed half-spaces in $\mathbb{R}^3$, H_1 containing the center of B_1, and H_2 containing that of B_2 (by construction, the set $B_1 \cup B_2$ is connected; see part (i)). Any compact subset K of $B_1 \cup B_2$ may thus be written as $K = K_1 \cup K_2$, where $K_1 := (K \cap H_1) \subset B_1$ and $K_2 := (K \cap H_2) \subset B_2$ (that the open sets found in part (i) may be chosen as *balls* thus play an essential rôle here). Hence

$$\lim_{n\to\infty} \|\boldsymbol{\Theta}_2^n - \boldsymbol{\Theta}\|_{3,K_1} = 0 \text{ and } \lim_{n\to\infty} \|\boldsymbol{\Theta}_2^n - \boldsymbol{\Theta}\|_{3,K_2} = 0,$$

the second relation following from the definition of the mapping $\boldsymbol{\Theta}_2^n$ on $B_2 \supset K_2$ and on the relations $\lim_{n\to\infty} \|\widetilde{\boldsymbol{\Theta}}_2^n - \boldsymbol{\Theta}\|_{3,K_2} = 0$ (part (ii)) and $\lim_{n\to\infty} \mathbf{Q}^n = \mathbf{I}$ and $\lim_{n\to\infty} \boldsymbol{c}^n = \mathbf{0}$ (part (iii)).

(iv) It remains to iterate the procedure described in parts (ii) and (iii). For some $r \geq 2$, assume that mappings $\mathbf{\Theta}_r^n \in \mathcal{C}^3(\bigcup_{s=1}^r B_s; \mathbf{E}^3)$, $n \geq 0$, have been found that satisfy

$$(\nabla \mathbf{\Theta}_r^n)^T \nabla \mathbf{\Theta}_r^n = \mathbf{C}^n \text{ in } \bigcup_{s=1}^r B_s,$$

$$\lim_{n\to\infty} \|\mathbf{\Theta}_r^n - \mathbf{\Theta}\|_{2,K} = 0 \text{ for all } K \Subset \bigcup_{s=1}^r B_s.$$

Since the restriction of $\mathbf{\Theta}$ to B_{r+1} is injective (part (i)), Theorem 1.8-2 shows that there exist mappings $\widetilde{\mathbf{\Theta}}_{r+1}^n \in \mathcal{C}^3(B_{r+1}; \mathbf{E}^3)$, $n \geq 0$, that satisfy

$$(\nabla \widetilde{\mathbf{\Theta}}_{r+1}^n)^T \nabla \widetilde{\mathbf{\Theta}}_{r+1}^n = \mathbf{C}^n \text{ in } B_{r+1},$$

$$\lim_{n\to\infty} \|\widetilde{\mathbf{\Theta}}_{r+1}^n - \mathbf{\Theta}\|_{3,K} = 0 \text{ for all } K \Subset B_{r+1},$$

and since the set $\bigcup_{s=1}^{r+1} B_s$ is connected (part (i)), Theorem 1.7-1 shows that there exist vectors $\boldsymbol{c}^n \in \mathbf{E}^3$ and matrices $\mathbf{Q}^n \in \mathbb{O}^3$, $n \geq 0$, such that

$$\widetilde{\mathbf{\Theta}}_{r+1}^n(x) = \boldsymbol{c}^n + \mathbf{Q}^n \mathbf{\Theta}_r^n(x) \text{ for all } x \in \Big(\bigcup_{s=1}^r B_s\Big) \cap B_{r+1}.$$

Then an argument similar to that used in part (ii) shows that $\lim_{n\to\infty} \mathbf{Q}^n = \mathbf{I}$ and $\lim_{n\to\infty} \boldsymbol{c}^n = \mathbf{0}$, and an argument similar to that used in part (iii) (note that the ball B_{r+1} may intersect more than one of the balls B_s, $1 \leq s \leq r$) shows that the mappings $\mathbf{\Theta}_{r+1}^n \in \mathcal{C}^3(\bigcup_{s=1}^r B_s; \mathbf{E}^3)$, $n \geq 0$, defined by

$$\mathbf{\Theta}_{r+1}^n(x) := \mathbf{\Theta}_r^n(x) \text{ for all } x \in \bigcup_{s=1}^r B_s,$$

$$\mathbf{\Theta}_{r+1}^n(x) := (\mathbf{Q}^n)^T(\widetilde{\mathbf{\Theta}}_r^n(x) - \boldsymbol{c}^n) \text{ for all } x \in B_{r+1},$$

satisfy

$$\lim_{n\to\infty} \|\mathbf{\Theta}_{r+1}^n - \mathbf{\Theta}\|_{3,K} = 0 \text{ for all } K \Subset \bigcup_{s=1}^r B_s.$$

Then the mappings $\mathbf{\Theta}^n : \Omega \to \mathbf{E}^3$, $n \geq 0$, defined by

$$\mathbf{\Theta}^n(x) := \mathbf{\Theta}_r^n(x) \text{ for all } x \in \bigcup_{s=1}^r B_s,\ r \geq 1,$$

possess all the required properties: They are unambiguously defined since for all $s > r$, $\mathbf{\Theta}_s^n(x) = \mathbf{\Theta}_r^n(x)$ for all $x \in \bigcup_{s=1}^r B_s$ by construction; they are of class $\mathcal{C}^3$ since the mappings $\mathbf{\Theta}_r^n : \bigcup_{s=1}^r B_s \to \mathbf{E}^3$ are themselves of class $\mathcal{C}^3$; they satisfy $(\nabla \mathbf{\Theta}^n)^T \nabla \mathbf{\Theta}^n = \mathbf{C}^n$ in Ω since the mappings $\mathbf{\Theta}_r^n$ satisfy the same relations in $\bigcup_{s=1}^r B_s$; and finally, they satisfy $\lim_{n\to\infty} \|\mathbf{\Theta}^n - \mathbf{\Theta}\|_{3,K} = 0$ for all $K \Subset \Omega$ since any compact subset of Ω is contained in $\bigcup_{s=1}^r B_s$ for r large enough. This completes the proof. $\square$

It is easily seen that the assumptions $R_{qijk} = 0$ in Ω are in fact superfluous in Theorem 1.8-3 (as shown in the next proof, these relations are consequences of the assumptions $R^n_{qijk} = 0$ in $\Omega, n \geq 0$, and $\lim_{n\to\infty} \|\mathbf{C}^n - \mathbf{C}\|_{2,K} = 0$ for all $K \Subset \Omega$). This observation gives rise to the following corollary to Theorem 1.8-3, in the form of another *sequential continuity result*, of interest by itself. The novelties are that the assumptions are now made on the immersions $\mathbf{\Theta}^n$, $n \geq 0$, and that this result also provides the existence of a "limit" immersion $\mathbf{\Theta}$.

Theorem 1.8-4. *Let Ω be a connected and simply-connected open subset of $\mathbb{R}^3$. Let there be given immersions $\mathbf{\Theta}^n \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$, $n \geq 0$, and a matrix field $\mathbf{C} \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ such that*

$$\lim_{n\to\infty} \|(\nabla\mathbf{\Theta}^n)^T\nabla\mathbf{\Theta}^n - \mathbf{C}\|_{2,K} = 0 \text{ for all } K \Subset \Omega.$$

Then there exist immersions $\widetilde{\mathbf{\Theta}}^n \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$, $n \geq 0$, of the form

$$\widetilde{\mathbf{\Theta}}^n = \boldsymbol{c}^n + \mathbf{Q}^n\mathbf{\Theta}^n, \text{ with } \boldsymbol{c}^n \in \mathbf{E}^3 \text{ and } \mathbf{Q}^n \in \mathbb{O}^3,$$

which thus satisfy $(\nabla\widetilde{\mathbf{\Theta}}^n)^T\nabla\widetilde{\mathbf{\Theta}}^n = (\nabla\mathbf{\Theta}^n)^T\nabla\mathbf{\Theta}^n$ in Ω for all $n \geq 0$, and there exists an immersion $\mathbf{\Theta} \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ such that

$$\nabla\mathbf{\Theta}^T\nabla\mathbf{\Theta} = \mathbf{C} \text{ in } \Omega \text{ and } \lim_{n\to\infty} \|\widetilde{\mathbf{\Theta}}^n - \mathbf{\Theta}\|_{3,K} = 0 \text{ for all } K \Subset \Omega.$$

Proof. Let the functions R^n_{qijk}, $n \geq 0$, and R_{qijk} be constructed from the components g^n_{ij} and g_{ij} of the matrix fields $\mathbf{C}^n := (\nabla\mathbf{\Theta}^n)^T\nabla\mathbf{\Theta}^n$ and $\mathbf{C}$ in the usual way (see, e.g., Theorem 1.6-1). Then $R^n_{qijk} = 0$ in Ω for all $n \geq 0$, since these relations are simply the necessary conditions of Theorem 1.5-1.

We now show that $R_{qijk} = 0$ in Ω. To this end, let K be any compact subset of Ω. The relations

$$\mathbf{C}^n = \mathbf{C}(\mathbf{I} + \mathbf{C}^{-1}(\mathbf{C}^n - \mathbf{C})),\ n \geq 0,$$

together with the inequalities $\|\mathbf{AB}\|_{2,K} \leq 4\|\mathbf{A}\|_{2,K}\|\mathbf{B}\|_{2,K}$ valid for any matrix fields $\mathbf{A}, \mathbf{B} \in \mathcal{C}^2(\Omega; \mathbb{M}^3)$, show that there exists $n_0 = n_0(K)$ such that the matrix fields $(\mathbf{I} + \mathbf{C}^{-1}(\mathbf{C}^n - \mathbf{C}))(x)$ are invertible at all $x \in K$ for all $n \geq n_0$. The same relations also show that there exists a constant M such that $\|(\mathbf{C}^n)^{-1}\|_{2,K} \leq M$ for all $n \geq n_0$. Hence the relations

$$(\mathbf{C}^n)^{-1} - \mathbf{C}^{-1} = \mathbf{C}^{-1}(\mathbf{C} - \mathbf{C}^n)(\mathbf{C}^n)^{-1},\ n \geq n_0,$$

together with the assumptions $\lim_{n\to\infty} \|\mathbf{C}^n - \mathbf{C}\|_{2,K} = 0$, in turn imply that the components $g^{ij,n}$, $n \geq n_0$, and g^{ij} of the matrix fields $(\mathbf{C}^n)^{-1}$ and $\mathbf{C}^{-1}$ satisfy

$$\lim_{n\to\infty} \|g^{ij,n} - g^{ij}\|_{2,K} = 0.$$

With self-explanatory notations, it thus follows that

$$\lim_{n\to\infty} \|\Gamma^n_{ijq} - \Gamma_{ijq}\|_{1,K} = 0 \text{ and } \lim_{n\to\infty} \|\Gamma^{p,n}_{ij} - \Gamma^p_{ij}\|_{1,K} = 0,$$

hence that $\lim_{n\to\infty} \|R^n_{qijk} - R_{qijk}\|_{0,K} = 0$. This shows that $R_{qijk} = 0$ in K, hence that $R_{qijk} = 0$ in Ω since K is an arbitrary compact subset of Ω.

By the fundamental existence theorem (Theorem 1.6-1), there thus exists a mapping $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ such that $\boldsymbol{\nabla}\boldsymbol{\Theta}^T\boldsymbol{\nabla}\boldsymbol{\Theta} = \mathbf{C}$ in Ω. Theorem 1.8-3 can now be applied, showing that there exist mappings $\widetilde{\boldsymbol{\Theta}}^n \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ such that

$$(\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^n)^T \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^n = \mathbf{C}^n \text{ in } \Omega,\ n \geq 0, \text{ and } \lim_{n\to\infty} \|\widetilde{\boldsymbol{\Theta}}^n - \boldsymbol{\Theta}\|_{3,K} \text{ for all } K \Subset \Omega.$$

Finally, the rigidity theorem (Theorem 1.7-1) shows that, for each $n \geq 0$, there exist $\boldsymbol{c}^n \in \mathbf{E}^3$ and $\mathbf{Q}^n \in \mathbb{O}^3$ such that $\widetilde{\boldsymbol{\Theta}}^n = \boldsymbol{c}^n + \mathbf{Q}^n\boldsymbol{\Theta}^n$ in Ω because the mappings $\widetilde{\boldsymbol{\Theta}}^n$ and $\boldsymbol{\Theta}^n$ share the same metric tensor field and the set Ω is connected. $\square$

It remains to show how the *sequential continuity* established in Theorem 1.8-3 implies the *continuity of a deformation as a function of its metric tensor* for *ad hoc* topologies.

Let Ω be an open subset of $\mathbb{R}^3$. For any integers $\ell \geq 0$ and $d \geq 1$, the space $\mathcal{C}^\ell(\Omega; \mathbb{R}^d)$ becomes a *locally convex topological space* when it is equipped with the *Fréchet topology* defined by the family of semi-norms $\|\cdot\|_{\ell,K}$, $K \Subset \Omega$, defined earlier. Then a sequence $(\boldsymbol{\Theta}^n)_{n\geq 0}$ converges to $\boldsymbol{\Theta}$ with respect to this topology if and only if

$$\lim_{n\to\infty} \|\boldsymbol{\Theta}^n - \boldsymbol{\Theta}\|_{\ell,K} = 0 \text{ for all } K \Subset \Omega.$$

Furthermore, this topology is *metrizable*: Let $(K_i)_{i\geq 0}$ be any sequence of subsets of Ω that satisfy

$$K_i \Subset \Omega \text{ and } K_i \subset \operatorname{int} K_{i+1} \text{ for all } i \geq 0, \text{ and } \Omega = \bigcup_{i=0}^{\infty} K_i.$$

Then

$$\lim_{n\to\infty} \|\boldsymbol{\Theta}^n - \boldsymbol{\Theta}\|_{\ell,K} = 0 \text{ for all } K \Subset \Omega \iff \lim_{n\to\infty} d_\ell(\boldsymbol{\Theta}^n, \boldsymbol{\Theta}) = 0,$$

where

$$d_\ell(\boldsymbol{\psi}, \boldsymbol{\Theta}) := \sum_{i=0}^{\infty} \frac{1}{2^i} \frac{\|\boldsymbol{\psi} - \boldsymbol{\Theta}\|_{\ell,K_i}}{1 + \|\boldsymbol{\psi} - \boldsymbol{\Theta}\|_{\ell,K_i}}.$$

For details about Fréchet topologies, see, e.g., Yosida [1966, Chapter 1].

Let $\dot{\mathcal{C}}^3(\Omega; \mathbf{E}^3) := \mathcal{C}^3(\Omega; \mathbf{E}^3)/\mathcal{R}$ denote the quotient set of $\mathcal{C}^3(\Omega; \mathbf{E}^3)$ by the equivalence relation $\mathcal{R}$, where $(\boldsymbol{\Theta}, \widetilde{\boldsymbol{\Theta}}) \in \mathcal{R}$ means that $\boldsymbol{\Theta}$ and $\widetilde{\boldsymbol{\Theta}}$ are *isometrically equivalent* (Section 1.7), i.e., that there exist a vector $\boldsymbol{c} \in \mathbf{E}^3$ and a matrix $\mathbf{Q} \in \mathbb{O}^3$ such that $\boldsymbol{\Theta}(x) = \boldsymbol{c} + \mathbf{Q}\widetilde{\boldsymbol{\Theta}}(x)$ for all $x \in \Omega$. Then it is easily verified that the set $\dot{\mathcal{C}}^3(\Omega; \mathbf{E}^3)$ becomes a *metric space* when it is equipped with the distance $\dot{d}_3$ defined by

$$\dot{d}_3(\dot{\boldsymbol{\Theta}}, \dot{\boldsymbol{\psi}}) = \inf_{\begin{cases}\boldsymbol{\kappa}\in\dot{\boldsymbol{\Theta}}\\ \boldsymbol{\chi}\in\dot{\boldsymbol{\psi}}\end{cases}} d_3(\boldsymbol{\kappa}, \boldsymbol{\chi}) = \inf_{\begin{cases}\boldsymbol{c}\in\mathbf{E}^3\\ \mathbf{Q}\in\mathbb{O}^3\end{cases}} d_3(\boldsymbol{\Theta}, \boldsymbol{c} + \mathbf{Q}\boldsymbol{\psi}),$$

where $\dot{\boldsymbol{\Theta}}$ denotes the equivalence class of $\boldsymbol{\Theta}$ *modulo* $\mathcal{R}$.

We now show that the announced continuity of an immersion as a function of its metric tensor is a corollary to Theorem 1.8-1. If d is a metric defined on a set X, the associated metric space is denoted $\{X; d\}$.

Theorem 1.8-5. *Let Ω be a connected and simply-connected open subset of $\mathbb{R}^3$. Let*

$$\mathcal{C}_0^2(\Omega;\mathbb{S}^3_>) := \{(g_{ij}) \in \mathcal{C}^2(\Omega;\mathbb{S}^3_>);\ R_{qijk} = 0 \text{ in } \Omega\},$$

and, given any matrix field $\mathbf{C} = (g_{ij}) \in \mathcal{C}_0^2(\Omega;\mathbb{S}^3_>)$, let $\mathcal{F}(\mathbf{C}) \in \dot{\mathcal{C}}^3(\Omega;\mathbf{E}^3)$ denote the equivalence class modulo $\mathcal{R}$ *of any $\mathbf{\Theta} \in \mathcal{C}^3(\Omega;\mathbf{E}^3)$ that satisfies $\boldsymbol{\nabla}\mathbf{\Theta}^T\boldsymbol{\nabla}\mathbf{\Theta} = \mathbf{C}$ in Ω. Then the mapping*

$$\mathcal{F} : \{\mathcal{C}_0^2(\Omega;\mathbb{S}^3_>); d_2\} \longrightarrow \{\dot{\mathcal{C}}^3(\Omega;\mathbf{E}^3); \dot{d}_3\}$$

defined in this fashion is continuous.

Proof. Since $\{\mathcal{C}_0^2(\Omega;\mathbb{S}^3_>); d_2\}$ and $\{\dot{\mathcal{C}}^3(\Omega;\mathbf{E}^3); \dot{d}_3\}$ are both metric spaces, it suffices to show that convergent sequences are mapped through $\mathcal{F}$ into convergent sequences.

Let then $\mathbf{C} \in \mathcal{C}_0^2(\Omega;\mathbb{S}^3_>)$ and $\mathbf{C}^n \in \mathcal{C}_0^2(\Omega;\mathbb{S}^3_>)$, $n \geq 0$, be such that

$$\lim_{n\to\infty} d_2(\mathbf{C}^n, \mathbf{C}) = 0,$$

i.e., such that $\lim_{n\to\infty} \|\mathbf{C}^n - \mathbf{C}\|_{2,K} = 0$ for all $K \Subset \Omega$. Given any $\mathbf{\Theta} \in \mathcal{F}(\mathbf{C})$, Theorem 1.8-3 shows that there exist $\mathbf{\Theta}^n \in \mathcal{F}(\mathbf{C}^n)$, $n \geq 0$, such that $\lim_{n\to\infty} \|\mathbf{\Theta}^n - \mathbf{\Theta}\|_{3,K} = 0$ for all $K \Subset \Omega$, i.e., such that

$$\lim_{n\to\infty} d_3(\mathbf{\Theta}^n, \mathbf{\Theta}) = 0.$$

Consequently,

$$\lim_{n\to\infty} \dot{d}_3(\mathcal{F}(\mathbf{C}^n), \mathcal{F}(\mathbf{C})) = 0.$$

$\square$

As shown by Ciarlet & C. Mardare [2004b], the above continuity result can be extended "*up to the boundary of the set* Ω", as follows. If Ω is bounded and has a Lipschitz-continuous boundary, the mapping $\mathcal{F}$ of Theorem 1.8-5 can be extended to a mapping that is locally Lipschitz-continuous with respect to the topologies of the Banach spaces $\mathcal{C}^2(\overline{\Omega};\mathbb{S}^3)$ for the continuous extensions of the symmetric matrix fields $\mathbf{C}$, and $\mathcal{C}^3(\overline{\Omega};\mathbf{E}^3)$ for the continuous extensions of the immersions $\mathbf{\Theta}$ (the existence of such continuous extensions is briefly commented upon at the end of Section 1.6).

Another extension, again motivated by nonlinear three-dimensional elasticity, is the following: Let Ω be a bounded and connected subset of $\mathbb{R}^3$, and let $\mathcal{B}$ be an elastic body with Ω as its *reference configuration.* Thanks mostly to the landmark existence theory of Ball [1977], it is now customary in nonlinear three-dimensional elasticity to view any mapping $\mathbf{\Theta} \in H^1(\Omega;\mathbf{E}^3)$ that is almost-everywhere injective and satisfies $\det \boldsymbol{\nabla}\mathbf{\Theta} > 0$ a.e. in Ω as a possible *deformation* of $\mathcal{B}$ when $\mathcal{B}$ is subjected to *ad hoc* applied forces and boundary conditions. The almost-everywhere injectivity of $\boldsymbol{\Theta}$ (understood in the sense of Ciarlet & Nečas [1987]) and the restriction on the sign of $\det \boldsymbol{\nabla\Theta}$ mathematically express (in an arguably weak way) the *non-interpenetrability* and *orientation-preserving conditions* that any physically realistic deformation should satisfy.

As mentioned earlier, the *Cauchy-Green tensor field* $\boldsymbol{\nabla\Theta}^T\boldsymbol{\nabla\Theta} \in L^1(\Omega; \mathbb{S}^3)$ associated with a deformation $\boldsymbol{\Theta} \in H^1(\Omega; \mathbf{E}^3)$ pervades the mathematical modeling of three-dimensional nonlinear elasticity. Conceivably, an alternative approach to the existence theory in three-dimensional elasticity could thus regard *the Cauchy-Green tensor as the primary unknown*, instead of the deformation itself as is usually the case.

Clearly, *the Cauchy-Green tensors depend continuously on the deformations*, since the Cauchy-Schwarz inequality immediately shows that the mapping

$$\boldsymbol{\Theta} \in H^1(\Omega; \mathbf{E}^3) \to \boldsymbol{\nabla\Theta}^T\boldsymbol{\nabla\Theta} \in L^1(\Omega; \mathbb{S}^3)$$

is continuous (irrespectively of whether the mappings $\boldsymbol{\Theta}$ are almost-everywhere injective and orientation-preserving).

Then Ciarlet & C. Mardare [2004c] have shown that, *under appropriate smoothness and orientation-preserving assumptions, the converse holds, i.e., the deformations depend continuously on their Cauchy-Green tensors, the topologies being those of the same spaces* $H^1(\Omega; \mathbf{E}^3)$ *and* $L^1(\Omega; \mathbb{S}^3)$ (by contrast with the orientation-preserving condition, the issue of non-interpenetrability turns out to be irrelevant to this issue). In fact, this continuity result holds in an arbitrary dimension d, at no extra cost in its proof; so it will be stated below in this more general setting. The notation $\mathbf{E}^d$ then denotes a d-dimensional Euclidean space and $\mathbb{S}^d$ denotes the space of all symmetric matrices of order d.

This continuity result is itself a simple consequence of the following **nonlinear Korn inequality**, which constitutes the main result of *ibid.*: Let Ω be a bounded and connected open subset of $\mathbb{R}^d$ with a Lipschitz-continuous boundary and let $\boldsymbol{\Theta} \in \mathcal{C}^1(\overline{\Omega}; \mathbf{E}^d)$ be a mapping satisfying $\det \boldsymbol{\nabla\Theta} > 0$ in $\overline{\Omega}$. Then there exists a constant $C(\boldsymbol{\Theta})$ with the following property: For each orientation-preserving mapping $\boldsymbol{\Phi} \in H^1(\Omega; \mathbf{E}^d)$, there exist a proper orthogonal matrix $\mathbf{R} = \mathbf{R}(\boldsymbol{\Phi}, \boldsymbol{\Theta})$ of order d (i.e., an orthogonal matrix of order d with a determinant equal to one) and a vector $\boldsymbol{b} = \boldsymbol{b}(\boldsymbol{\Phi}, \boldsymbol{\Theta})$ in $\mathbf{E}^d$ such that

$$\|\boldsymbol{\Phi} - (\boldsymbol{b} + \mathbf{R}\boldsymbol{\Theta})\|_{H^1(\Omega;\mathbf{E}^d)} \leq C(\boldsymbol{\Theta})\|\boldsymbol{\nabla\Phi}^T\boldsymbol{\nabla\Phi} - \boldsymbol{\nabla\Theta}^T\boldsymbol{\nabla\Theta}\|^{1/2}_{L^1(\Omega;\mathbb{S}^d)}.$$

That a vector $\boldsymbol{b}$ and an orthogonal matrix $\mathbf{R}$ should appear in the left-hand side of such an inequality is of course reminiscent of the classical *rigidity theorem* (Theorem 1.7-1), which asserts that, if two mappings $\widetilde{\boldsymbol{\Theta}} \in \mathcal{C}^1(\Omega; \mathbf{E}^d)$ and $\boldsymbol{\Theta} \in \mathcal{C}^1(\Omega; \mathbf{E}^d)$ satisfying $\det \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}} > 0$ and $\det \boldsymbol{\nabla\Theta} > 0$ in an open connected subset Ω of $\mathbb{R}^d$ have the same Cauchy-Green tensor field, then the two mappings are *isometrically equivalent*, i.e., there exist a vector $\boldsymbol{b}$ in $\mathbf{E}^d$ and an orthogonal matrix $\mathbf{R}$ of order d such that $\widetilde{\boldsymbol{\Theta}}(x) = \boldsymbol{b} + \mathbf{R}\boldsymbol{\Theta}(x)$ for all $x \in \Omega$.

More generally, we shall say that two orientation-preserving mappings $\widetilde{\boldsymbol{\Theta}} \in H^1(\Omega; \mathbf{E}^d)$ and $\boldsymbol{\Theta} \in H^1(\Omega; \mathbf{E}^d)$ are *isometrically equivalent* if there exist a vector $\boldsymbol{b}$ in $\mathbf{E}^d$ and an orthogonal matrix $\mathbf{R}$ of order d (a proper one in this case) such that

$$\widetilde{\boldsymbol{\Theta}}(x) = \boldsymbol{b} + \mathbf{R}\boldsymbol{\Theta}(x) \text{ for almost all } x \in \Omega.$$

One application of the above key inequality is the following *sequential continuity property*: Let $\boldsymbol{\Theta}^k \in H^1(\Omega; \mathbf{E}^d)$, $k \geq 1$, and $\boldsymbol{\Theta} \in \mathcal{C}^1(\overline{\Omega}; \mathbf{E}^d)$ be orientation-preserving mappings. Then there exist a constant $C(\boldsymbol{\Theta})$ and orientation-preserving mappings $\widetilde{\boldsymbol{\Theta}}^k \in H^1(\Omega; \mathbf{E}^d)$, $k \geq 1$, that are isometrically equivalent to $\boldsymbol{\Theta}^k$ such that

$$\|\widetilde{\boldsymbol{\Theta}}^k - \boldsymbol{\Theta}\|_{H^1(\Omega;\mathbf{E}^d)} \leq C(\boldsymbol{\Theta}) \|(\boldsymbol{\nabla}\boldsymbol{\Theta}^k)^T \boldsymbol{\nabla}\boldsymbol{\Theta}^k - \boldsymbol{\nabla}\boldsymbol{\Theta}^T \boldsymbol{\nabla}\boldsymbol{\Theta}\|^{1/2}_{L^1(\Omega;\mathbb{S}^d)}.$$

Hence *the sequence* $(\widetilde{\boldsymbol{\Theta}}^k)_{k=1}^\infty$ *converges to* $\boldsymbol{\Theta}$ *in* $H^1(\Omega; \mathbf{E}^d)$ *as* $k \to \infty$ *if the sequence* $((\boldsymbol{\nabla}\boldsymbol{\Theta}^k)^T \boldsymbol{\nabla}\boldsymbol{\Theta}^k)_{k=1}^\infty$ *converges to* $\boldsymbol{\nabla}\boldsymbol{\Theta}^T \boldsymbol{\nabla}\boldsymbol{\Theta}$ *in* $L^1(\Omega; \mathbb{S}^d)$ *as* $k \to \infty$.

Should the Cauchy-Green strain tensor be viewed as the primary unknown (as suggested above), such a sequential continuity could thus prove to be useful when considering *infimizing sequences* of the total energy, in particular for handling the part of the energy that takes into account the applied forces and the boundary conditions, which are both naturally expressed in terms of the deformation itself.

They key inequality is first established in the special case where $\boldsymbol{\Theta}$ is the identity mapping of the set Ω, by making use in particular of a fundamental "*geometric rigidity lemma*" recently proved by Friesecke, James & Müller [2002]. It is then extended to an arbitrary mapping $\boldsymbol{\Theta} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{R}^n)$ satisfying $\det \boldsymbol{\nabla}\boldsymbol{\Theta} > 0$ in $\overline{\Omega}$, thanks in particular to a methodology that bears some similarity with that used in Theorems 1.8-2 and 1.8-3.

Such results are to be compared with the earlier, pioneering estimates of John [1961], John [1972] and Kohn [1982], which implied *continuity at rigid body deformations*, i.e., at a mapping $\boldsymbol{\Theta}$ that is isometrically equivalent to the identity mapping of Ω. The recent and noteworthy continuity result of Reshetnyak [2003] for *quasi-isometric mappings* is in a sense complementary to the above one (it also deals with Sobolev type norms).

Chapter 2

DIFFERENTIAL GEOMETRY OF SURFACES

INTRODUCTION

We saw in Chapter 1 that an *open set* $\mathbf{\Theta}(\Omega)$ *in* $\mathbf{E}^3$, where Ω is an open set in $\mathbb{R}^3$ and $\mathbf{\Theta} : \Omega \to \mathbf{E}^3$ is a smooth injective immersion, is unambiguously defined (up to isometries of $\mathbf{E}^3$) by a *single* tensor field, the *metric tensor* field, whose covariant components $g_{ij} = g_{ji} : \Omega \to \mathbb{R}$ are given by $g_{ij} = \partial_i \mathbf{\Theta} \cdot \partial_j \mathbf{\Theta}$.

Consider instead a *surface* $\widehat{\omega} = \boldsymbol{\theta}(\omega)$ *in* $\mathbf{E}^3$, where ω is a two-dimensional open set in $\mathbb{R}^2$ and $\boldsymbol{\theta} : \omega \to \mathbf{E}^3$ is a smooth injective immersion. Then by contrast, such a "two-dimensional manifold" equipped with the coordinates of the points of ω as its *curvilinear coordinates*, requires *two* tensor fields for its definition (this time up to proper isometries of $\mathbf{E}^3$), the *first and second fundamental forms* of $\widehat{\omega}$. Their covariant components $a_{\alpha\beta} = a_{\beta\alpha} : \omega \to \mathbb{R}$ and $b_{\alpha\beta} = b_{\beta\alpha} : \omega \to \mathbb{R}$ are respectively given by (Greek indices or exponents take their values in $\{1, 2\}$):

$$a_{\alpha\beta} = \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta \text{ and } b_{\alpha\beta} = \boldsymbol{a}_3 \cdot \partial_\alpha \boldsymbol{a}_\beta,$$

where $\boldsymbol{a}_\alpha = \partial_\alpha \boldsymbol{\theta}$ and $\boldsymbol{a}_3 = \dfrac{\boldsymbol{a}_1 \wedge \boldsymbol{a}_2}{|\boldsymbol{a}_1 \wedge \boldsymbol{a}_2|}$.

The vector fields $\boldsymbol{a}_i : \omega \to \mathbb{R}^3$ defined in this fashion constitute the *covariant bases* along the surface $\widehat{\omega}$, while the vector fields $\boldsymbol{a}^i : \omega \to \mathbb{R}^3$ defined by the relations $\boldsymbol{a}^i \cdot \boldsymbol{a}_j = \delta^i_j$ constitute the *contravariant bases* along $\widehat{\omega}$.

These two fundamental forms are introduced and studied in Sections 2.1 to 2.5. In particular, it is shown how *areas* and *lengths*, i.e., "*metric notions*", *on* the surface $\widehat{\omega}$ are computed in terms of its curvilinear coordinates by means of the components $a_{\alpha\beta}$ of the first fundamental form (Theorem 2.3-1). It is also shown how the *curvature of a curve on* $\widehat{\omega}$ can be similarly computed, this time by means of the components of both fundamental forms (Theorem 2.4-1). Other classical notions about "curvature", such as the *principal curvatures* and the *Gaussian curvature*, are introduced and briefly discussed in Section 2.5.

We next introduce in Section 2.6 the fundamental notion of *covariant derivatives* $\eta_{i|\alpha}$ *of a vector field* $\eta_i \boldsymbol{a}^i : \omega \to \mathbb{R}^3$ *on* $\widehat{\omega}$, thus defined here by means of its covariant components η_i over the contravariant bases $\boldsymbol{a}^i$. We establish in this process the *formulas of Gauß and Weingarten* (Theorem 2.6-1). As illustrated in Chapter 4, covariant derivatives of vector fields on a surface (typically, the unknown displacement vector field of the middle surface of a shell) pervade the equations of shell theory.

It is a basic fact that the symmetric and positive definite matrix field $(a_{\alpha\beta})$ and the symmetric matrix field $(b_{\alpha\beta})$ defined on ω in this fashion cannot be arbitrary. More specifically, their components and some of their partial derivatives must satisfy *necessary conditions* taking the form of the following relations (meant to hold for all $\alpha, \beta, \sigma, \tau \in \{1, 2\}$), which respectively constitute the *Gauß*, and *Codazzi-Mainardi, equations* (Theorem 2.7-1): Let the functions $\Gamma_{\alpha\beta\tau}$ and $\Gamma^\sigma_{\alpha\beta}$ be defined by

$$\Gamma_{\alpha\beta\tau} = \frac{1}{2}(\partial_\beta a_{\alpha\tau} + \partial_\alpha a_{\beta\tau} - \partial_\tau a_{\alpha\beta}) \text{ and } \Gamma^\sigma_{\alpha\beta} = a^{\sigma\tau}\Gamma_{\alpha\beta\tau}, \text{ where } (a^{\sigma\tau}) := (a_{\alpha\beta})^{-1}.$$

Then, necessarily,

$$\begin{aligned} \partial_\beta \Gamma_{\alpha\sigma\tau} - \partial_\sigma \Gamma_{\alpha\beta\tau} + \Gamma^\mu_{\alpha\beta}\Gamma_{\sigma\tau\mu} - \Gamma^\mu_{\alpha\sigma}\Gamma_{\beta\tau\mu} &= b_{\alpha\sigma}b_{\beta\tau} - b_{\alpha\beta}b_{\sigma\tau} \text{ in } \omega, \\ \partial_\beta b_{\alpha\sigma} - \partial_\sigma b_{\alpha\beta} + \Gamma^\mu_{\alpha\sigma} b_{\beta\mu} - \Gamma^\mu_{\alpha\beta} b_{\sigma\mu} &= 0 \text{ in } \omega. \end{aligned}$$

The functions $\Gamma_{\alpha\beta\tau}$ and $\Gamma^\sigma_{\alpha\beta}$ are the *Christoffel symbols* of the *first*, and *second, kind*.

We also establish in passing (Theorem 2.7-2) the celebrated *Theorema Egregium of Gauß*: At each point of a surface, the Gaussian curvature is a given function (the same for any surface) of the components of the first fundamental form and their partial derivatives of order ≤ 2 at the same point.

We then turn to the reciprocal questions:

Given an open subset ω of $\mathbb{R}^2$ and a smooth enough symmetric and positive definite matrix field $(a_{\alpha\beta})$ together with a smooth enough symmetric matrix field $(b_{\alpha\beta})$ defined over ω, when are they the first and second fundamental forms of a surface $\boldsymbol{\theta}(\omega) \subset \mathbf{E}^3$, i.e., when does there exist an immersion $\boldsymbol{\theta} : \omega \to \mathbf{E}^3$ such that

$$a_{\alpha\beta} = \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} \text{ and } b_{\alpha\beta} = \partial_{\alpha\beta}\boldsymbol{\theta} \cdot \left\{ \frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|} \right\} \text{ in } \omega?$$

If such an immersion exists, to what extent is it unique?

As shown in Theorems 2.8-1 and 2.9-1 (like those of their "three-dimensional counterparts" in Sections 1.6 and 1.7, their proofs are by no means easy, especially that of the existence), the answers turn out to be remarkably simple: *Under the assumption that ω is simply-connected, the necessary conditions expressed by the Gauß and Codazzi-Mainardi equations are also sufficient for the existence of such an immersion $\boldsymbol{\theta}$.*

Besides, *if ω is connected, this immersion is unique up to proper isometries in* $\mathbf{E}$. This means that, if $\widetilde{\boldsymbol{\theta}} : \omega \to \mathbf{E}^3$ is any other smooth immersion satisfying

$$a_{\alpha\beta} = \partial_\alpha \widetilde{\boldsymbol{\theta}} \cdot \partial_\beta \widetilde{\boldsymbol{\theta}} \text{ and } b_{\alpha\beta} = \partial_{\alpha\beta}\widetilde{\boldsymbol{\theta}} \cdot \left\{ \frac{\partial_1 \widetilde{\boldsymbol{\theta}} \wedge \partial_2 \widetilde{\boldsymbol{\theta}}}{|\partial_1 \widetilde{\boldsymbol{\theta}} \wedge \partial_2 \widetilde{\boldsymbol{\theta}}|} \right\} \text{ in } \omega$$

there then exist a vector $\boldsymbol{c} \in \mathbf{E}^3$ and a proper orthogonal matrix $\mathbf{Q}$ of order three such that

$$\boldsymbol{\theta}(y) = \boldsymbol{c} + \mathbf{Q}\widetilde{\boldsymbol{\theta}}(y) \text{ for all } y \in \omega.$$

Together, the above existence and uniqueness theorems constitute the *fundamental theorem of surface theory*, another important special case of the *fundamental theorem of Riemannian geometry* already alluded to in Chapter 1. As such, they constitute the core of Chapter 2.

We conclude this chapter by showing (Theorem 2.10-3) that *the equivalence class of $\boldsymbol{\theta}$, defined in this fashion* modulo *proper isometries of $\mathbf{E}^3$, depends continuously on the matrix fields $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$ with respect to appropriate Fréchet topologies.*

2.1 CURVILINEAR COORDINATES ON A SURFACE

In addition to the rules governing Latin indices that we set in Section 1.1, we henceforth require that *Greek* indices and exponents vary in the set $\{1, 2\}$ and that the summation convention be systematically used in conjunction with these rules. For instance, the relation

$$\partial_\alpha(\eta_i \boldsymbol{a}^i) = (\eta_{\beta|\alpha} - b_{\alpha\beta}\eta_3)\boldsymbol{a}^\beta + (\eta_{3|\alpha} + b_\alpha^\beta \eta_\beta)\boldsymbol{a}^3$$

means that, for $\alpha = 1, 2$,

$$\partial_\alpha\Big(\sum_{i=1}^{3} \eta_i \boldsymbol{a}^i\Big) = \sum_{\beta=1}^{2}(\eta_{\beta|\alpha} - b_{\alpha\beta}\eta_3)\boldsymbol{a}^\beta + \Big(\eta_{3|\alpha} + \sum_{\beta=1}^{2} b_\alpha^\beta \eta_\beta\Big)\boldsymbol{a}^3.$$

Kronecker's symbols are designated by $\delta_\alpha^\beta, \delta_{\alpha\beta}$, or $\delta^{\alpha\beta}$ according to the context.

Let there be given as in Section 1.1 a *three-dimensional Euclidean space* $\mathbf{E}^3$, equipped with an orthonormal basis consisting of three vectors $\widehat{\boldsymbol{e}}^i = \widehat{\boldsymbol{e}}_i$, and let $\boldsymbol{a} \cdot \boldsymbol{b}, |\boldsymbol{a}|$, and $\boldsymbol{a} \wedge \boldsymbol{b}$ denote the Euclidean inner product, the Euclidean norm, and the vector product of vectors $\boldsymbol{a}, \boldsymbol{b}$ in the space $\mathbf{E}^3$.

In addition, let there be given a *two-dimensional vector space*, in which two vectors $\boldsymbol{e}^\alpha = \boldsymbol{e}_\alpha$ form a basis. *This space will be identified with* $\mathbb{R}^2$. Let y_α denote the coordinates of a point $y \in \mathbb{R}^2$ and let $\partial_\alpha := \partial/\partial y_\alpha$ and $\partial_{\alpha\beta} := \partial^2/\partial y_\alpha \partial y_\beta$.

Finally, let there be given an *open* subset ω of $\mathbb{R}^2$ and a smooth enough mapping $\boldsymbol{\theta} : \omega \to \mathbf{E}^3$ (specific smoothness assumptions on $\boldsymbol{\theta}$ will be made later, according to each context). The set

$$\widehat{\omega} := \boldsymbol{\theta}(\omega)$$

is called a **surface** in $\mathbf{E}^3$.

If the mapping $\boldsymbol{\theta} : \omega \to \mathbf{E}^3$ is injective, each point $\widehat{y} \in \widehat{\omega}$ can be unambiguously written as

$$\widehat{y} = \boldsymbol{\theta}(y), \quad y \in \omega,$$

and the two coordinates y_α of y are called the **curvilinear coordinates** of $\widehat{y}$ (Figure 2.1-1). Well-known *examples* of surfaces and of curvilinear coordinates and their corresponding coordinate lines (defined in Section 2.2) are given in Figures 2.1-2 and 2.1-3.

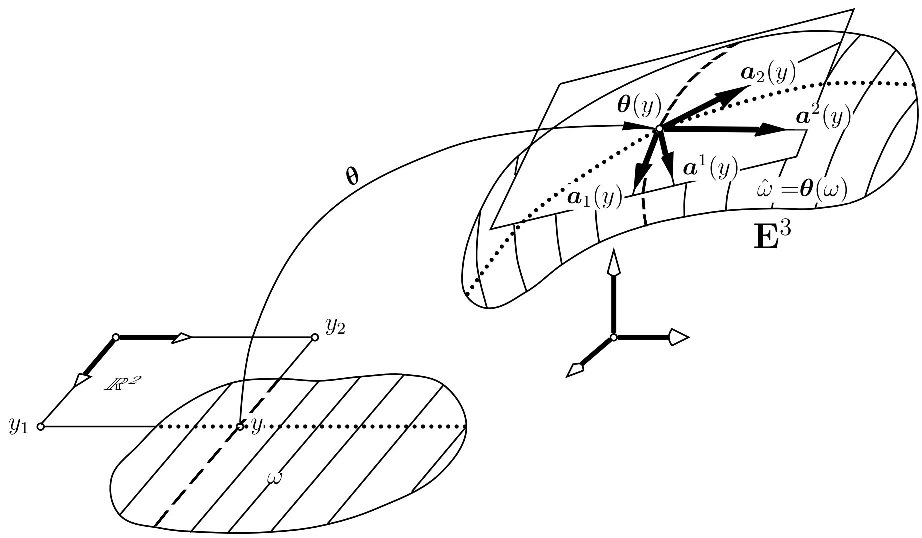

Figure 2.1-1: *Curvilinear coordinates on a surface and covariant and contravariant bases of the tangent plane.* Let $\widehat{\omega} = \boldsymbol{\theta}(\omega)$ be a surface in $\mathbf{E}^3$. The two coordinates y_1, y_2 of $y \in \omega$ are the curvilinear coordinates of $\widehat{y} = \boldsymbol{\theta}(y) \in \widehat{\omega}$. If the two vectors $\boldsymbol{a}_\alpha(y) = \partial_\alpha \boldsymbol{\theta}(y)$ are linearly independent, they are tangent to the coordinate lines passing through $\widehat{y}$ and they form the covariant basis of the tangent plane to $\widehat{\omega}$ at $\widehat{y} = \boldsymbol{\theta}(y)$. The two vectors $\boldsymbol{a}^\alpha(y)$ from this tangent plane defined by $\boldsymbol{a}^\alpha(y) \cdot \boldsymbol{a}_\beta(y) = \delta^\alpha_\beta$ form its contravariant basis.

Naturally, once a surface $\widehat{\omega}$ is defined as $\widehat{\omega} = \boldsymbol{\theta}(\omega)$, *there are infinitely many other ways of defining curvilinear coordinates on* $\widehat{\omega}$, depending on how the domain ω and the mapping $\boldsymbol{\theta}$ are chosen. For instance, a portion $\widehat{\omega}$ of a sphere may be represented by means of *Cartesian coordinates, spherical coordinates*, or *stereographic coordinates* (Figure 2.1-3). Incidentally, this example illustrates the variety of restrictions that have to be imposed on $\widehat{\omega}$ according to which kind of curvilinear coordinates it is equipped with!

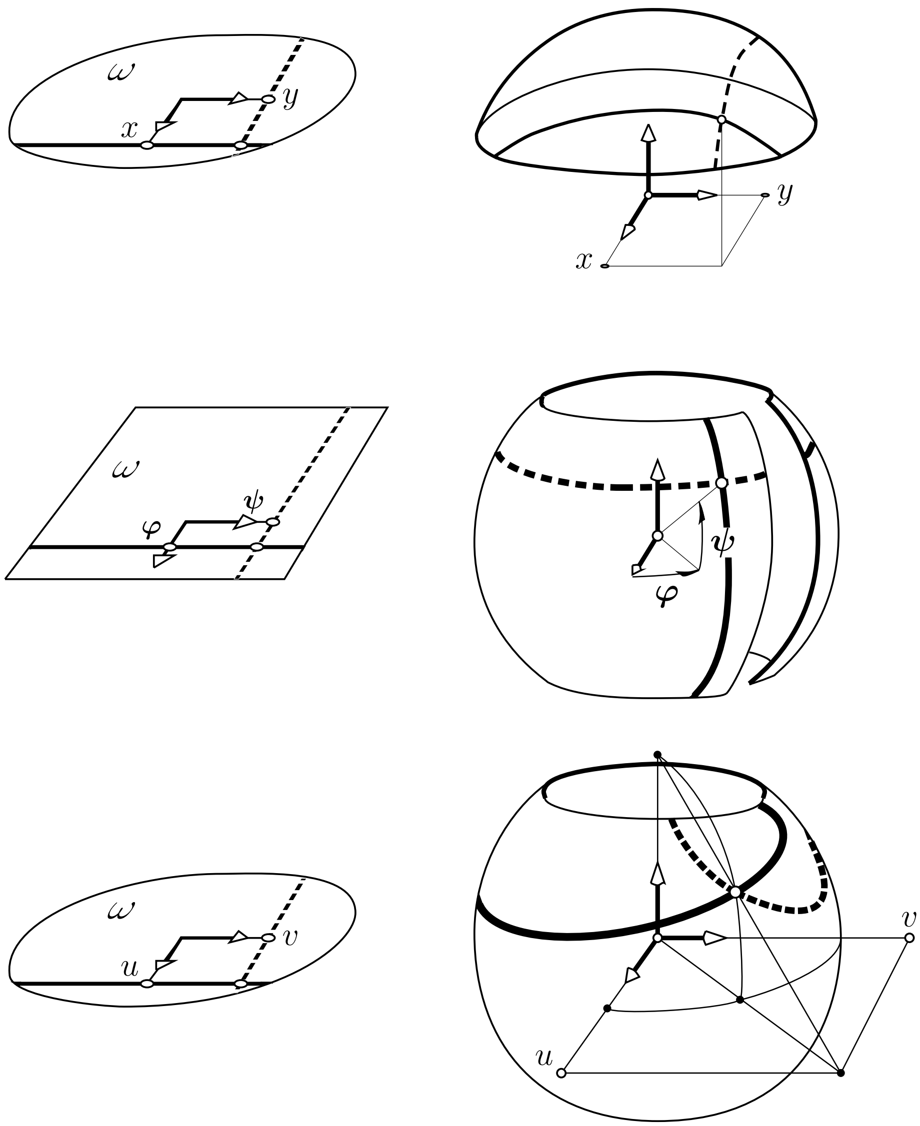

Figure 2.1-2: *Several systems of curvilinear coordinates on a sphere.* Let Σ be a sphere of radius R. A portion of Σ contained "in the northern hemisphere" can be represented by means of Cartesian coordinates, with a mapping $\boldsymbol{\theta}$ of the form:

$$\boldsymbol{\theta} : (x, y) \in \omega \to (x, y, \{R^2 - (x^2 + y^2)\}^{1/2}) \in \mathbf{E}^3.$$

A portion of Σ that excludes a neighborhood of both "poles" and of a "meridian" (to fix ideas) can be represented by means of spherical coordinates, with a mapping $\boldsymbol{\theta}$ of the form:

$$\boldsymbol{\theta} : (\varphi, \psi) \in \omega \to (R\cos\psi\cos\varphi, R\cos\psi\sin\varphi, R\sin\psi) \in \mathbf{E}^3.$$

A portion of Σ that excludes a neighborhood of the "North pole" can be represented by means of stereographic coordinates, with a mapping $\boldsymbol{\theta}$ of the form:

$$\boldsymbol{\theta} : (u, v) \in \omega \to \Big(\frac{2R^2 u}{u^2 + v^2 + R^2}, \frac{2R^2 v}{u^2 + v^2 + R^2}, R\frac{u^2 + v^2 - R^2}{u^2 + v^2 + R^2}\Big) \in \mathbf{E}^3.$$

The corresponding coordinate lines are represented in each case, with self-explanatory graphical conventions.

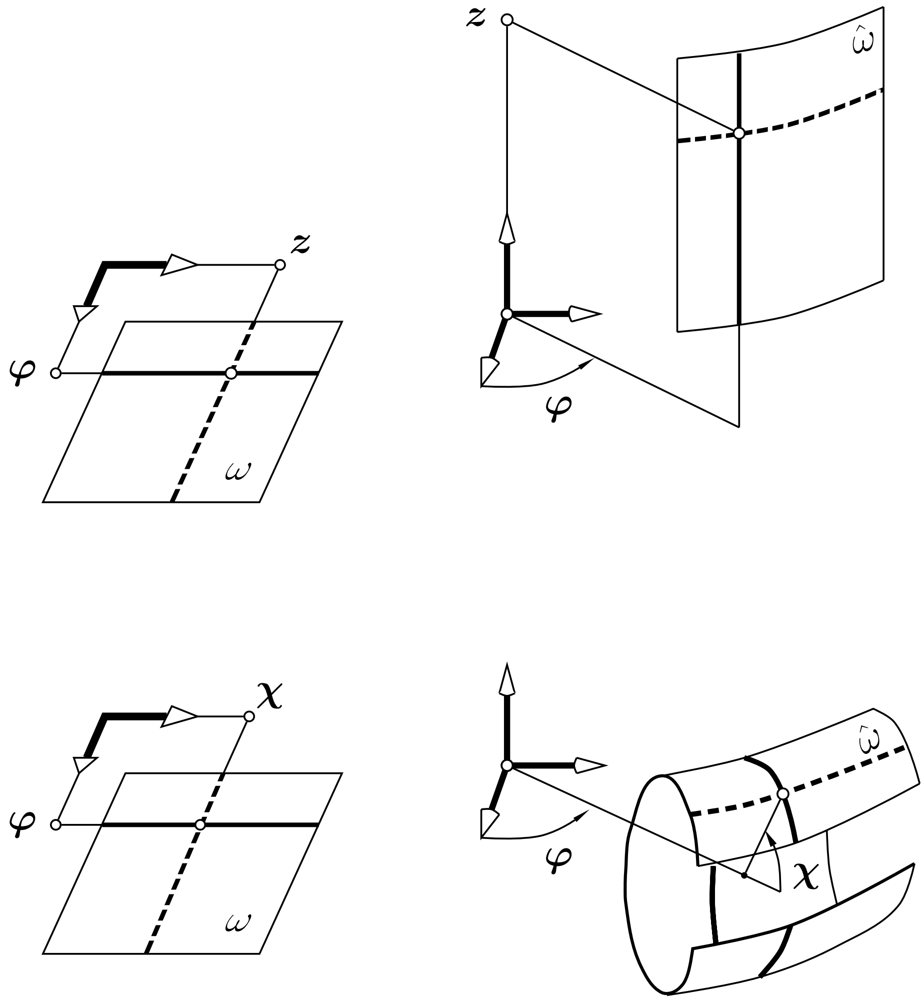

Figure 2.1-3: *Two familiar examples of surfaces and curvilinear coordinates.* A portion $\widehat{\omega}$ of a circular cylinder of radius R can be represented by a mapping $\boldsymbol{\theta}$ of the form

$$\boldsymbol{\theta} : (\varphi, z) \in \omega \to (R\cos\varphi, R\sin\varphi, z) \in \mathbf{E}^3.$$

A portion $\widehat{\omega}$ of a torus can be represented by a mapping $\boldsymbol{\theta}$ of the form

$$\boldsymbol{\theta} : (\varphi, \chi) \in \omega \to ((R + r\cos\chi)\cos\varphi, (R + r\cos\chi)\sin\varphi, r\sin\chi) \in \mathbf{E}^3,$$

with $R > r$.

The corresponding coordinate lines are represented in each case, with self-explanatory graphical conventions.

2.2 FIRST FUNDAMENTAL FORM

Let ω be an open subset of $\mathbb{R}^2$ and let

$$\boldsymbol{\theta} = \theta_i \widehat{\boldsymbol{e}}^i : \omega \subset \mathbb{R}^2 \to \boldsymbol{\theta}(\omega) = \widehat{\omega} \subset \mathbf{E}^3$$

be a mapping that is *differentiable at a point* $y \in \omega$. If $\boldsymbol{\delta y}$ is such that $(y+\boldsymbol{\delta y}) \in \omega$, then

$$\boldsymbol{\theta}(y + \boldsymbol{\delta y}) = \boldsymbol{\theta}(y) + \boldsymbol{\nabla\theta}(y)\boldsymbol{\delta y} + o(\boldsymbol{\delta y}),$$

where the 3×2 matrix $\boldsymbol{\nabla\theta}(y)$ and the column vector $\boldsymbol{\delta y}$ are defined by

$$\boldsymbol{\nabla\theta}(y) := \begin{pmatrix} \partial_1\theta_1 & \partial_2\theta_1 \\ \partial_1\theta_2 & \partial_2\theta_2 \\ \partial_1\theta_3 & \partial_2\theta_3 \end{pmatrix}(y) \text{ and } \boldsymbol{\delta y} = \begin{pmatrix} \delta y_1 \\ \delta y_2 \end{pmatrix}.$$

Let the two vectors $\boldsymbol{a}_\alpha(y) \in \mathbb{R}^3$ be defined by

$$\boldsymbol{a}_\alpha(y) := \partial_\alpha\boldsymbol{\theta}(y) = \begin{pmatrix} \partial_\alpha\theta_1 \\ \partial_\alpha\theta_2 \\ \partial_\alpha\theta_3 \end{pmatrix}(y),$$

i.e., $\boldsymbol{a}_\alpha(y)$ *is the* α-th *column vector of the matrix* $\boldsymbol{\nabla\theta}(y)$. Then the expansion of $\boldsymbol{\theta}$ about y may be also written as

$$\boldsymbol{\theta}(y + \boldsymbol{\delta y}) = \boldsymbol{\theta}(y) + \delta y^\alpha \boldsymbol{a}_\alpha(y) + o(\boldsymbol{\delta y}).$$

If in particular $\boldsymbol{\delta y}$ is of the form $\boldsymbol{\delta y} = \delta t \boldsymbol{e}_\alpha$, where $\delta t \in \mathbb{R}$ and $\boldsymbol{e}_\alpha$ is one of the basis vectors in $\mathbb{R}^2$, this relation reduces to

$$\boldsymbol{\theta}(y + \delta t \boldsymbol{e}_\alpha) = \boldsymbol{\theta}(y) + \delta t \boldsymbol{a}_\alpha(y) + o(\delta t).$$

A mapping $\boldsymbol{\theta} : \omega \to \mathbf{E}^3$ is an **immersion at** $y \in \omega$ if it is differentiable at y and the 3×2 matrix $\boldsymbol{\nabla\theta}(y)$ is of rank two, or equivalently if the two vectors $\boldsymbol{a}_\alpha(y) = \partial_\alpha\boldsymbol{\theta}(y)$ are linearly independent.

Assume from now on in this section that *the mapping* $\boldsymbol{\theta}$ *is an immersion at* y. In this case, the last relation shows that *each vector* $\boldsymbol{a}_\alpha(y)$ *is tangent to the* α-th **coordinate line** *passing through* $\widehat{y} = \boldsymbol{\theta}(y)$, defined as the image by $\boldsymbol{\theta}$ of the points of ω that lie on a line parallel to $\boldsymbol{e}_\alpha$ passing through y (there exist t_0 and t_1 with $t_0 < 0 < t_1$ such that the α-*th* coordinate line is given by $t \in \,]t_0, t_1[\, \to \boldsymbol{f}_\alpha(t) := \boldsymbol{\theta}(y + t\boldsymbol{e}_\alpha)$ in a neighborhood of $\widehat{y}$; hence $\boldsymbol{f}'_\alpha(0) = \partial_\alpha\boldsymbol{\theta}(y) = \boldsymbol{a}_\alpha(y)$); see Figures 2.1-1, 2.1-2, and 2.1-3.

The vectors $\boldsymbol{a}_\alpha(y)$, which thus span the *tangent plane* to the surface $\widehat{\omega}$ at $\widehat{y} = \boldsymbol{\theta}(y)$, form the **covariant basis of the tangent plane** to $\widehat{\omega}$ at $\widehat{y}$; see Figure 2.1-1.

Returning to a general increment $\boldsymbol{\delta y} = \delta y^\alpha \boldsymbol{e}_\alpha$, we also infer from the expansion of $\boldsymbol{\theta}$ about y that ($\boldsymbol{\delta y}^T$ and $\boldsymbol{\nabla\theta}(y)^T$ respectively designate the transpose of the column vector $\boldsymbol{\delta y}$ and the transpose of the matrix $\boldsymbol{\nabla\theta}(y)$)

$$\begin{aligned} |\boldsymbol{\theta}(y + \boldsymbol{\delta y}) - \boldsymbol{\theta}(y)|^2 &= \boldsymbol{\delta y}^T \boldsymbol{\nabla\theta}(y)^T \boldsymbol{\nabla\theta}(y) \boldsymbol{\delta y} + o(|\boldsymbol{\delta y}|^2) \\ &= \delta y^\alpha \boldsymbol{a}_\alpha(y) \cdot \boldsymbol{a}_\beta(y) \delta y^\beta + o(|\boldsymbol{\delta y}|^2). \end{aligned}$$

In other words, the principal part with respect to $\boldsymbol{\delta y}$ of the length between the points $\boldsymbol{\theta}(y+\boldsymbol{\delta y})$ and $\boldsymbol{\theta}(y)$ is $\{\delta y^\alpha \boldsymbol{a}_\alpha(y)\cdot\boldsymbol{a}_\beta(y)\delta y^\beta\}^{1/2}$. This observation suggests to define a matrix $(a_{\alpha\beta}(y))$ of order two by letting

$$a_{\alpha\beta}(y) := \boldsymbol{a}_\alpha(y)\cdot\boldsymbol{a}_\beta(y) = \big(\boldsymbol{\nabla\theta}(y)^T\boldsymbol{\nabla\theta}(y)\big)_{\alpha\beta}.$$

The elements $a_{\alpha\beta}(y)$ of this symmetric matrix are called the **covariant components of the first fundamental form**, also called the **metric tensor**, *of the surface* $\widehat{\omega}$ at $\widehat{y}=\boldsymbol{\theta}(y)$.

Note that *the matrix* $(a_{\alpha\beta}(y))$ *is positive definite* since the vectors $\boldsymbol{a}_\alpha(y)$ are assumed to be linearly independent.

The two vectors $\boldsymbol{a}_\alpha(y)$ being thus defined, *the four relations*

$$\boldsymbol{a}^\alpha(y)\cdot\boldsymbol{a}_\beta(y)=\delta^\alpha_\beta$$

unambiguously define two linearly independent vectors $\boldsymbol{a}^\alpha(y)$ *in the tangent plane*. To see this, let *a priori* $\boldsymbol{a}^\alpha(y)=Y^{\alpha\sigma}(y)\boldsymbol{a}_\sigma(y)$ in the relations $\boldsymbol{a}^\alpha(y)\cdot\boldsymbol{a}_\beta(y)=\delta^\alpha_\beta$. This gives $Y^{\alpha\sigma}(y)a_{\sigma\beta}(y)=\delta^\alpha_\beta$; hence $Y^{\alpha\sigma}(y)=a^{\alpha\sigma}(y)$, where

$$(a^{\alpha\beta}(y)) := (a_{\alpha\beta}(y))^{-1}.$$

Hence $\boldsymbol{a}^\alpha(y)=a^{\alpha\sigma}(y)\boldsymbol{a}_\sigma(y)$. These relations in turn imply that

$$\begin{aligned}\boldsymbol{a}^\alpha(y)\cdot\boldsymbol{a}^\beta(y) &= a^{\alpha\sigma}(y)a^{\beta\tau}(y)\boldsymbol{a}_\sigma(y)\cdot\boldsymbol{a}_\tau(y)\\ &= a^{\alpha\sigma}(y)a^{\beta\tau}(y)a_{\sigma\tau}(y) \;=\; a^{\alpha\sigma}(y)\delta^\beta_\sigma \;=\; a^{\alpha\beta}(y),\end{aligned}$$

and thus the vectors $\boldsymbol{a}^\alpha(y)$ are linearly independent since the matrix $(a^{\alpha\beta}(y))$ is positive definite. We would likewise establish that $\boldsymbol{a}_\alpha(y)=a_{\alpha\beta}(y)\boldsymbol{a}^\beta(y)$.

The two vectors $\boldsymbol{a}^\alpha(y)$ form the **contravariant basis of the tangent plane** to the surface $\widehat{\omega}$ at $\widehat{y}=\boldsymbol{\theta}(y)$ (Figure 2.1-1) and the elements $a^{\alpha\beta}(y)$ of the symmetric matrix $(a^{\alpha\beta}(y))$ are called the **contravariant components of the first fundamental form**, or **metric tensor**, of the surface $\widehat{\omega}$ at $\widehat{y}=\boldsymbol{\theta}(y)$.

Let us record for convenience the fundamental relations that exist between the vectors of the covariant and contravariant bases of the tangent plane and the covariant and contravariant components of the first fundamental tensor:

$$\begin{aligned}a_{\alpha\beta}(y)=\boldsymbol{a}_\alpha(y)\cdot\boldsymbol{a}_\beta(y) \quad &\text{and}\quad a^{\alpha\beta}(y)=\boldsymbol{a}^\alpha(y)\cdot\boldsymbol{a}^\beta(y),\\ \boldsymbol{a}_\alpha(y)=a_{\alpha\beta}(y)\boldsymbol{a}^\beta(y) \quad &\text{and}\quad \boldsymbol{a}^\alpha(y)=a^{\alpha\beta}(y)\boldsymbol{a}_\beta(y).\end{aligned}$$

A mapping $\boldsymbol{\theta}:\omega\to\mathbf{E}^3$ is an **immersion** if it is an immersion at each point in ω, i.e., if $\boldsymbol{\theta}$ is differentiable in ω and the two vectors $\partial_\alpha\boldsymbol{\theta}(y)$ are linearly independent at each $y\in\omega$.

If $\boldsymbol{\theta}:\omega\to\mathbf{E}^3$ is an immersion, the vector fields $\boldsymbol{a}_\alpha:\omega\to\mathbb{R}^3$ and $\boldsymbol{a}^\alpha:\omega\to\mathbb{R}^3$ respectively form the **covariant**, and **contravariant, bases of the tangent planes**.

A word of caution. The presentation in this section closely follows that of Section 1.2, the mapping $\boldsymbol{\theta} : \omega \subset \mathbb{R}^2 \to \mathbf{E}^3$ "replacing" the mapping $\boldsymbol{\Theta} : \Omega \subset \mathbb{R}^3 \to \mathbf{E}^3$. There are indeed strong *similarities* between the two presentations, such as the way the metric tensor is defined in both cases, but there are also sharp *differences*. In particular, the matrix $\boldsymbol{\nabla\theta}(y)$ is *not* a square matrix, while the matrix $\boldsymbol{\nabla\Theta}(x)$ *is* square! $\square$

2.3 AREAS AND LENGTHS ON A SURFACE

We now review fundamental formulas expressing *area* and *length elements* at a point $\widehat{y} = \boldsymbol{\theta}(y)$ of the surface $\widehat{\omega} = \boldsymbol{\theta}(\omega)$ in terms of the matrix $(a_{\alpha\beta}(y))$; see Figure 2.3-1.

These formulas highlight in particular the crucial rôle played by the matrix $(a_{\alpha\beta}(y))$ for computing "metric" notions at $\widehat{y} = \boldsymbol{\theta}(y)$. Indeed, the first fundamental form well deserves "metric tensor" as its *alias*!

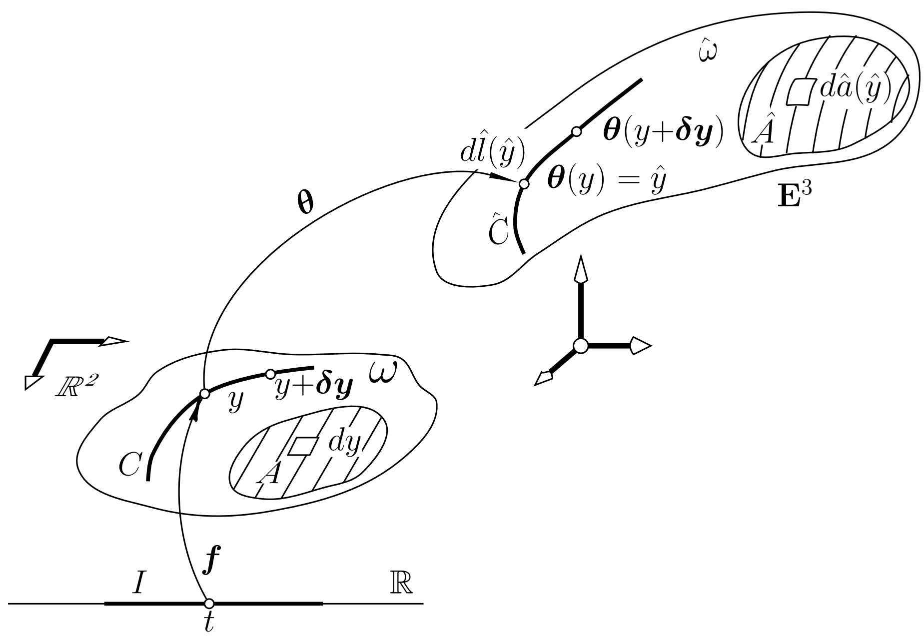

Figure 2.3-1: *Area and length elements on a surface.* The elements $\mathrm{d}\widehat{a}(\widehat{y})$ and $\mathrm{d}\widehat{\ell}(\widehat{y})$ at $\widehat{y} = \boldsymbol{\theta}(y) \in \widehat{\omega}$ are related to $\mathrm{d}y$ and $\boldsymbol{\delta y}$ by means of the covariant components of the metric tensor of the surface $\widehat{\omega}$; cf. Theorem 2.3-1. The corresponding relations are used for computing the area of a surface $\widehat{A} = \boldsymbol{\theta}(A) \subset \widehat{\omega}$ and the length of a curve $\widehat{C} = \boldsymbol{\theta}(C) \subset \widehat{\omega}$, where $C = \boldsymbol{f}(I)$ and I is a compact interval of $\mathbb{R}$.

Theorem 2.3-1. *Let ω be an open subset of $\mathbb{R}^2$, let $\boldsymbol{\theta} : \omega \to \mathbf{E}^3$ be an injective and smooth enough immersion, and let $\widehat{\omega} = \boldsymbol{\theta}(\omega)$.*

(a) *The area element* $\mathrm{d}\widehat{a}(\widehat{y})$ *at* $\widehat{y} = \boldsymbol{\theta}(y) \in \widehat{\omega}$ *is given in terms of the area element* $\mathrm{d}y$ *at* $y \in \omega$ *by*

$$\mathrm{d}\widehat{a}(\widehat{y}) = \sqrt{a(y)}\,\mathrm{d}y, \text{ where } a(y) := \det(a_{\alpha\beta}(y)).$$

(b) *The length element* $\mathrm{d}\widehat{\ell}(\widehat{y})$ *at* $\widehat{y} = \boldsymbol{\theta}(y) \in \widehat{\omega}$ *is given by*

$$\mathrm{d}\widehat{\ell}(\widehat{y}) = \left\{\delta y^\alpha a_{\alpha\beta}(y)\delta y^\beta\right\}^{1/2}.$$

Proof. The relation (a) between the area elements is well known. It can also be deduced directly from the relation between the area elements $\mathrm{d}\widehat{\Gamma}(\widehat{x})$ and $\mathrm{d}\Gamma(x)$ given in Theorem 1.3-1 (b) by means of an *ad hoc* "three-dimensional extension" of the mapping $\boldsymbol{\theta}$.

The expression of the length element in (b) recalls that $\mathrm{d}\widehat{\ell}(\widehat{y})$ is by definition the principal part with respect to $\boldsymbol{\delta y} = \delta y^\alpha \boldsymbol{e}_\alpha$ of the length $|\boldsymbol{\theta}(y + \boldsymbol{\delta y}) - \boldsymbol{\theta}(y)|$, whose expression precisely led to the introduction of the matrix $(a_{\alpha\beta}(y))$. $\square$

The relations found in Theorem 2.3-1 are used for computing *surface integrals* and *lengths* on the surface $\widehat{\omega}$ by means of integrals inside ω, i.e., in terms of the *curvilinear coordinates* used for defining the surface $\widehat{\omega}$ (see again Figure 2.3-1).

Let A be a domain in $\mathbb{R}^2$ such that $\overline{A} \subset \omega$ (a domain in $\mathbb{R}^2$ is a bounded, open, and connected subset of $\mathbb{R}^2$ with a Lipschitz-continuous boundary; cf. Section 1.3), let $\widehat{A} := \boldsymbol{\theta}(A)$, and let $\widehat{f} \in L^1(\widehat{A})$ be given. Then

$$\int_{\widehat{A}} \widehat{f}(\widehat{y})\,\mathrm{d}\widehat{a}(\widehat{y}) = \int_A (\widehat{f} \circ \boldsymbol{\theta})(y)\sqrt{a(y)}\,\mathrm{d}y.$$

In particular, the *area* of $\widehat{A}$ is given by

$$\mathit{area}\,\widehat{A} := \int_{\widehat{A}} \mathrm{d}\widehat{a}(\widehat{y}) = \int_A \sqrt{a(y)}\,\mathrm{d}y.$$

Consider next a *curve* $C = \boldsymbol{f}(I)$ in ω, where I is a compact interval of $\mathbb{R}$ and $\boldsymbol{f} = f^\alpha \boldsymbol{e}_\alpha : I \to \omega$ is a smooth enough injective mapping. Then the *length* of the *curve* $\widehat{C} := \boldsymbol{\theta}(C) \subset \widehat{\omega}$ is given by

$$\mathit{length}\,\widehat{C} := \int_I \left|\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\theta} \circ \boldsymbol{f})(t)\right| \mathrm{d}t = \int_I \sqrt{a_{\alpha\beta}(\boldsymbol{f}(t))\frac{\mathrm{d}f^\alpha}{\mathrm{d}t}(t)\frac{\mathrm{d}f^\beta}{\mathrm{d}t}(t)}\,\mathrm{d}t.$$

The last relation shows in particular that the lengths of curves inside the surface $\boldsymbol{\theta}(\omega)$ are precisely those induced by the Euclidean metric of the space $\mathbf{E}^3$. For this reason, the surface $\boldsymbol{\theta}(\omega)$ is said to be **isometrically imbedded** in $\mathbf{E}^3$.

2.4 SECOND FUNDAMENTAL FORM; CURVATURE ON A SURFACE

While the image $\boldsymbol{\Theta}(\Omega) \subset \mathbf{E}^3$ of a *three-dimensional* open set $\Omega \subset \mathbb{R}^3$ by a smooth enough immersion $\boldsymbol{\Theta} : \Omega \subset \mathbb{R}^3 \to \mathbf{E}^3$ is well defined by its "metric", uniquely up to isometries in $\mathbf{E}^3$ (provided *ad hoc* compatibility conditions are satisfied by the covariant components $g_{ij} : \Omega \to \mathbb{R}$ of its *metric tensor*; cf. Theorems 1.6-1 and 1.7-1), a surface given as the image $\boldsymbol{\theta}(\omega) \subset \mathbf{E}^3$ of a *two-dimensional* open set $\omega \subset \mathbb{R}^2$ by a smooth enough immersion $\boldsymbol{\theta} : \omega \subset \mathbb{R}^2 \to \mathbf{E}^3$ *cannot* be defined by its metric alone.

As intuitively suggested by Figure 2.4-1, the missing information is provided by the "curvature" of a surface. A natural way to give substance to this otherwise vague notion consists in specifying how the *curvature of a curve on a surface* can be computed. As shown in this section, solving this question relies on the knowledge of the *second fundamental form* of a surface, which naturally appears for this purpose through its covariant components (Theorem 2.4-1).

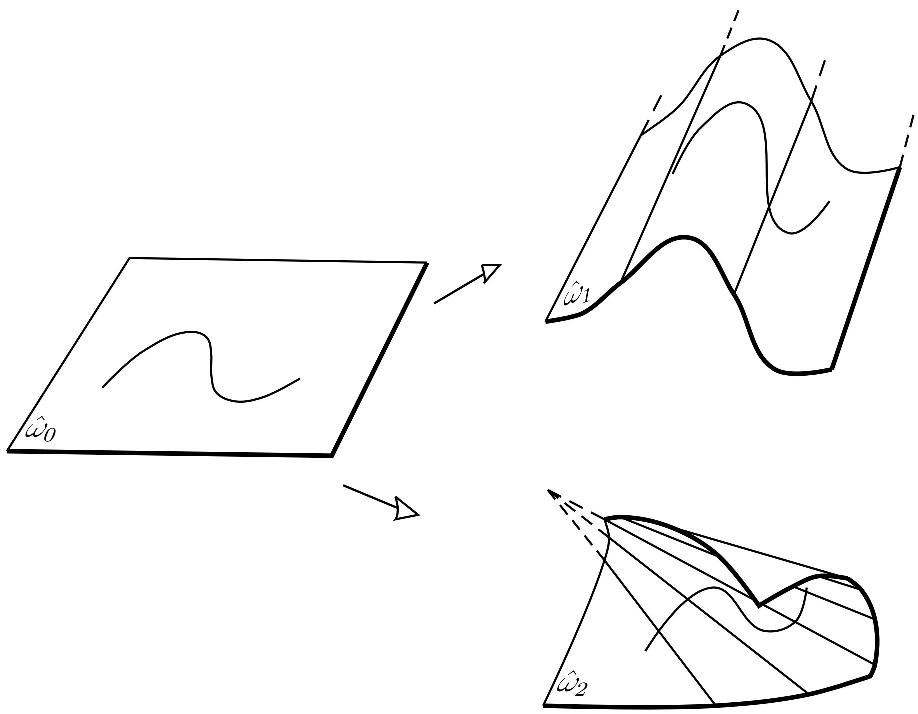

Figure 2.4-1: *A metric alone does not define a surface in* $\mathbf{E}^3$. A flat surface $\widehat{\omega}_0$ may be deformed into a portion $\widehat{\omega}_1$ of a cylinder or a portion $\widehat{\omega}_2$ of a cone without altering the length of any curve drawn on it (cylinders and cones are instances of "developable surfaces"; cf. Section 2.5). Yet it should be clear that in general $\widehat{\omega}_0$ and $\widehat{\omega}_1$, or $\widehat{\omega}_0$ and $\widehat{\omega}_2$, or $\widehat{\omega}_1$ and $\widehat{\omega}_2$, are not identical surfaces *modulo* an isometry of $\mathbf{E}^3$!

Consider as in Section 2.1 a *surface* $\widehat{\omega} = \boldsymbol{\theta}(\omega)$ in $\mathbf{E}^3$, where ω is an open subset of $\mathbb{R}^2$ and $\boldsymbol{\theta} : \omega \subset \mathbb{R}^2 \to \mathbf{E}^3$ is a smooth enough immersion. For each $y \in \omega$, the vector

$$\boldsymbol{a}_3(y) := \frac{\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)}{|\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)|}$$

is thus well defined, has Euclidean norm one, and is normal to the surface $\widehat{\omega}$ at the point $\widehat{y} = \boldsymbol{\theta}(y)$.

Remark. The denominator in the definition of $\boldsymbol{a}_3(y)$ may be also written as

$$|\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)| = \sqrt{a(y)},$$

where $a(y) := \det(a_{\alpha\beta}(y))$. This relation, which holds in fact even if $a(y) = 0$, will be established in the course of the proof of Theorem 4.2-2. □

Fix $y \in \omega$ and consider a *plane P normal to $\widehat{\omega}$ at $\widehat{y} = \boldsymbol{\theta}(y)$*, i.e., a plane that contains the vector $\boldsymbol{a}_3(y)$. The intersection $\widehat{C} = P \cap \widehat{\omega}$ is thus a *planar curve* on $\widehat{\omega}$.

As shown in Theorem 2.4-1, it is remarkable that the *curvature* of $\widehat{C}$ at $\widehat{y}$ can be computed by means of the covariant components $a_{\alpha\beta}(y)$ of the first fundamental form of the surface $\widehat{\omega} = \boldsymbol{\theta}(\omega)$ introduced in Section 2.2, together with the covariant components $b_{\alpha\beta}(y)$ of the *"second" fundamental form* of $\widehat{\omega}$. The definition of the **curvature** *of a planar curve* is recalled in Figure 2.4-2.

If the algebraic curvature of $\widehat{C}$ at $\widehat{y}$ is $\neq 0$, it can be written as $\dfrac{1}{R}$, and R is then called the **algebraic radius of curvature** of the curve $\widehat{C}$ at $\widehat{y}$. This means that the **center of curvature** of the curve $\widehat{C}$ at $\widehat{y}$ is the point $(\widehat{y} + R\boldsymbol{a}_3(y))$; see Figure 2.4-3. While R is *not* intrinsically defined, as its sign changes in any system of curvilinear coordinates where the normal vector $\boldsymbol{a}_3(y)$ is replaced by its opposite, *the center of curvature is intrinsically defined*.

If the curvature of $\widehat{C}$ at $\widehat{y}$ is 0, the radius of curvature of the curve $\widehat{C}$ at $\widehat{y}$ is said to be *infinite*; for this reason, it is customary to still write the curvature as $\dfrac{1}{R}$ in this case.

Note that *the real number $\dfrac{1}{R}$ is always well defined* by the formula given in the next theorem, since the symmetric matrix $(a_{\alpha\beta}(y))$ is positive definite. This implies in particular that *the radius of curvature never vanishes along a curve on a surface $\boldsymbol{\theta}(\omega)$ defined by a mapping $\boldsymbol{\theta}$ satisfying the assumptions of the next theorem, hence in particular of class $\mathcal{C}^2$ on ω.*

It is intuitively clear that if $R = 0$, the mapping $\boldsymbol{\theta}$ "cannot be too smooth". Think of a surface made of two portions of planes intersecting along a segment, which thus constitutes a *fold* on the surface. Or think of a surface $\boldsymbol{\theta}(\omega)$ with $0 \in \omega$ and $\boldsymbol{\theta}(y_1, y_2) = |y_1|^{1+\alpha}$ for some $0 < \alpha < 1$, so that $\boldsymbol{\theta} \in \mathcal{C}^1(\omega; \mathbf{E}^3)$ but $\boldsymbol{\theta} \notin \mathcal{C}^2(\omega; \mathbf{E}^3)$: The radius of curvature of a curve corresponding to a constant y_2 vanishes at $y_1 = 0$.

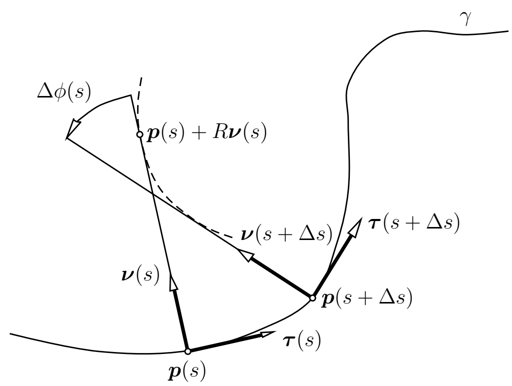

Figure 2.4-2: *Curvature of a planar curve.* Let γ be a smooth enough planar curve, parametrized by its curvilinear abscissa s. Consider two points $\boldsymbol{p}(s)$ and $\boldsymbol{p}(s + \Delta s)$ with curvilinear abscissae s and $s + \Delta s$ and let $\Delta\phi(s)$ be the algebraic angle between the two normals $\boldsymbol{\nu}(s)$ and $\boldsymbol{\nu}(s + \Delta s)$ (oriented in the usual way) to γ at those points. When $\Delta s \to 0$, the ratio $\dfrac{\Delta\phi(s)}{\Delta s}$ has a limit, called the "curvature" of γ at $\boldsymbol{p}(s)$. If this limit is non-zero, its inverse R is called the "algebraic radius of curvature" of γ at $\boldsymbol{p}(s)$ (the sign of R depends on the orientation chosen on γ).

The point $\boldsymbol{p}(s) + R\boldsymbol{\nu}(s)$, which is intrinsically defined, is called the "center of curvature" of γ at $\boldsymbol{p}(s)$: It is the center of the "osculating circle" at $\boldsymbol{p}(s)$, i.e., the limit as $\Delta s \to 0$ of the circle tangent to γ at $\boldsymbol{p}(s)$ that passes through the point $\boldsymbol{p}(s+\Delta s)$. The center of curvature is also the limit as $\Delta s \to 0$ of the intersection of the normals $\boldsymbol{\nu}(s)$ and $\boldsymbol{\nu}(s+\Delta s)$. Consequently, the centers of curvature of γ lie on a curve (dashed on the figure), called "la développée" in French, that is tangent to the normals to γ.

Theorem 2.4-1. *Let ω be an open subset of $\mathbb{R}^2$, let $\boldsymbol{\theta} \in \mathcal{C}^2(\omega; \mathbf{E}^3)$ be an injective immersion, and let $y \in \omega$ be fixed.*

Consider a plane P normal to $\widehat{\omega} = \boldsymbol{\theta}(\omega)$ at the point $\widehat{y} = \boldsymbol{\theta}(y)$. The intersection $P \cap \widehat{\omega}$ is a curve $\widehat{C}$ on $\widehat{\omega}$, which is the image $\widehat{C} = \boldsymbol{\theta}(C)$ of a curve C in the set $\overline{\omega}$. Assume that, in a sufficiently small neighborhood of y, the restriction of C to this neighborhood is the image $\boldsymbol{f}(I)$ of an open interval $I \subset \mathbb{R}$, where $\boldsymbol{f} = f^\alpha \boldsymbol{e}_\alpha : I \to \mathbb{R}$ is a smooth enough injective mapping that satisfies $\dfrac{\mathrm{d}f^\alpha}{\mathrm{d}t}(t)\,\boldsymbol{e}_\alpha \neq \mathbf{0}$, where $t \in I$ is such that $y = \boldsymbol{f}(t)$ (Figure 2.4-3).

Then the curvature $\dfrac{1}{R}$ of the planar curve $\widehat{C}$ at $\widehat{y}$ is given by the ratio

$$\frac{1}{R} = \frac{b_{\alpha\beta}(\boldsymbol{f}(t))\dfrac{\mathrm{d}f^\alpha}{\mathrm{d}t}(t)\,\dfrac{\mathrm{d}f^\beta}{\mathrm{d}t}(t)}{a_{\alpha\beta}(\boldsymbol{f}(t))\dfrac{\mathrm{d}f^\alpha}{\mathrm{d}t}(t)\,\dfrac{\mathrm{d}f^\beta}{\mathrm{d}t}(t)},$$

where $a_{\alpha\beta}(y)$ are the covariant components of the first fundamental form of $\widehat{\omega}$ at y (Section 2.1) *and*

$$b_{\alpha\beta}(y) := \boldsymbol{a}_3(y) \cdot \partial_\alpha \boldsymbol{a}_\beta(y) = -\partial_\alpha \boldsymbol{a}_3(y) \cdot \boldsymbol{a}_\beta(y) = b_{\beta\alpha}(y).$$

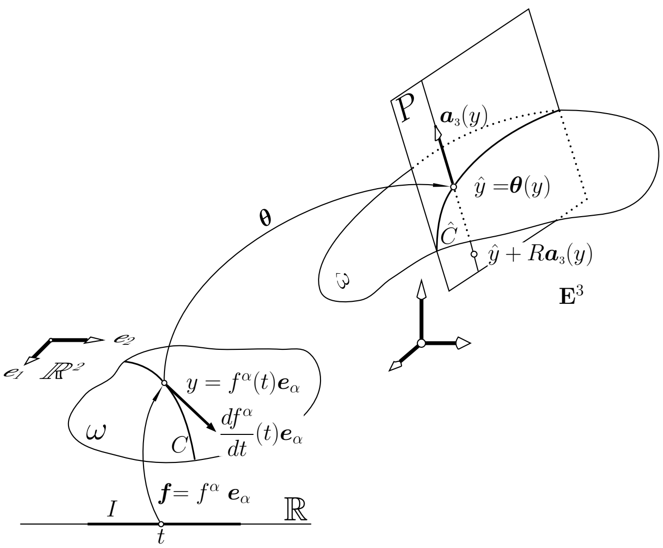

Figure 2.4-3: *Curvature on a surface.* Let P be a plane containing the vector $\boldsymbol{a}_3(y) = \dfrac{\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)}{|\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)|}$, which is normal to the surface $\widehat{\omega} = \boldsymbol{\theta}(\omega)$. The algebraic curvature $\dfrac{1}{R}$ of the planar curve $\widehat{C} = P \cap \widehat{\omega} = \boldsymbol{\theta}(C)$ at $\widehat{y} = \boldsymbol{\theta}(y)$ is given by the ratio

$$\frac{1}{R} = \frac{b_{\alpha\beta}(\boldsymbol{f}(t)) \dfrac{\mathrm{d}f^\alpha}{\mathrm{d}t}(t) \dfrac{\mathrm{d}f^\beta}{\mathrm{d}t}(t)}{a_{\alpha\beta}(\boldsymbol{f}(t)) \dfrac{\mathrm{d}f^\alpha}{\mathrm{d}t}(t) \dfrac{\mathrm{d}f^\beta}{\mathrm{d}t}(t)},$$

where $a_{\alpha\beta}(y)$ and $b_{\alpha\beta}(y)$ are the covariant components of the first and second fundamental forms of the surface $\widehat{\omega}$ at $\widehat{y}$ and $\dfrac{\mathrm{d}f^\alpha}{\mathrm{d}t}(t)$ are the components of the vector tangent to the curve $C = \boldsymbol{f}(I)$ at $y = \boldsymbol{f}(t) = f^\alpha(t)\boldsymbol{e}_\alpha$. If $\dfrac{1}{R} \neq 0$, the center of curvature of the curve $\widehat{C}$ at $\widehat{y}$ is the point $(\widehat{y} + R\boldsymbol{a}_3(y))$, which is intrinsically defined in the Euclidean space $\mathbf{E}^3$.

Proof. (i) We first establish a well-known formula giving the curvature $\dfrac{1}{R}$ of a planar curve. Using the notations of Figure 2.4-2, we note that

$$\sin \Delta\phi(s) = \boldsymbol{\nu}(s) \cdot \boldsymbol{\tau}(s + \Delta s) = -\{\boldsymbol{\nu}(s + \Delta s) - \boldsymbol{\nu}(s)\} \cdot \boldsymbol{\tau}(s + \Delta s),$$

so that

$$\frac{1}{R} := \lim_{\Delta s \to 0} \frac{\Delta\phi(s)}{\Delta s} = \lim_{\Delta s \to 0} \frac{\sin \Delta\phi(s)}{\Delta s} = -\frac{\mathrm{d}\boldsymbol{\nu}}{\mathrm{d}s}(s) \cdot \boldsymbol{\tau}(s).$$

(ii) The curve $(\boldsymbol{\theta} \circ \boldsymbol{f})(I)$, which is *a priori* parametrized by $t \in I$, can be also parametrized by its curvilinear abscissa s in a neighborhood of the point $\widehat{y}$. There thus exist an interval $\tilde{I} \subset I$ and a mapping $\boldsymbol{p} : J \to P$, where $J \subset \mathbb{R}$ is an interval, such that

$$(\boldsymbol{\theta} \circ \boldsymbol{f})(t) = \boldsymbol{p}(s) \text{ and } (\boldsymbol{a}_3 \circ \boldsymbol{f})(t) = \boldsymbol{\nu}(s) \text{ for all } t \in \tilde{I},\ s \in J.$$

By (i), the curvature $\frac{1}{R}$ of $\widehat{C}$ is given by

$$\frac{1}{R} = -\frac{\mathrm{d}\boldsymbol{\nu}}{\mathrm{d}s}(s) \cdot \boldsymbol{\tau}(s),$$

where

$$\begin{aligned}
\frac{\mathrm{d}\boldsymbol{\nu}}{\mathrm{d}s}(s) &= \frac{\mathrm{d}(\boldsymbol{a}_3 \circ \boldsymbol{f})}{\mathrm{d}t}(t)\frac{\mathrm{d}t}{\mathrm{d}s} = \partial_\alpha \boldsymbol{a}_3(\boldsymbol{f}(t))\frac{\mathrm{d}f^\alpha}{\mathrm{d}t}(t)\,\frac{\mathrm{d}t}{\mathrm{d}s},\\
\boldsymbol{\tau}(s) &= \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}s}(s) = \frac{\mathrm{d}(\boldsymbol{\theta} \circ \boldsymbol{f})}{\mathrm{d}t}(t)\frac{\mathrm{d}t}{\mathrm{d}s}\\
&= \partial_\beta \boldsymbol{\theta}(\boldsymbol{f}(t))\frac{\mathrm{d}f^\beta}{\mathrm{d}t}(t)\,\frac{\mathrm{d}t}{\mathrm{d}s} = \boldsymbol{a}_\beta(\boldsymbol{f}(t))\frac{\mathrm{d}f^\beta}{\mathrm{d}t}(t)\,\frac{\mathrm{d}t}{\mathrm{d}s}.
\end{aligned}$$

Hence

$$\frac{1}{R} = -\partial_\alpha \boldsymbol{a}_3(\boldsymbol{f}(t)) \cdot \boldsymbol{a}_\beta(\boldsymbol{f}(t))\frac{\mathrm{d}f^\alpha}{\mathrm{d}t}(t)\,\frac{\mathrm{d}f^\beta}{\mathrm{d}t}(t)\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2.$$

To obtain the announced expression for $\frac{1}{R}$, it suffices to note that

$$-\partial_\alpha \boldsymbol{a}_3(\boldsymbol{f}(t)) \cdot \boldsymbol{a}_\beta(\boldsymbol{f}(t)) = b_{\alpha\beta}(\boldsymbol{f}(t)),$$

by definition of the functions $b_{\alpha\beta}$ and that (Theorem 2.3-1 (b))

$$\mathrm{d}s = \left\{\delta y^\alpha a_{\alpha\beta}(y)\delta y^\beta\right\}^{1/2} = \left\{a_{\alpha\beta}(\boldsymbol{f}(t))\frac{\mathrm{d}f^\alpha}{\mathrm{d}t}(t)\,\frac{\mathrm{d}f^\beta}{\mathrm{d}t}(t)\right\}^{1/2}\mathrm{d}t.$$

□

The knowledge of the curvatures of curves contained in planes *normal to* $\widehat{\omega}$ suffices for computing the curvature of *any* curve on $\widehat{\omega}$. More specifically, the radius of curvature $\widetilde{R}$ at $\widehat{y}$ of any smooth enough curve $\widetilde{C}$ (planar or not) on the surface $\widehat{\omega}$ is given by $\frac{\cos\varphi}{\widetilde{R}} = \frac{1}{R}$, where φ is the angle between the "principal normal" to $\widetilde{C}$ at $\widehat{y}$ and $\boldsymbol{a}_3(y)$ and $\frac{1}{R}$ is given in Theorem 2.4-1; see, e.g., Stoker [1969, Chapter 4, Section 12].

The elements $b_{\alpha\beta}(y)$ of the symmetric matrix $(b_{\alpha\beta}(y))$ defined in Theorem 2.4-1 are called the **covariant components of the second fundamental form** of the surface $\widehat{\omega} = \boldsymbol{\theta}(\omega)$ at $\widehat{y} = \boldsymbol{\theta}(y)$.

2.5 PRINCIPAL CURVATURES; GAUSSIAN CURVATURE

The analysis of the previous section suggests that precise information about the shape of a surface $\widehat{\omega} = \boldsymbol{\theta}(\omega)$ in a neighborhood of one of its points $\widehat{y} = \boldsymbol{\theta}(y)$ can be gathered by letting the plane P turn around the normal vector $\boldsymbol{a}_3(y)$

and by following in this process the variations of the curvatures at $\widehat{y}$ of the corresponding planar curves $P \cap \widehat{\omega}$, as given in Theorem 2.4-1.

As a first step in this direction, we show that *these curvatures span a compact interval of* $\mathbb{R}$. In particular then, they "stay away from infinity".

Note that this compact interval contains 0 if, and only if, the radius of curvature of the curve $P \cap \widehat{\omega}$ is infinite for at least one such plane P.

Theorem 2.5-1. (a) *Let the assumptions and notations be as in* Theorem 2.4-1. *For a fixed $y \in \omega$, consider the set $\mathcal{P}$ of all planes P normal to the surface $\widehat{\omega} = \boldsymbol{\theta}(\omega)$ at $\widehat{y} = \boldsymbol{\theta}(y)$. Then the set of curvatures of the associated planar curves $P \cap \widehat{\omega}$, $P \in \mathcal{P}$, is a compact interval of $\mathbb{R}$, denoted* $\left[\dfrac{1}{R_1(y)}, \dfrac{1}{R_2(y)}\right]$.

(b) *Let the matrix $(b_\alpha^\beta(y))$, α being the row index, be defined by*

$$b_\alpha^\beta(y) := a^{\beta\sigma}(y) b_{\alpha\sigma}(y),$$

where $(a^{\alpha\beta}(y)) = (a_{\alpha\beta}(y))^{-1}$ (Section 2.2) *and the matrix $(b_{\alpha\beta}(y))$ is defined as in* Theorem 2.4-1. *Then*

$$\frac{1}{R_1(y)} + \frac{1}{R_2(y)} = b_1^1(y) + b_2^2(y),$$

$$\frac{1}{R_1(y)R_2(y)} = b_1^1(y)b_2^2(y) - b_1^2(y)b_2^1(y) = \frac{\det(b_{\alpha\beta}(y))}{\det(a_{\alpha\beta}(y))}.$$

(c) *If* $\dfrac{1}{R_1(y)} \neq \dfrac{1}{R_2(y)}$, *there is a unique pair of orthogonal planes $P_1 \in \mathcal{P}$ and $P_2 \in \mathcal{P}$ such that the curvatures of the associated planar curves $P_1 \cap \widehat{\omega}$ and $P_2 \cap \widehat{\omega}$ are precisely* $\dfrac{1}{R_1(y)}$ *and* $\dfrac{1}{R_2(y)}$.

Proof. (i) Let $\Delta(P)$ denote the intersection of $P \in \mathcal{P}$ with the tangent plane T to the surface $\widehat{\omega}$ at $\widehat{y}$, and let $\widehat{C}(P)$ denote the intersection of P with $\widehat{\omega}$. Hence $\Delta(P)$ is tangent to $\widehat{C}(P)$ at $\widehat{y} \in \widehat{\omega}$.

In a sufficiently small neighborhood of $\widehat{y}$ the restriction of the curve $\widehat{C}(P)$ to this neighborhood is given by $\widehat{C}(P) = (\boldsymbol{\theta} \circ \boldsymbol{f}(P))(I(P))$, where $I(P) \subset \mathbb{R}$ is an open interval and $\boldsymbol{f}(P) = f^\alpha(P)\boldsymbol{e}_\alpha : I(P) \to \mathbb{R}^2$ is a smooth enough injective mapping that satisfies $\dfrac{\mathrm{d}f^\alpha(P)}{\mathrm{d}t}(t)\boldsymbol{e}_\alpha \neq \mathbf{0}$, where $t \in I(P)$ is such that $y = \boldsymbol{f}(P)(t)$. Hence the line $\Delta(P)$ is given by

$$\Delta(P) = \left\{\widehat{y} + \lambda \frac{\mathrm{d}(\boldsymbol{\theta} \circ \boldsymbol{f}(P))}{\mathrm{d}t}(t); \lambda \in \mathbb{R}\right\} = \{\widehat{y} + \lambda\xi^\alpha \boldsymbol{a}_\alpha(y); \lambda \in \mathbb{R}\},$$

where $\xi^\alpha := \dfrac{\mathrm{d}f^\alpha(P)}{\mathrm{d}t}(t)$ and $\xi^\alpha \boldsymbol{e}_\alpha \neq \mathbf{0}$ by assumption.

Since the line $\{y + \mu\xi^\alpha \boldsymbol{e}_\alpha; \mu \in \mathbb{R}\}$ is tangent to the curve $C(P) := \boldsymbol{\theta}^{-1}(\widehat{C}(P))$ at $y \in \omega$ (the mapping $\boldsymbol{\theta} : \omega \to \mathbb{R}^3$ is injective by assumption) for each such parametrizing function $\boldsymbol{f}(P) : I(P) \to \mathbb{R}^2$ and since the vectors $\boldsymbol{a}_\alpha(y)$ are linearly independent, there exists a bijection between the set of all lines $\Delta(P) \subset T$, $P \in \mathcal{P}$, and the set of all lines supporting the nonzero tangent vectors to the curve $C(P)$.

Hence Theorem 2.4-1 shows that when P varies in $\mathcal{P}$, the curvature of the corresponding curves $\widehat{C} = \widehat{C}(P)$ at $\widehat{y}$ takes the same values as does the ratio $\dfrac{b_{\alpha\beta}(y)\xi^\alpha\xi^\beta}{a_{\alpha\beta}(y)\xi^\alpha\xi^\beta}$ when $\boldsymbol{\xi} := \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}$ varies in $\mathbb{R}^2 - \{\mathbf{0}\}$.

(ii) Let the symmetric matrices $\mathbf{A}$ and $\mathbf{B}$ of order two be defined by

$$\mathbf{A} := (a_{\alpha\beta}(y)) \text{ and } \mathbf{B} := (b_{\alpha\beta}(y)).$$

Since $\mathbf{A}$ is positive definite, it has a (unique) *square root* $\mathbf{C}$, i.e., a symmetric positive definite matrix $\mathbf{C}$ such that $\mathbf{A} = \mathbf{C}^2$. Hence the ratio

$$\frac{b_{\alpha\beta}(y)\xi^\alpha\xi^\beta}{a_{\alpha\beta}(y)\xi^\alpha\xi^\beta} = \frac{\boldsymbol{\xi}^T\mathbf{B}\boldsymbol{\xi}}{\boldsymbol{\xi}^T\mathbf{A}\boldsymbol{\xi}} = \frac{\boldsymbol{\eta}^T\mathbf{C}^{-1}\mathbf{B}\mathbf{C}^{-1}\boldsymbol{\eta}}{\boldsymbol{\eta}^T\boldsymbol{\eta}}, \text{ where } \boldsymbol{\eta} = \mathbf{C}\boldsymbol{\xi},$$

is nothing but the *Rayleigh quotient* associated with the symmetric matrix $\mathbf{C}^{-1}\mathbf{B}\mathbf{C}^{-1}$. When $\boldsymbol{\eta}$ varies in $\mathbb{R}^2 - \{\mathbf{0}\}$, this Rayleigh quotient thus spans the compact interval of $\mathbb{R}$ whose end-points are the smallest and largest eigenvalue, respectively denoted $\dfrac{1}{R_1(y)}$ and $\dfrac{1}{R_2(y)}$, of the matrix $\mathbf{C}^{-1}\mathbf{B}\mathbf{C}^{-1}$ (for a proof, see, e.g., Ciarlet [1982, Theorem 1.3-1]). This proves (a).

Furthermore, the relation

$$\mathbf{C}(\mathbf{C}^{-1}\mathbf{B}\mathbf{C}^{-1})\mathbf{C}^{-1} = \mathbf{B}\mathbf{C}^{-2} = \mathbf{B}\mathbf{A}^{-1}$$

shows that the eigenvalues of the symmetric matrix $\mathbf{C}^{-1}\mathbf{B}\mathbf{C}^{-1}$ coincide with those of the (in general non-symmetric) matrix $\mathbf{B}\mathbf{A}^{-1}$. Note that $\mathbf{B}\mathbf{A}^{-1} = (b_\alpha^\beta(y))$ with $b_\alpha^\beta(y) := a^{\beta\sigma}(y)b_{\alpha\sigma}(y)$, α being the row index, since $\mathbf{A}^{-1} = (a^{\alpha\beta}(y))$.

Hence the relations in (b) simply express that the sum and the product of the eigenvalues of the matrix $\mathbf{B}\mathbf{A}^{-1}$ are respectively equal to its trace and to its determinant, which may be also written as $\dfrac{\det(b_{\alpha\beta}(y))}{\det(a_{\alpha\beta}(y))}$ since $\mathbf{B}\mathbf{A}^{-1} = (b_\alpha^\beta(y))$. This proves (b).

(iii) Let $\boldsymbol{\eta}_1 = \begin{pmatrix}\eta_1^1\\ \eta_1^2\end{pmatrix} = \mathbf{C}\boldsymbol{\xi}_1$ and $\boldsymbol{\eta}_2 = \begin{pmatrix}\eta_2^1\\ \eta_2^2\end{pmatrix} = \mathbf{C}\boldsymbol{\xi}_2$, with $\boldsymbol{\xi}_1 = \begin{pmatrix}\xi_1^1\\ \xi_1^2\end{pmatrix}$ and $\boldsymbol{\xi}_2 = \begin{pmatrix}\xi_2^1\\ \xi_2^2\end{pmatrix}$, be two orthogonal ($\boldsymbol{\eta}_1^T\boldsymbol{\eta}_2 = 0$) eigenvectors of the symmetric matrix $\mathbf{C}^{-1}\mathbf{B}\mathbf{C}^{-1}$, corresponding to the eigenvalues $\dfrac{1}{R_1(y)}$ and $\dfrac{1}{R_2(y)}$, respectively. Hence

$$0 = \boldsymbol{\eta}_1^T\boldsymbol{\eta}_2 = \boldsymbol{\xi}_1^T\mathbf{C}^T\mathbf{C}\boldsymbol{\xi}_2 = \boldsymbol{\xi}_1^T\mathbf{A}\boldsymbol{\xi}_2 = 0,$$

since $\mathbf{C}^T = \mathbf{C}$. By (i), the corresponding lines $\Delta(P_1)$ and $\Delta(P_2)$ of the tangent plane are parallel to the vectors $\xi_1^\alpha \boldsymbol{a}_\alpha(y)$ and $\xi_2^\beta \boldsymbol{a}_\beta(y)$, which are orthogonal since

$$\left\{\xi_1^\alpha \boldsymbol{a}_\alpha(y)\right\} \cdot \left\{\xi_2^\beta \boldsymbol{a}_\beta(y)\right\} = a_{\alpha\beta}(y)\xi_1^\alpha \xi_2^\beta = \boldsymbol{\xi}_1^T \mathbf{A} \boldsymbol{\xi}_2.$$

If $\dfrac{1}{R_1(y)} \neq \dfrac{1}{R_2(y)}$, the directions of the vectors $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ are uniquely determined and the lines $\Delta(P_1)$ and $\Delta(P_2)$ are likewise uniquely determined. This proves (c). □

We are now in a position to state several *fundamental definitions*:

The elements $b_\alpha^\beta(y)$ of the (in general non-symmetric) matrix $(b_\alpha^\beta(y))$ defined in Theorem 2.5-1 are called the **mixed components of the second fundamental form** of the surface $\widehat{\omega} = \boldsymbol{\theta}(\omega)$ at $\widehat{y} = \boldsymbol{\theta}(y)$.

The real numbers $\dfrac{1}{R_1(y)}$ and $\dfrac{1}{R_2(y)}$ (one or both possibly equal to 0) found in Theorem 2.5-1 are called the **principal curvatures** of $\widehat{\omega}$ at $\widehat{y}$.

If $\dfrac{1}{R_1(y)} = \dfrac{1}{R_2(y)}$, the curvatures of the planar curves $P \cap \widehat{\omega}$ are the same in all directions, i.e., for all $P \in \mathcal{P}$. If $\dfrac{1}{R_1(y)} = \dfrac{1}{R_2(y)} = 0$, the point $\widehat{y} = \boldsymbol{\theta}(y)$ is called a **planar point**. If $\dfrac{1}{R_1(y)} = \dfrac{1}{R_2(y)} \neq 0, \widehat{y}$ is called an **umbilical point**.

It is remarkable that, *if all the points of $\widehat{\omega}$ are planar, then $\widehat{\omega}$ is a portion of a plane*. Likewise, *if all the points of $\widehat{\omega}$ are umbilical, then $\widehat{\omega}$ is a portion of a sphere*. For proofs, see, e.g., Stoker [1969, p. 87 and p. 99].

Let $\widehat{y} = \boldsymbol{\theta}(y) \in \widehat{\omega}$ be a point that is neither planar nor umbilical; in other words, the principal curvatures at $\widehat{y}$ are not equal. Then the two orthogonal lines tangent to the planar curves $P_1 \cap \widehat{\omega}$ and $P_2 \cap \widehat{\omega}$ (Theorem 2.5-1 (c)) are called the **principal directions** at $\widehat{y}$.

A **line of curvature** is a curve on $\widehat{\omega}$ that is tangent to a principal direction at each one of its points. It can be shown that *a point that is neither planar nor umbilical possesses a neighborhood where two orthogonal families of lines of curvature can be chosen as coordinate lines*. See, e.g., Klingenberg [1973, Lemma 3.6.6].

If $\dfrac{1}{R_1(y)} \neq 0$ *and* $\dfrac{1}{R_2(y)} \neq 0$, the real numbers $R_1(y)$ and $R_2(y)$ are called the **principal radii of curvature** of $\widehat{\omega}$ at $\widehat{y}$. If, e.g., $\dfrac{1}{R_1(y)} = 0$, the corresponding radius of curvature $R_1(y)$ is said to be *infinite*, according to the convention made in Section 2.4. While the principal radii of curvature may simultaneously change their signs in another system of curvilinear coordinates, the associated *centers of curvature* are intrinsically defined.

The numbers $\dfrac{1}{2}\left(\dfrac{1}{R_1(y)} + \dfrac{1}{R_2(y)}\right)$ and $\dfrac{1}{R_1(y)R_2(y)}$, which are the principal invariants of the matrix $(b_\alpha^\beta(y))$ (Theorem 2.5-1), are respectively called the **mean curvature** and the **Gaussian**, or **total, curvature** of the surface $\widehat{\omega}$ at $\widehat{y}$.

A point on a surface is an **elliptic, parabolic**, or **hyperbolic, point** according as its *Gaussian curvature* is $> 0, = 0$ but it is not a planar point, or < 0; see Figure 2.5-1.

An **asymptotic line** is a curve on a surface that is everywhere tangent to a direction along which the radius of curvature is infinite; any point along an asymptotic line is thus either parabolic or hyperbolic. It can be shown that, *if all the points of a surface are hyperbolic, any point possesses a neighborhood where two intersecting families of asymptotic lines can be chosen as coordinate lines.* See, e.g., Klingenberg [1973, Lemma 3.6.12].

As intuitively suggested by Figure 2.4-1, a surface in $\mathbb{R}^3$ cannot be defined by its *metric* alone, i.e., through its first fundamental form alone, since its *curvature* must be in addition specified through its second fundamental form. But quite surprisingly, the *Gaussian curvature* at a point can also be expressed solely in terms of the functions $a_{\alpha\beta}$ and their derivatives! This is the celebrated *Theorema Egregium* ("astonishing theorem") of Gauß [1828]; see Theorem 2.6.2 in the next section.

Another striking result involving the *Gaussian curvature* is the equally celebrated **Gauß-Bonnet theorem**, so named after Gauß [1828] and Bonnet [1848] (for a "modern" proof, see, e.g., Klingenberg [1973, Theorem 6.3-5] or do Carmo [1994, Chapter 6, Theorem 1]): *Let S be a smooth enough, "closed", "orientable", and compact surface in* $\mathbb{R}^3$ (a "closed" surface is one "without boundary", such as a sphere or a torus; "orientable" surfaces, which exclude for instance Klein bottles, are defined in, e.g., Klingenberg [1973, Section 5.5]) *and let $K : S \to \mathbb{R}$ denote its Gaussian curvature. Then*

$$\int_S K(\widehat{y})\,\mathrm{d}\widehat{a}(\widehat{y}) = 2\pi(2 - 2g(S)),$$

where the **genus** *$g(S)$ is the number of "holes" of S* (for instance, a sphere has genus zero, while a torus has genus one). The integer $\chi(S)$ defined by $\chi(S) := (2 - 2g(S))$ is the **Euler characteristic** of $\widehat{\omega}$.

According to the definition of Stoker [1969, Chapter 5, Section 2], a **developable surface** is one whose *Gaussian curvature* vanishes everywhere. Developable surfaces are otherwise often defined as "ruled" surfaces whose Gaussian curvature vanishes everywhere, as in, e.g., Klingenberg [1973, Section 3.7]). A portion of a plane provides a first example, the only one of a developable surface *all* points of which are planar. Any developable surface *all* points of which are *parabolic* can be likewise fully described: It is *either* a portion of a cylinder, *or* a portion of a cone, *or* a portion of a surface spanned by the tangents to a skewed curve. The description of a developable surface comprising both planar and parabolic points is more subtle (although the above examples are in a sense the only ones possible, at least locally; see Stoker [1969, Chapter 5, Sections 2 to 6]).

The interest of developable surfaces is that they can be, at least locally, continuously "rolled out", or "developed" (hence their name), onto a *plane, without changing the metric* of the intermediary surfaces in the process.

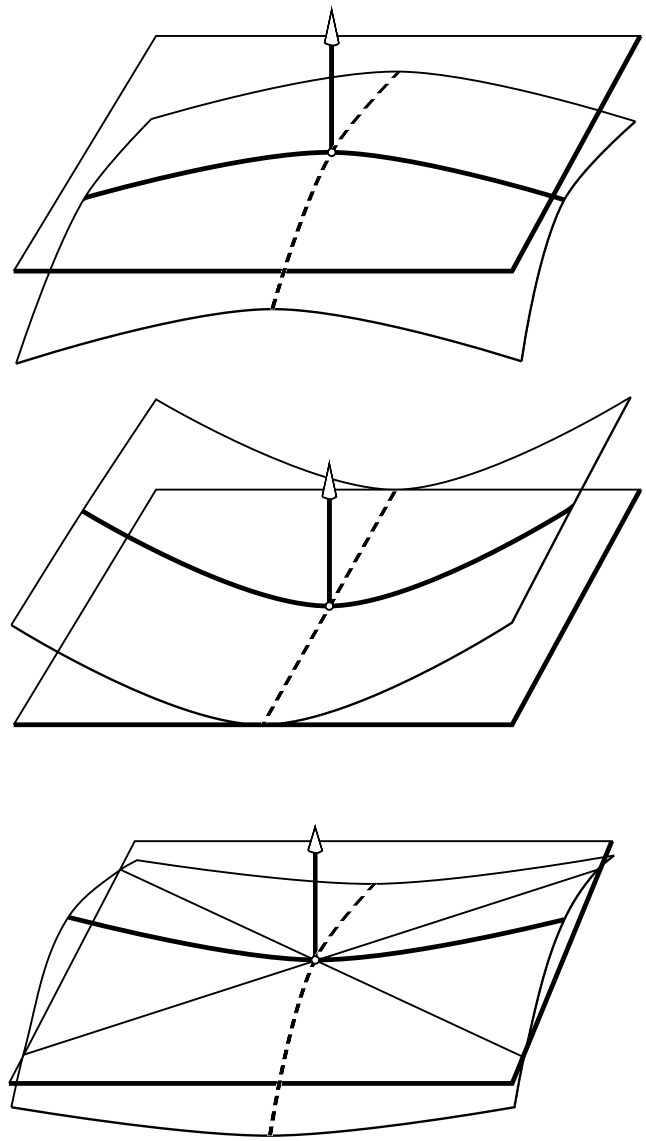

Figure 2.5-1: *Different kinds of points on a surface.* A point is elliptic if the Gaussian curvature is > 0 or equivalently, if the two principal radii of curvature are of the same sign; the surface is then locally on one side of its tangent plane. A point is parabolic if exactly one of the two principal radii of curvature is infinite; the surface is again locally on one side of its tangent plane. A point is hyperbolic if the Gaussian curvature is < 0 or equivalently, if the two principal radii of curvature are of different signs; the surface then intersects its tangent plane along two curves.

More details about these various notions are found in classic texts such as Stoker [1969], Klingenberg [1973], do Carmo [1976], Berger & Gostiaux [1987], Spivak [1999], or Kühnel [2002].

2.6 COVARIANT DERIVATIVES OF A VECTOR FIELD DEFINED ON A SURFACE; THE GAUSS AND WEINGARTEN FORMULAS

As in Sections 2.2 and 2.4, consider a surface $\widehat{\omega} = \boldsymbol{\theta}(\omega)$ in $\mathbf{E}^3$, where $\boldsymbol{\theta} : \omega \subset \mathbb{R}^2 \to \mathbf{E}^3$ is a smooth enough injective immersion, and let

$$\boldsymbol{a}_3(y) = \boldsymbol{a}^3(y) := \frac{\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)}{|\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)|}, \quad y \in \omega.$$

Then the vectors $\boldsymbol{a}_\alpha(y)$ (which form the covariant basis of the tangent plane to $\widehat{\omega}$ at $\widehat{y} = \boldsymbol{\theta}(y)$; see Figure 2.1-1) together with the vector $\boldsymbol{a}_3(y)$ (which is normal to $\widehat{\omega}$ and has Euclidean norm one) form the **covariant basis** *at* $\widehat{y}$.

Let the vectors $\boldsymbol{a}^\alpha(y)$ of the tangent plane to $\widehat{\omega}$ at $\widehat{y}$ be defined by the relations $\boldsymbol{a}^\alpha(y) \cdot \boldsymbol{a}_\beta(y) = \delta^\alpha_\beta$. Then the vectors $\boldsymbol{a}^\alpha(y)$ (which form the contravariant basis of the tangent plane at $\widehat{y}$; see again Figure 2.1-1) together with the vector $\boldsymbol{a}^3(y)$ form the **contravariant basis** *at* $\widehat{y}$; see Figure 2.6-1. Note that the vectors of the covariant and contravariant bases at $\widehat{y}$ satisfy

$$\boldsymbol{a}^i(y) \cdot \boldsymbol{a}_j(y) = \delta^i_j.$$

Suppose that a *vector field is defined on the surface* $\widehat{\omega}$. One way to define such a field in terms of the *curvilinear coordinates* used for defining the surface $\widehat{\omega}$ consists in writing it as $\eta_i \boldsymbol{a}^i : \omega \to \mathbb{R}^3$, i.e., in specifying its **covariant components** $\eta_i : \omega \to \mathbb{R}$ over the vector fields $\boldsymbol{a}^i$ formed by the *contravariant bases.* This means that $\eta_i(y)\boldsymbol{a}^i(y)$ is the vector at each point $\widehat{y} = \boldsymbol{\theta}(y) \in \widehat{\omega}$ (Figure 2.6-1).

Our objective in this section is to compute the *partial derivatives* $\partial_\alpha(\eta_i \boldsymbol{a}^i)$ of such a vector field. These are found in the next theorem, as immediate consequences of two basic formulas, those of *Gauß* and *Weingarten.* The *Christoffel symbols "on a surface"* and the *covariant derivatives of a vector field defined on a surface* are also naturally introduced in this process.

A word of caution. The Christoffel symbols "on a surface" introduced in this section and the next one, viz., $\Gamma^\sigma_{\alpha\beta}$ and $\Gamma_{\alpha\beta\tau}$, are thus denoted by the *same* symbols as the "three-dimensional" Christoffel symbols introduced in Sections 1.4 and 1.5. No confusion should arise, however. □

Theorem 2.6-1. *Let ω be an open subset of $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^2(\omega; \mathbf{E}^3)$ be an immersion.*

(a) *The derivatives of the vectors of the covariant and contravariant bases are given by*

$$\partial_\alpha \boldsymbol{a}_\beta = \Gamma^\sigma_{\alpha\beta} \boldsymbol{a}_\sigma + b_{\alpha\beta} \boldsymbol{a}_3 \text{ and } \partial_\alpha \boldsymbol{a}^\beta = -\Gamma^\beta_{\alpha\sigma} \boldsymbol{a}^\sigma + b^\beta_\alpha \boldsymbol{a}^3,$$
$$\partial_\alpha \boldsymbol{a}_3 = \partial_\alpha \boldsymbol{a}^3 = -b_{\alpha\beta} \boldsymbol{a}^\beta = -b^\sigma_\alpha \boldsymbol{a}_\sigma,$$

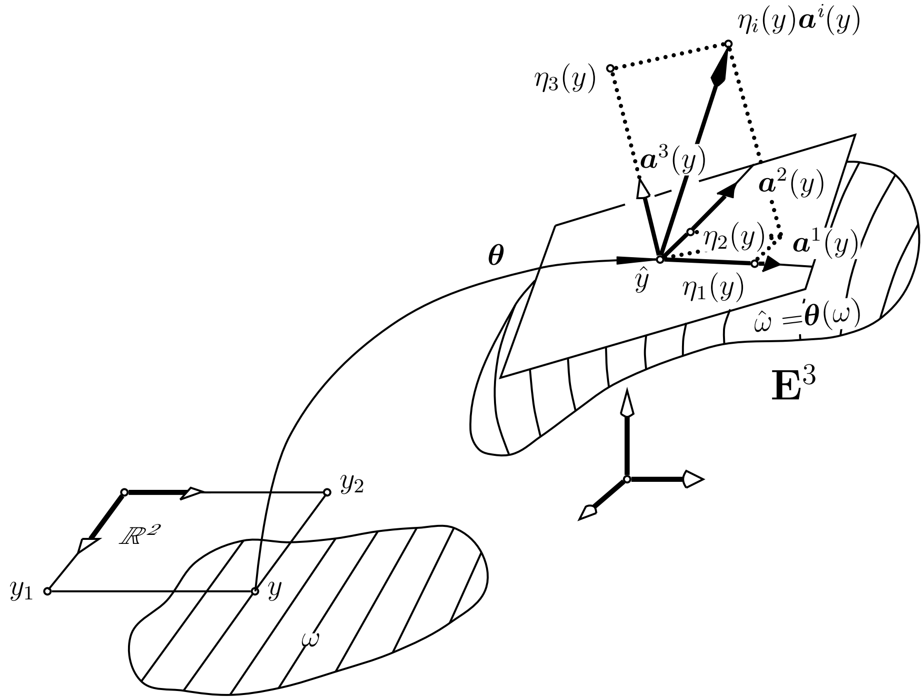

Figure 2.6-1: *Contravariant bases and vector fields along a surface.* At each point $\widehat{y} = \boldsymbol{\theta}(y) \in \widehat{\omega} = \boldsymbol{\theta}(\omega)$, the three vectors $\boldsymbol{a}^i(y)$, where $\boldsymbol{a}^\alpha(y)$ form the contravariant basis of the tangent plane to $\widehat{\omega}$ at $\widehat{y}$ (Figure 2.1-1) and $\boldsymbol{a}^3(y) = \dfrac{\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)}{|\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)|}$, form the contravariant basis at $\widehat{y}$. An arbitrary vector field defined on $\widehat{\omega}$ may then be defined by its covariant components $\eta_i : \omega \to \mathbb{R}$. This means that $\eta_i(y)\boldsymbol{a}^i(y)$ is the vector at the point $\widehat{y}$.

where the covariant and mixed components $b_{\alpha\beta}$ *and* b_α^β *of the second fundamental form of* $\widehat{\omega}$ *are defined in* Theorems 2.4-1 *and* 2.5-1 *and*

$$\Gamma^\sigma_{\alpha\beta} := \boldsymbol{a}^\sigma \cdot \partial_\alpha \boldsymbol{a}_\beta.$$

(b) *Let there be given a vector field* $\eta_i \boldsymbol{a}^i : \omega \to \mathbb{R}^3$ *with covariant components* $\eta_i \in \mathcal{C}^1(\omega)$. *Then* $\eta_i \boldsymbol{a}^i \in \boldsymbol{\mathcal{C}}^1(\omega)$ *and the partial derivatives* $\partial_\alpha(\eta_i \boldsymbol{a}^i) \in \boldsymbol{\mathcal{C}}^0(\omega)$ *are given by*

$$\begin{aligned}\partial_\alpha(\eta_i \boldsymbol{a}^i) &= (\partial_\alpha \eta_\beta - \Gamma^\sigma_{\alpha\beta}\eta_\sigma - b_{\alpha\beta}\eta_3)\boldsymbol{a}^\beta + (\partial_\alpha \eta_3 + b_\alpha^\beta \eta_\beta)\boldsymbol{a}^3 \\ &= (\eta_{\beta|\alpha} - b_{\alpha\beta}\eta_3)\boldsymbol{a}^\beta + (\eta_{3|\alpha} + b_\alpha^\beta \eta_\beta)\boldsymbol{a}^3,\end{aligned}$$

where

$$\eta_{\beta|\alpha} := \partial_\alpha \eta_\beta - \Gamma^\sigma_{\alpha\beta}\eta_\sigma \text{ and } \eta_{3|\alpha} := \partial_\alpha \eta_3.$$

Proof. Since any vector $\boldsymbol{c}$ in the tangent plane can be expanded as $\boldsymbol{c} = (\boldsymbol{c}\cdot\boldsymbol{a}_\beta)\boldsymbol{a}^\beta = (\boldsymbol{c}\cdot\boldsymbol{a}^\sigma)\boldsymbol{a}_\sigma$, since $\partial_\alpha\boldsymbol{a}^3$ is in the tangent plane ($\partial_\alpha\boldsymbol{a}^3\cdot\boldsymbol{a}^3 = \frac{1}{2}\partial_\alpha(\boldsymbol{a}^3\cdot\boldsymbol{a}^3) = 0$), and since $\partial_\alpha\boldsymbol{a}^3\cdot\boldsymbol{a}_\beta = -b_{\alpha\beta}$ (Theorem 2.4-1), it follows that

$$\partial_\alpha\boldsymbol{a}^3 = (\partial_\alpha\boldsymbol{a}^3\cdot\boldsymbol{a}_\beta)\boldsymbol{a}^\beta = -b_{\alpha\beta}\boldsymbol{a}^\beta.$$

This formula, together with the definition of the functions b_α^β (Theorem 2.5-1), implies in turn that

$$\partial_\alpha\boldsymbol{a}_3 = (\partial_\alpha\boldsymbol{a}_3\cdot\boldsymbol{a}^\sigma)\boldsymbol{a}_\sigma = -b_{\alpha\beta}(\boldsymbol{a}^\beta\cdot\boldsymbol{a}^\sigma)\boldsymbol{a}_\sigma = -b_{\alpha\beta}a^{\beta\sigma}\boldsymbol{a}_\sigma = -b_\alpha^\sigma\boldsymbol{a}_\sigma.$$

Any vector $\boldsymbol{c}$ can be expanded as $\boldsymbol{c} = (\boldsymbol{c}\cdot\boldsymbol{a}^i)\boldsymbol{a}_i = (\boldsymbol{c}\cdot\boldsymbol{a}_j)\boldsymbol{a}^j$. In particular,

$$\partial_\alpha\boldsymbol{a}_\beta = (\partial_\alpha\boldsymbol{a}_\beta\cdot\boldsymbol{a}^\sigma)\boldsymbol{a}_\sigma + (\partial_\alpha\boldsymbol{a}_\beta\cdot\boldsymbol{a}^3)\boldsymbol{a}_3 = \Gamma_{\alpha\beta}^\sigma\boldsymbol{a}_\sigma + b_{\alpha\beta}\boldsymbol{a}_3,$$

by definition of $\Gamma_{\alpha\beta}^\sigma$ and $b_{\alpha\beta}$. Finally,

$$\partial_\alpha\boldsymbol{a}^\beta = (\partial_\alpha\boldsymbol{a}^\beta\cdot\boldsymbol{a}_\sigma)\boldsymbol{a}^\sigma + (\partial_\alpha\boldsymbol{a}^\beta\cdot\boldsymbol{a}_3)\boldsymbol{a}^3 = -\Gamma_{\alpha\sigma}^\beta\boldsymbol{a}^\sigma + b_\alpha^\beta\boldsymbol{a}^3,$$

since

$$\partial_\alpha\boldsymbol{a}^\beta\cdot\boldsymbol{a}_3 = -\boldsymbol{a}^\beta\cdot\partial_\alpha\boldsymbol{a}_3 = b_\alpha^\sigma\boldsymbol{a}_\sigma\cdot\boldsymbol{a}^\beta = b_\alpha^\beta.$$

That $\eta_i\boldsymbol{a}^i \in \boldsymbol{\mathcal{C}}^1(\omega)$ if $\eta_i \in \mathcal{C}^1(\omega)$ is clear since $\boldsymbol{a}^i \in \boldsymbol{C}^1(\omega)$ if $\boldsymbol{\theta} \in \mathcal{C}^2(\omega;\mathbf{E}^3)$. The formulas established *supra* immediately lead to the announced expression of $\partial_\alpha(\eta_i\boldsymbol{a}^i)$. □

The relations (found in Theorem 2.6-1)

$$\partial_\alpha\boldsymbol{a}_\beta = \Gamma_{\alpha\beta}^\sigma\boldsymbol{a}_\sigma + b_{\alpha\beta}\boldsymbol{a}_3 \text{ and } \partial_\alpha\boldsymbol{a}^\beta = -\Gamma_{\alpha\sigma}^\beta\boldsymbol{a}^\sigma + b_\alpha^\beta\boldsymbol{a}^3$$

and

$$\partial_\alpha\boldsymbol{a}_3 = \partial_\alpha\boldsymbol{a}^3 = -b_{\alpha\beta}\boldsymbol{a}^\beta = -b_\alpha^\sigma\boldsymbol{a}_\sigma,$$

respectively constitute the **formulas of Gauß** and **Weingarten**. The functions (also found in Theorem 2.6-1)

$$\eta_{\beta|\alpha} = \partial_\alpha\eta_\beta - \Gamma_{\alpha\beta}^\sigma\eta_\sigma \text{ and } \eta_{3|\alpha} = \partial_\alpha\eta_3$$

are the **first-order covariant derivatives of the vector field** $\eta_i\boldsymbol{a}^i : \omega \to \mathbb{R}^3$, and the functions

$$\Gamma_{\alpha\beta}^\sigma := \boldsymbol{a}^\sigma\cdot\partial_\alpha\boldsymbol{a}_\beta = -\partial_\alpha\boldsymbol{a}^\sigma\cdot\boldsymbol{a}_\beta$$

are the **Christoffel symbols of the second kind** (the Christoffel symbols of the first kind are introduced in the next section).

Remark. The Christoffel symbols $\Gamma^\sigma_{\alpha\beta}$ can be also defined solely in terms of the covariant components of the first fundamental form; see the proof of Theorem 2.7-1 □

The definition of the covariant derivatives $\eta_{\alpha|\beta} = \partial_\beta \eta_\alpha - \Gamma^\sigma_{\alpha\beta}\eta_\sigma$ of a vector field defined on a surface $\boldsymbol{\theta}(\omega)$ given in Theorem 2.6-1 is highly reminiscent of the definition of the covariant derivatives $v_{i\|j} = \partial_j v_i - \Gamma^p_{ij} v_p$ of a vector field defined on an open set $\boldsymbol{\Theta}(\Omega)$ given in Section 1.4. However, the former are more subtle to apprehend than the latter. To see this, recall that the covariant derivatives $v_{i\|j} = \partial_j v_i - \Gamma^p_{ij} v_p$ may be also defined by the relations (Theorem 1.4-2)

$$v_{i\|j}\boldsymbol{g}^j = \partial_j(v_i \boldsymbol{g}^i).$$

By contrast, *even if only tangential vector fields $\eta_\alpha \boldsymbol{a}^\alpha$ on the surface $\boldsymbol{\theta}(\omega)$ are considered* (i.e., vector fields $\eta_i \boldsymbol{a}^i : \omega \to \mathbb{R}^3$ for which $\eta_3 = 0$), *their covariant derivatives $\eta_{\alpha|\beta} = \partial_\beta \eta_\alpha - \Gamma^\sigma_{\alpha\beta}\eta_\sigma$ satisfy only the relations*

$$\eta_{\alpha|\beta}\boldsymbol{a}^\alpha = \mathbf{P}\left\{\partial_\beta(\eta_\alpha \boldsymbol{a}^\alpha)\right\},$$

where $\mathbf{P}$ denotes the projection operator on the tangent plane in the direction of the normal vector (i.e., $\mathbf{P}(c_i \boldsymbol{a}^i) := c_\alpha \boldsymbol{a}^\alpha$), since

$$\partial_\beta(\eta_\alpha \boldsymbol{a}^\alpha) = \eta_{\alpha|\beta}\boldsymbol{a}^\alpha + b^\alpha_\beta \eta_\alpha \boldsymbol{a}^3$$

for such tangential fields by Theorem 2.6-1. The reason is that a surface has in general a *nonzero curvature*, manifesting itself here by the "extra term" $b^\alpha_\beta \eta_\alpha \boldsymbol{a}^3$. This term vanishes in ω if $\widehat{\omega}$ is a *portion of a plane*, since in this case $b^\alpha_\beta = b_{\alpha\beta} = 0$. Note that, *again in this case*, the formula giving the partial derivatives in Theorem 2.9-1 (b) reduces to

$$\partial_\alpha(\eta_i \boldsymbol{a}^i) = (\eta_{i|\alpha})\boldsymbol{a}^i.$$

2.7 NECESSARY CONDITIONS SATISFIED BY THE FIRST AND SECOND FUNDAMENTAL FORMS: THE GAUSS AND CODAZZI-MAINARDI EQUATIONS; GAUSS' THEOREMA EGREGIUM

It is remarkable that *the components $a_{\alpha\beta} = a_{\beta\alpha} : \omega \to \mathbb{R}$ and $b_{\alpha\beta} = b_{\beta\alpha} : \omega \to \mathbb{R}$ of the first and second fundamental forms of a surface $\boldsymbol{\theta}(\omega)$*, defined by a smooth enough immersion $\boldsymbol{\theta} : \omega \to \mathbf{E}^3$, *cannot be arbitrary functions.*

As shown in the next theorem, they must satisfy relations that take the form:

$$\partial_\beta \Gamma_{\alpha\sigma\tau} - \partial_\sigma \Gamma_{\alpha\beta\tau} + \Gamma^\mu_{\alpha\beta}\Gamma_{\sigma\tau\mu} - \Gamma^\mu_{\alpha\sigma}\Gamma_{\beta\tau\mu} = b_{\alpha\sigma}b_{\beta\tau} - b_{\alpha\beta}b_{\sigma\tau} \text{ in } \omega,$$
$$\partial_\beta b_{\alpha\sigma} - \partial_\sigma b_{\alpha\beta} + \Gamma^\mu_{\alpha\sigma}b_{\beta\mu} - \Gamma^\mu_{\alpha\beta}b_{\sigma\mu} = 0 \text{ in } \omega,$$

where the functions $\Gamma_{\alpha\beta\tau}$ and $\Gamma^\sigma_{\alpha\beta}$ have simple expressions in terms of the functions $a_{\alpha\beta}$ and of some of their partial derivatives (as shown in the next proof, it so happens that the functions $\Gamma^\sigma_{\alpha\beta}$ as defined in Theorem 2.7-1 coincide with the Christoffel symbols introduced in the previous section; this explains why they are denoted by the same symbol).

These relations, which are meant to hold for all $\alpha, \beta, \sigma, \tau \in \{1, 2\}$, respectively constitute the **Gauß**, and **Codazzi-Mainardi, equations**.

Theorem 2.7-1. *Let ω be an open subset of $\mathbb{R}^2$, let $\boldsymbol{\theta} \in \mathcal{C}^3(\omega; \mathbf{E}^3)$ be an immersion, and let*

$$a_{\alpha\beta} := \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} \text{ and } b_{\alpha\beta} := \partial_{\alpha\beta} \boldsymbol{\theta} \cdot \left\{ \frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|} \right\}$$

denote the covariant components of the first and second fundamental forms of the surface $\boldsymbol{\theta}(\omega)$. Let the functions $\Gamma_{\alpha\beta\tau} \in \mathcal{C}^1(\omega)$ and $\Gamma^\sigma_{\alpha\beta} \in \mathcal{C}^1(\omega)$ be defined by

$$\Gamma_{\alpha\beta\tau} := \frac{1}{2}(\partial_\beta a_{\alpha\tau} + \partial_\alpha a_{\beta\tau} - \partial_\tau a_{\alpha\beta}),$$
$$\Gamma^\sigma_{\alpha\beta} := a^{\sigma\tau} \Gamma_{\alpha\beta\tau} \text{ where } (a^{\sigma\tau}) := (a_{\alpha\beta})^{-1}.$$

Then, necessarily,

$$\partial_\beta \Gamma_{\alpha\sigma\tau} - \partial_\sigma \Gamma_{\alpha\beta\tau} + \Gamma^\mu_{\alpha\beta} \Gamma_{\sigma\tau\mu} - \Gamma^\mu_{\alpha\sigma} \Gamma_{\beta\tau\mu} = b_{\alpha\sigma} b_{\beta\tau} - b_{\alpha\beta} b_{\sigma\tau} \text{ in } \omega,$$
$$\partial_\beta b_{\alpha\sigma} - \partial_\sigma b_{\alpha\beta} + \Gamma^\mu_{\alpha\sigma} b_{\beta\mu} - \Gamma^\mu_{\alpha\beta} b_{\sigma\mu} = 0 \text{ in } \omega.$$

Proof. Let $\boldsymbol{a}_\alpha = \partial_\alpha \boldsymbol{\theta}$. It is then immediately verified that the functions $\Gamma_{\alpha\beta\tau}$ are also given by

$$\Gamma_{\alpha\beta\tau} = \partial_\alpha \boldsymbol{a}_\beta \cdot \boldsymbol{a}_\tau.$$

Let $\boldsymbol{a}_3 = \dfrac{\boldsymbol{a}_1 \wedge \boldsymbol{a}_2}{|\boldsymbol{a}_1 \wedge \boldsymbol{a}_2|}$ and, for each $y \in \omega$, let the three vectors $\boldsymbol{a}^j(y)$ be defined by the relations $\boldsymbol{a}^j(y) \cdot \boldsymbol{a}_i(y) = \delta^j_i$. Since we also have $\boldsymbol{a}^\beta = a^{\alpha\beta} \boldsymbol{a}_\alpha$ and $\boldsymbol{a}^3 = \boldsymbol{a}_3$, the last relations imply that $\Gamma^\sigma_{\alpha\beta} = \partial_\alpha \boldsymbol{a}_\beta \cdot \boldsymbol{a}^\sigma$, hence that

$$\partial_\alpha \boldsymbol{a}_\beta = \Gamma^\sigma_{\alpha\beta} \boldsymbol{a}_\sigma + b_{\alpha\beta} \boldsymbol{a}_3,$$

since $\partial_\alpha \boldsymbol{a}_\beta = (\partial_\alpha \boldsymbol{a}_\beta \cdot \boldsymbol{a}^\sigma) \boldsymbol{a}_\sigma + (\partial_\alpha \boldsymbol{a}_\beta \cdot \boldsymbol{a}^3) \boldsymbol{a}_3$. Differentiating the same relations yields

$$\partial_\sigma \Gamma_{\alpha\beta\tau} = \partial_{\alpha\sigma} \boldsymbol{a}_\beta \cdot \boldsymbol{a}_\tau + \partial_\alpha \boldsymbol{a}_\beta \cdot \partial_\sigma \boldsymbol{a}_\tau,$$

so that the above relations together give

$$\partial_\alpha \boldsymbol{a}_\beta \cdot \partial_\sigma \boldsymbol{a}_\tau = \Gamma^\mu_{\alpha\beta} \boldsymbol{a}_\mu \cdot \partial_\sigma \boldsymbol{a}_\tau + b_{\alpha\beta} \boldsymbol{a}_3 \cdot \partial_\sigma \boldsymbol{a}_\tau = \Gamma^\mu_{\alpha\beta} \Gamma_{\sigma\tau\mu} + b_{\alpha\beta} b_{\sigma\tau}.$$

Consequently,

$$\partial_{\alpha\sigma} \boldsymbol{a}_\beta \cdot \boldsymbol{a}_\tau = \partial_\sigma \Gamma_{\alpha\beta\tau} - \Gamma^\mu_{\alpha\beta} \Gamma_{\sigma\tau\mu} - b_{\alpha\beta} b_{\sigma\tau}.$$

Since $\partial_{\alpha\sigma} \boldsymbol{a}_\beta = \partial_{\alpha\beta} \boldsymbol{a}_\sigma$, we also have

$$\partial_{\alpha\sigma} \boldsymbol{a}_\beta \cdot \boldsymbol{a}_\tau = \partial_\beta \Gamma_{\alpha\sigma\tau} - \Gamma^\mu_{\alpha\sigma} \Gamma_{\beta\tau\mu} - b_{\alpha\sigma} b_{\beta\tau}.$$

Hence the Gauß equations immediately follow.

Since $\partial_\alpha \boldsymbol{a}_3 = (\partial_\alpha \boldsymbol{a}_3 \cdot \boldsymbol{a}_\sigma)\boldsymbol{a}^\sigma + (\partial_\alpha \boldsymbol{a}_3 \cdot \boldsymbol{a}_3)\boldsymbol{a}^3$ and $\partial_\alpha \boldsymbol{a}_3 \cdot \boldsymbol{a}_\sigma = -b_{\alpha\sigma} = -\partial_\alpha \boldsymbol{a}_\sigma \cdot \boldsymbol{a}_3$, we have

$$\partial_\alpha \boldsymbol{a}_3 = -b_{\alpha\sigma}\boldsymbol{a}^\sigma.$$

Differentiating the relations $b_{\alpha\beta} = \partial_\alpha \boldsymbol{a}_\beta \cdot \boldsymbol{a}_3$, we obtain

$$\partial_\sigma b_{\alpha\beta} = \partial_{\alpha\sigma}\boldsymbol{a}_\beta \cdot \boldsymbol{a}_3 + \partial_\alpha \boldsymbol{a}_\beta \cdot \partial_\sigma \boldsymbol{a}_3.$$

This relation and the relations $\partial_\alpha \boldsymbol{a}_\beta = \Gamma^\sigma_{\alpha\beta}\boldsymbol{a}_\sigma + b_{\alpha\beta}\boldsymbol{a}_3$ and $\partial_\alpha \boldsymbol{a}_3 = -b_{\alpha\sigma}\boldsymbol{a}^\sigma$ together imply that

$$\partial_\alpha \boldsymbol{a}_\beta \cdot \partial_\sigma \boldsymbol{a}_3 = -\Gamma^\mu_{\alpha\beta} b_{\sigma\mu}.$$

Consequently,

$$\partial_{\alpha\sigma}\boldsymbol{a}_\beta \cdot \boldsymbol{a}_3 = \partial_\sigma b_{\alpha\beta} + \Gamma^\mu_{\alpha\beta} b_{\sigma\mu}.$$

Since $\partial_{\alpha\sigma}\boldsymbol{a}_\beta = \partial_{\alpha\beta}\boldsymbol{a}_\sigma$, we also have

$$\partial_{\alpha\sigma}\boldsymbol{a}_\beta \cdot \boldsymbol{a}_3 = \partial_\beta b_{\alpha\sigma} + \Gamma^\mu_{\alpha\sigma} b_{\beta\mu}.$$

Hence the Codazzi-Mainardi equations immediately follow. □

Remark. The vectors $\boldsymbol{a}_\alpha$ and $\boldsymbol{a}^\beta$ introduced above respectively form the covariant and contravariant bases of the tangent plane to the surface $\boldsymbol{\theta}(\omega)$, the unit vector $\boldsymbol{a}_3 = \boldsymbol{a}^3$ is normal to the surface, and the functions $a^{\alpha\beta}$ are the contravariant components of the first fundamental form (Sections 2.2 and 2.3). □

As shown in the above proof, the Gauß and Codazzi-Mainardi equations thus simply constitute a re-writing of the relations $\partial_{\alpha\sigma}\boldsymbol{a}_\beta = \partial_{\alpha\beta}\boldsymbol{a}_\sigma$ in the form of the equivalent relations $\partial_{\alpha\sigma}\boldsymbol{a}_\beta \cdot \boldsymbol{a}_\tau = \partial_{\alpha\beta}\boldsymbol{a}_\sigma \cdot \boldsymbol{a}_\tau$ and $\partial_{\alpha\sigma}\boldsymbol{a}_\beta \cdot \boldsymbol{a}_3 = \partial_{\alpha\beta}\boldsymbol{a}_\sigma \cdot \boldsymbol{a}_3$.

The functions

$$\Gamma_{\alpha\beta\tau} = \frac{1}{2}(\partial_\beta a_{\alpha\tau} + \partial_\alpha a_{\beta\tau} - \partial_\tau a_{\alpha\beta}) = \partial_\alpha \boldsymbol{a}_\beta \cdot \boldsymbol{a}_\tau = \Gamma_{\beta\alpha\tau}$$

and

$$\Gamma^\sigma_{\alpha\beta} = a^{\sigma\tau}\Gamma_{\alpha\beta\tau} = \partial_\alpha \boldsymbol{a}_\beta \cdot \boldsymbol{a}^\sigma = \Gamma^\sigma_{\beta\alpha}$$

are the **Christoffel symbols of the first**, and **second, kind**. We recall that the Christoffel symbols of the second kind also naturally appeared in a different context (that of covariant differentiation; cf. Section 2.6).

Finally, the functions

$$S_{\tau\alpha\beta\sigma} := \partial_\beta \Gamma_{\alpha\sigma\tau} - \partial_\sigma \Gamma_{\alpha\beta\tau} + \Gamma^\mu_{\alpha\beta}\Gamma_{\sigma\tau\mu} - \Gamma^\mu_{\alpha\sigma}\Gamma_{\beta\tau\mu}$$

are the **covariant components of the Riemann curvature tensor of the surface $\boldsymbol{\theta}(\omega)$**.

The definitions of the functions $\Gamma^\sigma_{\alpha\beta}$ and $\Gamma_{\alpha\beta\tau}$ imply that *the sixteen Gauß equations are satisfied if and only if they are satisfied for* $\alpha = 1, \beta = 2,$

$\sigma = 1$, $\tau = 2$ and that *the Codazzi-Mainardi equations are satisfied if and only if they are satisfied* for $\alpha = 1$, $\beta = 2$, $\sigma = 1$ and $\alpha = 1$, $\beta = 2$, $\sigma = 2$ (other choices of indices with the same properties are clearly possible).

In other words, the Gauß equations and the Codazzi-Mainardi equations in fact respectively reduce to *one* and *two* equations.

Letting $\alpha = 2$, $\beta = 1$, $\sigma = 2$, $\tau = 1$ in the Gauß equations gives in particular

$$S_{1212} = \det(b_{\alpha\beta}).$$

Consequently, the *Gaussian curvature* at each point $\mathbf{\Theta}(y)$ of the surface $\boldsymbol{\theta}(\omega)$ can be written as

$$\frac{1}{R_1(y)R_2(y)} = \frac{S_{1212}(y)}{\det(a_{\alpha\beta}(y))},\ y \in \omega,$$

since $\dfrac{1}{R_1(y)R_2(y)} = \dfrac{\det(b_{\alpha\beta}(y)}{\det(a_{\alpha\beta}(y))}$ (Theorem 2.5-1). By inspection of the function S_{1212}, we thus reach the astonishing conclusion that, *at each point of the surface, a notion involving the "curvature" of the surface, viz., the Gaussian curvature, is entirely determined by the knowledge of the "metric" of the surface at the same point*, viz., the components of the first fundamental forms and their partial derivatives of order ≤ 2 at the same point! This startling conclusion naturally deserves a theorem:

Theorem 2.7-2. *Let ω be an open subset of $\mathbb{R}^2$, let $\boldsymbol{\theta} \in \mathcal{C}^3(\omega; \mathbf{E}^3)$ be an immersion, let $a_{\alpha\beta} = \partial_\alpha\boldsymbol{\theta} \cdot \partial_\beta\boldsymbol{\theta}$ denote the covariant components of the first fundamental form of the surface $\boldsymbol{\theta}(\omega)$, and let the functions $\Gamma_{\alpha\beta\tau}$ and S_{1212} be defined by*

$$\Gamma_{\alpha\beta\tau} := \frac{1}{2}(\partial_\beta a_{\alpha\tau} + \partial_\alpha a_{\beta\tau} - \partial_\tau a_{\alpha\beta}),$$
$$S_{1212} := \frac{1}{2}(2\partial_{12}a_{12} - \partial_{11}a_{22} - \partial_{22}a_{11}) + a^{\alpha\beta}(\Gamma_{12\alpha}\Gamma_{12\beta} - \Gamma_{11\alpha}\Gamma_{22\beta}).$$

Then, at each point $\boldsymbol{\theta}(y)$ of the surface $\boldsymbol{\theta}(\omega)$, the principal curvatures $\frac{1}{R_1(y)}$ and $\frac{1}{R_2(y)}$ satisfy

$$\frac{1}{R_1(y)R_2(y)} = \frac{S_{1212}(y)}{\det(a_{\alpha\beta}(y))},\ y \in \omega.$$

Theorem 2.7-2 constitutes the famed **Theorema Egregium** of Gauß [1828], so named by Gauß who had been himself astounded by his discovery.

2.8 EXISTENCE OF A SURFACE WITH PRESCRIBED FIRST AND SECOND FUNDAMENTAL FORMS

Let $\mathbb{M}^2, \mathbb{S}^2$, and $\mathbb{S}^2_>$ denote the sets of all square matrices of order two, of all symmetric matrices of order two, and of all symmetric, positive definite matrices of order two.

So far, we have considered that we are given an open set $\omega \subset \mathbb{R}^2$ and a smooth enough immersion $\boldsymbol{\theta} : \omega \to \mathbf{E}^3$, thus allowing us to define the fields $(a_{\alpha\beta}) : \omega \to \mathbb{S}^2_>$ and $(b_{\alpha\beta}) : \omega \to \mathbb{S}^2$, where $a_{\alpha\beta} : \omega \to \mathbb{R}$ and $b_{\alpha\beta} : \omega \to \mathbb{R}$ are the covariant components of the *first* and *second fundamental forms* of the surface $\boldsymbol{\theta}(\omega) \subset \mathbf{E}^3$.

Note that *the immersion* $\boldsymbol{\theta}$ *need not be injective* in order that these matrix fields be well defined.

We now turn to the reciprocal questions:

Given an open subset ω of $\mathbb{R}^2$ and two smooth enough matrix fields $(a_{\alpha\beta}) : \omega \to \mathbb{S}^2_>$ and $(b_{\alpha\beta}) : \omega \to \mathbb{S}^2$, when are they the first and second fundamental forms of a surface $\boldsymbol{\theta}(\omega) \subset \mathbf{E}^3$, i.e., when does there exist an immersion $\boldsymbol{\theta} : \omega \to \mathbf{E}^3$ such that

$$a_{\alpha\beta} = \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} \text{ and } b_{\alpha\beta} = \partial_{\alpha\beta} \boldsymbol{\theta} \cdot \left\{ \frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|} \right\} \text{ in } \omega?$$

If such an immersion exists, to what extent is it unique?

The answers to these questions turn out to be remarkably simple: *If* ω *is simply-connected, the necessary conditions of* Theorem 2.7-1, i.e., *the Gauß and Codazzi-Mainardi equations, are also sufficient for the existence of such an immersion. If* ω *is connected, this immersion is unique up to isometries in* $\mathbf{E}^3$.

Whether an immersion found in this fashion is *injective* is a different issue, which accordingly should be resolved by different means.

Following Ciarlet & Larsonneur [2001], we now give a self-contained, complete, and essentially elementary, proof of this well-known result. This proof amounts to showing that it can be established as a simple *corollary* to the *fundamental theorem on flat Riemannian manifolds* established in Theorems 1.6-1 and 1.7-1 when the manifold is an open set in $\mathbb{R}^3$.

This proof has also the merit to shed light on the analogies (which cannot remain unnoticed!) between the assumptions and conclusions of both *existence* results (compare Theorems 1.6-1 and 2.8-1) and both *uniqueness* results (compare Theorems 1.7-1 and 2.9-1).

A direct proof of the fundamental theorem of surface theory is given in Klingenberg [1973, Theorem 3.8.8], where the global existence of the mapping $\boldsymbol{\theta}$ is based on an existence theorem for ordinary differential equations, analogous to that used in part (ii) of the proof of Theorem 1.6-1. A proof of the "local" version of this theorem, which constitutes *Bonnet's theorem*, is found in, e.g., do Carmo [1976].

This result is another special case of the *fundamental theorem of Riemannian geometry* alluded to in Section 1.6. We recall that this theorem asserts that a simply-connected Riemannian manifold of dimension p can be isometrically immersed into a Euclidean space of dimension $(p+q)$ if and only if there exist tensors satisfying together generalized *Gauß*, and *Codazzi-Mainardi, equations* and that the corresponding isometric immersions are unique up to isometries in the Euclidean space. A substantial literature has been devoted to this theorem and its various proofs, which usually rely on basic notions of Riemannian geometry, such as connections or normal bundles, and on the theory of differential forms. See in particular the earlier papers of Janet [1926] and Cartan [1927] and the more recent references of Szczarba [1970], Tenenblat [1971], Jacobowitz [1982], and Szopos [2005].

Like the fundamental theorem of three-dimensional differential geometry, this theorem comprises two essentially distinct parts, a *global existence result* (Theorem 2.8-1) and a *uniqueness result* (Theorem 2.9-1), the latter being also called *rigidity theorem.* Note that these two results are established under *different assumptions* on the set ω and on the smoothness of the fields $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$.

These existence and uniqueness results together constitute the **fundamental theorem of surface theory**.

Theorem 2.8-1. *Let ω be a connected and simply-connected open subset of $\mathbb{R}^2$ and let $(a_{\alpha\beta}) \in \mathcal{C}^2(\omega; \mathbb{S}^2_>)$ and $(b_{\alpha\beta}) \in \mathcal{C}^2(\omega; \mathbb{S}^2)$ be two matrix fields that satisfy the Gauß and Codazzi-Mainardi equations, viz.,*

$$\partial_\beta \Gamma_{\alpha\sigma\tau} - \partial_\sigma \Gamma_{\alpha\beta\tau} + \Gamma^\mu_{\alpha\beta}\Gamma_{\sigma\tau\mu} - \Gamma^\mu_{\alpha\sigma}\Gamma_{\beta\tau\mu} = b_{\alpha\sigma}b_{\beta\tau} - b_{\alpha\beta}b_{\sigma\tau} \text{ in } \omega,$$
$$\partial_\beta b_{\alpha\sigma} - \partial_\sigma b_{\alpha\beta} + \Gamma^\mu_{\alpha\sigma} b_{\beta\mu} - \Gamma^\mu_{\alpha\beta} b_{\sigma\mu} = 0 \text{ in } \omega,$$

where

$$\Gamma_{\alpha\beta\tau} := \frac{1}{2}(\partial_\beta a_{\alpha\tau} + \partial_\alpha a_{\beta\tau} - \partial_\tau a_{\alpha\beta}),$$
$$\Gamma^\sigma_{\alpha\beta} := a^{\sigma\tau}\Gamma_{\alpha\beta\tau} \text{ where } (a^{\sigma\tau}) := (a_{\alpha\beta})^{-1}.$$

Then there exists an immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\omega; \mathbf{E}^3)$ such that

$$a_{\alpha\beta} = \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} \text{ and } b_{\alpha\beta} = \partial_{\alpha\beta}\boldsymbol{\theta} \cdot \left\{ \frac{\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}}{|\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}|} \right\} \text{ in } \omega.$$

Proof. The proof of this theorem as a corollary to Theorem 1.6-1 relies on the following elementary observation: Given a smooth enough immersion $\boldsymbol{\theta} : \omega \to \mathbf{E}^3$ and $\varepsilon > 0$, let the mapping $\boldsymbol{\Theta} : \omega \times \left]-\varepsilon, \varepsilon\right[\to \mathbf{E}^3$ be defined by

$$\boldsymbol{\Theta}(y, x_3) := \boldsymbol{\theta}(y) + x_3 \boldsymbol{a}_3(y) \text{ for all } (y, x_3) \in \omega \times \left]-\varepsilon, \varepsilon\right[,$$

where $\boldsymbol{a}_3 := \dfrac{\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}}{|\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}|}$, and let

$$g_{ij} := \partial_i \boldsymbol{\Theta} \cdot \partial_j \boldsymbol{\Theta}.$$

Then an immediate computation shows that

$$g_{\alpha\beta} = a_{\alpha\beta} - 2x_3 b_{\alpha\beta} + x_3^2 c_{\alpha\beta} \text{ and } g_{i3} = \delta_{i3} \text{ in } \omega \times \left]-\varepsilon, \varepsilon\right[,$$

where $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are the covariant components of the first and second fundamental forms of the surface $\boldsymbol{\theta}(\omega)$ and $c_{\alpha\beta} := a^{\sigma\tau} b_{\alpha\sigma} b_{\beta\tau}$.

Assume that the matrices (g_{ij}) constructed in this fashion are *invertible*, hence positive definite, over the set $\omega \times \left]-\varepsilon, \varepsilon\right[$ (they need not be, of course; but the resulting difficulty is easily circumvented; see parts (i) and (viii) below). Then the field $(g_{ij}) : \omega \times]-\varepsilon, \varepsilon[\to \mathbb{S}^3_>$ becomes a natural candidate for applying the "three-dimensional" existence result of Theorem 1.6-1, provided of course that *the "three-dimensional" sufficient conditions* of this theorem, viz.,

$$\partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma^p_{ij} \Gamma_{kqp} - \Gamma^p_{ik} \Gamma_{jqp} = 0 \text{ in } \Omega,$$

can be shown to hold, as consequences of the "two-dimensional" Gauß and Codazzi-Mainardi equations. That this is indeed the case is the essence of the present proof (see parts (i) to (vii)).

By Theorem 1.6-1, there then exists an immersion $\boldsymbol{\Theta} : \omega \times]-\varepsilon, \varepsilon[\to \mathbf{E}^3$ that satisfies $g_{ij} = \partial_i \boldsymbol{\Theta} \cdot \partial_j \boldsymbol{\Theta}$ in $\omega \times]-\varepsilon, \varepsilon[$. It thus remains to check that $\boldsymbol{\theta} := \boldsymbol{\Theta}(\cdot, 0)$ indeed satisfies (see part (ix))

$$a_{\alpha\beta} = \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} \text{ and } b_{\alpha\beta} = \partial_{\alpha\beta} \boldsymbol{\theta} \cdot \left\{ \frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|} \right\} \text{ in } \omega.$$

The actual implementation of this program essentially involves elementary, *albeit* sometimes lengthy, computations, which accordingly will be omitted for the most part; only the main intermediate results will be recorded. For clarity, the proof is broken into nine parts, numbered (i) to (ix).

To avoid confusion with the "three-dimensional" Christoffel symbols, those "on a surface" will be denoted $C^\sigma_{\alpha\beta}$ and $C_{\alpha\beta\tau}$ *in this proof* (and only in this proof).

(i) *Given two matrix fields* $(a_{\alpha\beta}) \in \mathcal{C}^2(\omega; \mathbb{S}^2_>)$ *and* $(b_{\alpha\beta}) \in \mathcal{C}^2(\omega; \mathbb{S}^2)$*, let the matrix field* $(g_{ij}) \in \mathcal{C}^2(\omega \times \mathbb{R}; \mathbb{S}^3)$ *be defined by*

$$g_{\alpha\beta} := a_{\alpha\beta} - 2x_3 b_{\alpha\beta} + x_3^2 c_{\alpha\beta} \text{ and } g_{i3} := \delta_{i3} \text{ in } \omega \times \mathbb{R}$$

(the variable $y \in \omega$ is omitted; x_3 designates the variable in $\mathbb{R}$), *where*

$$c_{\alpha\beta} := b^\tau_\alpha b_{\beta\tau} \text{ and } b^\tau_\alpha := a^{\sigma\tau} b_{\alpha\sigma} \text{ in } \omega.$$

Let ω_0 *be an open subset of* $\mathbb{R}^2$ *such that* $\overline{\omega}_0$ *is a compact subset of* ω*. Then there exists* $\varepsilon_0 = \varepsilon_0(\omega_0) > 0$ *such that the symmetric matrices* (g_{ij}) *are positive definite at all points in* $\overline{\Omega}_0$*, where*

$$\Omega_0 := \omega_0 \times]-\varepsilon_0, \varepsilon_0[\,.$$

Besides, the elements of the inverse matrix (g^{pq}) *are given in* $\overline{\Omega}_0$ *by*

$$g^{\alpha\beta} = \sum_{n \geq 0} (n+1) x_3^n a^{\alpha\sigma} (\mathbf{B}^n)^\beta_\sigma \text{ and } g^{i3} = \delta^{i3},$$

where

$$(\mathbf{B})^\beta_\sigma := b^\beta_\sigma \text{ and } (\mathbf{B}^n)^\beta_\sigma := b^{\sigma_1}_\sigma \cdots b^\beta_{\sigma_{n-1}} \text{ for } n \geq 2,$$

i.e., $(\mathbf{B}^n)^\beta_\sigma$ *designates for any* $n \geq 0$ *the element at the* σ-th *row and* β-th *column of the matrix* $\mathbf{B}^n$*. The above series are absolutely convergent in the space* $\mathcal{C}^2(\overline{\Omega}_0)$*.*

Let *a priori* $g^{\alpha\beta} = \sum_{n\geq 0} x_3^n h_n^{\alpha\beta}$ where $h_n^{\alpha\beta}$ are functions of $y \in \overline{\omega}_0$ only, so that the relations $g^{\alpha\beta}g_{\beta\tau} = \delta_\tau^\alpha$ read

$$h_0^{\alpha\beta}a_{\beta\tau} + x_3(h_1^{\alpha\beta}a_{\beta\tau} - 2h_0^{\alpha\beta}b_{\beta\tau}) + \sum_{n\geq 2} x_3^n(h_n^{\alpha\beta}a_{\beta\tau} - 2h_{n-1}^{\alpha\beta}b_{\beta\tau} + h_{n-2}^{\alpha\beta}c_{\beta\tau}) = \delta_\tau^\alpha.$$

It is then easily verified that the functions $h_n^{\alpha\beta}$ are given by

$$h_n^{\alpha\beta} = (n+1)a^{\alpha\sigma}(\mathbf{B}^n)_\sigma^\beta,\ n \geq 0,$$

so that

$$g^{\alpha\beta} = \sum_{n\geq 0}(n+1)x_3^n a^{\alpha\sigma} b_\sigma^{\sigma_1} \cdots b_{\sigma_{n-1}}^\beta.$$

It is clear that such a series is absolutely convergent in the space $\mathcal{C}^2(\overline{\omega}_0 \times [-\varepsilon_0, \varepsilon_0])$ if $\varepsilon_0 > 0$ is small enough.

(ii) *The functions $C_{\alpha\beta}^\sigma$ being defined by*

$$C_{\alpha\beta}^\sigma := a^{\sigma\tau}C_{\alpha\beta\tau},$$

where

$$(a^{\sigma\tau}) := (a_{\alpha\beta})^{-1} \text{ and } C_{\alpha\beta\tau} := \frac{1}{2}(\partial_\beta a_{\alpha\tau} + \partial_\alpha a_{\beta\tau} - \partial_\tau a_{\alpha\beta}),$$

define the functions

$$b_\alpha^\tau|_\beta := \partial_\beta b_\alpha^\tau + C_{\beta\mu}^\tau b_\alpha^\mu - C_{\alpha\beta}^\mu b_\mu^\tau,$$
$$b_{\alpha\beta|\sigma} := \partial_\sigma b_{\alpha\beta} - C_{\alpha\sigma}^\mu b_{\beta\mu} - C_{\beta\sigma}^\mu b_{\alpha\mu} = b_{\beta\alpha|\sigma}.$$

Then

$$b_\alpha^\tau|_\beta = a^{\sigma\tau}b_{\alpha\sigma|\beta} \text{ and } b_{\alpha\sigma|\beta} = a_{\sigma\tau}b_\alpha^\tau|_\beta.$$

Furthermore, the assumed Codazzi-Mainardi equations imply that

$$b_\alpha^\tau|_\beta = b_\beta^\tau|_\alpha \text{ and } b_{\alpha\sigma|\beta} = b_{\alpha\beta|\sigma}.$$

The above relations follow from straightforward computations based on the definitions of the functions $b_\alpha^\tau|_\beta$ and $b_{\alpha\beta|\sigma}$. They are recorded here because they play a pervading rôle in the subsequent computations.

(iii) *The functions $g_{ij} \in \mathcal{C}^2(\overline{\Omega}_0)$ and $g^{ij} \in \mathcal{C}^2(\overline{\Omega}_0)$ being defined as in* part (i), *define the functions $\Gamma_{ijq} \in \mathcal{C}^1(\overline{\Omega}_0)$ and $\Gamma_{ij}^p \in \mathcal{C}^1(\overline{\Omega}_0)$ by*

$$\Gamma_{ijq} := \frac{1}{2}(\partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij}) \text{ and } \Gamma_{ij}^p := g^{pq}\Gamma_{ijq}.$$

Then the functions $\Gamma_{ijq} = \Gamma_{jiq}$ *and* $\Gamma^p_{ij} = \Gamma^p_{ji}$ *have the following expressions*:

$$\begin{aligned}
&\Gamma_{\alpha\beta\sigma} = C_{\alpha\beta\sigma} - x_3(b^\tau_\alpha|_\beta a_{\tau\sigma} + 2C^\tau_{\alpha\beta} b_{\tau\sigma}) + x_3^2(b^\tau_\alpha|_\beta b_{\tau\sigma} + C^\tau_{\alpha\beta} c_{\tau\sigma}),\\
&\Gamma_{\alpha\beta 3} = -\Gamma_{\alpha 3\beta} = b_{\alpha\beta} - x_3 c_{\alpha\beta},\\
&\Gamma_{\alpha 33} = \Gamma_{3\beta 3} = \Gamma_{33q} = 0,\\
&\Gamma^\sigma_{\alpha\beta} = C^\sigma_{\alpha\beta} - \sum\nolimits_{n\geq 0} x_3^{n+1} b^\tau_\alpha|_\beta (\mathbf{B}^n)^\sigma_\tau,\\
&\Gamma^3_{\alpha\beta} = b_{\alpha\beta} - x_3 c_{\alpha\beta},\\
&\Gamma^\beta_{\alpha 3} = -\sum\nolimits_{n\geq 0} x_3^n (\mathbf{B}^{n+1})^\beta_\alpha,\\
&\Gamma^3_{3\beta} = \Gamma^p_{33} = 0,
\end{aligned}$$

where the functions $c_{\alpha\beta}$, $(\mathbf{B}^n)^\sigma_\tau$, *and* $b^\tau_\alpha|_\beta$ *are defined as in* parts (i) *and* (ii).

All computations are straightforward. We simply point out that the assumed *Codazzi-Mainardi equations* are needed to conclude that the factor of x_3 in the function $\Gamma_{\alpha\beta\sigma}$ is indeed that announced above. We also note that the computation of the factor of x_3^2 in $\Gamma_{\alpha\beta\sigma}$ relies in particular on the easily established relations

$$\partial_\alpha c_{\beta\sigma} = b^\tau_\beta|_\alpha b_{\sigma\tau} + b^\mu_\sigma|_\alpha b_{\mu\beta} + C^\mu_{\alpha\beta} c_{\sigma\mu} + C^\mu_{\alpha\sigma} c_{\beta\mu}.$$

(iv) *The functions* $\Gamma_{ijq} \in \mathcal{C}^1(\overline{\Omega}_0)$ *and* $\Gamma^p_{ij} \in \mathcal{C}^1(\overline{\Omega}_0)$ *being defined as in* part (iii), *define the functions* $R_{qijk} \in \mathcal{C}^0(\overline{\Omega}_0)$ *by*

$$R_{qijk} := \partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma^p_{ij}\Gamma_{kqp} - \Gamma^p_{ik}\Gamma_{jqp}.$$

Then, in order that the relations

$$R_{qijk} = 0 \text{ in } \overline{\Omega}_0$$

hold, it is sufficient that

$$R_{1212} = 0, \quad R_{\alpha 2\beta 3} = 0, \quad R_{\alpha 3\beta 3} = 0 \text{ in } \overline{\Omega}_0.$$

The above definition of the functions R_{qijk} and that of the functions Γ_{ijq} and Γ^p_{ij} (part (iii)) together imply that, for all i, j, k, q,

$$\begin{aligned}
R_{qijk} &= R_{jkqi} = -R_{qikj},\\
R_{qijk} &= 0 \text{ if } j = k \text{ or } q = i.
\end{aligned}$$

Consequently, the relation $R_{1212} = 0$ implies that $R_{\alpha\beta\sigma\tau} = 0$, the relations $R_{\alpha 2\beta 3} = 0$ imply that $R_{qijk} = 0$ if exactly one index is equal to 3, and finally, the relations $R_{\alpha 3\beta 3} = 0$ imply that $R_{qijk} = 0$ if exactly two indices are equal to 3.

(v) *The functions*

$$R_{\alpha 3\beta 3} := \partial_\beta \Gamma_{33\alpha} - \partial_3 \Gamma_{3\beta\alpha} + \Gamma^p_{3\beta}\Gamma_{3\alpha p} - \Gamma^p_{33}\Gamma_{\beta\alpha p}$$

satisfy

$$R_{\alpha 3\beta 3} = 0 \text{ in } \overline{\Omega}_0.$$

These relations immediately follow from the expressions found in part (iii) for the functions Γ_{ijq} and Γ^p_{ij}. Note that neither the Gauß equations nor the Codazzi-Mainardi equations are needed here.

(vi) *The functions*

$$R_{\alpha 2\beta 3} := \partial_\beta \Gamma_{23\alpha} - \partial_3 \Gamma_{2\beta\alpha} + \Gamma^p_{2\beta}\Gamma_{3\alpha p} - \Gamma^p_{23}\Gamma_{\beta\alpha p}$$

satisfy

$$R_{\alpha 2\beta 3} = 0 \text{ in } \overline{\Omega}_0.$$

The definitions of the functions $g_{\alpha\beta}$ (part (i)) and Γ_{ijq} (part (iii)) show that

$$\partial_\beta \Gamma_{23\alpha} - \partial_3 \Gamma_{2\beta\alpha} = (\partial_2 b_{\alpha\beta} - \partial_\alpha b_{2\beta}) + x_3(\partial_\alpha c_{2\beta} - \partial_2 c_{\alpha\beta}).$$

Then the expressions found in part (iii) show that

$$\begin{aligned}
\Gamma^p_{2\beta}\Gamma_{3\alpha p} - \Gamma^p_{23}\Gamma_{\beta\alpha p} &= \Gamma^\sigma_{3\alpha}\Gamma_{2\beta\sigma} - \Gamma^\sigma_{23}\Gamma_{\alpha\beta\sigma} \\
&= C^\sigma_{\alpha\beta} b_{2\sigma} - C^\sigma_{2\beta} b_{\alpha\sigma} \\
&\quad + x_3(b^\sigma_2|_\beta b_{\alpha\sigma} - b^\sigma_\alpha|_\beta b_{2\sigma} + C^\sigma_{2\beta} c_{\alpha\sigma} - C^\sigma_{\alpha\beta} c_{2\sigma}),
\end{aligned}$$

and the relations $R_{\alpha 3\beta 3} = 0$ follow by making use of the relations

$$\partial_\alpha c_{\beta\sigma} = b^\tau_\beta|_\alpha b_{\sigma\tau} + b^\mu_\sigma|_\alpha b_{\mu\beta} + C^\mu_{\alpha\beta} c_{\sigma\mu} + C^\mu_{\alpha\sigma} c_{\beta\mu}$$

together with the relations

$$\partial_2 b_{\alpha\beta} - \partial_\alpha b_{2\beta} + C^\sigma_{\alpha\beta} b_{2\sigma} - C^\sigma_{2\beta} b_{\alpha\sigma} = 0,$$

which are special cases of the assumed *Codazzi-Mainardi equations*.

(vii) *The function*

$$R_{1212} := \partial_1 \Gamma_{221} - \partial_2 \Gamma_{211} + \Gamma^p_{21}\Gamma_{21p} - \Gamma^p_{22}\Gamma_{11p}$$

satisfies

$$R_{1212} = 0 \text{ in } \overline{\Omega}_0.$$

The computations leading to this relation are fairly lengthy and they require some care. We simply record the main intermediary steps, which consist in evaluating separately the various terms occurring in the function R_{1212} rewritten as

$$R_{1212} = (\partial_1 \Gamma_{221} - \partial_2 \Gamma_{211}) + (\Gamma^\sigma_{12}\Gamma_{12\sigma} - \Gamma^\sigma_{11}\Gamma_{22\sigma}) + (\Gamma_{123}\Gamma_{123} - \Gamma_{113}\Gamma_{223}).$$

First, the expressions found in part (iii) for the functions $\Gamma_{\alpha\beta 3}$ easily yield

$$\begin{aligned}
\Gamma_{123}\Gamma_{123} - \Gamma_{113}\Gamma_{223} &= (b^2_{12} - b_{11}b_{22}) \\
&+ x_3(b_{11}c_{22} - 2b_{12}c_{12} + b_{22}c_{11}) + x^2_3(c^2_{12} - c_{11}c_{22}).
\end{aligned}$$

Second, the expressions found in part (iii) for the functions $\Gamma_{\alpha\beta\sigma}$ and $\Gamma^\sigma_{\alpha\beta}$ yield, after some manipulations:

$$\begin{aligned}\Gamma^\sigma_{12}\Gamma_{12\sigma} - \Gamma^\sigma_{11}\Gamma_{22\sigma} = {} & (C^\sigma_{12}C^\tau_{12} - C^\sigma_{11}C^\tau_{22})a_{\sigma\tau} \\ & + x_3\{(C^\sigma_{11}b^\tau_2|_2 - 2C^\sigma_{12}b^\tau_1|_2 + C^\sigma_{22}b^\tau_1|_1)a_{\sigma\tau} \\ & + 2(C^\sigma_{11}C^\tau_{22} - C^\sigma_{12}C^\tau_{12})b_{\sigma\tau}\} \\ & + x_3^2\{b^\sigma_1|_1 b^\tau_2|_2 - b^\sigma_1|_2 b^\tau_1|_2)a_{\sigma\tau} \\ & + (C^\sigma_{11}b^\tau_2|_2 - 2C^\sigma_{12}b^\tau_1|_2 + C^\sigma_{22}b^\tau_1|_1)b_{\sigma\tau} \\ & + (C^\sigma_{11}C^\tau_{22} - C^\sigma_{12}C^\tau_{12})c_{\sigma\tau}\}.\end{aligned}$$

Third, after somewhat delicate computations, which in particular make use of the relations established in part (ii) about the functions $b^\tau_\alpha|_\beta$ and $b_{\alpha\beta|\sigma}$, it is found that

$$\begin{aligned}\partial_1\Gamma_{221} - \partial_2\Gamma_{211} = {} & \partial_1 C_{221} - \partial_2 C_{211} \\ & - x_3\{S_{1212}b^\alpha_\alpha + (C^\sigma_{11}b^\tau_2|_2 - 2C^\sigma_{12}b^\tau_1|_2 + C^\sigma_{22}b^\tau_1|_1)a_{\sigma\tau} \\ & + 2(C^\sigma_{11}C^\tau_{22} - C^\sigma_{12}C^\tau_{12})b_{\sigma\tau}\} \\ & + x_3^2\{S_{\sigma\tau 12}b^\sigma_1 b^\tau_2 + (b^\sigma_1|_1 b^\tau_2|_2 - b^\sigma_1|_2 b^\tau_1|_2)a_{\sigma\tau} \\ & + (C^\sigma_{11}b^\tau_2|_2 - 2C^\sigma_{12}b^\tau_1|_2 + C^\sigma_{22}b^\tau_1|_1)b_\sigma \\ & + (C^\sigma_{11}C^\tau_{22} - C^\sigma_{12}C^\tau_{12})c_{\sigma\tau}\},\end{aligned}$$

where the functions

$$S_{\tau\alpha\beta\sigma} := \partial_\beta C_{\alpha\sigma\tau} - \partial_\sigma C_{\alpha\beta\tau} + C^\mu_{\alpha\beta}C_{\sigma\tau\mu} - C^\mu_{\alpha\sigma}C_{\beta\tau\mu}$$

are precisely those appearing in the left-hand sides of the Gauß equations.

It is then easily seen that the above equations together yield

$$\begin{aligned}R_{1212} = {} & \{S_{1212} - (b_{11}b_{22} - b_{12}b_{12})\} \\ & - x_3\{S_{1212} - (b_{11}b_{22} - b_{12}b_{12})b^\alpha_\alpha\} \\ & + x_3^2\{S_{\sigma\tau 12}b^\sigma_1 b^\tau_2 + (c_{12}c_{12} - c_{11}c_{22})\}.\end{aligned}$$

Since

$$\begin{aligned}S_{\sigma\tau 12}b^\sigma_1 b^\tau_2 &= S_{1212}(b^1_1 b^2_2 - b^2_1 b^1_2), \\ c_{12}c_{12} - c_{11}c_{22} &= (b_{11}b_{12} - b_{11}b_{22})(b^1_1 b^2_2 - b^2_1 b^1_2),\end{aligned}$$

it is finally found that the function R_{1212} has the following remarkable expression:

$$R_{1212} = \{S_{1212} - (b_{11}b_{22} - b_{12}b_{12})\}\{1 - x_3(b^1_1 + b^2_2) + x_3^2(b^1_1 b^2_2 - b^2_1 b^1_2)\}.$$

By the assumed *Gauß equations*,

$$S_{1212} = b_{11}b_{22} - b_{12}b_{12}.$$

Hence $R_{1212} = 0$ as announced.

(viii) *Let ω be a connected and simply-connected open subset of $\mathbb{R}^2$. Then there exist open subsets $\omega_\ell, \ell \geq 0$, of $\mathbb{R}^2$ such that $\overline{\omega}_\ell$ is a compact subset of ω for each $\ell \geq 0$ and*

$$\omega = \bigcup_{\ell \geq 0} \omega_\ell.$$

Furthermore, for each $\ell \geq 0$, there exists $\varepsilon_\ell = \varepsilon_\ell(\omega_\ell) > 0$ such that the symmetric matrices (g_{ij}) are positive definite at all points in $\overline{\Omega}_\ell$, where

$$\Omega_\ell := \omega_\ell \times \left]-\varepsilon_\ell, \varepsilon_\ell\right[.$$

Finally, the open set

$$\Omega := \bigcup_{\ell \geq 0} \Omega_\ell$$

is connected and simply-connected.

Let ω_ℓ, $\ell \geq 0$, be open subsets of ω with compact closures $\overline{\omega}_\ell \subset \omega$ such that $\omega = \bigcup_{\ell \geq 0} \omega_\ell$. For each ℓ, a set $\Omega_\ell := \omega_\ell \times \left]-\varepsilon_\ell, \varepsilon_\ell\right[$ can then be constructed in the same way that the set Ω_0 was constructed in part (i).

It is clear that the set $\Omega := \bigcup_{\ell \geq 0} \Omega_\ell$ is connected. To show that Ω is simply-connected, let $\boldsymbol{\gamma}$ be a *loop in* Ω, i.e., a mapping $\boldsymbol{\gamma} \in \mathcal{C}^0([0,1]; \mathbb{R}^3)$ that satisfies

$$\boldsymbol{\gamma}(0) = \boldsymbol{\gamma}(1) \text{ and } \boldsymbol{\gamma}(t) \in \Omega \text{ for all } 0 \leq t \leq 1.$$

Let the projection operator $\boldsymbol{\pi} : \Omega \to \omega$ be defined by $\boldsymbol{\pi}(y, x_3) = y$ for all $(y, x_3) \in \Omega$, and let the mapping $\boldsymbol{\varphi}_0 : [0,1] \times [0,1] \to \mathbb{R}^3$ be defined by

$$\boldsymbol{\varphi}_0(t, \lambda) := (1 - \lambda)\boldsymbol{\gamma}(t) + \lambda \boldsymbol{\pi}(\boldsymbol{\gamma}(t)) \text{ for all } 0 \leq t \leq 1,\ 0 \leq \lambda \leq 1.$$

Then $\boldsymbol{\varphi}_0$ is a continuous mapping such that $\boldsymbol{\varphi}_0([0,1] \times [0,1]) \subset \Omega$, by definition of the set Ω. Furthermore, $\boldsymbol{\varphi}_0(t, 0) = \boldsymbol{\gamma}(t)$ and $\boldsymbol{\varphi}_0(t, 1) = \boldsymbol{\pi}(\boldsymbol{\gamma}(t))$ for all $t \in [0,1]$.

The mapping

$$\widetilde{\gamma} := \boldsymbol{\pi} \circ \boldsymbol{\gamma} \in \mathcal{C}^0([0,1]; \mathbb{R}^2)$$

is a *loop in* ω since $\widetilde{\gamma}(0) = \boldsymbol{\pi}(\boldsymbol{\gamma}(0)) = \boldsymbol{\pi}(\boldsymbol{\gamma}(1)) = \widetilde{\gamma}(1)$ and $\widetilde{\gamma}(t) \in \omega$ for all $0 \leq t \leq 1$. Since ω is simply connected, there exist a mapping $\boldsymbol{\varphi}_1 \in \mathcal{C}^0([0,1] \times [0,1]; \mathbb{R}^2)$ and a point $y^0 \in \omega$ such that

$$\boldsymbol{\varphi}_1(t, 1) = \widetilde{\gamma} \text{ and } \boldsymbol{\varphi}_1(t, 2) = y^0 \text{ for all } 0 \leq t \leq 1$$

and

$$\boldsymbol{\varphi}_1(t, \lambda) \in \omega \text{ for all } 0 \leq t \leq 1,\ 1 \leq \lambda \leq 2.$$

Then the mapping $\boldsymbol{\varphi} \in \mathcal{C}^0([0,1] \times [0,2]; \mathbb{R}^3)$ defined by

$$\begin{aligned} \boldsymbol{\varphi}(t, \lambda) &= \boldsymbol{\varphi}_0(t, \lambda) \quad \text{for all} \quad 0 \leq t \leq 1,\ 0 \leq \lambda \leq 1, \\ \boldsymbol{\varphi}(t, \lambda) &= \boldsymbol{\varphi}_1(t, \lambda) \quad \text{for all} \quad 0 \leq t \leq 1,\ 1 \leq \lambda \leq 2, \end{aligned}$$

is a *homotopy in* Ω that reduces the loop $\boldsymbol{\gamma}$ to the point $(y^0, 0) \in \Omega$. Hence the set Ω is simply-connected.

(ix) *By* parts (iv) *to* (viii), *the functions* $\Gamma_{ijq} \in \mathcal{C}^1(\Omega)$ *and* $\Gamma_{ij}^p \in \mathcal{C}^1(\Omega)$ *constructed as in* part (iii) *satisfy*

$$\partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma_{ij}^p \Gamma_{kqp} - \Gamma_{ik}^p \Gamma_{jqp} = 0$$

in the connected and simply-connected open set Ω. *By* Theorem 1.6-1, *there thus exists an immersion* $\mathbf{\Theta} \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ *such that*

$$g_{ij} = \partial_i \mathbf{\Theta} \cdot \partial_j \mathbf{\Theta} \text{ in } \Omega,$$

where the matrix field $(g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ *is defined by*

$$g_{\alpha\beta} = a_{\alpha\beta} - 2x_3 b_{\alpha\beta} + x_3^2 c_{\alpha\beta} \text{ and } g_{i3} = \delta_{i3} \text{ in } \Omega.$$

Then the mapping $\boldsymbol{\theta} \in \mathcal{C}^3(\omega; \mathbf{E}^3)$ *defined by*

$$\boldsymbol{\theta}(y) = \mathbf{\Theta}(y, 0) \text{ for all } y \in \omega,$$

satisfies

$$a_{\alpha\beta} = \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} \textit{ and } b_{\alpha\beta} = \partial_{\alpha\beta} \boldsymbol{\theta} \cdot \left\{ \frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|} \right\} \text{ in } \omega.$$

Let $\boldsymbol{g}_i := \partial_i \mathbf{\Theta}$. Then $\partial_{33} \mathbf{\Theta} = \partial_3 \boldsymbol{g}_3 = \Gamma_{33}^p \boldsymbol{g}_p = \mathbf{0}$; cf. part (iii). Hence there exists a mapping $\boldsymbol{\theta}^1 \in \mathcal{C}^3(\omega; \mathbf{E}^3)$ such that

$$\mathbf{\Theta}(y, x_3) = \boldsymbol{\theta}(y) + x_3 \boldsymbol{\theta}^1(y) \text{ for all } (y, x_3) \in \Omega,$$

and consequently, $\boldsymbol{g}_\alpha = \partial_\alpha \boldsymbol{\theta} + x_3 \partial_\alpha \boldsymbol{\theta}^1$ and $\boldsymbol{g}_3 = \boldsymbol{\theta}^1$. The relations $g_{i3} = \boldsymbol{g}_i \cdot \boldsymbol{g}_3 = \delta_{i3}$ (cf. part (i)) then show that

$$(\partial_\alpha \boldsymbol{\theta} + x_3 \partial_\alpha \boldsymbol{\theta}^1) \cdot \boldsymbol{\theta}^1 = 0 \text{ and } \boldsymbol{\theta}^1 \cdot \boldsymbol{\theta}^1 = 1.$$

These relations imply that $\partial_\alpha \boldsymbol{\theta} \cdot \boldsymbol{\theta}^1 = 0$. Hence either $\boldsymbol{\theta}^1 = \boldsymbol{a}_3$ or $\boldsymbol{\theta}^1 = -\boldsymbol{a}_3$ in ω, where

$$\boldsymbol{a}_3 := \frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|}.$$

But $\boldsymbol{\theta}^1 = -\boldsymbol{a}_3$ is ruled out since

$$\{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}\} \cdot \boldsymbol{\theta}^1 = \det(g_{ij})|_{x_3=0} > 0.$$

Noting that

$$\partial_\alpha \boldsymbol{\theta} \cdot \boldsymbol{a}_3 = 0 \text{ implies } \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{a}_3 = -\partial_{\alpha\beta} \boldsymbol{\theta} \cdot \boldsymbol{a}_3,$$

we obtain, on the one hand,

$$\begin{aligned} g_{\alpha\beta} &= (\partial_\alpha \boldsymbol{\theta} + x_3 \partial_\alpha \boldsymbol{a}_3) \cdot (\partial_\beta \boldsymbol{\theta} + x_3 \partial_\beta \boldsymbol{a}_3) \\ &= \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} - 2x_3 \partial_{\alpha\beta} \boldsymbol{\theta} \cdot \boldsymbol{a}_3 + x_3^2 \partial_\alpha \boldsymbol{a}_3 \cdot \partial_\beta \boldsymbol{a}_3 \text{ in } \Omega. \end{aligned}$$

Since, on the other hand,

$$g_{\alpha\beta} = a_{\alpha\beta} - 2x_3 b_{\alpha\beta} + x_3^2 c_{\alpha\beta} \text{ in } \Omega$$

by part (i), we conclude that

$$a_{\alpha\beta} = \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} \text{ and } b_{\alpha\beta} = \partial_{\alpha\beta} \boldsymbol{\theta} \cdot \boldsymbol{a}_3 \text{ in } \omega,$$

as desired. This completes the proof. □

Remarks. (1) The functions $c_{\alpha\beta} = b_\alpha^\tau b_{\beta\tau} = \partial_\alpha \boldsymbol{a}_3 \cdot \partial_\beta \boldsymbol{a}_3$ introduced in part (i) are the covariant components of the *third fundamental form* of the surface $\boldsymbol{\theta}(\omega)$.

(2) The series expansion $g^{\alpha\beta} = \sum_{n\geq 0}(n+1)x_3^n a^{\alpha\sigma}(\mathbf{B}^n)_\sigma^\beta$ found in part (i) is known; cf., e.g., Naghdi [1972].

(3) The functions $b_\alpha^\tau|_\beta$ and $b_{\alpha\beta|\sigma}$ introduced in part (ii) will be identified later as *covariant derivatives* of the second fundamental form of the surface $\boldsymbol{\theta}(\omega)$; see Section 4.2.

(4) The *Gauß equations* are used only *once* in the above proof, for showing that $R_{1212} = 0$ in part (vii). □

The *regularity assumptions* made in Theorem 2.8-1 on the matrix fields $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$ can be significantly relaxed in several ways. First, Cristinel Mardare has shown by means of an *ad hoc*, but not trivial, modification of the proof given here, that the existence of an immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\omega; \mathbf{E}^3)$ still holds under the weaker (but certainly more natural, in view of the regularity of the resulting immersion $\boldsymbol{\theta}$) assumption that $(b_{\alpha\beta}) \in \mathcal{C}^1(\omega; \mathbb{S}^2)$, all other assumptions of Theorem 2.8-1 holding *verbatim*.

In fact, Hartman & Wintner [1950] had already shown the stronger result that the existence theorem still holds if $(a_{\alpha\beta}) \in \mathcal{C}^1(\omega; \mathbb{S}^2_>)$ and $(b_{\alpha\beta}) \in \mathcal{C}^0(\omega; \mathbb{S}^2)$, with a resulting mapping $\boldsymbol{\theta}$ in the space $\mathcal{C}^2(\omega; \mathbf{E}^3)$. Their result has been itself superseded by that of S. Mardare [2003b], who established that if $(a_{\alpha\beta}) \in W^{1,\infty}_{\text{loc}}(\omega; \mathbb{S}^2_>)$ and $(b_{\alpha\beta}) \in L^\infty_{\text{loc}}(\omega; \mathbb{S}^2)$ are two matrix fields that satisfy the Gauß and Codazzi-Mainardi equations in the sense of distributions, then there exists a mapping $\boldsymbol{\theta} \in W^{2,\infty}_{\text{loc}}(\omega; \mathbf{E}^3)$ such that $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$ are the fundamental forms of the surface $\boldsymbol{\theta}(\omega)$. As of June 2005, the last word in this direction seems to belong to S. Mardare [2005], who was able to further reduce these regularities, to those of the spaces $W^{1,p}_{\text{loc}}(\omega; \mathbb{S}^2_>)$ and $L^p_{\text{loc}}(\omega; \mathbb{S}^2)$ for any $p > 2$, with a resulting mapping $\boldsymbol{\theta}$ in the space $W^{2,p}_{\text{loc}}(\omega; \mathbf{E}^3)$.

2.9 UNIQUENESS UP TO PROPER ISOMETRIES OF SURFACES WITH THE SAME FUNDAMENTAL FORMS

In Section 2.8, we have established the *existence* of an immersion $\boldsymbol{\theta} : \omega \subset \mathbb{R}^2 \to \mathbf{E}^3$ giving rise to a surface $\boldsymbol{\theta}(\omega)$ with prescribed first and second fundamental forms, provided these forms satisfy *ad hoc* sufficient conditions. We now turn to the question of *uniqueness* of such immersions.

This is the object of the next theorem, which constitutes another *rigidity theorem*, called the **rigidity theorem for surfaces**. It asserts that, if two immersions $\widetilde{\boldsymbol{\theta}} \in \mathcal{C}^2(\omega; \mathbf{E}^3)$ and $\boldsymbol{\theta} \in \mathcal{C}^2(\omega; \mathbf{E}^3)$ share the same fundamental forms, then the surface $\boldsymbol{\theta}(\omega)$ is obtained by subjecting the surface $\widetilde{\boldsymbol{\theta}}(\omega)$ to a *rotation* (represented by an orthogonal matrix $\mathbf{Q}$ with $\det \mathbf{Q} = 1$), then by subjecting the rotated surface to a *translation* (represented by a vector $\boldsymbol{c}$).

Such a geometric transformation of the surface $\widetilde{\boldsymbol{\theta}}(\omega)$ is sometimes called a "*rigid transformation*", to remind that it corresponds to the idea of a "rigid" one in $\mathbf{E}^3$. This observation motivates the terminology "rigidity theorem".

As shown by Ciarlet & Larsonneur [2001] (whose proof is adapted here), the issue of uniqueness can be resolved as a corollary to its "three-dimensional counterpart", like the issue of existence. We recall that $\mathbb{O}^3$ denotes the set of all orthogonal matrices of order three and that $\mathbb{O}^3_+ = \{\mathbf{Q} \in \mathbb{O}^3; \det \mathbf{Q} = 1\}$ denotes the set of all proper orthogonal matrices of order three.

Theorem 2.9-1. *Let ω be a connected open subset of $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^2(\omega; \mathbf{E}^3)$ and $\widetilde{\boldsymbol{\theta}} \in \mathcal{C}^2(\omega; \mathbf{E}^3)$ be two immersions such that their associated first and second fundamental forms satisfy* (with self-explanatory notations)

$$a_{\alpha\beta} = \widetilde{a}_{\alpha\beta} \text{ and } b_{\alpha\beta} = \widetilde{b}_{\alpha\beta} \text{ in } \omega.$$

Then there exist a vector $\boldsymbol{c} \in \mathbf{E}^3$ and a matrix $\mathbf{Q} \in \mathbb{O}^3_+$ such that

$$\boldsymbol{\theta}(y) = \boldsymbol{c} + \mathbf{Q}\widetilde{\boldsymbol{\theta}}(y) \text{ for all } y \in \omega.$$

Proof. Arguments similar to those used in parts (i) and (viii) of the proof of Theorem 2.8-1 show that there exist open subsets ω_ℓ of ω and real numbers $\varepsilon_\ell > 0$, $\ell \geq 0$, such that the symmetric matrices (g_{ij}) defined by

$$g_{\alpha\beta} := a_{\alpha\beta} - 2x_3 b_{\alpha\beta} + x_3^2 c_{\alpha\beta} \text{ and } g_{i3} = \delta_{i3},$$

where $c_{\alpha\beta} := a^{\sigma\tau} b_{\alpha\sigma} b_{\beta\tau}$, are positive definite in the set

$$\Omega := \bigcup_{\ell \geq 0} \omega_\ell \times \left]-\varepsilon_\ell, \varepsilon_\ell\right[.$$

The two immersions $\boldsymbol{\Theta} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ and $\widetilde{\boldsymbol{\Theta}} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ defined by (with self-explanatory notations)

$$\boldsymbol{\Theta}(y, x_3) := \boldsymbol{\theta}(y) + x_3 \boldsymbol{a}_3(y) \text{ and } \widetilde{\boldsymbol{\Theta}}(y, x_3) := \widetilde{\boldsymbol{\theta}}(y) + x_3 \widetilde{\boldsymbol{a}}_3(y)$$

for all $(y, x_3) \in \Omega$ therefore satisfy

$$g_{ij} = \widetilde{g}_{ij} \text{ in } \Omega.$$

By Theorem 1.7-1, there exist a vector $\boldsymbol{c} \in \mathbf{E}^3$ and an orthogonal matrix $\mathbf{Q}$ such that

$$\boldsymbol{\Theta}(y, x_3) = \boldsymbol{c} + \mathbf{Q}\widetilde{\boldsymbol{\Theta}}(y, x_3) \text{ for all } (y, x_3) \in \Omega.$$

Hence, on the one hand,

$$\det \boldsymbol{\nabla}\boldsymbol{\Theta}(y, x_3) = \det \mathbf{Q} \det \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}(y, x_3) \text{ for all } (y, x_3) \in \Omega.$$

On the other hand, a simple computation shows that

$$\det \boldsymbol{\nabla}\boldsymbol{\Theta}(y, x_3) = \sqrt{\det(a_{\alpha\beta}(y)}\{1 - x_3(b_1^1 + b_2^2)(y) + x_3^2(b_1^1 b_2^2 - b_1^2 b_2^1)(y)\}$$

for all $(y, x_3) \in \Omega$, where

$$b_\alpha^\beta(y) := a^{\beta\sigma}(y) b_{\alpha\sigma}(y),\ y \in \omega,$$

so that

$$\det \boldsymbol{\nabla}\boldsymbol{\Theta}(y, x_3) = \det \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}(y, x_3) \text{ for all } (y, x_3) \in \Omega.$$

Therefore $\det \mathbf{Q} = 1$, which shows that the orthogonal matrix $\mathbf{Q}$ is in fact proper. The conclusion then follows by letting $x_3 = 0$ in the relation

$$\boldsymbol{\Theta}(y, x_3) = \boldsymbol{c} + \mathbf{Q}\widetilde{\boldsymbol{\Theta}}(y, x_3) \text{ for all } (y, x_3) \in \Omega.$$

□

A **proper isometry of $\mathbf{E}^3$** is a mapping $\mathbf{J}_+ : \mathbf{E}^3 \to \mathbf{E}^3$ of the form $\mathbf{J}_+(x) = \boldsymbol{c} + \mathbf{Q}\boldsymbol{ox}$ for all $x \in \mathbf{E}^3$, with $\boldsymbol{c} \in \mathbf{E}^3$ and $\mathbf{Q} \in \mathbb{O}_+^3$. Theorem 2.9-1 thus asserts that *two immersions $\boldsymbol{\theta} \in \mathcal{C}^1(\omega; \mathbf{E}^3)$ and $\widetilde{\boldsymbol{\theta}} \in \mathcal{C}^1(\omega; \mathbf{E}^3)$ share the same fundamental forms over an open connected subset ω of $\mathbb{R}^3$ if and only if $\boldsymbol{\theta} = \mathbf{J}_+ \circ \boldsymbol{\theta}$, where $\mathbf{J}_+$ is a proper isometry of $\mathbf{E}^3$.*

Remark. By contrast, the "three-dimensional" rigidity theorem (Theorem 1.7-1) involves isometries of $\mathbf{E}^3$ that may not be proper. □

Theorem 2.9-1 constitutes the "classical" rigidity theorem for surfaces, in the sense that both immersions $\boldsymbol{\theta}$ and $\widetilde{\boldsymbol{\theta}}$ are assumed to be in the space $\mathcal{C}^2(\omega; \mathbf{E}^3)$.

As a preparation to our next result, we note that the second fundamental form of the surface $\boldsymbol{\theta}(\omega)$ can still be defined under the weaker assumptions that $\boldsymbol{\theta} \in \mathcal{C}^1(\omega; \mathbf{E}^3)$ and $\boldsymbol{a}_3 = \dfrac{\boldsymbol{a}_1 \wedge \boldsymbol{a}_2}{|\boldsymbol{a}_1 \wedge \boldsymbol{a}_2|} \in \mathcal{C}^1(\omega; \mathbf{E}^3)$, by means of the definition

$$b_{\alpha\beta} := -\boldsymbol{a}_\alpha \cdot \partial_\beta \boldsymbol{a}_3,$$

which evidently coincides with the usual one when $\boldsymbol{\theta} \in \mathcal{C}^2(\omega; \mathbf{E}^3)$.

Following Ciarlet & C. Mardare [2004a], we now show that *a similar result holds under the assumptions that $\widetilde{\boldsymbol{\theta}} \in H^1(\omega; \mathbf{E}^3)$ and $\widetilde{\boldsymbol{a}}_3 := \dfrac{\widetilde{\boldsymbol{a}}_1 \wedge \widetilde{\boldsymbol{a}}_2}{|\widetilde{\boldsymbol{a}}_1 \wedge \widetilde{\boldsymbol{a}}_2|} \in H^1(\omega; \mathbf{E}^3)$* (with self-explanatory notations). Naturally, our first task will be to verify that the vector field $\widetilde{\boldsymbol{a}}_3$, which is not necessarily well defined a.e. in ω for an arbitrary mapping $\widetilde{\boldsymbol{\theta}} \in H^1(\omega; \mathbf{E}^3)$, is nevertheless well defined a.e. in ω for those mappings $\widetilde{\boldsymbol{\theta}}$ that satisfy the assumptions of the next theorem. This fact will in turn imply that the functions $\widetilde{b}_{\alpha\beta} := -\widetilde{\boldsymbol{a}}_\alpha \cdot \partial_\beta \widetilde{\boldsymbol{a}}_3$ are likewise well defined a.e. in ω.

Theorem 2.9-2. *Let ω be a connected open subset of $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^1(\omega; \mathbf{E}^3)$ be an immersion that satisfies $\boldsymbol{a}_3 \in \mathcal{C}^1(\omega; \mathbf{E}^3)$. Assume that there exists a vector field $\widetilde{\boldsymbol{\theta}} \in H^1(\omega; \mathbf{E}^3)$ that satisfies*

$$\widetilde{a}_{\alpha\beta} = a_{\alpha\beta} \text{ a.e. in } \omega, \quad \widetilde{\boldsymbol{a}}_3 \in H^1(\omega; \mathbf{E}^3), \quad \text{and} \quad \widetilde{b}_{\alpha\beta} = b_{\alpha\beta} \text{ a.e. in } \omega.$$

Then there exist a vector $\boldsymbol{c} \in \mathbf{E}^3$ and a matrix $\mathbf{Q} \in \mathbb{O}^3_+$ such that

$$\widetilde{\boldsymbol{\theta}}(y) = \boldsymbol{c} + \mathbf{Q}\boldsymbol{\theta}(y) \text{ for almost all } y \in \omega.$$

Proof. The proof essentially relies on the extension to a Sobolev space setting of the "three-dimensional" rigidity theorem established in Theorem 1.7-3.

(i) To begin with, we record several *technical preliminaries*.

First, we observe that the relations $\widetilde{a}_{\alpha\beta} = a_{\alpha\beta}$ a.e. in ω and the assumption that $\boldsymbol{\theta} \in \mathcal{C}^1(\omega; \mathbf{E}^3)$ is an immersion together imply that

$$|\widetilde{\boldsymbol{a}}_1 \wedge \widetilde{\boldsymbol{a}}_2| = \sqrt{\det(\widetilde{a}_{\alpha\beta})} = \sqrt{\det(a_{\alpha\beta})} > 0 \text{ a.e. in } \omega.$$

Consequently, *the vector field $\widetilde{\boldsymbol{a}}_3$, and thus the functions $\widetilde{b}_{\alpha\beta}$, are well defined a.e. in ω.*

Second, we establish that

$$b_{\alpha\beta} = b_{\beta\alpha} \text{ in } \omega \text{ and } \widetilde{b}_{\alpha\beta} = \widetilde{b}_{\beta\alpha} \text{ a.e. in } \omega,$$

i.e., that $\boldsymbol{a}_\alpha \cdot \partial_\beta \boldsymbol{a}_3 = \boldsymbol{a}_\beta \cdot \partial_\alpha \boldsymbol{a}_3$ in ω and $\widetilde{\boldsymbol{a}}_\alpha \cdot \partial_\beta \widetilde{\boldsymbol{a}}_3 = \widetilde{\boldsymbol{a}}_\beta \cdot \partial_\alpha \widetilde{\boldsymbol{a}}_3$ a.e. in ω. To this end, we note that either the assumptions $\boldsymbol{\theta} \in \mathcal{C}^1(\omega; \mathbf{E}^3)$ and $\boldsymbol{a}_3 \in \mathcal{C}^1(\omega; \mathbf{E}^3)$ together, or the assumptions $\boldsymbol{\theta} \in H^1(\omega; \mathbf{E}^3)$ and $\boldsymbol{a}_3 \in H^1(\omega; \mathbf{E}^3)$ together, imply that $\boldsymbol{a}_\alpha \cdot \partial_\beta \boldsymbol{a}_3 = \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{a}_3 \in L^1_{\text{loc}}(\omega)$, hence that $\partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{a}_3 \in \mathcal{D}'(\omega)$.

Given any $\varphi \in \mathcal{D}(\omega)$, let U denote an open subset of $\mathbb{R}^2$ such that $\operatorname{supp} \varphi \subset U$ and $\overline{U}$ is a compact subset of ω. Denoting by ${}_{X'}\langle \cdot, \cdot \rangle_X$ the duality pairing between a topological vector space X and its dual X', we have

$$\begin{aligned} {}_{\mathcal{D}'(\omega)}\langle \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{a}_3, \varphi \rangle_{\mathcal{D}(\omega)} &= \int_\omega \varphi \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{a}_3 \, \mathrm{d}y \\ &= \int_\omega \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta (\varphi \boldsymbol{a}_3) \, \mathrm{d}y - \int_\omega (\partial_\beta \varphi) \partial_\alpha \boldsymbol{\theta} \cdot \boldsymbol{a}_3 \, \mathrm{d}y. \end{aligned}$$

Observing that $\partial_\alpha \boldsymbol{\theta} \cdot \boldsymbol{a}_3 = 0$ a.e. in ω and that

$$\begin{aligned} -\int_\omega \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta (\varphi \boldsymbol{a}_3) \, \mathrm{d}y &= -\int_U \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta (\varphi \boldsymbol{a}_3) \, \mathrm{d}y \\ &= {}_{H^{-1}(U; \mathbb{E}^3)}\langle \partial_\beta (\partial_\alpha \boldsymbol{\theta}), \, \varphi \boldsymbol{a}_3 \rangle_{H^1_0(U; \mathbb{E}^3)}, \end{aligned}$$

we reach the conclusion that the expression ${}_{\mathcal{D}'(\omega)}\langle \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{a}_3,\, \varphi\rangle_{\mathcal{D}(\omega)}$ is symmetric with respect to α and β since $\partial_{\alpha\beta}\boldsymbol{\theta} = \partial_{\beta\alpha}\boldsymbol{\theta}$ in $\mathcal{D}'(U)$. Hence $\partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{a}_3 = \partial_\beta \boldsymbol{\theta} \cdot \partial_\alpha \boldsymbol{a}_3$ in $L^1_{\mathrm{loc}}(\omega)$, and the announced symmetries are established.

Third, let

$$\widetilde{c}_{\alpha\beta} := \partial_\alpha \widetilde{\boldsymbol{a}}_3 \cdot \partial_\beta \widetilde{\boldsymbol{a}}_3 \text{ and } c_{\alpha\beta} := \partial_\alpha \boldsymbol{a}_3 \cdot \partial_\beta \boldsymbol{a}_3.$$

Then we claim that $\widetilde{c}_{\alpha\beta} = c_{\alpha\beta}$ a.e. in ω. To see this, we note that the matrix fields $(\widetilde{a}^{\alpha\beta}) := (\widetilde{a}_{\alpha\beta})^{-1}$ and $(a^{\alpha\beta}) := (a_{\alpha\beta})^{-1}$ are well defined and equal a.e. in ω since $\boldsymbol{\theta}$ is an immersion and $\widetilde{a}_{\alpha\beta} = a_{\alpha\beta}$ a.e. in ω. The formula of Weingarten (Section 2.6) can thus be applied a.e. in ω, showing that $\widetilde{c}_{\alpha\beta} = \widetilde{a}^{\sigma\tau}\widetilde{b}_{\sigma\alpha}\widetilde{b}_{\tau\beta}$ a.e. in ω.

The assertion then follows from the assumptions $\widetilde{b}_{\alpha\beta} = b_{\alpha\beta}$ a.e. in ω.

(ii) *Starting from the set* ω *and the mapping* $\boldsymbol{\theta}$ (as given in the statement of Theorem 2.9-2), *we next construct a set* Ω *and a mapping* $\boldsymbol{\Theta}$ *that satisfy the assumptions of* Theorem 1.7-2. More precisely, let

$$\boldsymbol{\Theta}(y, x_3) := \boldsymbol{\theta}(y) + x_3 \boldsymbol{a}_3(y) \text{ for all } (y, x_3) \in \omega \times \mathbb{R}.$$

Then the mapping $\boldsymbol{\Theta} := \omega \times \mathbb{R} \to \mathbb{E}^3$ defined in this fashion is clearly continuously differentiable on $\omega \times \mathbb{R}$ and

$$\det \boldsymbol{\nabla\Theta}(y, x_3) = \sqrt{\det(a_{\alpha\beta}(y))}\{1 - x_3(b_1^1 + b_2^2)(y) + x_3^2(b_1^1 b_2^2 - b_1^2 b_2^1)(y)\}$$

for all $(y, x_3) \in \omega \times \mathbb{R}$, where

$$b_\alpha^\beta(y) := a^{\beta\sigma}(y) b_{\alpha\sigma}(y), y \in \omega.$$

Let ω_n, $n \geq 0$, be open subsets of $\mathbb{R}^2$ such that $\overline{\omega}_n$ is a compact subset of ω and $\omega = \bigcup_{n\geq 0} \omega_n$. Then the continuity of the functions $a_{\alpha\beta}, a^{\alpha\beta}, b_{\alpha\beta}$ and the assumption that $\boldsymbol{\theta}$ is an immersion together imply that, for each $n \geq 0$, there exists $\varepsilon_n > 0$ such that

$$\det \boldsymbol{\nabla\Theta}(y, x_3) > 0 \text{ for all } (y, x_3) \in \overline{\omega}_n \times [-\varepsilon_n, \varepsilon_n].$$

Besides, there is no loss of generality in assuming that $\varepsilon_n \leq 1$ (this property will be used in part (iii)).

Let then

$$\Omega := \bigcup_{n\geq 0} (\omega_n \times \,]-\varepsilon_n, \varepsilon_n[).$$

Then it is clear that Ω *is a connected open subset of* $\mathbb{R}^3$ and that *the mapping* $\boldsymbol{\Theta} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ *satisfies* $\det \boldsymbol{\nabla\Theta} > 0$ *in* Ω.

Finally, note that the covariant components $g_{ij} \in \mathcal{C}^0(\Omega)$ of the metric tensor field associated with the mapping $\boldsymbol{\Theta}$ are given by (the symmetries $b_{\alpha\beta} = b_{\beta\alpha}$ established in (i) are used here)

$$g_{\alpha\beta} = a_{\alpha\beta} - 2x_3 b_{\alpha\beta} + x_3^2 c_{\alpha\beta}, \quad g_{\alpha 3} = 0, \quad g_{33} = 1.$$

(iii) Starting with the mapping $\widetilde{\boldsymbol{\theta}}$ (as given in the statement of Theorem 2.9-2), we *construct a mapping* $\widetilde{\boldsymbol{\Theta}}$ *that satisfies the assumptions of* Theorem 1.7-2. To this end, we define a mapping $\widetilde{\boldsymbol{\Theta}} : \Omega \to \mathbb{E}^3$ by letting

$$\widetilde{\boldsymbol{\Theta}}(y, x_3) := \widetilde{\boldsymbol{\theta}}(y) + x_3 \widetilde{\boldsymbol{a}}_3(y) \text{ for all } (y, x_3) \in \Omega,$$

where the set Ω is defined as in (ii). Hence $\widetilde{\boldsymbol{\Theta}} \in H^1(\Omega; \mathbf{E}^3)$, since $\Omega \subset \omega \times]-1, 1[$. Besides, $\det \boldsymbol{\nabla} \widetilde{\boldsymbol{\Theta}} = \det \boldsymbol{\nabla} \boldsymbol{\Theta}$ a.e. in Ω since the functions $\widetilde{b}_\alpha^\beta := \widetilde{a}^{\beta\sigma} \widetilde{b}_{\alpha\sigma}$, which are well defined a.e. in ω, are equal, again a.e. in ω, to the functions b_α^β. Likewise, the components $\widetilde{g}_{ij} \in L^1(\Omega)$ of the metric tensor field associated with the mapping $\widetilde{\boldsymbol{\Theta}}$ satisfy $\widetilde{g}_{ij} = g_{ij}$ a.e. in Ω since $\widetilde{a}_{\alpha\beta} = a_{\alpha\beta}$ and $\widetilde{b}_{\alpha\beta} = b_{\alpha\beta}$ a.e. in ω by assumption and $\widetilde{c}_{\alpha\beta} = c_{\alpha\beta}$ a.e. in ω by part (i).

(iv) By Theorem 1.7-2, there exist a vector $\boldsymbol{c} \in \mathbf{E}^3$ and a matrix $\mathbf{Q} \in \mathbb{O}^3_+$ such that

$$\widetilde{\boldsymbol{\theta}}(y) + x_3 \widetilde{\boldsymbol{a}}_3(y) = \boldsymbol{c} + \mathbf{Q}(\boldsymbol{\theta}(y) + x_3 \boldsymbol{a}_3(y)) \text{ for almost all } (y, x_3) \in \Omega.$$

Differentiating with respect to x_3 in this equality between functions in $H^1(\Omega; \mathbf{E}^3)$ shows that $\widetilde{\boldsymbol{a}}_3(y) = \mathbf{Q}\boldsymbol{a}_3(y)$ for almost all $y \in \omega$. Hence $\widetilde{\boldsymbol{\theta}}(y) = \boldsymbol{c} + \mathbf{Q}\boldsymbol{\theta}(y)$ for almost all $y \in \omega$ as announced. □

Remarks. (1) The existence of $\widetilde{\boldsymbol{\theta}} \in H^1(\omega; \mathbf{E}^3)$ satisfying the assumptions of Theorem 2.9-2 implies that $\widetilde{\boldsymbol{\theta}} \in \mathcal{C}^1(\omega; \mathbf{E}^3)$ and $\widetilde{\boldsymbol{a}}_3 \in \mathcal{C}^1(\omega; \mathbf{E}^3)$, and that $\boldsymbol{\theta} \in H^1(\omega; \mathbf{E}^3)$ and $\boldsymbol{a}_3 \in H^1(\omega; \mathbf{E}^3)$.

(2) It is easily seen that the conclusion of Theorem 2.9-2 is still valid if the assumptions $\widetilde{\boldsymbol{\theta}} \in H^1(\omega; \mathbf{E}^3)$ and $\widetilde{\boldsymbol{a}}_3 \in H^1(\omega; \mathbf{E}^3)$ are replaced by the weaker assumptions $\widetilde{\boldsymbol{\theta}} \in H^1_{\text{loc}}(\omega; \mathbf{E}^3)$ and $\widetilde{\boldsymbol{a}}_3 \in H^1_{\text{loc}}(\omega; \mathbf{E}^3)$. □

2.10 CONTINUITY OF A SURFACE AS A FUNCTION OF ITS FUNDAMENTAL FORMS

Let ω be a connected and simply-connected open subset of $\mathbb{R}^2$. Together, Theorems 2.8-1 and 2.9-1 establish the existence of a mapping F that associates to any pair of matrix fields $(a_{\alpha\beta}) \in \mathcal{C}^2(\omega; \mathbb{S}^2_>)$ and $(b_{\alpha\beta}) \in \mathcal{C}^2(\omega; \mathbb{S}^2)$ satisfying the Gauß and Codazzi-Mainardi equations in ω a well-defined element $F((a_{\alpha\beta}), (b_{\alpha\beta}))$ in the quotient set $\mathcal{C}^3(\omega; \mathbf{E}^3)/R$, where $(\boldsymbol{\theta}, \widetilde{\boldsymbol{\theta}}) \in R$ means that there exists a vector $\boldsymbol{c} \in \mathbf{E}^3$ and a matrix $\mathbf{Q} \in \mathbb{O}^3_+$ such that $\boldsymbol{\theta}(y) = \boldsymbol{c} + \mathbf{Q}\widetilde{\boldsymbol{\theta}}(y)$ for all $y \in \omega$.

A natural question thus arises as to whether there exist *ad hoc* topologies on the space $\mathcal{C}^2(\omega; \mathbb{S}^2) \times \mathcal{C}^2(\omega; \mathbb{S}^2)$ and on the quotient set $\mathcal{C}^3(\omega; \mathbf{E}^3)/R$ such that the mapping F defined in this fashion is *continuous*.

Equivalently, *is a surface a continuous function of its fundamental forms?*

The purpose of this section, which is based on Ciarlet [2003], is to provide an affirmative answer to the above question, through a proof that relies in an essential way on the solution to the *analogous problem in dimension three* given in Section 1.8.

Such a question is not only relevant to surface theory, but it also finds its source in two-dimensional *nonlinear shell theories*, where the *stored energy functions* are often functions of the *first and second fundamental forms of the unknown deformed middle surface*. For instance, the well-known stored energy function w_K proposed by Koiter [1966, Equations (4.2), (8.1), and (8.3)] for modeling nonlinearly elastic shells made with a homogeneous and isotropic elastic material takes the form:

$$w_K = \frac{\varepsilon}{2} a^{\alpha\beta\sigma\tau}(\widetilde{a}_{\sigma\tau} - a_{\sigma\tau})(\widetilde{a}_{\alpha\beta} - a_{\alpha\beta}) + \frac{\varepsilon^3}{6} a^{\alpha\beta\sigma\tau}(\widetilde{b}_{\sigma\tau} - b_{\sigma\tau})(\widetilde{b}_{\alpha\beta} - b_{\alpha\beta}),$$

where 2ε is the thickness of the shell,

$$a^{\alpha\beta\sigma\tau} := \frac{4\lambda\mu}{\lambda + 2\mu} a^{\alpha\beta} a^{\sigma\tau} + 2\mu(a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma}),$$

$\lambda > 0$ and $\mu > 0$ are the two *Lamé constants* of the constituting material, $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are the covariant components of the first and second fundamental forms of the given undeformed middle surface, $(a^{\alpha\beta}) = (a_{\alpha\beta})^{-1}$, and finally $\widetilde{a}_{\alpha\beta}$ and $\widetilde{b}_{\alpha\beta}$ are the covariant components of the first and second fundamental forms of the *unknown deformed middle surface* (see Section 4.1 for a more detailed description of Koiter's equations for a nonlinearly elastic shell).

An inspection of the above stored energy functions thus suggests a tempting approach to shell theory, where the functions $\widetilde{a}_{\alpha\beta}$ *and* $\widetilde{b}_{\alpha\beta}$ *would be regarded as the primary unknowns* in lieu of the customary (Cartesian or curvilinear) components of the displacement. In such an approach, the unknown components $\widetilde{a}_{\alpha\beta}$ and $\widetilde{b}_{\alpha\beta}$ must naturally satisfy the classical *Gauß* and *Codazzi-Mainardi equations* in order that they actually define a surface.

To begin with, we introduce the following two-dimensional analogs to the notations used in Section 1.8. Let ω be an open subset of $\mathbb{R}^3$. The notation $\kappa \Subset \omega$ means that κ is a compact subset of ω. If $f \in \mathcal{C}^\ell(\omega;\mathbb{R})$ or $\boldsymbol{\theta} \in \mathcal{C}^\ell(\omega;\mathbf{E}^3)$, $\ell \geq 0$, and $\kappa \Subset \omega$, we let

$$\|f\|_{\ell,\kappa} := \sup_{\substack{y\in\kappa \\ |\alpha|\leq\ell}} |\partial^\alpha f(y)| \quad , \quad \|\boldsymbol{\theta}\|_{\ell,\kappa} := \sup_{\substack{y\in\kappa \\ |\alpha|\leq\ell}} |\partial^\alpha \boldsymbol{\theta}(y)|,$$

where ∂^α stands for the standard multi-index notation for partial derivatives and $|\cdot|$ denotes the Euclidean norm in the latter definition. If $\mathbf{A} \in \mathcal{C}^\ell(\omega;\mathbb{M}^3)$, $\ell \geq 0$, and $\kappa \Subset \omega$, we likewise let

$$\|\mathbf{A}\|_{\ell,\kappa} = \sup_{\substack{y\in\kappa \\ |\alpha|\leq\ell}} |\partial^\alpha \mathbf{A}(y)|,$$

where $|\cdot|$ denotes the matrix spectral norm.

The next *sequential continuity* result constitutes the key step towards establishing the continuity of a surface as a function of its two fundamental forms in *ad hoc* metric spaces (see Theorem 2.10-2).

Theorem 2.10-1. *Let ω be a connected and simply-connected open subset of $\mathbb{R}^2$. Let $(a_{\alpha\beta}) \in \mathcal{C}^2(\omega; \mathbb{S}^2_>)$ and $(b_{\alpha\beta}) \in \mathcal{C}^2(\omega; \mathbb{S}^2)$ be matrix fields satisfying the Gauß and Codazzi-Mainardi equations in ω and let $(a^n_{\alpha\beta}) \in \mathcal{C}^2(\omega; \mathbb{S}^2_>)$ and $(b^n_{\alpha\beta}) \in \mathcal{C}^2(\omega; \mathbb{S}^2)$ be matrix fields satisfying for each $n \geq 0$ the Gauß and Codazzi-Mainardi equations in ω. Assume that these matrix fields satisfy*

$$\lim_{n\to\infty} \|a^n_{\alpha\beta} - a_{\alpha\beta}\|_{2,\kappa} = 0 \text{ and } \lim_{n\to\infty} \|b^n_{\alpha\beta} - b_{\alpha\beta}\|_{2,\kappa} = 0 \text{ for all } \kappa \Subset \omega.$$

Let $\boldsymbol{\theta} \in \mathcal{C}^3(\omega; \mathbf{E}^3)$ be any immersion that satisfies

$$a_{\alpha\beta} = \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} \text{ and } b_{\alpha\beta} = \partial_{\alpha\beta} \boldsymbol{\theta} \cdot \left\{ \frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|} \right\} \text{ in } \omega$$

(such immersions exist by Theorem 2.8-1). *Then there exist immersions $\boldsymbol{\theta}^n \in \mathcal{C}^3(\omega; \mathbf{E}^3)$ satisfying*

$$a^n_{\alpha\beta} = \partial_\alpha \boldsymbol{\theta}^n \cdot \partial_\beta \boldsymbol{\theta}^n \text{ and } b^n_{\alpha\beta} = \partial_{\alpha\beta} \boldsymbol{\theta}^n \cdot \left\{ \frac{\partial_1 \boldsymbol{\theta}^n \wedge \partial_2 \boldsymbol{\theta}^n}{|\partial_1 \boldsymbol{\theta}^n \wedge \partial_2 \boldsymbol{\theta}^n|} \right\} \text{ in } \omega,\ n \geq 0,$$

such that

$$\lim_{n\to\infty} \|\boldsymbol{\theta}^n - \boldsymbol{\theta}\|_{3,\kappa} = 0 \text{ for all } \kappa \Subset \omega.$$

Proof. For clarity, the proof is broken into five parts.

(i) *Let the matrix fields $(g_{ij}) \in \mathcal{C}^2(\omega \times \mathbb{R}; \mathbb{S}^3)$ and $(g^n_{ij}) \in \mathcal{C}^2(\omega \times \mathbb{R}; \mathbb{S}^3)$, $n \geq 0$, be defined by*

$$\begin{aligned} g_{\alpha\beta} &:= a_{\alpha\beta} - 2x_3 b_{\alpha\beta} + x_3^2 c_{\alpha\beta} \quad \text{and} \quad g_{i3} := \delta_{i3}, \\ g^n_{\alpha\beta} &:= a^n_{\alpha\beta} - 2x_3 b^n_{\alpha\beta} + x_3^2 c^n_{\alpha\beta} \quad \text{and} \quad g^n_{i3} := \delta_{i3}, n \geq 0 \end{aligned}$$

(the variable $y \in \omega$ is omitted, x_3 designates the variable in $\mathbb{R}$), *where*

$$\begin{aligned} c_{\alpha\beta} := b^\tau_\alpha b_{\beta\tau}, \quad b^\tau_\alpha := a^{\sigma\tau} b_{\alpha\sigma}, \quad (a^{\sigma\tau}) := (a_{\alpha\beta})^{-1}, \\ c^n_{\alpha\beta} := b^{\tau,n}_\alpha b^n_{\beta\tau}, \quad b^{\tau,n}_\alpha := a^{\sigma\tau,n} b^n_{\alpha\sigma}, \quad (a^{\sigma\tau,n}) := (a^n_{\alpha\beta})^{-1}, n \geq 0. \end{aligned}$$

Let ω_0 be an open subset of $\mathbb{R}^2$ such that $\overline{\omega}_0 \Subset \omega$. Then there exists $\varepsilon_0 = \varepsilon_0(\omega_0) > 0$ such that the symmetric matrices

$$\mathbf{C}(y, x_3) := (g_{ij}(y, x_3)) \text{ and } \mathbf{C}^n(y, x_3) := (g^n_{ij}(y, x_3)),\ n \geq 0,$$

are positive definite at all points $(y, x_3) \in \overline{\Omega}_0$, where

$$\Omega_0 := \omega_0 \times \left]-\varepsilon_0, \varepsilon_0\right[.$$

The matrices $\mathbf{C}(y, x_3) \in \mathbb{S}^3$ and $\mathbf{C}^n(y, x_3) \in \mathbb{S}^3$ are of the form (the notations are self-explanatory):

$$\begin{aligned} \mathbf{C}(y, x_3) &= \mathbf{C}_0(y) + x_3 \mathbf{C}_1(y) + x_3^2 \mathbf{C}_2(y), \\ \mathbf{C}^n(y, x_3) &= \mathbf{C}^n_0(y) + x_3 \mathbf{C}^n_1(y) + x_3^2 \mathbf{C}^n_2(y),\ n \geq 0. \end{aligned}$$

First, it is easily deduced from the matrix identity $\mathbf{B} = \mathbf{A}(\mathbf{I} + \mathbf{A}^{-1}(\mathbf{B}-\mathbf{A}))$ and the assumptions $\lim_{n\to\infty} \|a^n_{\alpha\beta} - a_{\alpha\beta}\|_{0,\overline{\omega}_0} = 0$ and $\lim_{n\to\infty} \|b^n_{\alpha\beta} - b_{\alpha\beta}\|_{0,\overline{\omega}_0} = 0$ that there exists a constant M such that

$$\|(\mathbf{C}^n_0)^{-1}\|_{0,\overline{\omega}_0} + \|\mathbf{C}^n_1\|_{0,\overline{\omega}_0} + \|\mathbf{C}^n_2\|_{0,\overline{\omega}_0} \le M \text{ for all } n \ge 0.$$

This uniform bound and the relations

$$\begin{aligned}\mathbf{C}(y, x_3) &= \mathbf{C}_0(y)\{\mathbf{I} + (\mathbf{C}_0(y))^{-1}(-2x_3\mathbf{C}_1(y) + x_3^2\mathbf{C}_2(y))\},\\ \mathbf{C}^n(y, x_3) &= \mathbf{C}^n_0(y)\{\mathbf{I} + (\mathbf{C}^n_0(y))^{-1}(-2x_3\mathbf{C}^n_1(y) + x_3^2\mathbf{C}^n_2(y))\},\ n \ge 0,\end{aligned}$$

together imply that there exists $\varepsilon_0 = \varepsilon_0(\omega_0) > 0$ such that the matrices $\mathbf{C}(y, x_3)$ and $\mathbf{C}^n(y, x_3)$, $n \ge 0$, are invertible for all $(y, x_3) \in \overline{\omega}_0 \times [-\varepsilon_0, \varepsilon_0]$.

These matrices are positive definite for $x_3 = 0$ by assumption. Hence they remain so for all $x_3 \in [-\varepsilon_0, \varepsilon_0]$ since they are invertible.

(ii) *Let ω_ℓ, $\ell \ge 0$, be open subsets of $\mathbb{R}^2$ such that $\overline{\omega}_\ell \Subset \omega$ for each ℓ and $\omega = \bigcup_{\ell\ge 0}\omega_\ell$. By* (i), *there exist numbers $\varepsilon_\ell = \varepsilon_\ell(\omega_\ell) > 0$, $\ell \ge 0$, such that the symmetric matrices $\mathbf{C}(x) = (g_{ij}(x))$ and $\mathbf{C}^n(x) = (g^n_{ij}(x))$, $n \ge 0$, defined for all $x = (y, x_3) \in \omega \times \mathbb{R}$ as in* (i), *are positive definite at all points $x = (y, x_3) \in \overline{\Omega}_\ell$, where $\Omega_\ell := \omega_\ell \times]-\varepsilon_\ell, \varepsilon_\ell[$, hence at all points $x = (y, x_3)$ of the open set*

$$\Omega := \bigcup_{\ell\ge 0} \Omega_\ell,$$

which is connected and simply connected. Let the functions $R_{qijk} \in \mathcal{C}^0(\Omega)$ be defined from the matrix fields $(g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ by

$$R_{qijk} := \partial_j\Gamma_{ikq} - \partial_k\Gamma_{ijq} + \Gamma^p_{ij}\Gamma_{kqp} - \Gamma^p_{ik}\Gamma_{jqp}$$

where

$$\Gamma_{ijq} := \frac{1}{2}(\partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij}) \text{ and } \Gamma^p_{ij} := g^{pq}\Gamma_{ijq}, \text{ with } (g^{pq}) := (g_{ij})^{-1},$$

and let the functions $R^n_{qijk} \in \mathcal{C}^0(\Omega)$, $n \ge 0$ be similarly defined from the matrix fields $(g^n_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$, $n \ge 0$. Then

$$R_{qijk} = 0 \text{ in } \Omega \text{ and } R^n_{qijk} = 0 \text{ in } \Omega \text{ for all } n \ge 0.$$

That Ω is connected and simply-connected is established in part (viii) of the proof of Theorem 2.8-1. That $R_{qijk} = 0$ in Ω, and similarly that $R^n_{qijk} = 0$ in Ω for all $n \ge 0$, is established as in parts (iv) to (viii) of the same proof.

(iii) *The matrix fields $\mathbf{C} = (g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ and $\mathbf{C}^n = (g^n_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ defined in* (ii) *satisfy* (the notations used here are those of Section 1.8)

$$\lim_{n\to\infty} \|\mathbf{C}^n - \mathbf{C}\|_{2,K} = 0 \text{ for all } K \Subset \Omega.$$

Given any compact subset K of Ω, there exists a finite set Λ_K of integers such that $K \subset \bigcup_{\ell\in\Lambda_K} \Omega_\ell$. Since by assumption,

$$\lim_{n\to\infty} \|a^n_{\alpha\beta}-a_{\alpha\beta}\|_{2,\overline{\omega}_\ell} = 0 \text{ and } \lim_{n\to\infty} \|b^n_{\alpha\beta}-b_{\alpha\beta}\|_{2,\overline{\omega}_\ell} = 0,\ \ell \in \Lambda_K,$$

it follows that

$$\lim_{n\to\infty} \|\mathbf{C}^n_p-\mathbf{C}_p\|_{2,\overline{\omega}_\ell} = 0,\ \ell \in \Lambda_k,\ p = 0, 1, 2,$$

where the matrices $\mathbf{C}_p$ and $\mathbf{C}^n_p$, $n \geq 0$, $p = 0, 1, 2$, are those defined in the proof of part (i). The definition of the norm $\|\cdot\|_{2,\overline{\Omega}_\ell}$ then implies that

$$\lim_{n\to\infty} \|\mathbf{C}^n-\mathbf{C}\|_{2,\overline{\Omega}_\ell} = 0,\ \ell \in \Lambda_K.$$

The conclusion then follows from the finiteness of the set Λ_K.

(iv) *Conclusion.*

Given any mapping $\boldsymbol{\theta} \in \mathcal{C}^3(\omega; \mathbf{E}^3)$ that satisfies

$$a_{\alpha\beta} = \partial_\alpha\boldsymbol{\theta} \cdot \partial_\beta\boldsymbol{\theta} \text{ and } b_{\alpha\beta} = \partial_{\alpha\beta}\boldsymbol{\theta} \cdot \left\{ \frac{\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}}{|\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}|} \right\} \text{ in } \omega,$$

let the mapping $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$ be defined by

$$\boldsymbol{\Theta}(y, x_3) := \boldsymbol{\theta}(y) + x_3\boldsymbol{a}_3(y) \text{ for all } (y, x_3) \in \Omega,$$

where $\boldsymbol{a}_3 := \dfrac{\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}}{|\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}|}$, and let

$$g_{ij} := \partial_i\boldsymbol{\Theta} \cdot \partial_j\boldsymbol{\Theta}.$$

Then an immediate computation shows that

$$g_{\alpha\beta} = a_{\alpha\beta} - 2x_3 b_{\alpha\beta} + x_3^2 c_{\alpha\beta} \text{ and } g_{i3} = \delta_{i3} \text{ in } \Omega,$$

where $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are the covariant components of the first and second fundamental forms of the surface $\boldsymbol{\theta}(\omega)$ and $c_{\alpha\beta} = a^{\sigma\tau} b_{\alpha\sigma} b_{\beta\tau}$.

In other words, *the matrices* (g_{ij}) *constructed in this fashion coincide over the set* Ω *with those defined in* part (i). Since parts (ii) and (iii) of the above proof together show that all the assumptions of Theorem 1.8-3 are satisfied by the fields $\mathbf{C} = (g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ and $\mathbf{C}^n = (g^n_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$, there exist mappings $\boldsymbol{\Theta}^n \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ satisfying $(\boldsymbol{\nabla}\boldsymbol{\Theta}^n)^T\boldsymbol{\nabla}\boldsymbol{\Theta}^n = \mathbf{C}^n$ in Ω, $n \geq 0$, such that

$$\lim_{n\to\infty} \|\boldsymbol{\Theta}^n-\boldsymbol{\Theta}\|_{3,K} = 0 \text{ for all } K \Subset \Omega.$$

We now show that the mappings

$$\boldsymbol{\theta}^n(\cdot) := \boldsymbol{\Theta}^n(\cdot, 0) \in \mathcal{C}^3(\omega; \mathbf{E}^3)$$

indeed satisfy

$$a^n_{\alpha\beta} = \partial_\alpha \boldsymbol{\theta}^n \cdot \partial_\beta \boldsymbol{\theta}^n \text{ and } b^n_{\alpha\beta} = \partial_{\alpha\beta} \boldsymbol{\theta}^n \cdot \left\{ \frac{\partial_1 \boldsymbol{\theta}^n \wedge \partial_2 \boldsymbol{\theta}^n}{|\partial_1 \boldsymbol{\theta}^n \wedge \partial_2 \boldsymbol{\theta}^n|} \right\} \text{ in } \omega.$$

Dropping the exponent n for notational convenience in this part of the proof, let $\boldsymbol{g}_i := \partial_i \boldsymbol{\Theta}$. Then $\partial_{33}\boldsymbol{\Theta} = \partial_3 \boldsymbol{g}_3 = \Gamma^p_{33} \boldsymbol{g}_p = \boldsymbol{0}$, since it is easily verified that the functions Γ^p_{33}, constructed from the functions g_{ij} as indicated in part (ii), vanish in Ω. Hence there exists a mapping $\boldsymbol{\theta}^1 \in \mathcal{C}^3(\omega; \mathbf{E}^3)$ such that

$$\boldsymbol{\Theta}(y, x_3) = \boldsymbol{\theta}(y) + x_3 \boldsymbol{\theta}^1(y) \text{ for all } (y, x_3) \in \Omega.$$

Consequently, $\boldsymbol{g}_\alpha = \partial_\alpha \boldsymbol{\theta} + x_3 \partial_\alpha \boldsymbol{\theta}^1$ and $\boldsymbol{g}_3 = \boldsymbol{\theta}^1$. The relations $g_{i3} = \boldsymbol{g}_i \cdot \boldsymbol{g}_3 = \delta_{i3}$ then show that

$$(\partial_\alpha \boldsymbol{\theta} + x_3 \partial_\alpha \boldsymbol{\theta}^1) \cdot \boldsymbol{\theta}^1 = 0 \text{ and } \boldsymbol{\theta}^1 \cdot \boldsymbol{\theta}^1 = 1.$$

These relations imply that $\partial_\alpha \boldsymbol{\theta} \cdot \boldsymbol{\theta}^1 = 0$. Hence either $\boldsymbol{\theta}^1 = \boldsymbol{a}_3$ or $\boldsymbol{\theta}^1 = -\boldsymbol{a}_3$ in ω. But $\boldsymbol{\theta}^1 = -\boldsymbol{a}_3$ is ruled out since we must have

$$\{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}\} \cdot \boldsymbol{\theta}^1 = \det(g_{ij})|_{x_3=0} > 0.$$

Noting that

$$\partial_\alpha \boldsymbol{\theta} \cdot \boldsymbol{a}_3 = 0 \text{ implies } \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{a}_3 = -\partial_{\alpha\beta} \boldsymbol{\theta} \cdot \boldsymbol{a}_3,$$

we obtain, on the one hand,

$$\begin{aligned} g_{\alpha\beta} &= (\partial_\alpha \boldsymbol{\theta} + x_3 \partial_\alpha \boldsymbol{a}_3) \cdot (\partial_\beta \boldsymbol{\theta} + x_3 \partial_\beta \boldsymbol{a}_3) \\ &= \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} - 2x_3 \partial_{\alpha\beta} \boldsymbol{\theta} \cdot \boldsymbol{a}_3 + x_3^2 \partial_\alpha \boldsymbol{a}_3 \cdot \partial_\beta \boldsymbol{a}_3 \text{ in } \Omega. \end{aligned}$$

Since, on the other hand,

$$g_{\alpha\beta} = a_{\alpha\beta} - 2x_3 b_{\alpha\beta} + x_3^2 c_{\alpha\beta} \text{ in } \Omega,$$

we conclude that

$$a_{\alpha\beta} = \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} \text{ and } b_{\alpha\beta} = \partial_{\alpha\beta} \boldsymbol{\theta} \cdot \boldsymbol{a}_3 \text{ in } \omega,$$

as desired.

It remains to verify that

$$\lim_{n \to \infty} \|\boldsymbol{\theta}^n - \boldsymbol{\theta}\|_{3,\kappa} = 0 \text{ for all } \kappa \Subset \omega.$$

But these relations immediately follow from the relations

$$\lim_{n \to \infty} \|\boldsymbol{\Theta}^n - \boldsymbol{\Theta}\|_{3,K} = 0 \text{ for all } K \Subset \Omega,$$

combined with the observations that a compact subset of ω is also one of Ω, that $\boldsymbol{\Theta}(\cdot, 0) = \boldsymbol{\theta}$ and $\boldsymbol{\Theta}^n(\cdot, 0) = \boldsymbol{\theta}^n$, and finally, that

$$\|\boldsymbol{\theta}^n - \boldsymbol{\theta}\|_{3,\kappa} \leq \|\boldsymbol{\Theta}^n - \boldsymbol{\Theta}\|_{3,\kappa}.$$

□

Remark. At first glance, it seems that Theorem 2.10-1 could be established by a proof similar to that of its "three-dimensional counterpart", viz., Theorem 1.8-3. A quick inspection reveals, however, that the proof of Theorem 1.8-2 does not carry over to the present situation. □

In fact, it is not necessary to assume in Theorem 2.10-1 that the "limit" matrix fields $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$ satisfy the Gauß and Codazzi-Mainardi equations (see the proof of the next theorem). More specifically, another *sequential continuity* result can be derived from Theorem 2.10-1. Its interest is that the assumptions are now made on the immersions $\boldsymbol{\theta}^n$ that define the surfaces $\boldsymbol{\theta}^n(\omega)$ for all $n \geq 0$; besides the existence of a "limit" surface $\boldsymbol{\theta}(\omega)$ is also established.

Theorem 2.10-2. *Let ω be a connected and simply-connected open subset of $\mathbb{R}^2$. For each $n \geq 0$, let there be given immersions $\boldsymbol{\theta}^n \in \mathcal{C}^3(\omega; \mathbf{E}^3)$, let $a^n_{\alpha\beta}$ and $b^n_{\alpha\beta}$ denote the covariant components of the first and second fundamental forms of the surface $\boldsymbol{\theta}^n(\omega)$, and assume that $b^n_{\alpha\beta} \in \mathcal{C}^2(\omega)$. Let there be also given matrix fields $(a_{\alpha\beta}) \in \mathcal{C}^2(\omega; \mathbb{S}^2_>)$ and $(b_{\alpha\beta}) \in \mathcal{C}^2(\omega; \mathbb{S}^2)$ with the property that*

$$\lim_{n\to\infty} \|a^n_{\alpha\beta} - a_{\alpha\beta}\|_{2,\kappa} = 0 \text{ and } \lim_{n\to\infty} \|b^n_{\alpha\beta} - b_{\alpha\beta}\|_{2,\kappa} = 0 \text{ for all } \kappa \Subset \omega.$$

Then there exist immersions $\widetilde{\boldsymbol{\theta}}^n \in \mathcal{C}^3(\omega; \mathbf{E}^3)$ of the form

$$\widetilde{\boldsymbol{\theta}}^n = \boldsymbol{c}^n + \mathbf{Q}^n \boldsymbol{\theta}^n, \text{ with } \boldsymbol{c}^n \in \mathbf{E}^3 \text{ and } \mathbf{Q}^n \in \mathbb{O}^3_+$$

(hence the first and second fundamental forms of the surfaces $\widetilde{\boldsymbol{\theta}}^n(\omega)$ and $\boldsymbol{\theta}^n(\omega)$ are the same for all $n \geq 0$), *and there exists an immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\omega, \mathbf{E}^3)$ such that $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are the covariant components of the first and second fundamental forms of the surface $\boldsymbol{\theta}(\omega)$. Besides,*

$$\lim_{n\to\infty} \|\widetilde{\boldsymbol{\theta}}^n - \boldsymbol{\theta}\|_{3,\kappa} = 0 \text{ for all } \kappa \Subset \omega.$$

Proof. An argument similar to that used in the proof of Theorem 1.8-4 shows that passing to the limit as $n \to \infty$ is allowed in the Gauß and Codazzi-Mainardi equations, which are satisfied in the spaces $\mathcal{C}^0(\omega)$ and $\mathcal{C}^1(\omega)$ respectively by the functions $a^n_{\alpha\beta}$ and $b^n_{\alpha\beta}$ for each $n \geq 0$ (as necessary conditions; cf. Theorem 2.7-1). Hence the limit functions $a_{\alpha\beta}$ and $b_{\alpha\beta}$ also satisfy the Gauß and Codazzi-Mainardi equations.

By the fundamental existence theorem (Theorem 2.8-1), there thus exists an immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\omega; \mathbf{E}^3)$ such that

$$a_{\alpha\beta} = \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} \text{ and } b_{\alpha\beta} = \partial_{\alpha\beta} \boldsymbol{\theta} \cdot \Big\{ \frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|} \Big\}.$$

Theorem 2.10-1 can now be applied, showing that there exist mappings (now denoted) $\widetilde{\boldsymbol{\theta}}^n \in \mathcal{C}^3(\omega; \mathbf{E}^3)$ such that

$$a^n_{\alpha\beta} = \partial_\alpha \widetilde{\boldsymbol{\theta}}^n \cdot \partial_\beta \widetilde{\boldsymbol{\theta}}^n \text{ and } b^n_{\alpha\beta} = \partial_{\alpha\beta} \widetilde{\boldsymbol{\theta}}^n \cdot \Big\{ \frac{\partial_1 \widetilde{\boldsymbol{\theta}}^n \wedge \partial_2 \widetilde{\boldsymbol{\theta}}^n}{|\partial_1 \widetilde{\boldsymbol{\theta}}^n \wedge \partial_2 \widetilde{\boldsymbol{\theta}}^n|} \Big\} \text{ in } \omega,\ n \geq 0,$$

and

$$\lim_{n\to\infty} \|\widetilde{\boldsymbol{\theta}}^n - \boldsymbol{\theta}\|_{3,\kappa} = 0 \text{ for all } \kappa \Subset \omega.$$

Finally, the rigidity theorem for surfaces (Theorem 2.9-1) shows that, for each $n \geq 0$, there exist $\boldsymbol{c}^n \in \mathbf{E}^3$ and $\mathbf{Q}^n \in \mathbb{O}^3_+$ such that

$$\widetilde{\boldsymbol{\theta}}^n = \boldsymbol{c}^n + \mathbf{Q}^n \boldsymbol{\theta}^n \text{ in } \omega,$$

since the surfaces $\widetilde{\boldsymbol{\theta}}^n(\omega)$ and $\boldsymbol{\theta}^n(\omega)$ share the same fundamental forms and the set ω is connected. $\square$

It remains to show how the *sequential continuity* established in Theorem 2.10-1 implies the *continuity of a surface as a function of its fundamental forms* for *ad hoc* topologies.

Let ω be an open subset of $\mathbb{R}^2$. We recall (see Section 1.8) that, for any integers $\ell \geq 0$ and $d \geq 1$, the space $\mathcal{C}^\ell(\omega; \mathbb{R}^d)$ becomes a *locally convex topological space* when it is equipped with the *Fréchet topology* defined by the family of semi-norms $\|\cdot\|_{\ell,\kappa}$, $\kappa \Subset \omega$. Then a sequence $(\boldsymbol{\theta}^n)_{n\geq 0}$ converges to $\boldsymbol{\theta}$ with respect to this topology if and only if

$$\lim_{n\to\infty} \|\boldsymbol{\theta}^n - \boldsymbol{\theta}\|_{\ell,\kappa} = 0 \text{ for all } \kappa \Subset \omega.$$

Furthermore, this topology is *metrizable*: Let $(\kappa_i)_{i\geq 0}$ be any sequence of subsets of ω that satisfy

$$\kappa_i \Subset \omega \text{ and } \kappa_i \subset \operatorname{int} \kappa_{i+1} \text{ for all } i \geq 0, \text{ and } \omega = \bigcup_{i=0}^{\infty} \kappa_i.$$

Then

$$\lim_{n\to\infty} \|\boldsymbol{\theta}^n - \boldsymbol{\theta}\|_{\ell,\kappa} = 0 \text{ for all } \kappa \Subset \omega \Longleftrightarrow \lim_{n\to\infty} d_\ell(\boldsymbol{\theta}^n, \boldsymbol{\theta}) = 0,$$

where

$$d_\ell(\boldsymbol{\psi}, \boldsymbol{\theta}) := \sum_{i=0}^{\infty} \frac{1}{2^i} \frac{\|\boldsymbol{\psi} - \boldsymbol{\theta}\|_{\ell,\kappa_i}}{1 + \|\boldsymbol{\psi} - \boldsymbol{\theta}\|_{\ell,\kappa_i}}.$$

Let $\dot{\mathcal{C}}^3(\omega; \mathbf{E}^3) := \mathcal{C}^3(\omega; \mathbf{E}^3)/R$ denote the quotient set of $\mathcal{C}^3(\omega; \mathbf{E}^3)$ by the equivalence relation R, where $(\boldsymbol{\theta}, \widetilde{\boldsymbol{\theta}}) \in R$ means that there exist a vector $\boldsymbol{c} \in \mathbf{E}^3$ and a matrix $\mathbf{Q} \in \mathbb{O}^3_+$ such that $\boldsymbol{\theta}(y) = \boldsymbol{c} + \mathbf{Q}\widetilde{\boldsymbol{\theta}}(y)$ for all $y \in \omega$. Then the set $\dot{\mathcal{C}}^3(\omega; \mathbf{E}^3)$ becomes a *metric space* when it is equipped with the distance $\dot{d}_3$ defined by

$$\dot{d}_3(\dot{\boldsymbol{\theta}}, \dot{\boldsymbol{\psi}}) := \inf_{\begin{cases}\boldsymbol{\kappa} \in \dot{\boldsymbol{\theta}} \\ \boldsymbol{\chi} \in \dot{\boldsymbol{\psi}}\end{cases}} d_3(\boldsymbol{\kappa}, \boldsymbol{\chi}) = \inf_{\begin{cases}\boldsymbol{c} \in \mathbf{E}^3 \\ \mathbf{Q} \in \mathbb{O}^3\end{cases}} d_3(\boldsymbol{\theta}, \boldsymbol{c} + \mathbf{Q}\boldsymbol{\psi}),$$

where $\dot{\boldsymbol{\theta}}$ denotes the equivalence class of $\boldsymbol{\theta}$ *modulo* R.

The announced continuity of a surface as a function of its fundamental forms is then a corollary to Theorem 2.10-1. If d is a metric defined on a set X, the associated metric space is denoted $\{X; d\}$.

Theorem 2.10-3. *Let ω be connected and simply connected open subset of $\mathbb{R}^2$. Let*

$$\begin{aligned}\mathcal{C}_0^2(\omega;\mathbb{S}^2_> \times \mathbb{S}^2) := \{((a_{\alpha\beta}),(b_{\alpha\beta})) \in \mathcal{C}^2(\omega;\mathbb{S}^2_>) \times \mathcal{C}^2(\omega;\mathbb{S}^2);\\ \partial_\beta C_{\alpha\sigma\tau} - \partial_\sigma C_{\alpha\beta\tau} + C^\mu_{\alpha\beta} C_{\sigma\tau\mu} - C^\mu_{\alpha\sigma} C_{\beta\tau\mu} &= b_{\alpha\sigma} b_{\beta\tau} - b_{\alpha\beta} b_{\sigma\tau} \text{ in } \omega,\\ \partial_\beta b_{\alpha\sigma} - \partial_\sigma b_{\alpha\beta} + C^\mu_{\alpha\sigma} b_{\beta\mu} - C^\mu_{\alpha\beta} b_{\sigma\mu} &= 0 \text{ in } \omega\}.\end{aligned}$$

Given any element $((a_{\alpha\beta}),(b_{\alpha\beta})) \in \mathcal{C}_0^2(\omega;\mathbb{S}^2_> \times \mathbb{S}^2)$, let $F(((a_{\alpha\beta}),(b_{\alpha\beta}))) \in \dot{\mathcal{C}}^3(\omega;\mathbf{E}^3)$ denote the equivalence class modulo R *of any immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\omega;\mathbf{E}^3)$ that satisfies*

$$a_{\alpha\beta} = \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} \text{ and } b_{\alpha\beta} = \partial_{\alpha\beta}\boldsymbol{\theta} \cdot \left\{ \frac{\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}}{|\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}|} \right\} \text{ in } \omega.$$

Then the mapping

$$F : \{\mathcal{C}_0^2(\omega;\mathbb{S}^2_> \times \mathbb{S}^2); d_2\} \to \{\dot{\mathcal{C}}^3(\omega;\mathbf{E}^3); \dot{d}^3\}$$

defined in this fashion is continuous.

Proof. Since $\{\mathcal{C}_0^2(\omega;\mathbb{S}^2_> \times \mathbb{S}); d_2\}$ and $\{\dot{\mathcal{C}}^3(\omega;\mathbf{E}^3); \dot{d}^3\}$ are both metric spaces, it suffices to show that convergent sequences are mapped through F into convergent sequences.

Let then $((a_{\alpha\beta}),(b_{\alpha\beta})) \in \mathcal{C}_0^2(\omega;\mathbb{S}^2_> \times \mathbb{S}^2)$ and $((a^n_{\alpha\beta}),(b^n_{\alpha\beta})) \in \mathcal{C}_0^2(\omega;\mathbb{S}^2_> \times \mathbb{S}^2)$, $n \geq 0$, be such that

$$\lim_{n\to\infty} d_2(((a^n_{\alpha\beta}),(b^n_{\alpha\beta})),((a_{\alpha\beta}),(b_{\alpha\beta}))) = 0,$$

i.e., such that

$$\lim_{n\to\infty} \|a^n_{\alpha\beta} - a_{\alpha\beta}\|_{2,\kappa} = 0 \text{ and } \lim_{n\to\infty} \|b^n_{\alpha\beta} - b_{\alpha\beta}\|_{2,\kappa} = 0 \text{ for all } \kappa \Subset \omega.$$

Let there be given any $\boldsymbol{\theta} \in F(((a_{\alpha\beta}),(b_{\alpha\beta})))$. Then Theorem 2.10-1 shows that there exist $\boldsymbol{\theta}^n \in F(((a^n_{\alpha\beta}),(b^n_{\alpha\beta})))$, $n \geq 0$, such that

$$\lim_{n\to\infty} \|\boldsymbol{\theta}^n - \boldsymbol{\theta}\|_{3,\kappa} = 0 \text{ for all } \kappa \Subset \omega,$$

i.e., such that

$$\lim_{n\to\infty} d_3(\boldsymbol{\theta}^n,\boldsymbol{\theta}) = 0.$$

Consequently,

$$\lim_{n\to\infty} \dot{d}_3(F(((a^n_{\alpha\beta}),(b^n_{\alpha\beta}))), F(((a_{\alpha\beta}),(b_{\alpha\beta})))) = 0,$$

and the proof is complete. □

The above continuity results have been extended "up to the boundary of the set ω" by Ciarlet & C. Mardare [2005].

Chapter 3

APPLICATIONS TO THREE-DIMENSIONAL ELASTICITY IN CURVILINEAR COORDINATES

INTRODUCTION

The *raison d'être* of equations of three-dimensional elasticity directly expressed in *curvilinear coordinates* is twofold. First and foremost, they constitute an inevitable point of departure for the justification of most two-dimensional shell theories (such as those studied in the next chapter). Second, they are clearly more convenient than their Cartesian counterparts for modeling bodies with specific geometries, e.g., spherical or cylindrical.

Consider a *nonlinear elastic body*, whose *reference configuration* is of the form $\boldsymbol{\Theta}(\overline{\Omega})$, where Ω is a domain in $\mathbb{R}^3$ and $\boldsymbol{\Theta} : \overline{\Omega} \to \mathbf{E}^3$ is a smooth enough immersion. Let $\Gamma_0 \cup \Gamma_1$ denote a partition of the boundary $\partial\Omega$ such that *area* $\Gamma_0 > 0$. The body is subjected to *applied body forces* in its interior $\boldsymbol{\Theta}(\Omega)$, to *applied surface forces* on the portion $\boldsymbol{\Theta}(\Gamma_1)$ of its boundary, and to a homogeneous *boundary condition of place* on the remaining portion $\boldsymbol{\Theta}(\Gamma_0)$ of its boundary (this means that the displacement vanishes there).

We first review in Section 3.1 the associated *equations of nonlinear three-dimensional elasticity in Cartesian coordinates*, i.e., expressed in terms of the Cartesian coordinates of the set $\boldsymbol{\Theta}(\overline{\Omega})$.

We then examine in Sections 3.2 to 3.5 how these equations are transformed when they are expressed in terms of *curvilinear coordinates*, i.e., in terms of the coordinates of the set $\overline{\Omega}$.

To this end, we put to use in particular the notion of *covariant derivatives of a vector field* introduced in Chapter 1. In this fashion, we show (Theorem 3.3-1) that the variational equations of the *principle of virtual work in curvilinear coordinates* take the following form:

$$\int_\Omega \sigma^{ij}(E'_{i\|j}(\boldsymbol{u})\boldsymbol{v})\sqrt{g}\,\mathrm{d}x = \int_\Omega f^i v_i \sqrt{g}\,\mathrm{d}x + \int_{\Gamma_1} h^i v_i \sqrt{g}\,\mathrm{d}\Gamma$$

for all $\boldsymbol{v} = (v_i) \in \mathbf{W}(\Omega)$. In these equations, the functions $\sigma^{ij} = \sigma^{ji} : \overline{\Omega} \to \mathbb{R}$

are the contravariant components of the *second Piola-Kirchhoff stress tensor field*; the functions

$$E_{i\|j}(\boldsymbol{u}) = \frac{1}{2}(u_{i\|j} + u_{j\|i} + g^{mn} u_{m\|i} u_{n\|j}) : \overline{\Omega} \to \mathbb{R}$$

are the covariant components of the *Green-St Venant strain tensor* field associated with a displacement vector field $u_i \boldsymbol{g}^i : \overline{\Omega} \to \mathbb{R}^3$ of the reference configuration $\boldsymbol{\Theta}(\overline{\Omega})$; the functions $f^i : \Omega \to \mathbb{R}$ and $h^i : \Gamma_1 \to \mathbb{R}$ are the contravariant components of the applied body and surface forces; $\Gamma_0 \cup \Gamma_1$ denotes a partition of the boundary of Ω; finally, $\mathbf{W}(\Omega)$ denotes a space of sufficiently smooth vector fields $\boldsymbol{v} = (v_i) : \overline{\Omega} \to \mathbb{R}^3$ that vanish on Γ_0.

We also show (Theorem 3.3-1) that the above principle of virtual work is formally equivalent to the following *equations of equilibrium in curvilinear coordinates*:

$$-(\sigma^{ij} + \sigma^{kj} g^{i\ell} u_{\ell\|k})\|_j = f^i \text{ in } \Omega,$$
$$(\sigma^{ij} + \sigma^{kj} g^{i\ell} u_{\ell\|k}) n_j = h^i \text{ on } \Gamma_1,$$

where functions such as

$$t^{ij}\|_j = \partial_j t^{ij} + \Gamma^i_{pj} t^{pj} + \Gamma^j_{jq} t^{iq},$$

which *naturally* appear in the derivation of these equations, provide instances of *first-order covariant derivatives of a tensor field*.

These equations must be complemented by the *constitutive equation* of the elastic material, which in general takes the form

$$\sigma^{ij}(x) = \mathcal{R}^{ij}(x, (E_{m\|n}(\boldsymbol{u})(x))) \text{ for all } x \in \overline{\Omega}$$

for *ad hoc* functions $\mathcal{R}^{ij}$ that characterize the elastic material constituting the body. In the important special case where the elastic material is *homogeneous* and *isotropic* and the *reference configuration* $\boldsymbol{\Theta}(\overline{\Omega})$ *is a natural state*, the functions $\mathcal{R}^{ij}$ are of the specific form

$$\mathcal{R}^{ij}(x, \mathbf{E}) = A^{ijk\ell}(x) E_{k\ell} + o(\mathbf{E}) \text{ for all } x \in \overline{\Omega} \text{ and } \mathbf{E} = (E_{k\ell}) \in \mathbb{S}^3,$$

where the functions

$$A^{ijk\ell} = \lambda g^{ij} g^{k\ell} + \mu(g^{ik} g^{j\ell} + g^{i\ell} g^{jk})$$

designate the contravariant components of the *elasticity tensor* of the elastic material and the constants λ and μ, which satisfy the inequalities $3\lambda + 2\mu > 0$ and $\mu > 0$, are the *Lamé constants* of the material (Theorem 3.4-1).

Together with boundary conditions such as

$$u_i = 0 \text{ on } \Gamma_0,$$

the equations of equilibrium and the constitutive equation constitute the *boundary value problem of nonlinear three-dimensional elasticity in curvilinear coordinates* (Section 3.5). Its unknown is the vector field $\boldsymbol{u} = (u_i) : \overline{\Omega} \to \mathbb{R}^3$, where the functions $u_i : \overline{\Omega} \to \mathbb{R}$ are the covariant components of the *unknown displacement field* $u_i \boldsymbol{g}^i : \overline{\Omega} \to \mathbb{R}^3$ of the reference configuration $\boldsymbol{\Theta}(\overline{\Omega})$.

We then derive by means of a formal linearization procedure the equations that constitute the *boundary value problem of linearized three-dimensional elasticity in curvilinear coordinates* (Section 3.6). This problem is studied in detail in the rest of this chapter.

The *variational*, or *weak, formulation* of this linear boundary value problem consists in seeking a vector field

$$\boldsymbol{u} = (u_i) \in \mathbf{V}(\Omega) = \{\boldsymbol{v} = (v_i) \in \mathbf{H}^1(\Omega); \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_0\}$$

such that

$$\int_\Omega A^{ijk\ell} e_{k\|\ell}(\boldsymbol{u}) e_{i\|j}(\boldsymbol{v}) \sqrt{g}\, \mathrm{d}x = \int_\Omega f^i v_i \sqrt{g}\, \mathrm{d}x + \int_{\Gamma_1} h^i v_i \sqrt{g}\, \mathrm{d}\Gamma$$

for all $\boldsymbol{v} = (v_i) \in \mathbf{V}(\Omega)$, where $u_i \boldsymbol{g}^i$ is now to be interpreted as a "linearized approximation" of the unknown displacement vector field of the reference configuration. The functions $e_{i\|j}(\boldsymbol{v}) \in L^2(\Omega)$, which are defined for each $\boldsymbol{v} = (v_i) \in \mathbf{H}^1(\Omega)$ by

$$e_{i\|j}(\boldsymbol{v}) = \frac{1}{2}(v_{i\|j} + v_{j\|i})$$

are the covariant components of the *linearized strain tensor in curvilinear coordinates*. Equivalently, the vector field $\boldsymbol{u} \in \mathbf{V}(\Omega)$ minimizes the functional $J : \mathbf{V}(\Omega) \to \mathbb{R}$ defined by

$$J(\boldsymbol{v}) = \frac{1}{2} \int_\Omega A^{ijk\ell} e_{k\|\ell}(\boldsymbol{v}) e_{i\|j}(\boldsymbol{v}) \sqrt{g}\, \mathrm{d}x - \Big\{ \int_\Omega f^i v_i \sqrt{g}\, \mathrm{d}x + \int_{\Gamma_1} h^i v_i \sqrt{g}\, \mathrm{d}\Gamma \Big\}$$

for all $\boldsymbol{v} \in \mathbf{H}^1(\Omega)$.

We then show how a fundamental *lemma of J.L. Lions* (Theorem 3.7-1) can be put to use for directly establishing a *Korn inequality in curvilinear coordinates*. When *area* $\Gamma_0 > 0$, this key inequality asserts the existence of a constant C such that (Theorem 3.8-3)

$$\|\boldsymbol{v}\|_{1,\Omega} \leq C \Big\{ \sum_{i,j} \|e_{i\|j}(\boldsymbol{v})\|^2_{0,\Omega} \Big\}^{1/2} \text{ for all } \boldsymbol{v} \in \mathbf{V}(\Omega).$$

Together with the *uniform positive-definiteness of the elasticity tensor*, which holds under the assumptions $3\lambda + 2\mu > 0$ and $\mu > 0$ (Theorem 3.9-1), this inequality in turn yields (Theorem 3.9-2) the *existence* and *uniqueness* of a solution to the variational formulation of the equations of linearized three-dimensional elasticity, again directly in *curvilinear coordinates.*

In this chapter, expressions such as "equations of nonlinear elasticity", "Korn's inequality", etc., are meant to be understood as "equations of nonlinear *three-dimensional* elasticity", "*three-dimensional* Korn's inequality", etc.

3.1 THE EQUATIONS OF NONLINEAR ELASTICITY IN CARTESIAN COORDINATES

We briefly review in this section the equations of nonlinear elasticity. All details needed about the various notions introduced in this section may be found in Ciarlet [1988], to which more specific references are also provided below. Additional references are provided at the end of this section.

Latin indices or exponents range in the set $\{1,2,3\}$, except when they are used for indexing sequences. Let $\mathbf{E}^3$ denote a three-dimensional Euclidean space, let $\boldsymbol{a}\cdot\boldsymbol{b}$ denote the Euclidean inner product of $\boldsymbol{a},\boldsymbol{b}\in\mathbf{E}^3$, and let $|\boldsymbol{a}|$ denote the Euclidean norm of $\boldsymbol{a}\in\mathbf{E}^3$. Let $\widehat{\boldsymbol{e}}_i=\widehat{\boldsymbol{e}}^i$ denote the orthonormal basis vectors of $\mathbf{E}^3$, let $\widehat{x}_i=\widehat{x}^i$ denote the *Cartesian coordinates* of $\widehat{x}=\widehat{x}_i\widehat{\boldsymbol{e}}^i=\widehat{x}^i\widehat{\boldsymbol{e}}_i\in\mathbf{E}^3$ and let $\widehat{\partial}_i:=\partial/\partial\widehat{x}_i$. Let $\mathbb{M}^3$ and $\mathbb{S}^3$ denote the space of all matrices of order three and that of all symmetric matrices of order three, let $\mathbf{A}:\mathbf{B}:=\operatorname{tr}\mathbf{A}^T\mathbf{B}$ denote the matrix inner product of $\mathbf{A},\mathbf{B}\in\mathbb{M}^3$, and let $\|\mathbf{A}\|:=\sqrt{\mathbf{A}:\mathbf{A}}$ denote the associated matrix norm of $\mathbf{A}\in\mathbb{M}^3$.

We recall that a *domain* in $\mathbf{E}^3$ is a bounded, open, and connected subset Ω of $\mathbf{E}^3$ with a Lipschitz-continuous boundary, the set Ω being locally on the same side of its boundary (see Nečas [1967] or Adams [1975]).

Let $\widehat{\Omega}$ be a domain in $\mathbf{E}^3$, let $d\widehat{x}$ denote the *volume element* in $\widehat{\Omega}$, let $d\widehat{\Gamma}$ denote the *area element* along the boundary $\widehat{\Gamma}$ of $\widehat{\Omega}$, and let $\widehat{\boldsymbol{n}}=\widehat{n}_i\widehat{\boldsymbol{e}}^i$ denote the *unit* ($|\widehat{\boldsymbol{n}}|=1$) *outer normal vector field* along $\widehat{\Gamma}$.

Remark. The reason we use in this section notations such as $\widehat{\Omega},\widehat{\Gamma}$, or $\widehat{\boldsymbol{n}}$ is to later afford a proper distinction between equations written in terms of the *Cartesian coordinates* $\widehat{x}_i$ of the points $\widehat{x}$ of the set $\widehat{\Omega}$ on the one hand as in this section, and equations written in terms of *curvilinear coordinates* x_i on the other hand, the points $x=(x_i)$ then varying in a domain Ω with boundary Γ and unit outer normal vector field $\boldsymbol{n}$ (see Section 3.5). □

If $\widehat{\boldsymbol{v}}=(\widehat{v}_i):\{\widehat{\Omega}\}^-\to\mathbb{R}^3$ is a smooth enough vector field, its *gradient* $\widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{v}}:\{\widehat{\Omega}\}^-\to\mathbb{M}^3$ is the matrix field defined by

$$\widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{v}}:=\begin{pmatrix}\widehat{\partial}_1\widehat{v}_1 & \widehat{\partial}_2\widehat{v}_1 & \widehat{\partial}_3\widehat{v}_1\\ \widehat{\partial}_1\widehat{v}_2 & \widehat{\partial}_2\widehat{v}_2 & \widehat{\partial}_3\widehat{v}_2\\ \widehat{\partial}_1\widehat{v}_3 & \widehat{\partial}_2\widehat{v}_3 & \widehat{\partial}_3\widehat{v}_3\end{pmatrix}.$$

If $\widehat{\mathbf{T}}=(\hat{t}^{ij}):\{\widehat{\Omega}\}^-\to\mathbb{M}^3$ is a smooth enough matrix field (the first exponent i is the row index), its *divergence* $\widehat{\mathbf{div}}\,\widehat{\mathbf{T}}:\{\widehat{\Omega}\}^-\to\mathbb{M}^3$ is the vector field defined by

$$\widehat{\mathbf{div}}\,\widehat{\mathbf{T}}:=\begin{pmatrix}\widehat{\partial}_j\hat{t}^{1j}\\ \widehat{\partial}_j\hat{t}^{2j}\\ \widehat{\partial}_j\hat{t}^{3j}\end{pmatrix}.$$

If $\widehat{W}:\{\widehat{\Omega}\}^-\times\mathbb{M}^3\to\mathbb{R}$ is a smooth enough function, its *partial derivative*

$\dfrac{\partial \widehat{W}}{\partial \mathbf{F}}(x, \mathbf{F}) \in \mathbb{M}^3$ at a point $(x, \mathbf{F}) = (x, (F_{ij})) \in \{\widehat{\Omega}\} \times \mathbb{M}^3$ is the matrix that satisfies

$$\widehat{W}(x, \mathbf{F} + \mathbf{G}) = \widehat{W}(x, \mathbf{F}) + \frac{\partial \widehat{W}}{\partial \mathbf{F}}(x, \mathbf{F}) : \mathbf{G} + o(\|\mathbf{G}\|).$$

Equivalently,

$$\frac{\partial \widehat{W}}{\partial \mathbf{F}}(x, \mathbf{F}) := \begin{pmatrix} \partial\widehat{W}/\partial F_{11} & \partial\widehat{W}/\partial F_{12} & \partial\widehat{W}/\partial F_{13} \\ \partial\widehat{W}/\partial F_{21} & \partial\widehat{W}/\partial F_{22} & \partial\widehat{W}/\partial F_{23} \\ \partial\widehat{W}/\partial F_{31} & \partial\widehat{W}/\partial F_{32} & \partial\widehat{W}/\partial F_{33} \end{pmatrix} (x, \mathbf{F}).$$

Let there be given a *body*, which occupies the set $\{\widehat{\Omega}\}^-$ in the absence of applied forces. The set $\{\widehat{\Omega}\}^-$ is called the **reference configuration** of the body. Let $\widehat{\Gamma} = \widehat{\Gamma}_0 \cup \widehat{\Gamma}_1$ be a $d\widehat{\Gamma}$-measurable partition ($\widehat{\Gamma}_0 \cap \widehat{\Gamma}_1 = \phi$) of the boundary $\widehat{\Gamma}$ of $\widehat{\Omega}$.

The body is subjected to **applied body forces** in its interior $\widehat{\Omega}$, of *density* $\widehat{\boldsymbol{f}} = \widehat{f}^i \widehat{\boldsymbol{e}}_i : \widehat{\Omega} \to \mathbb{R}^3$ per unit volume, and to **applied surface forces** on the portion $\widehat{\Gamma}_1$ of its boundary, of *density* $\widehat{\boldsymbol{h}} = \widehat{h}^i \boldsymbol{e}_i : \widehat{\Gamma}_1 \to \mathbb{R}^3$ per unit area. We assume that these densities do not depend on the unknown, i.e., that the applied forces considered here are *dead loads* (cf. Ciarlet [1988, Section 2.7]).

The *unknown* is the **displacement field** $\widehat{\boldsymbol{u}} = \widehat{u}_i \widehat{\boldsymbol{e}}^i = \widehat{u}^i \widehat{\boldsymbol{e}}_i : \{\widehat{\Omega}\}^- \to \mathbf{E}^3$, where the three functions $\widehat{u}_i = \widehat{u}^i : \{\widehat{\Omega}\}^- \to \mathbb{R}$ are the *Cartesian components* of the displacement that the body undergoes when it is subjected to applied forces. This means that $\widehat{\boldsymbol{u}}(\widehat{x}) = \widehat{u}_i(\widehat{x})\widehat{\boldsymbol{e}}^i$ is the *displacement of the point* $\widehat{x} \in \{\widehat{\Omega}\}^-$ (see Figure 1.4-1).

It is assumed that *the displacement field vanishes on the set* $\widehat{\Gamma}_0$, i.e., that it satisfies the (homogeneous) **boundary condition of place**

$$\widehat{\boldsymbol{u}} = \mathbf{0} \text{ on } \widehat{\Gamma}_0.$$

Let $\boldsymbol{id}_{\{\widehat{\Omega}\}}$ denote the identity mapping of the space $\mathbf{E}^3$. The mapping $\widehat{\boldsymbol{\varphi}} : \{\widehat{\Omega}\}^- \to \mathbb{R}^3$ defined by

$$\widehat{\boldsymbol{\varphi}} := \boldsymbol{id}_{\{\widehat{\Omega}\}^-} + \widehat{\boldsymbol{u}},$$

i.e., by $\widehat{\boldsymbol{\varphi}}(\widehat{x}) = \boldsymbol{o}\widehat{\boldsymbol{x}} + \widehat{\boldsymbol{u}}(\widehat{x})$ for all $\widehat{x} \in \{\widehat{\Omega}\}^-$, is called a **deformation** of the reference configuration $\{\widehat{\Omega}\}^-$ and the set $\widehat{\boldsymbol{\varphi}}\{\widehat{\Omega}\}^-$ is called a **deformed configuration**. Since the approach in this section is for its most part formal, we assume throughout that the requirements that the deformation $\widehat{\boldsymbol{\varphi}}$ should satisfy in order to be physically admissible (orientation-preserving character and injectivity; cf. *ibid.*, Section 1.4) are satisfied. Naturally, the deformation $\boldsymbol{\varphi}$ may be equivalently considered as the unknown instead of the displacement field $\widehat{\boldsymbol{u}}$.

The following **equations of equilibrium in Cartesian coordinates** (cf. *ibid.*, Sections 2.5 and 2.6) are then satisfied in the reference configuration $\{\widehat{\Omega}\}^-$: There exists a matrix field $\widehat{\mathbf{\Sigma}} = (\widehat{\sigma}^{ij}) : \{\widehat{\Omega}\}^- \to \mathbb{M}^3$, called the **second Piola-Kirchhoff stress tensor field**, such that

$$
\begin{aligned}
-\widehat{\mathbf{div}}\{(\mathbf{I} + \widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{u}})\widehat{\mathbf{\Sigma}}\} &= \widehat{\boldsymbol{f}} \text{ in } \widehat{\Omega},\\
(\mathbf{I} + \widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{u}})\widehat{\mathbf{\Sigma}}\widehat{\boldsymbol{n}} &= \widehat{\boldsymbol{h}} \text{ on } \widehat{\Gamma}_1,\\
\widehat{\mathbf{\Sigma}} &= \widehat{\mathbf{\Sigma}}^T \text{ in } \{\widehat{\Omega}\}^-.
\end{aligned}
$$

Componentwise, the equations of equilibrium thus read:

$$
\begin{aligned}
-\widehat{\partial}_j(\widehat{\sigma}^{ij} + \widehat{\sigma}^{kj}\widehat{\sigma}_k\widehat{u}^i) &= \widehat{f}^i \text{ in } \widehat{\Omega},\\
(\widehat{\sigma}^{ij} + \widehat{\sigma}^{kj}\widehat{\partial}_k\widehat{u}^i)\widehat{n}_j &= \widehat{h}^i \text{ on } \widehat{\Gamma}_1,\\
\widehat{\sigma}^{ij} &= \widehat{\sigma}^{ji} \text{ in } \{\widehat{\Omega}\}^-.
\end{aligned}
$$

Note that *the symmetry of the matrix field $\widehat{\mathbf{\Sigma}}$ is part of the equations of equilibrium.*

The components $\widehat{\sigma}^{ij}$ of the field $\widehat{\mathbf{\Sigma}}$ are called the **second Piola-Kirchhoff stresses**. The boundary conditions on the set $\widehat{\Gamma}_1$ constitute a **boundary condition of traction**.

The matrix field $\widehat{\mathbf{T}} = (\hat{t}^{ij}) : \{\widehat{\Omega}\}^- \to \mathbb{M}^3$, where

$$
\widehat{\mathbf{T}} := (\mathbf{I} + \widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{u}})\widehat{\mathbf{\Sigma}} = \widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{\varphi}}\widehat{\mathbf{\Sigma}}
$$

is called the **first Piola-Kirchhoff stress tensor field**. Its components $\hat{t}^{ij} = (\delta^{ij} + \partial_k u^i)\sigma^{kj}$ are called the **first Piola-Kirchhoff stresses**.

While the *equations of equilibrium* are thus expressed in a simpler manner in terms of the *first* Piola-Kirchhoff stress tensor, it turns out that the *constitutive equation* of an elastic material (see below) is more naturally expressed in terms of the *second* Piola-Kirchhoff stress tensor. This is why the equations of equilibrium were directly written here in terms of the latter stress tensor.

Let $\mathbf{W}(\widehat{\Omega})$ denote a space of sufficiently smooth vector fields $\widehat{\boldsymbol{v}} = \widehat{v}_i\widehat{\boldsymbol{e}}^i = \widehat{v}^i\widehat{\boldsymbol{e}}_i : \{\widehat{\Omega}\}^- \to \mathbf{E}^3$ *that vanish on* $\widehat{\Gamma}_0$. The following *Green's formula* is then easily established: For any smooth enough matrix field $\widehat{\mathbf{T}} : \{\widehat{\Omega}\}^- \to \mathbb{M}^3$, and vector field $\widehat{\boldsymbol{v}} \in \mathbf{W}(\widehat{\Omega})$,

$$
\int_{\widehat{\Omega}} \widehat{\mathbf{div}}\, \widehat{\mathbf{T}} \cdot \widehat{\boldsymbol{v}}\, d\widehat{x} = -\int_{\Omega} \widehat{\mathbf{T}} : \widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{v}}\, d\widehat{x} + \int_{\widehat{\Gamma}_1} \widehat{\mathbf{T}}\widehat{\boldsymbol{n}} \cdot \widehat{\boldsymbol{v}}\, d\widehat{\Gamma}.
$$

If the unknown vector field $\widehat{\boldsymbol{u}} = (\widehat{u}_i)$ and the fields $\widehat{\boldsymbol{f}}$ and $\widehat{\boldsymbol{h}}$ are smooth enough, it is immediately established, because of the above Green's formula, that *the first and second equations of equilibrium are formally equivalent to the* **principle of virtual work in Cartesian coordinates**, which states that:

$$
\int_{\widehat{\Omega}} (\mathbf{I} + \widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{u}})\widehat{\mathbf{\Sigma}} : \widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{v}}\, d\widehat{x} = \int_{\widehat{\Omega}} \widehat{\boldsymbol{f}} \cdot \widehat{\boldsymbol{v}}\, d\widehat{x} + \int_{\widehat{\Gamma}_1} \widehat{\boldsymbol{h}} \cdot \widehat{\boldsymbol{v}}\, d\widehat{\Gamma} \text{ for all } \widehat{\boldsymbol{v}} \in \mathbf{W}(\widehat{\Omega}).
$$

Componentwise, the principle of virtual work thus reads:

$$\int_{\widehat{\Omega}} (\widehat{\sigma}^{ij} + \widehat{\sigma}^{kj} \widehat{\partial}_k \widehat{u}^i) \widehat{\partial}_j \widehat{v}_i \, \mathrm{d}\widehat{x} = \int_{\widehat{\Omega}} \widehat{f}^i \widehat{v}_i \, \mathrm{d}\widehat{x} + \int_{\widehat{\Gamma}_1} \widehat{h}^i \widehat{v}_i \, \mathrm{d}\widehat{\Gamma}$$

for all $\widehat{v}_i \widehat{\boldsymbol{e}}^i \in \mathbf{W}(\widehat{\Omega})$.

The principle of virtual work is thus nothing but the *weak*, or *variational*, form of the equations of equilibrium. The vector fields $\widehat{\boldsymbol{v}} \in \mathbf{W}(\widehat{\Omega})$ that enter it are "variations" around the deformation $\widehat{\boldsymbol{\varphi}} = \boldsymbol{id}_{\{\widehat{\Omega}\}^-} + \widehat{\boldsymbol{u}}$ (cf. *ibid.*, Section 2.6).

Let the **Green-St Venant strain tensor field** associated with an arbitrary displacement field $\widehat{\boldsymbol{v}} = \widehat{v}_i \widehat{\boldsymbol{e}}^i = \widehat{v}^i \widehat{\boldsymbol{e}}_i : \{\widehat{\Omega}\}^- \to \mathbf{E}^3$ of the reference configuration $\{\widehat{\Omega}\}^-$ be defined by (cf. *ibid.*, Section 1.8):

$$\widehat{\mathbf{E}}(\widehat{\boldsymbol{v}}) := \frac{1}{2}\left(\widehat{\boldsymbol{\nabla}} \widehat{\boldsymbol{v}}^T + \widehat{\boldsymbol{\nabla}} \widehat{\boldsymbol{v}} + \widehat{\boldsymbol{\nabla}} \widehat{\boldsymbol{v}}^T \widehat{\boldsymbol{\nabla}} \widehat{\boldsymbol{v}}\right) = (\widehat{E}_{ij}(\widehat{\boldsymbol{v}})),$$

where the matrix field $\boldsymbol{\nabla v}$ denotes the corresponding **displacement gradient field**. The components

$$\widehat{E}_{ij}(\widehat{\boldsymbol{v}}) = \frac{1}{2}(\widehat{\partial}_i \widehat{v}_j + \widehat{\partial}_j \widehat{v}_i + \widehat{\partial}_i \widehat{v}_m \widehat{\partial}_j \widehat{v}^m)$$

of the matrix field $\widehat{\mathbf{E}}(\widehat{\boldsymbol{v}})$ are called the **Green-St Venant strains**.

Let

$$\widehat{\boldsymbol{\psi}} := \boldsymbol{id}_{\{\widehat{\Omega}\}^-} + \widehat{\boldsymbol{v}} = \widehat{\psi}_i \widehat{\boldsymbol{e}}^i = \widehat{\psi}^i \widehat{\boldsymbol{e}}_i$$

denote the associated deformation of the reference configuration $\{\widehat{\Omega}\}^-$ and assume that the mapping $\widehat{\boldsymbol{\psi}} : \widehat{\Omega} \to \mathbf{E}^3$ is an injective immersion, so that the set $\widehat{\boldsymbol{\psi}}(\widehat{\Omega})$ can be considered as being equipped with the Cartesian coordinates $\widehat{x}_i$ in $\mathbf{E}^3$ as its *curvilinear coordinates*. In this interpretation the *covariant components of the metric tensor of the set* $\widehat{\boldsymbol{\psi}}(\widehat{\Omega})$ are thus given by (Section 1.2)

$$(\widehat{\boldsymbol{\nabla}} \widehat{\boldsymbol{\psi}}^T \widehat{\boldsymbol{\nabla}} \widehat{\boldsymbol{\psi}})_{ij} = \widehat{\partial}_i \widehat{\psi}_m \widehat{\partial}_j \widehat{\psi}^m = \delta_{ij} + 2\widehat{E}_{ij}(\widehat{\boldsymbol{v}}) = (\mathbf{I} + 2\widehat{\mathbf{E}}(\widehat{\boldsymbol{v}}))_{ij}.$$

In the context of nonlinear three-dimensional elasticity, the matrix field $\widehat{\boldsymbol{\nabla}} \widehat{\boldsymbol{\psi}}^T \widehat{\boldsymbol{\nabla}} \widehat{\boldsymbol{\psi}}$ is called the **Cauchy-Green tensor field**.

Since the constant functions δ_{ij} are the covariant components of the metric tensor of the set $\widehat{\Omega}$ (which corresponds to the particular deformation $\boldsymbol{id}_{\{\widehat{\Omega}\}^-}$), *the components* $\widehat{E}_{ij}(\widehat{\boldsymbol{v}})$ *measure the differences between the covariant components of the metric tensor of the deformed configuration and those of the reference configuration.* This is why the field

$$\widehat{\mathbf{E}}(\widehat{\boldsymbol{v}}) = \frac{1}{2}(\boldsymbol{\nabla} \boldsymbol{\psi}^T \boldsymbol{\nabla} \boldsymbol{\psi} - \mathbf{I})$$

is also aptly called the **change of metric tensor field** associated with the displacement field $\widehat{\boldsymbol{v}}$. This also explains why "strain" means "change of metric".

Remark. The *linearized strain tensor*

$$\widehat{\boldsymbol{e}}(\widehat{\boldsymbol{v}}) = (\widehat{e}_{ij}(\widehat{\boldsymbol{v}})) := \frac{1}{2}(\{\widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{v}}\}^T + \widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{v}}),$$

where

$$\widehat{e}_{ij}(\widehat{\boldsymbol{v}}) = \frac{1}{2}(\widehat{\partial}_j\widehat{v}_i + \widehat{\partial}_i\widehat{v}_j),$$

that naturally arises in *linearized elasticity in Cartesian coordinates* (see Section 3.6) is thus exactly the linear part with respect to $\widehat{\boldsymbol{v}}$ in the strain tensor $\widehat{\mathbf{E}}(\widehat{\boldsymbol{v}})$. □

A material is **elastic** if, at each point $\widehat{x}$ of the reference configuration $\{\widehat{\Omega}\}^-$, *the stress tensor* $\widehat{\boldsymbol{\Sigma}}(\widehat{x})$ *is a known function* (that characterizes the material) *of the displacement gradient* $\widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{u}}(\widehat{x})$ *at the same point*. The consideration of the fundamental *principle of material frame-indifference* further implies (cf. *ibid.*, Theorem 3.6-2) that $\widehat{\boldsymbol{\Sigma}}(\widehat{x})$ is in fact a function of $\widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{u}}(\widehat{x})$ by means of the *Green-St Venant strain tensor* $\mathbf{E}(\boldsymbol{u}(x))$ *at* $\widehat{x}$.

Equivalently, there exists a **response function** $\widehat{\boldsymbol{\mathcal{R}}} = (\widehat{\mathcal{R}}^{ji}) : \{\widehat{\Omega}\}^- \times \mathbb{S}^3 \to \mathbb{S}^3$ such that, at each point $\widehat{x}$ of the reference configuration, the stress tensor $\widehat{\boldsymbol{\Sigma}}(\widehat{x})$ is given by the relation

$$\widehat{\boldsymbol{\Sigma}}(x) = \widehat{\boldsymbol{\mathcal{R}}}(\widehat{x}, \widehat{\mathbf{E}}(\widehat{\boldsymbol{u}})(\widehat{x})),$$

which is called the **constitutive equation in Cartesian coordinates** of the material.

If the material is elastic, the unknown vector field $\widehat{\boldsymbol{u}} = \widehat{u}_i\widehat{\boldsymbol{e}}^i = \widehat{u}^i\widehat{\boldsymbol{e}}_i : \{\widehat{\Omega}\}^- \to \mathbb{R}^3$ should thus satisfy the following **boundary value problem of nonlinear elasticity in Cartesian coordinates**:

$$\begin{aligned}
-\widehat{\mathbf{div}}\{(\mathbf{I} + \widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{u}})\widehat{\boldsymbol{\Sigma}}\} &= \widehat{\boldsymbol{f}} \text{ in } \widehat{\Omega},\\
\widehat{\boldsymbol{u}} &= \widehat{\mathbf{0}} \text{ on } \widehat{\Gamma}_0,\\
(\mathbf{I} + \widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{u}})\widehat{\boldsymbol{\Sigma}}\widehat{\boldsymbol{n}} &= \widehat{\boldsymbol{h}} \text{ on } \widehat{\Gamma}_1,\\
\widehat{\boldsymbol{\Sigma}} &= \widehat{\boldsymbol{\mathcal{R}}}(\cdot, \widehat{\mathbf{E}}(\widehat{\boldsymbol{u}})) \text{ in } \{\widehat{\Omega}\}^-,\\
\mathbf{E}(\boldsymbol{u}) &= \frac{1}{2}(\boldsymbol{\nabla}\widetilde{\boldsymbol{u}}^T + \boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}\boldsymbol{u}^T\boldsymbol{\nabla}\boldsymbol{u}) \text{ in } \{\widehat{\Omega}\}^-.
\end{aligned}$$

Componentwise, this boundary value problem reads:

$$\begin{aligned}
-\widehat{\partial}_j(\widehat{\sigma}^{ij} + \widehat{\sigma}^{kj}\widehat{\partial}_k\widehat{u}^i) &= \widehat{f}^i \text{ in } \widehat{\Omega},\\
\widehat{u}^i &= 0 \text{ on } \Gamma_0,\\
(\widehat{\sigma}^{ij} + \widehat{\sigma}^{kj}\widehat{\partial}_k\widehat{u}^i)\widehat{n}_j &= \widehat{h}^i \text{ on } \Gamma_1,\\
\widehat{\sigma}^{ij} &= \widehat{\mathcal{R}}^{ij}(\cdot, (\widehat{E}_{k\ell}(\widehat{\boldsymbol{u}}))) \text{ in } \{\widehat{\Omega}\}^-,\\
\widehat{E}_{k\ell}(\widehat{\boldsymbol{u}}) &= \frac{1}{2}(\partial_k\widehat{u}_\ell + \partial_\ell\widehat{u}_k + \partial_k\widehat{u}_m\partial_\ell\widehat{u}^m) \text{ in } \{\widehat{\Omega}\}^-.
\end{aligned}$$

If *area* $\widehat{\Gamma}_0 > 0$ and *area* $\widehat{\Gamma}_1 > 0$, the above boundary value problem is called a **displacement-traction problem**, by reference to the boundary conditions. For the same reason, it is called a **pure displacement problem** if $\widehat{\Gamma}_1 = \phi$, or a **pure traction problem** if $\widehat{\Gamma}_0 = \phi$.

Thanks to the symmetry of the tensor field $\widehat{\mathbf{\Sigma}}$ and to the relations $\mathbf{AB} : \mathbf{C} = \mathbf{B} : \mathbf{A}^T\mathbf{C} = \mathbf{B}^T : \mathbf{C}^T\mathbf{A}$ valid for any $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{M}^3$, the integrand appearing in the left-hand side of the principle of virtual work can be re-written as

$$\begin{aligned}
&(\mathbf{I} + \widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{u}})\widehat{\mathbf{\Sigma}} : \widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{v}} = \widehat{\mathbf{\Sigma}} : \widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{v}} + (\widehat{\boldsymbol{\nabla}}\boldsymbol{u}\widehat{\mathbf{\Sigma}}) : \widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{v}} \\
&= \frac{1}{2}\Big(\widehat{\mathbf{\Sigma}} : \widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{v}} + \widehat{\mathbf{\Sigma}}^T : \widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{v}} + (\widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{u}}\widehat{\mathbf{\Sigma}}) : \widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{v}} + (\widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{u}}\widehat{\mathbf{\Sigma}}^T) : \widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{v}}\Big) \\
&= \frac{1}{2}\widehat{\mathbf{\Sigma}} : \Big(\widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{v}} + \widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{v}}^T + \widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{u}}^T\widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{v}} + \widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{v}}^T\widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{u}}\Big) \\
&= \widehat{\mathbf{\Sigma}} : \widehat{\mathbf{E}}'(\widehat{\boldsymbol{u}})\widehat{\boldsymbol{v}},
\end{aligned}$$

where

$$\widehat{\mathbf{E}}'(\widehat{\boldsymbol{u}})\widehat{\boldsymbol{v}} = \frac{1}{2}\Big(\widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{v}} + \widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{v}}^T + \widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{u}}^T\widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{v}} + \widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{v}}^T\widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{u}}\Big)$$

denotes the *Gâteaux derivative* of the mapping $\widehat{\mathbf{E}} : \widehat{\boldsymbol{v}} \in \mathbf{W}(\widehat{\Omega}) \to \mathbb{S}^3$ (it is immediately verified that $\widehat{\mathbf{E}}'(\widehat{\boldsymbol{u}})\widehat{\boldsymbol{v}}$ is indeed the linear part with respect to $\widehat{\boldsymbol{v}}$ in the difference $\widehat{\mathbf{E}}(\widehat{\boldsymbol{u}} + \widehat{\boldsymbol{v}}) - \widehat{\mathbf{E}}(\widehat{\boldsymbol{u}})$). Naturally, *ad hoc* topologies must be specified insuring that the mapping $\widehat{\mathbf{E}}$ is differentiable (for instance, it is so if it is considered as a mapping from the space $\mathbf{W}^{1,4}(\widehat{\Omega})$ into the space $L^2(\Omega; \mathbb{S}^3)$). Consequently, the left-hand of the principle of virtual work takes the form

$$\begin{aligned}
\int_{\widehat{\Omega}} (\mathbf{I} + \widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{u}})\widehat{\mathbf{\Sigma}} : \widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{v}}\,\mathrm{d}x &= \int_{\widehat{\Omega}} \widehat{\mathbf{\Sigma}} : (\widehat{\mathbf{E}}'(\widehat{\boldsymbol{u}})\widehat{\boldsymbol{v}})\,\mathrm{d}\widehat{x} \\
&= \int_{\widehat{\Omega}} \widehat{\boldsymbol{\mathcal{R}}}(\cdot, \widehat{\mathbf{E}}(\widehat{\boldsymbol{u}})) : (\widehat{\mathbf{E}}'(\widehat{\boldsymbol{u}})\widehat{\boldsymbol{v}})\,\mathrm{d}\widehat{x},
\end{aligned}$$

or, componentwise,

$$\begin{aligned}
\int_{\widehat{\Omega}} (\widehat{\sigma}^{ij} + \widehat{\sigma}^{kj}\widehat{\partial}_k\widehat{u}^i)\widehat{\partial}_j\widehat{v}_i\,\mathrm{d}\widehat{x} &= \int_{\widehat{\Omega}} \widehat{\sigma}^{ij}(E'_{ij}(\widehat{\boldsymbol{u}})\widehat{\boldsymbol{v}})\,\mathrm{d}\widehat{x} \\
&= \int_{\widehat{\Omega}} \mathcal{R}^{ij}(\widehat{x}, \widehat{\mathbf{E}}(\widehat{\boldsymbol{u}}))(\widehat{E}'_{ij}(\widehat{\boldsymbol{u}})\widehat{\boldsymbol{v}})\,\mathrm{d}\widehat{x}.
\end{aligned}$$

This observation motivates the following definition: An elastic material is **hyperelastic** if there exists a **stored energy function** $\widehat{\mathcal{W}} : \{\widehat{\Omega}\}^- \times \mathbb{S}^3 \to \mathbb{R}$ such that

$$\widehat{\boldsymbol{\mathcal{R}}}(\widehat{x}, \widehat{\mathbf{E}}) = \frac{\partial \widehat{\mathcal{W}}}{\partial \widehat{\mathbf{E}}}(\widehat{x}, \widehat{\mathbf{E}}) \text{ for all } (\widehat{x}, \widehat{\mathbf{E}}) \in \{\widehat{\Omega}\}^- \times \mathbb{S}^3,$$

or equivalently, such that

$$\widehat{\mathcal{R}}^{ij}(\widehat{x}, \widehat{\mathbf{E}}) = \frac{\partial \widehat{\mathcal{W}}}{\partial \widehat{E}_{ij}}(\widehat{x}, \widehat{E}) \text{ for all } (\widehat{x}, \widehat{\mathbf{E}}) = (\widehat{x}, (\widehat{E}_{ij})) \in \{\widehat{\Omega}\}^- \times \mathbb{S}^3.$$

If the elastic material is hyperelastic, the principle of virtual work thus takes the form

$$\int_{\widehat{\Omega}} \frac{\partial \widehat{\mathcal{W}}}{\partial \mathbf{E}}(\cdot, \widehat{\mathbf{E}}(\widehat{\boldsymbol{u}})) : (\widehat{\mathbf{E}}'(\widehat{\boldsymbol{u}})\widehat{\boldsymbol{v}})\, \mathrm{d}\widehat{x} = \int_{\widehat{\Omega}} \widehat{\boldsymbol{f}} \cdot \widehat{\boldsymbol{v}}\, \mathrm{d}\widehat{x} + \int_{\widehat{\Gamma}_1} \widehat{\boldsymbol{h}} \cdot \widehat{\boldsymbol{v}}\, \mathrm{d}\widehat{\Gamma}$$

for all $\widehat{\boldsymbol{v}} \in \mathbf{W}(\widehat{\Omega})$, so that it becomes formally equivalent to the equations

$$\widehat{J}'(\widehat{\boldsymbol{u}})\widehat{\boldsymbol{v}} = 0 \text{ for all } \widehat{\boldsymbol{v}} \in \mathbf{W}(\widehat{\Omega}),$$

where $\widehat{J}'(\widehat{\boldsymbol{u}})\widehat{\boldsymbol{v}}$ denotes the *Gâteaux derivative* at $\widehat{\boldsymbol{u}}$ of the **energy** $\widehat{J} : \mathbf{W}(\widehat{\Omega}) \to \mathbb{R}$ defined for all $\widehat{\boldsymbol{v}} \in \mathbf{W}(\widehat{\Omega})$ by

$$\widehat{J}(\widehat{\boldsymbol{v}}) = \int_{\widehat{\Omega}} \widehat{\mathcal{W}}(\cdot, \widehat{\mathbf{E}}(\widehat{\boldsymbol{v}}))\, \mathrm{d}\widehat{x} - \Big\{ \int_{\widehat{\Omega}} \widehat{\boldsymbol{f}} \cdot \widehat{\boldsymbol{v}}\, \mathrm{d}\widehat{x} + \int_{\widehat{\Gamma}_1} \widehat{\boldsymbol{h}} \cdot \widehat{\boldsymbol{v}}\, \mathrm{d}\widehat{\Gamma} \Big\}.$$

Naturally, an *ad hoc* topology in the space $\mathbf{W}(\widehat{\Omega})$ must be again specified, so that the energy $\widehat{J}$ is differentiable on that space.

To sum up, *if the elastic material is hyperelastic, finding the unknown displacement field* $\widehat{\mathbf{u}}$ *amounts, at least formally, to finding the stationary points* (hence in particular, the minimizers) *of the energy* $\widehat{J}$ *over an* ad hoc *space* $\mathbf{W}(\widehat{\Omega})$ *of vector fields* $\widehat{\boldsymbol{v}}$ *satisfying the boundary conditions* $\widehat{\boldsymbol{v}} = \mathbf{0}$ *on* $\widehat{\Gamma}_0$, i.e., to *finding those vector fields* $\widehat{\mathbf{u}} \in \mathbf{W}(\widehat{\Omega})$ *such that the Fréchet derivative* $J'(\widehat{\boldsymbol{u}})$ *vanishes.*

In other words, *the boundary value problem of three-dimensional nonlinear elasticity becomes, at least formally, equivalent to a problem in the calculus of variations if the material is hyperelastic.*

This observation was put to a beautiful use when Ball [1977] established in a landmark paper the existence of *minimizers of the energy* for hyperelastic materials whose stored energy is *polyconvex* and satisfies *ad hoc growth conditions* (the notion of polyconvexity, which is due to John Ball, plays a fundamental role in the calculus of variations). This theory accommodates non-smooth boundaries and boundary conditions such as those considered here and is not restricted to "small enough" forces. However, it does not provide the existence of a solution to the corresponding variational problem (let alone to the original boundary value problem), because the energy is *not* differentiable in the spaces where the minimizers are found (a detailed account of John Ball's theory is also found in Ciarlet [1988, Chapter 7]).

If the elastic material is *homogeneous* and *isotropic* and the reference configuration $\{\widehat{\Omega}\}^-$ is a **natural state**, i.e., is *"stress-free"* (these notions are defined in *ibid.*, Chapter 3), the response function takes a remarkably simple form "for small deformations" (*ibid.*, Theorem 3.8-1): There exist two *constants* λ and μ, called the **Lamé constants** of the material, such that

$$\widehat{\boldsymbol{\mathcal{R}}}(\widehat{\mathbf{E}}) = \lambda(\operatorname{tr}\widehat{\mathbf{E}})\mathbf{I} + 2\mu\widehat{\mathbf{E}} + o(\widehat{\mathbf{E}}) \text{ for all } \mathbf{E} \in \mathbb{S}^3$$

(the response function no longer depends on $\widehat{x} \in \widehat{\Omega}$ by virtue of the assumption of homogeneity). Equivalently,

$$\widehat{\mathcal{R}}^{ij}(\widehat{\mathbf{E}}) = \widehat{A}^{ijk\ell}\widehat{E}_{k\ell} + o(\widehat{\mathbf{E}}) \text{ for all } \widehat{\mathbf{E}} = (\widehat{E}_{k\ell}) \in \mathbb{S}^3,$$

where the constants

$$\widehat{A}^{ijk\ell} = \lambda\delta^{ij}\delta^{k\ell} + \mu(\delta^{ik}\delta^{j\ell} + \delta^{i\ell}\delta^{jk})$$

are called the **Cartesian components of the elasticity tensor** (characterizing the elastic body under consideration). Experimental evidence shows that the Lamé constants of actual elastic materials satisfy $3\lambda + 2\mu > 0$ and $\mu > 0$.

For such materials, an *existence theory* is available that relies on the *implicit function theorem*. It is, however, restricted to smooth boundaries, to "small enough" applied forces, and to special classes of boundary conditions, which do *not* include those of the displacement-traction problems considered here (save exceptional cases). See Ciarlet [1988, Section 6.7] and Valent [1988].

Detailed expositions of the *modeling* of *three-dimensional nonlinear elasticity* are found in Truesdell & Noll [1965], Germain [1972], Wang & Truesdell [1973], Germain [1981], Marsden & Hughes [1983, Chapters 1–5], and Ciarlet [1988, Chapters 1–5]. Its *mathematical theory* is exposed in Ball [1977], Marsden & Hughes [1983], Valent [1988], and Ciarlet [1988, Chapters 6 and 7].

3.2 PRINCIPLE OF VIRTUAL WORK IN CURVILINEAR COORDINATES

Our first objective is to transform the principle of virtual work expressed in *Cartesian* coordinates (Section 3.1) into similar equations, but now expressed in arbitrary *curvilinear* coordinates.

Remark. Our point of departure could be as well the formally equivalent equations of equilibrium in Cartesian coordinates. It has been instead preferred here to derive the equations of equilibrium in curvilinear coordinates as natural corollaries to the principle of virtual work in curvilinear coordinates; see Section 3.3. □

Our *point of departure* thus consists of the variational equations

$$\int_{\widehat{\Omega}} \widehat{\sigma}^{ij}(\widehat{E}'_{ij}(\widehat{\boldsymbol{u}})\widehat{\boldsymbol{v}})\,\mathrm{d}\widehat{x} = \int_{\widehat{\Omega}} \widehat{f}^i\widehat{v}_i\,\mathrm{d}\widehat{x} + \int_{\widehat{\Gamma}_1} \widehat{h}^i\widehat{v}_i\,\mathrm{d}\widehat{\Gamma},$$

which are satisfied for all $\widehat{\boldsymbol{v}} = \widehat{v}_i\widehat{\boldsymbol{e}}^i \in \mathbf{W}(\widehat{\Omega})$. We recall that $\widehat{\Omega}$ is a domain in $\mathbf{E}^3$ with its boundary $\widehat{\Gamma}$ partitioned as $\widehat{\Gamma} = \widehat{\Gamma}_0 \cup \widehat{\Gamma}_1$, $\mathbf{W}(\widehat{\Omega})$ is a space of sufficiently smooth vector fields $\widehat{\boldsymbol{v}} = (\widehat{v}_i)$ that vanish on $\widehat{\Gamma}_0$, $\widehat{\boldsymbol{u}} = (\widehat{u}_i) \in \mathbf{W}(\widehat{\Omega})$ is the displacement field of the reference configuration $\{\widehat{\Omega}\}^-$, the functions $\widehat{\sigma}^{ij} = \widehat{\sigma}^{ji} : \{\widehat{\Omega}\}^- \to \mathbb{R}$ are the second Piola-Kirchhoff stresses, the functions

$$\widehat{E}'_{ij}(\widehat{\boldsymbol{u}})\widehat{\boldsymbol{v}} = \frac{1}{2}(\widehat{\partial}_j\widehat{v}_i + \widehat{\partial}_i\widehat{v}_j + \widehat{\partial}_i\widehat{u}_m\widehat{\partial}_j\widehat{v}^m + \widehat{\partial}_j\widehat{u}_m\widehat{\partial}_i\widehat{v}^m)$$

are the Gâteaux derivatives of the Green-St Venant strains $\widehat{E}_{ij} : \mathbf{W}(\widehat{\Omega}) \to \mathbb{R}$ defined by

$$\widehat{E}_{ij}(\widehat{\boldsymbol{v}}) = \frac{1}{2}(\widehat{\partial}_i \widehat{v}_j + \widehat{\partial}_j \widehat{v}_i + \widehat{\partial}_i \widehat{v}_m \widehat{\partial}_j \widehat{v}^m),$$

and finally, $(\widehat{f}^i) : \widehat{\Omega} \to \mathbb{R}^3$ and $(\widehat{h}^i) : \widehat{\Gamma}_1 \to \mathbb{R}^3$ are the densities of the applied forces.

The above equations are expressed in terms of the *Cartesian coordinates* $\widehat{x}_i$ of the points $\widehat{x} = (\widehat{x}_i) \in \{\widehat{\Omega}\}^-$ and of the *Cartesian components* of the functions $\widehat{\sigma}^{ij}, \widehat{u}_i, \widehat{v}_i, \widehat{f}^i$, and $\widehat{h}^i$.

Assume that we are also given a domain Ω in $\mathbb{R}^3$ and a smooth enough injective immersion $\boldsymbol{\Theta} : \overline{\Omega} \to \mathbf{E}^3$ such that $\boldsymbol{\Theta}(\overline{\Omega}) = \{\widehat{\Omega}\}^-$. Our objective consists in expressing the equations of the principle of virtual work in terms of the *curvilinear coordinates* x_i of the points $\widehat{x} = \boldsymbol{\Theta}(x) \in \{\widehat{\Omega}\}^-$, where $x = (x_i) \in \overline{\Omega}$.

In other words, we wish to carry out a *change of variables*, from the "old" variables $\widehat{x}_i$ to the "new" variables x_i, in each one of the integrals appearing in the above variational equations, which we thus wish to write as

$$\int_{\widehat{\Omega}} \cdots \, d\widehat{x} = \int_{\Omega} \cdots \, dx \quad \text{and} \quad \int_{\widehat{\Gamma}_1} \cdots \, d\widehat{\Gamma} = \int_{\Gamma_1} \cdots \, d\Gamma,$$

where Γ_1 is the subset of the boundary of Ω that satisfies $\boldsymbol{\Theta}(\Gamma_1) = \widehat{\Gamma}_1$. As expected, we shall make an extensive use of notions introduced in Chapter 1 in this process.

A word of caution. From now on, we shall freely extend *without notice* all the definitions given (for instance, that of an immersion), or properties established, in Chapter 1 on *open sets* to their analogs on *closures of domains*. In particular, the definition of m-th order differentiability, $m \geq 1$, can be extended as follows for functions defined over the closure of a domain. Let Ω be an open subset of $\mathbb{R}^n$, $n \geq 2$. For any integer $m \geq 1$, define the space $\mathcal{C}^m(\overline{\Omega})$ as the *subspace of the space* $\mathcal{C}^m(\Omega)$ consisting of all functions $f \in \mathcal{C}^m(\Omega)$ that, together with all their partial derivatives of order $\leq m$, possess continuous extensions to the closure $\overline{\Omega}$. Because, in particular, of a deep extension theorem of Whitney [1934], one can then show that, *if the boundary of* Ω *is Lipschitz-continuous*, the space $\mathcal{C}^m(\overline{\Omega})$ can be defined equivalently as

$$\mathcal{C}^m(\overline{\Omega}) = \{f|_\Omega;\ f \in \mathcal{C}^m(\mathbb{R}^n)\}$$

(irrespective of whether Ω is bounded); for a proof, see, e.g., Ciarlet & C. Mardare [2004a, Theorem 4.2]. For further results in this direction, see Stein [1970]. □

Because the "old" unknowns $\widehat{u}_i : \{\widehat{\Omega}\}^- \to \mathbb{R}$ are the components of a *vector field*, some care evidently must be exercised in the definition of the "new" unknowns, which must reflect the "physical invariance" of the *displacement vector* $\widehat{u}_i(\widehat{x})\widehat{\boldsymbol{e}}^i$ at each point $\widehat{x} \in \{\widehat{\Omega}\}^-$. Accordingly, we proceed as in Section 1.4.

Since the mapping $\boldsymbol{\Theta} : \overline{\Omega} \to \mathbf{E}^3$ considered above is assumed to be an *immersion*, the three vectors $\boldsymbol{g}_i(x) := \partial_i \boldsymbol{\Theta}(x)$, which are linearly independent at all points $x \in \overline{\Omega}$, thus form the *covariant basis* at $\widehat{x} = \boldsymbol{\Theta}(x) \in \{\widehat{\Omega}\}^-$ (Section 1.2). We may thus unambiguously define three *new unknowns* $u_i : \overline{\Omega} \to \mathbb{R}$ by requiring that

$$u_i(x)\boldsymbol{g}^i(x) := \widehat{u}_i(\widehat{x})\widehat{\boldsymbol{e}}^i \text{ for all } \widehat{x} = \boldsymbol{\Theta}(x),\ x \in \overline{\Omega},$$

where the three vectors $\boldsymbol{g}^i(x)$ form the *contravariant basis* at $\widehat{x} = \boldsymbol{\Theta}(x) \in \{\widehat{\Omega}\}^-$ (see Figure 1.4-2). Using the relations $\boldsymbol{g}^i(x) \cdot \boldsymbol{g}_j(x) = \delta^i_j$ and $\widehat{\boldsymbol{e}}^i \cdot \widehat{\boldsymbol{e}}_j = \delta^i_j$, we note (again as in Section 1.4) that the old and new unknowns are related by

$$u_j(x) = \widehat{u}_i(\widehat{x})\widehat{\boldsymbol{e}}^i \cdot \boldsymbol{g}_j(x) \text{ and } \widehat{u}_i(\widehat{x}) = u_j(x)\boldsymbol{g}^j(x) \cdot \widehat{\boldsymbol{e}}_i.$$

Let

$$[\boldsymbol{g}_j(x)]^i := \boldsymbol{g}_j(x) \cdot \widehat{\boldsymbol{e}}^i \text{ and } [\boldsymbol{g}^j(x)]_i := \boldsymbol{g}^j(x) \cdot \widehat{\boldsymbol{e}}_i,$$

i.e., $[\boldsymbol{g}_j(x)]^i$ denotes the i-th component of the vector $\boldsymbol{g}_j(x)$, and $[\boldsymbol{g}^j(x)]_i$ denotes the i-th component of the vector $\boldsymbol{g}^j(x)$, over the basis $\{\widehat{\boldsymbol{e}}^1, \widehat{\boldsymbol{e}}^2, \widehat{\boldsymbol{e}}^3\} = \{\widehat{\boldsymbol{e}}_1, \widehat{\boldsymbol{e}}_2, \widehat{\boldsymbol{e}}_3\}$ of the Euclidean space $\mathbf{E}^3$. In terms of these notations, the preceding relations thus become

$$u_j(x) = \widehat{u}_i(\widehat{x})[\boldsymbol{g}_j(x)]^i \text{ and } \widehat{u}_i(\widehat{x}) = u_j(x)[\boldsymbol{g}^j(x)]_i \text{ for all } \widehat{x} = \boldsymbol{\Theta}(x),\ x \in \overline{\Omega}.$$

The three components $u_i(x)$ are called the **covariant components of the displacement vector** $u_i(x)\boldsymbol{g}^i(x)$ **at** $\widehat{x}$, and the three functions $u_i : \overline{\Omega} \to \mathbb{R}$ defined in this fashion are called the **covariant components of the displacement field** $u_i\boldsymbol{g}^i : \overline{\Omega} \to \mathbb{R}^3$.

A word of caution. While the "old" unknown vector $(\widehat{u}_i(\widehat{x})) \in \mathbb{R}^3$ can be, *and will henceforth be*, justifiably identified with the *displacement vector* $\widehat{\boldsymbol{u}}(\widehat{x}) = \widehat{u}_i(\widehat{x})\widehat{\boldsymbol{e}}^i$ since the basis $\{\widehat{\boldsymbol{e}}^1, \widehat{\boldsymbol{e}}^2, \widehat{\boldsymbol{e}}^3\}$ is fixed in $\mathbf{E}^3$, *this is no longer true for the "new" unknown vector* $(u_i(x)) \in \mathbb{R}^3$, since its components $u_i(x)$ now represent the components of the displacement vector over the basis $\{\boldsymbol{g}^1(x), \boldsymbol{g}^2(x), \boldsymbol{g}^3(x)\}$, *which varies with* $x \in \overline{\Omega}$.

In the same vein, *the vector fields*

$$\boldsymbol{u} = (u_i) \text{ and } \widetilde{\boldsymbol{u}} := u_i\boldsymbol{g}^i,$$

which are both defined on $\overline{\Omega}$, *must be carefully distinguished*! While the latter has an *intrinsic character*, the former has not; it only provides a means of recovering the field $\widetilde{\boldsymbol{u}}$ *via* its covariant components u_i. □

We likewise associate "new" functions $v_i : \overline{\Omega} \to \mathbb{R}$ with the "old" functions $\widehat{v}_i : \{\widehat{\Omega}\}^- \to \mathbb{R}$ appearing in the equations of the principle of virtual work by letting

$$v_i(x)\boldsymbol{g}^i(x) := \widehat{v}_i(\widehat{x})\widehat{\boldsymbol{e}}^i \text{ for all } \widehat{x} = \boldsymbol{\Theta}(x),\ x \in \overline{\Omega}.$$

We begin the change of variables by considering the integrals found in the *right-hand side* of the variational equations of the principle of virtual work, i.e., those corresponding to the applied forces. With the Cartesian components $\widehat{f}^i : \widehat{\Omega} = \boldsymbol{\Theta}(\Omega) \to \mathbb{R}$ of the applied body force density, let there be associated its **contravariant components** $f^i : \Omega \to \mathbb{R}$, defined by

$$f^i(x)\boldsymbol{g}_i(x) := \widehat{f}^i(\widehat{x})\widehat{\boldsymbol{e}}_i \text{ for all } \widehat{x} = \boldsymbol{\Theta}(x),\ x \in \Omega.$$

This definition shows that

$$f^i(x) = \widehat{f}^j(\widehat{x})[\boldsymbol{g}^i(x)]_j,$$

and consequently that

$$\begin{aligned}\widehat{f}^i(\widehat{x})\widehat{v}_i(\widehat{x}) &= (\widehat{f}^i(\widehat{x})\widehat{\boldsymbol{e}}_i) \cdot (\widehat{v}_j(\widehat{x})\widehat{\boldsymbol{e}}^j)\\ &= (f^i(x)\boldsymbol{g}_i(x)) \cdot (v_j(x)\boldsymbol{g}^j(x)) = f^i(x)v_i(x)\end{aligned}$$

for all $\widehat{x} = \boldsymbol{\Theta}(x)$, $x \in \Omega$. Hence

$$\widehat{f}^i(\widehat{x})\widehat{v}_i(\widehat{x})\,\mathrm{d}\widehat{x} = f^i(x)v_i(x)\sqrt{g(x)}\,\mathrm{d}x \text{ for all } \widehat{x} = \boldsymbol{\Theta}(x),\ x \in \Omega,$$

since $\mathrm{d}\widehat{x} = \sqrt{g(x)}\,\mathrm{d}x$ (Theorem 1.3-1 (a)), and thus

$$\int_{\widehat{\Omega}} \widehat{f}^i\widehat{v}_i\,\mathrm{d}\widehat{x} = \int_{\Omega} f^i v_i \sqrt{g}\,\mathrm{d}x.$$

Remark. What has just been proved is in effect the *invariance of the number* $f^i(x)v_i(x)$ *with respect to changes of curvilinear coordinates*, provided one vector (here $f^i(x)\boldsymbol{g}_i(x)$) appears by means of its contravariant components (i.e., over the covariant basis) and the other vector (here $v_i(x)\boldsymbol{g}^i(x)$) appears by means of its covariant components (i.e., over the contravariant basis). Naturally, this number is nothing but the *Euclidean inner product of the two vectors.* □

With the Cartesian components $\widehat{h}^i : \widehat{\Gamma}_1 = \boldsymbol{\Theta}(\Gamma_1) \to \mathbb{R}$ of the applied surface force density, let there likewise be associated its **contravariant components** $h^i : \Gamma_1 \to \mathbb{R}$, defined by

$$h^i(x)\boldsymbol{g}_i(x)\sqrt{g(x)}\,\mathrm{d}\Gamma(x) := \widehat{h}^i(\widehat{x})\widehat{\boldsymbol{e}}_i\,\mathrm{d}\widehat{\Gamma}(\widehat{x}) \text{ for all } \widehat{x} = \boldsymbol{\Theta}(x),\ x \in \Gamma_1,$$

where the area elements $\mathrm{d}\widehat{\Gamma}(\widehat{x})$ at $\widehat{x} = \boldsymbol{\Theta}(x) \in \widehat{\Gamma}_1$ and $\mathrm{d}\Gamma(x)$ at $x \in \Gamma_1$ are related by

$$\mathrm{d}\widehat{\Gamma}(\widehat{x}) = \sqrt{g(x)}\sqrt{n_k(x)g^{k\ell}(x)n_\ell(x)}\,\mathrm{d}\Gamma(x),$$

where the functions $n_k : \Gamma_1 \to \mathbb{R}$ denote the components of the unit outer normal vector field along the boundary of Ω (Theorem 1.3-1 (b)).

This definition shows that

$$\widehat{h}^i(\widehat{x})\widehat{\boldsymbol{e}}_i = \left\{n_k(x)g^{k\ell}(x)n_\ell(x)\right\}^{-1/2} h^i(x)\boldsymbol{g}_i(x) \text{ for all } \widehat{x} = \boldsymbol{\Theta}(x),\ x \in \Gamma_1,$$

and that

$$h^i(x) = \sqrt{n_k(x)g^{k\ell}(x)n_\ell(x)}\,\widehat{h}^j(\widehat{x})[\boldsymbol{g}^i(x)]_j.$$

The factor $\sqrt{g}$ introduced in the definition of the functions h^i implies that

$$\int_{\widehat{\Gamma}_1} \widehat{h}^i \widehat{v}_i \,\mathrm{d}\widehat{\Gamma} = \int_{\Gamma_1} h^i v_i \sqrt{g}\,\mathrm{d}\Gamma.$$

As a result, the *same* factor $\sqrt{g}$ appears in both integrals $\int_\Omega f^i v_i \sqrt{g}\,\mathrm{d}x$ and $\int_{\Gamma_1} h^i v_i \sqrt{g}\,\mathrm{d}\Gamma$. This common factor in turn gives rise to a more "natural" boundary condition on Γ_1 when the variational equations of the principle of virtual work are used to derive the *equilibrium equations in curvilinear coordinates* (Theorem 3.3-1).

Since our treatment in this section is essentially *formal* (as in Section 3.1), the functions $\widehat{u}_i, \widehat{v}_i, \widehat{\sigma}^{ij}, \widehat{f}^i$, and $\widehat{h}^i$ appearing in the next theorem are again assumed to be smooth enough, so as to insure that all the computations involved make sense.

Transforming the integrals appearing in the *left-hand side* of the variational equations of the principle of virtual work relies in particular on the fundamental notion of *covariant differentiation of a vector field*, introduced in Section 1.4.

Theorem 3.2-1. *Let $\widehat{\Omega}$ be a domain in $\mathbf{E}^3$, let Ω be a domain in $\mathbb{R}^3$, and let $\boldsymbol{\Theta} : \overline{\Omega} \to \mathbf{E}^3$ be an injective immersion that is also a $\mathcal{C}^2$-diffeomorphism from $\overline{\Omega}$ onto $\{\widehat{\Omega}\}^- = \boldsymbol{\Theta}(\overline{\Omega})$.*

Define the vector fields $\boldsymbol{u} = (u_i) : \overline{\Omega} \to \mathbb{R}^3$ and $\boldsymbol{v} = (v_i) : \overline{\Omega} \to \mathbb{R}^3$ and the symmetric matrix fields $(\sigma^{ij}) : \overline{\Omega} \to \mathbb{S}^3$ and $(E_{ij}(\boldsymbol{v})) : \overline{\Omega} \to \mathbb{S}^3$ by

$$u_i(x) := \widehat{u}_k(\widehat{x})[\boldsymbol{g}_i(x)]^k \text{ and } v_i(x) := \widehat{v}_k(\widehat{x})[\boldsymbol{g}_i(x)]^k, \text{ for all } \widehat{x} = \boldsymbol{\Theta}(x),\ x \in \overline{\Omega},$$
$$\sigma^{ij}(x) := \widehat{\sigma}^{k\ell}(\widehat{x})[\boldsymbol{g}^i(x)]_k[\boldsymbol{g}^j(x)]_\ell \text{ for all } \widehat{x} = \boldsymbol{\Theta}(x),\ x \in \overline{\Omega},$$
$$E_{ij}(\boldsymbol{v})(x) := \left(\widehat{E}_{k\ell}(\widehat{\boldsymbol{v}})(\widehat{x})\right)[\boldsymbol{g}_i(x)]^k[\boldsymbol{g}_j(x)]^\ell \text{ for all } \widehat{x} = \boldsymbol{\Theta}(x),\ x \in \overline{\Omega}.$$

Then the functions $E_{ij} = E_{ji} : \mathbf{W}(\Omega) \to \mathbb{R}$ are also given by

$$E_{ij}(\boldsymbol{v}) := \frac{1}{2}(v_{i\|j} + v_{j\|i} + g^{mn} v_{m\|i} v_{n\|j}) \text{ for all } \boldsymbol{v} = (v_i) : \overline{\Omega} \to \mathbb{R}^3,$$

where the functions $v_{i\|j} := \partial_j v_i - \Gamma^p_{ij} v_p$, where $\Gamma^p_{ij} := \boldsymbol{g}^p \cdot \partial_i \boldsymbol{g}_j$, are the covariant derivatives of the vector field $v_i \boldsymbol{g}^i : \overline{\Omega} \to \mathbb{R}^3$. The Gâteaux derivatives $E'_{ij}(\widehat{\boldsymbol{u}})\widehat{\boldsymbol{v}}$ are then related to the Gâteaux derivatives

$$E'_{ij}(\boldsymbol{u})\boldsymbol{v} := \frac{1}{2}\big(v_{i\|j} + v_{j\|i} + g^{mn}\{u_{m\|i} v_{n\|j} + u_{n\|j} v_{m\|i}\}\big)$$

of the functions $E_{i\|j} : \mathbf{W}(\Omega) \to \mathbb{R}$ by the relations

$$\left(\widehat{E}'_{ij}(\widehat{\boldsymbol{u}})\widehat{\boldsymbol{v}}\right)(\widehat{x}) = \left(E'_{k\ell}(\boldsymbol{u})\boldsymbol{v}[\boldsymbol{g}^k]_i[\boldsymbol{g}^\ell]_j\right)(x) \text{ at all } \widehat{x} \in \boldsymbol{\Theta}(x),\ x \in \overline{\Omega}.$$

Let $\mathbf{W}(\Omega)$ *denote a space of sufficiently smooth vector fields* $\boldsymbol{v} = (v_i) : \overline{\Omega} \to \mathbb{R}^3$ *that vanish on* Γ_0. *Then the above matrix and vector fields satisfy the following variational equations:*

$$\int_\Omega \sigma^{ij} \left(E'_{ij}(\boldsymbol{u})\boldsymbol{v}\right) \sqrt{g}\, \mathrm{d}x = \int_\Omega f^i v_i \sqrt{g}\, \mathrm{d}x + \int_{\Gamma_1} h^i v_i \sqrt{g}\, \mathrm{d}\Gamma$$

for all $\boldsymbol{v} = (v_i) \in \mathbf{W}(\Omega)$, *where* $\Gamma_0 := \boldsymbol{\Theta}^{-1}(\widehat{\Gamma}_0)$ *and* $\Gamma_1 := \boldsymbol{\Theta}^{-1}(\widehat{\Gamma}_1)$.

Proof. The following conventions hold throughout this proof: The simultaneous appearance of $\widehat{x}$ and x in an equality means that they are related by $\widehat{x} = \boldsymbol{\Theta}(x)$ and that the equality in question holds for all $x \in \overline{\Omega}$. The appearance of x alone in a relation means that this relation holds for all $x \in \overline{\Omega}$.

(i) *Expression of the matrix field* $(\widehat{\sigma}^{ij}) : \{\widehat{\Omega}\}^- \to \mathbb{S}^3$ *in terms of the matrix field* $(\sigma^{ij}) : \overline{\Omega} \to \mathbb{S}^3$ *as defined in the statement of the theorem.*

In part (i) of the proof of Theorem 1.4-1, it was shown that $[\boldsymbol{g}^i(x)]_k = \widehat{\partial}_k \widehat{\Theta}^i(\widehat{x})$, where $\widehat{\boldsymbol{\Theta}} = \widehat{\Theta}^i \boldsymbol{e}_i : \{\widehat{\Omega}\}^- \to \mathbb{R}^3$ denotes the inverse mapping of $\boldsymbol{\Theta} = \Theta^k \widehat{\boldsymbol{e}}_k : \overline{\Omega} \to \mathbf{E}^3$. Since $\partial_j \Theta^\ell(x) = [\boldsymbol{g}_j(x)]^\ell$, the relation $\boldsymbol{\nabla}\boldsymbol{\Theta}(x)\widehat{\boldsymbol{\nabla}}\widehat{\boldsymbol{\Theta}}(\widehat{x}) = \mathbf{I}$ shows that

$$[\boldsymbol{g}^p(x)]_k [\boldsymbol{g}_p(x)]^i = \delta^i_k.$$

We thus have

$$\begin{aligned}
\widehat{\sigma}^{ij}(\widehat{x}) &= \widehat{\sigma}^{k\ell}(\widehat{x})\delta^i_k \delta^j_\ell \\
&= \widehat{\sigma}^{k\ell}(\widehat{x})[\boldsymbol{g}^p(x)]_k [\boldsymbol{g}_p(x)]^i [\boldsymbol{g}^q(x)]_\ell [\boldsymbol{g}_q(x)]^j \\
&= \sigma^{pq}(x)[\boldsymbol{g}_p(x)]^i [\boldsymbol{g}_q(x)]^j,
\end{aligned}$$

since, by definition of the functions σ^{pq},

$$\widehat{\sigma}^{k\ell}(\widehat{x})[\boldsymbol{g}^p(x)]_k [\boldsymbol{g}^q(x)]_\ell = \sigma^{pq}(x).$$

(ii) *Expressions of the functions* $\widehat{E}_{ij}(\widehat{v}) : \{\widehat{\Omega}\}^- \to \mathbb{R}$ *in terms of the functions* $E_{ij}(\boldsymbol{v}) : \overline{\Omega} \to \mathbb{R}$ *as defined in the statement of the theorem.*

We likewise have

$$\begin{aligned}
\widehat{E}_{ij}(\widehat{\boldsymbol{v}})(\widehat{x}) &= \widehat{E}_{pq}(\widehat{\boldsymbol{v}})(\widehat{x})\delta^p_i \delta^q_j \\
&= \widehat{E}_{pq}(\widehat{\boldsymbol{v}})(\widehat{x})[\boldsymbol{g}_k(x)]^p [\boldsymbol{g}_\ell(x)]^q [\boldsymbol{g}^k(x)]_i [\boldsymbol{g}^\ell(x)]_j \\
&= \Big(E_{k\ell}(\boldsymbol{v})[\boldsymbol{g}^k]_i [\boldsymbol{g}^\ell]_j\Big)(x),
\end{aligned}$$

since, by definition of the functions $E_{k\ell}$,

$$\Big(\widehat{E}_{pq}(\widehat{\boldsymbol{v}})(\widehat{x})\Big)[\boldsymbol{g}_k(x)]^p [\boldsymbol{g}_\ell(x)]^q = E_{k\ell}(\boldsymbol{v})(x).$$

(iii) *Expressions of the functions $E_{ij}(\boldsymbol{v})$ in terms of the fields $\boldsymbol{v} = (v_i) : \overline{\Omega} \to \mathbb{R}^3$ as defined in the statement of the theorem.*

In Theorem 1.4-1, it was established that

$$\widehat{\partial}_j \widehat{v}_i(\widehat{x}) = \Big(v_{k\|\ell}[\boldsymbol{g}^k]_i[\boldsymbol{g}^\ell]_j\Big)(x).$$

Then this relation and the relation

$$[\boldsymbol{g}^k(x)]_m[\boldsymbol{g}^\ell(x)]_m = \boldsymbol{g}^k(x)\cdot\boldsymbol{g}^\ell(x) = g^{k\ell}(x)$$

together imply that

$$\begin{aligned}\widehat{E}_{ij}(\widehat{\boldsymbol{v}})(\widehat{x}) &= \frac{1}{2}\left(\widehat{\partial}_i\widehat{v}_j + \widehat{\partial}_j\widehat{v}_i + \widehat{\partial}_i\widehat{v}_m\widehat{\partial}_j\widehat{v}^m\right)(\widehat{x})\\ &= \frac{1}{2}\Big((v_{k\|\ell} + v_{\ell\|k} + g^{mn}v_{m\|k}v_{n\|\ell})[\boldsymbol{g}^k]_i[\boldsymbol{g}^\ell]_j\Big)(x).\end{aligned}$$

Because of the equivalence (with self-explanatory notations)

$$\widehat{A}_{ij}(\widehat{x}) = \Big(A_{k\ell}[\boldsymbol{g}^k]_i[\boldsymbol{g}^\ell]_j\Big)(x) \iff A_{k\ell}(x) = \widehat{A}_{pq}(\widehat{x})\Big([\boldsymbol{g}_k]^p[\boldsymbol{g}_\ell]^q\Big)(x),$$

we thus conclude from part (ii) that the functions $E_{ij}(\boldsymbol{v})$ are also given by

$$E_{ij}(\boldsymbol{v}) = \frac{1}{2}(v_{i\|j} + v_{j\|i} + g^{mn}v_{m\|i}v_{n\|j}) \text{ for all } \boldsymbol{v} = (v_i) : \overline{\Omega} \to \mathbb{R}^3.$$

(iv) *Expression of the Gâteaux derivatives $\widehat{E}'_{ij}(\widehat{\boldsymbol{u}})\widehat{\boldsymbol{v}}$ in terms of the Gâteaux derivatives $E'_{ij}(\boldsymbol{u})(\boldsymbol{v})$.*

We likewise have

$$\begin{aligned}\left(\widehat{E}'_{ij}(\widehat{\boldsymbol{u}})\widehat{\boldsymbol{v}}\right)(\widehat{x}) &= \Big(\frac{1}{2}(\widehat{\partial}_j\widehat{v}_i + \widehat{\partial}_i\widehat{v}_j + \widehat{\partial}_i\widehat{u}_m\widehat{\partial}_j\widehat{v}^m + \widehat{\partial}_j\widehat{u}_m\widehat{\partial}_i\widehat{v}^m)\Big)(x)\\ &= \Big(\frac{1}{2}(v_{k\|\ell} + v_{\ell\|k} + g^{mn}\{u_{m\|k}v_{n\|\ell} + u_{n\|\ell}v_{m\|k}\})[\boldsymbol{g}^k]_i[\boldsymbol{g}^\ell]_j\Big)(x)\\ &= \Big((E'_{k\ell}(\boldsymbol{u})\boldsymbol{v})[\boldsymbol{g}^k]_i[\boldsymbol{g}^\ell]_j\Big)(x),\end{aligned}$$

since it is immediately verified that $E'_{ij}(\boldsymbol{u})\boldsymbol{v}$ is indeed given by (as the linear part with respect to $\boldsymbol{v}$ in the difference $\{E_{ij}(\boldsymbol{u}+\boldsymbol{v}) - E_{ij}(\boldsymbol{u})\}$):

$$E'_{ij}(\boldsymbol{u})\boldsymbol{v} = \frac{1}{2}\big(v_{i\|j} + v_{j\|i} + g^{mn}\{u_{m\|i}v_{n\|j} + u_{n\|j}v_{m\|i}\}\big).$$

(v) *Conclusions:* Parts (i) and (iv) together show that

$$\begin{aligned}\left(\widehat{\sigma}^{ij}(\widehat{E}'_{ij}(\widehat{\boldsymbol{u}})\widehat{\boldsymbol{v}})\right)(\widehat{x}) &= \big(\sigma^{pq}(E'_{k\ell}(\boldsymbol{u})\boldsymbol{v})[\boldsymbol{g}_p]^i[\boldsymbol{g}_q]^j[\boldsymbol{g}^k]_i[\boldsymbol{g}^\ell]_j\big)(x)\\ &= \big(\sigma^{ij}(E'_{ij}(\boldsymbol{u})\boldsymbol{v})\big)(x),\end{aligned}$$

since

$$[\boldsymbol{g}_p(x)]^i[\boldsymbol{g}^k(x)]_i = \boldsymbol{g}_p(x)\cdot\boldsymbol{g}^k(x) = \delta_p^k \text{ and } [\boldsymbol{g}_q(x)]^j[\boldsymbol{g}^\ell(x)]_j = \boldsymbol{g}_q(x)\cdot\boldsymbol{g}^\ell(x) = \delta_q^\ell.$$

Finally, $\mathrm{d}\widehat{x} = \sqrt{g(x)}\,\mathrm{d}x$ (Theorem 1.3-1 (a)). Therefore,

$$\int_{\widehat{\Omega}} \widehat{\sigma}^{ij}\left(\widehat{E}'_{ij}(\widehat{\boldsymbol{u}})\widehat{\boldsymbol{v}}\right)\mathrm{d}\widehat{x} = \int_\Omega \sigma^{ij}(E'_{ij}(\boldsymbol{u})\boldsymbol{v})\sqrt{g}\,\mathrm{d}x,$$

on the one hand. At the beginning of this section, it was also shown that the definition of the functions f^i and h^i implies that

$$\int_{\widehat{\Omega}} \widehat{f}^i\widehat{v}_i\,\mathrm{d}\widehat{x} = \int_\Omega f^i v_i\sqrt{g}\,\mathrm{d}x \text{ and } \int_{\widehat{\Gamma}_1} \widehat{h}^i\widehat{v}_i\,\mathrm{d}\widehat{\Gamma} = \int_{\Gamma_1} h^i v_i\sqrt{g}\,\mathrm{d}\Gamma,$$

on the other hand. Thus the proof is complete. □

Naturally, if $\mathbf{E}^3$ is identified with $\mathbb{R}^3$ and $\boldsymbol{\Theta} = \boldsymbol{id}_{\mathbb{R}^3}$, each vector $\boldsymbol{g}_i(x)$ is equal to $\widehat{\boldsymbol{e}}_i$ and thus $\boldsymbol{g}^i(x) = \widehat{\boldsymbol{e}}^i$, $g(x) = 1$, $g^{ij}(x) = \delta^{ij}$, and $\Gamma_{ij}^p(x) = 0$ for all $x \in \overline{\Omega}$. Consequently, the fields (σ^{ij}) and $(\widehat{\sigma}^{ij})$, (u_i) and $(\widehat{u}_i)$, (f^i) and $(\widehat{f}^i)$, and (h^i) and $(\widehat{h}^i)$ coincide in this case.

The variational problem found in Theorem 3.2-1 constitutes the **principle of virtual work in curvilinear coordinates** and the functions $\sigma^{ij} : \overline{\Omega} \to \mathbb{R}$ are the **contravariant components of the second Piola-Kirchhoff stress tensor field**. They are also called the **second Piola-Kirchhoff stresses in curvilinear coordinates**.

The functions $E_{ij}(\boldsymbol{v}) : \overline{\Omega} \to \mathbb{R}$ are the **covariant components of the Green-St Venant strain tensor field** associated with an arbitrary displacement field $v_i\boldsymbol{g}^i : \overline{\Omega} \to \mathbb{R}$ of the reference configuration $\boldsymbol{\Theta}(\overline{\Omega})$. They are also called the **Green-St Venant strains in curvilinear coordinates**.

We saw in Section 3.1 that the Green-St Venant strain tensor in *Cartesian coordinates* is indeed a "*change of metric*" tensor, since it may also be written as

$$\widehat{\mathbf{E}}(\widehat{\boldsymbol{v}}) = \frac{1}{2}\Big(\{\boldsymbol{\nabla}(\boldsymbol{id}_{\{\widehat{\Omega}\}^-} + \widehat{\boldsymbol{v}})\}^T\boldsymbol{\nabla}(\boldsymbol{id}_{\{\widehat{\Omega}\}^-} + \widehat{\boldsymbol{v}}) - \mathbf{I}\Big).$$

Naturally, this interpretation holds as well in *curvilinear coordinates*. Indeed, a straightforward computation shows that the Green-St Venant strains in curvilinear coordinates may also be written as

$$E_{ij}(\boldsymbol{v}) = \frac{1}{2}(g_{ij}(\boldsymbol{v}) - g_{ij}),$$

where

$$g_{ij}(\boldsymbol{v}) := \partial_i(\boldsymbol{\Theta} + v_k\boldsymbol{g}^k)\cdot\partial_j(\boldsymbol{\Theta} + v_\ell\boldsymbol{g}^\ell) \text{ and } g_{ij} = \partial_i\boldsymbol{\Theta}\cdot\partial_j\boldsymbol{\Theta}$$

respectively denote the covariant components of the metric tensors of the deformed configuration $(\boldsymbol{\Theta} + v_k\boldsymbol{g}^k)(\overline{\Omega})$ associated with a displacement field $v_k\boldsymbol{g}^k$, and of the reference configuration $\boldsymbol{\Theta}(\overline{\Omega})$.

3.3 EQUATIONS OF EQUILIBRIUM IN CURVILINEAR COORDINATES; COVARIANT DERIVATIVES OF A TENSOR FIELD

While deriving the equations of equilibrium, i.e., the boundary value problem that is formally equivalent to the variational equations of the principle of virtual work, simply amounts in *Cartesian coordinates* to applying the fundamental Green formula, doing so in *curvilinear coordinates* is more subtle. As we next show, it relies in particular on the notion of *covariant differentiation of a tensor field*.

As in Sections 3.1 and 3.2, our treatment is *formal*, in that *ad hoc* smoothness is implicitly assumed throughout. We recall that the space $\mathbf{W}(\Omega)$ denotes a space of sufficiently smooth vector fields $\boldsymbol{v} : \overline{\Omega} \to \mathbb{R}^3$ that vanish on Γ_0.

Theorem 3.3-1. *Let Ω be a domain in $\mathbb{R}^3$, let $\Gamma = \Gamma_0 \cup \Gamma_1$ be a partition of the boundary Γ of Ω, let n_i denote the components of the unit outer normal vector field along $\partial\Omega$, and let $f^i : \Omega \to \mathbb{R}$ and $h^i : \Gamma_1 \to \mathbb{R}$ be given functions. Then a symmetric matrix field $(\sigma^{ij}) : \overline{\Omega} \to \mathbb{S}^3$ and a vector field $\boldsymbol{u} = (u_i) : \overline{\Omega} \to \mathbb{R}^3$ satisfy the principle of virtual work in curvilinear coordinates found in* Theorem 3.2-1, *viz.,*

$$\int_\Omega \sigma^{ij}(E'_{ij}(\boldsymbol{u})\boldsymbol{v})\sqrt{g}\,\mathrm{d}x = \int_\Omega f^i v_i \sqrt{g}\,\mathrm{d}x + \int_{\Gamma_1} h^i v_i \sqrt{g}\,\mathrm{d}\Gamma$$

for all $\boldsymbol{v} = (v_i) \in \mathbf{W}(\Omega)$ if and only if these fields satisfy the following boundary value problem:

$$\begin{aligned} -(\sigma^{ij} + \sigma^{kj} g^{i\ell} u_{\ell\|k})\|_j &= f^i \text{ in } \Omega, \\ (\sigma^{ij} + \sigma^{kj} g^{i\ell} u_{\ell\|k}) n_j &= h^i \text{ on } \Gamma_1, \end{aligned}$$

where, for an arbitrary tensor field with smooth enough contravariant components $t^{ij} : \overline{\Omega} \to \mathbb{R}$,

$$t^{ij}\|_j := \partial_j t^{ij} + \Gamma^i_{pj} t^{pj} + \Gamma^j_{jq} t^{iq}.$$

Proof. (i) We first establish the relations

$$\partial_j \sqrt{g} = \sqrt{g}\,\Gamma^q_{qj}.$$

To this end, we recall that $\sqrt{g} = |\det \boldsymbol{\nabla\Theta}|$, that the column vectors of the matrix $\boldsymbol{\nabla\Theta}$ are $\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3$ (in this order), and that the vectors $\boldsymbol{g}_i$ are linearly independent at all points in $\overline{\Omega}$. Assume for instance that $\det \boldsymbol{\nabla\Theta} > 0$, so that

$$\sqrt{g} = \det \boldsymbol{\nabla\Theta} = \det(\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3) \text{ in } \overline{\Omega}.$$

Then

$$\begin{aligned} \partial_j \sqrt{g} &= \det(\partial_j \boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3) + \det(\boldsymbol{g}_1, \partial_j \boldsymbol{g}_2, \boldsymbol{g}_3) + \det(\boldsymbol{g}_1, \boldsymbol{g}_2, \partial_j \boldsymbol{g}_3) \\ &= \Gamma^p_{1j} \det(\boldsymbol{g}_p, \boldsymbol{g}_2, \boldsymbol{g}_3) + \Gamma^p_{2j} \det(\boldsymbol{g}_1, \boldsymbol{g}_p, \boldsymbol{g}_3) + \Gamma^p_{3j} \det(\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_p) \end{aligned}$$

since $\partial_j \boldsymbol{g}_q = \Gamma^p_{qj} \boldsymbol{g}_p$ (Theorem 1.4-2 (a)); hence

$$\partial_j \sqrt{g} = \left(\Gamma^1_{1j} + \Gamma^2_{2j} + \Gamma^3_{3j}\right) \det(\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3) = \Gamma^q_{qj} \sqrt{g}.$$

The proof is similar if $\sqrt{g} = -\det \boldsymbol{\nabla\Theta}$ in $\overline{\Omega}$.

(ii) Because of the symmetries $\sigma^{ij} = \sigma^{ji}$, the integrand in the left-hand side of the principle of virtual work (Theorem 3.2-1) can be re-written as

$$\sigma^{ij}(E'_{ij}(\boldsymbol{u})\boldsymbol{v}) = \sigma^{ij}(v_{i\|j} + g^{mn} u_{m\|i} v_{n\|j}).$$

Taking into account the relations $\partial_j \sqrt{g} = \sqrt{g}\, \Gamma^q_{qj}$ (part (i)) and using the fundamental Green formula

$$\int_\Omega (\partial_j v) w \,\mathrm{d}x = -\int_\Omega v \partial_j w \,\mathrm{d}x + \int_\Gamma v w n_j \,\mathrm{d}\Gamma,$$

we obtain:

$$\begin{aligned}
\int_\Omega \sigma^{ij} v_{i\|j} \sqrt{g} \,\mathrm{d}x &= \int_\Omega \sqrt{g} \sigma^{ij} \partial_j v_i \,\mathrm{d}x - \int_\Omega \sqrt{g} \sigma^{ij} \Gamma^p_{ij} v_p \,\mathrm{d}x \\
&= -\int_\Omega \partial_j \big(\sqrt{g} \sigma^{ij}\big) v_i \,\mathrm{d}x + \int_\Gamma \sqrt{g} \sigma^{ij} n_j v_i \,\mathrm{d}\Gamma - \int_\Omega \sqrt{g} \sigma^{pj} \Gamma^i_{pj} v_i \,\mathrm{d}x \\
&= -\int_\Omega \sqrt{g} \big(\partial_j \sigma^{ij} + \Gamma^i_{pj} \sigma^{pj} + \Gamma^j_{jq} \sigma^{iq}\big) v_i \,\mathrm{d}x + \int_\Gamma \sqrt{g} \sigma^{ij} n_j v_i \,\mathrm{d}\Gamma \\
&= -\int_\Omega (\sigma^{ij}\|_j) v_i \sqrt{g} \,\mathrm{d}x + \int_\Gamma \sigma^{ij} n_j v_i \sqrt{g} \,\mathrm{d}\Gamma
\end{aligned}$$

for all vector fields $\boldsymbol{v} = (v_i) : \overline{\Omega} \to \mathbb{R}^3$. We likewise obtain:

$$\begin{aligned}
\int_\Omega \sigma^{ij} g^{mn} u_{m\|i} v_{n\|j} \sqrt{g} \,\mathrm{d}x = &-\int_\Omega \sqrt{g} \big\{(\sigma^{kj} g^{i\ell} u_{\ell\|k})\|_j\big\} v_i \,\mathrm{d}x \\
&+ \int_\Gamma \sqrt{g} (\sigma^{kj} g^{i\ell} u_{\ell\|k}) n_j v_i \,\mathrm{d}\Gamma
\end{aligned}$$

for all vector fields $\boldsymbol{v} = (v_i) : \overline{\Omega} \to \mathbb{R}^3$. Hence the variational equations of the principle of virtual work imply that

$$\begin{aligned}
&\int_\Omega \sqrt{g} \big\{(\sigma^{ij} + \sigma^{kj} g^{i\ell} u_{\ell\|k})\|_j + f^i\big\} v_i \,\mathrm{d}x \\
&\qquad = \int_{\Gamma_1} \sqrt{g} \big\{(\sigma^{ij} + \sigma^{kj} g^{i\ell} u_{\ell\|k}) n_j - h^i\big\} v_i \,\mathrm{d}\Gamma
\end{aligned}$$

for all $(v_i) \in \mathbf{W}(\Omega)$. Letting (v_i) vary first in $(\mathcal{D}(\Omega))^3$, then in $\mathbf{W}(\Omega)$, yields the announced boundary value problem.

The converse is established by means of the same integration by parts formulas. □

The equations

$$
\begin{aligned}
-(\sigma^{ij} + \sigma^{kj} g^{i\ell} u_{\ell\|k})\|_j &= f^i \text{ in } \Omega, \\
(\sigma^{ij} + \sigma^{kj} g^{i\ell} u_{\ell\|k}) n_j &= h^i \text{ on } \Gamma_1 \\
\sigma^{ij} &= \sigma^{ji} \text{ in } \Omega,
\end{aligned}
$$

constitute the **equations of equilibrium in curvilinear coordinates**.

The functions

$$t^{ij} = \sigma^{ij} + \sigma^{kj} g^{i\ell} u_{\ell\|k}$$

are the **contravariant components of the first Piola-Kirchhoff stress tensor.**

Finally, the functions

$$t^{ij}\|_j := \partial_j t^{ij} + \Gamma^i_{pj} t^{pj} + \Gamma^j_{jq} t^{iq}$$

are instances of **first-order covariant derivatives of a tensor field**, defined here by means of its *contravariant* components $t^{ij} : \overline{\Omega} \to \mathbb{R}$ (for a more systematic derivation, see, e.g., Lebedev & Cloud [2003, Chapter 4]).

We emphasize that, like their Cartesian special cases, *the principle of virtual work and the equations of equilibrium are valid for any continuum* (see, e.g., Ciarlet [1988, Chapter 2]), i.e., regardless of the *nature* of the continuum.

The object of the next section is precisely to take this last aspect into account.

3.4 CONSTITUTIVE EQUATION IN CURVILINEAR COORDINATES

Because of Theorem 3.2-1, the *constitutive equation* of an elastic material (cf. Section 3.1) can be easily converted in terms of curvilinear coordinates.

Theorem 3.4-1. *Let the notations and assumptions be as in* Theorem 3.2-1. *Let there be given functions* $\widehat{\mathcal{R}}^{ij} : \{\widehat{\Omega}\}^- \times \mathbb{S}^3 \to \mathbb{R}$ *such that*

$$\widehat{\sigma}^{ij}(\widehat{x}) = \widehat{\mathcal{R}}^{ij}(\widehat{x}, (\widehat{E}_{k\ell}(\widehat{\boldsymbol{u}})(\widehat{x}))) \text{ for all } \widehat{x} \in \{\widehat{\Omega}\}^-$$

is the constitutive equation of an elastic material with $\{\widehat{\Omega}\}^-$ *as its reference configuration.*

Then there exist functions $\mathcal{R}^{ij} : \overline{\Omega} \times \mathbb{S}^3 \to \mathbb{R}$, *depending only on the functions* $\widehat{\mathcal{R}}^{ij}$ *and on the mapping* $\boldsymbol{\Theta} : \overline{\Omega} \to \mathbf{E}^3$, *such that the second Piola-Kirchhoff stresses in curvilinear coordinates* $\sigma^{ij} : \overline{\Omega} \to \mathbb{R}$ *are given in terms of the covariant components of the Green-St Venant strain tensor as*

$$\sigma^{ij}(x) = \mathcal{R}^{ij}(x, (E_{k\ell}(\boldsymbol{u})(x))) \text{ for all } x \in \overline{\Omega}.$$

Assume that, in addition, the elastic material is homogeneous and isotropic and that its reference configuration $\{\widehat{\Omega}\}^-$ *is a natural state, in which case the functions* $\widehat{\mathcal{R}}^{ij}$ *satisfy*

$$\widehat{\mathcal{R}}^{ij}(\widehat{\mathbf{E}}) = \widehat{A}^{ijk\ell} \widehat{E}_{k\ell} + o(\widehat{\mathbf{E}}) \text{ for all } \widehat{\mathbf{E}} = (\widehat{E}_{k\ell}) \in \mathbb{S}^3,$$

with

$$\widehat{A}^{ijk\ell} = \lambda \delta^{ij} \delta^{k\ell} + \mu(\delta^{ik}\delta^{j\ell} + \delta^{i\ell}\delta^{jk}).$$

Then, in this case, the functions $\mathcal{R}^{ij}$ satisfy

$$\mathcal{R}^{ij}(x, \mathbf{E}) = A^{ijk\ell}(x) E_{k\ell} + o(\mathbf{E}) \text{ for all } \mathbf{E} = (E_{k\ell}) \in \mathbb{S}^3$$

at each $x \in \overline{\Omega}$, where

$$A^{ijk\ell} := \lambda g^{ij} g^{k\ell} + \mu(g^{ik} g^{j\ell} + g^{i\ell} g^{jk}) = A^{jik\ell} = A^{k\ell ij}.$$

Proof. Let $\widehat{\boldsymbol{\Theta}} = \widehat{\Theta}^i \boldsymbol{e}_i : \{\widehat{\Omega}\}^- \to \mathbb{R}^3$ denote the inverse mapping of the $\mathcal{C}^2$-diffeomorphism $\boldsymbol{\Theta} : \overline{\Omega} \to \mathbf{E}^3$. Theorem 3.2-1 and its proof show that, for all $x = \widehat{\boldsymbol{\Theta}}(\widehat{x}) \in \overline{\Omega}$,

$$\begin{aligned}
\sigma^{ij}(x) &= \widehat{\mathcal{R}}^{k\ell}(\widehat{x}, (\widehat{E}_{pq}(\widehat{\boldsymbol{u}})(\widehat{x})))[\boldsymbol{g}^i(x)]_k [\boldsymbol{g}^j(x)]_\ell, \\
\widehat{E}_{pq}(\widehat{\boldsymbol{u}})(\widehat{x}) &= (E_{mn}(\boldsymbol{v})[\boldsymbol{g}^m]_p [\boldsymbol{g}^n]_q)(x), \\
[\boldsymbol{g}^i(x)]_k &= \widehat{\partial}_k \widehat{\Theta}^i(\widehat{x}).
\end{aligned}$$

The announced conclusions immediately follow from these relations. □

Given an elastic material with $\boldsymbol{\Theta}(\overline{\Omega})$ as its reference configuration, the relations

$$\sigma^{ij}(x) = \mathcal{R}^{ij}(x, (E_{k\ell}(\boldsymbol{u})(x))) \text{ for all } x \in \overline{\Omega}$$

form its **constitutive equation in curvilinear coordinates**, the matrix field $(\mathcal{R}^{ij}) : \overline{\Omega} \times \mathbb{S}^3 \to \mathbb{S}^3$ being its **response function in curvilinear coordinates**. The functions $A^{ijk\ell} = \lambda g^{ij} g^{k\ell} + \mu(g^{ik} g^{j\ell} + g^{i\ell} g^{jk})$ are the **contravariant components of the three-dimensional elasticity tensor in curvilinear coordinates** (characterizing a specific elastic body).

3.5 THE EQUATIONS OF NONLINEAR ELASTICITY IN CURVILINEAR COORDINATES

Assembling the various relations found in Sections 3.2, 3.3, and 3.4, we are in a position to describe the basic **boundary value problem of nonlinear elasticity in curvilinear coordinates**. We are given:

– *a domain* Ω *in* $\mathbb{R}^3$ whose boundary Γ is partitioned as $\Gamma = \Gamma_0 \cup \Gamma_1$ and an *immersion* $\boldsymbol{\Theta} : \overline{\Omega} \to \mathbf{E}^3$ that is also a $\mathcal{C}^2$-diffeomorphism from $\overline{\Omega}$ onto its image;

– a matrix field $(\mathcal{R}^{ij}) : \overline{\Omega} \times \mathbb{S}^3 \to \mathbb{S}^3$, which is the *response function of an elastic material with the set* $\boldsymbol{\Theta}(\overline{\Omega}) \subset \mathbf{E}^3$ *as its reference configuration*;

– a vector field $(f^i) : \Omega \to \mathbb{R}^3$ and a vector field $(h^i) : \Gamma_1 \to \mathbb{R}^3$, whose components are the contravariant components of the *applied body and surface force* densities.

We are seeking a vector field $\boldsymbol{u} = (u_i) : \overline{\Omega} \to \mathbb{R}^3$, whose components are the covariant components of the *displacement field* $u_i \boldsymbol{g}^i$ *of the set* $\boldsymbol{\Theta}(\overline{\Omega})$, that satisfies the following boundary value problem:

$$\begin{aligned} -(\sigma^{ij} + \sigma^{kj} g^{i\ell} u_{\ell\|k})\|_j &= f^i \text{ in } \Omega, \\ u_i &= 0 \text{ on } \Gamma_0, \\ (\sigma^{ij} + \sigma^{kj} g^{i\ell} u_{\ell\|k}) n_j &= h^i \text{ on } \Gamma_1, \\ \sigma^{ij} &= \mathcal{R}^{ij}(\cdot, (E_{k\ell}(\boldsymbol{u}))) \text{ in } \overline{\Omega}, \end{aligned}$$

where

$$\begin{aligned} t^{ij}\|_j &:= \partial_j t^{ij} + \Gamma^i_{pj} t^{pj} + \Gamma^j_{jq} t^{iq}, \\ u_{\ell\|k} &:= \partial_k u_\ell - \Gamma^p_{ij} u_p, \\ E_{k\ell}(\boldsymbol{u}) &:= \frac{1}{2}(u_{k\|\ell} + u_{\ell\|k} + g^{mn} u_{m\|k} u_{n\|\ell}). \end{aligned}$$

If in addition the elastic material is homogeneous and isotropic and its reference configuration $\boldsymbol{\Theta}(\overline{\Omega})$ is a natural state, the functions $\mathcal{R}^{ij}$ satisfy

$$\mathcal{R}^{ij}(x, \mathbf{E}) = A^{ijk\ell}(x) E_{k\ell} + o(\mathbf{E}) \text{ for all } \mathbf{E} = (E_{k\ell}) \in \mathbb{S}^3$$

at each $x \in \overline{\Omega}$, where

$$A^{ijk\ell} = \lambda g^{ij} g^{k\ell} + \mu(g^{ik} g^{j\ell} + g^{i\ell} g^{jk}).$$

The above nonlinear boundary value problem is a **pure displacement problem** if $\Gamma_0 = \Gamma$, a **pure traction problem** if $\Gamma_1 = \Gamma$, and a **displacement-traction problem** if $area\,\Gamma_0 > 0$ and $area\,\Gamma_1 > 0$.

Let $\mathbf{W}(\Omega)$ denote a space of sufficiently smooth vector fields $\boldsymbol{v} = (v_i) : \overline{\Omega} \to \mathbb{R}^3$ that vanish on Γ_0. Then *the unknown vector field* $\boldsymbol{u} \in \mathbf{W}(\Omega)$ *satisfies the above boundary value problem if and only if it satisfies, at least formally, the following variational equations*:

$$\int_\Omega \mathcal{R}^{ij}(\cdot, (E_{k\ell}(\boldsymbol{u}))) \left(E'_{ij}(\boldsymbol{u})\boldsymbol{v}\right) \sqrt{g}\,\mathrm{d}x = \int_\Omega f^i v_i \sqrt{g}\,\mathrm{d}x + \int_{\Gamma_1} h^i v_i \sqrt{g}\,\mathrm{d}\Gamma$$

for all $\boldsymbol{v} = (v_i) \in \mathbf{W}(\Omega)$, where

$$E'_{ij}(\boldsymbol{u})\boldsymbol{v} := \frac{1}{2}\big(v_{i\|j} + v_{j\|i} + g^{mn}\{u_{m\|i} v_{n\|j} + u_{n\|j} v_{m\|i}\}\big)$$

are the Gâteaux derivatives of the functions $E_{ij} : \mathbf{W}(\Omega) \to \mathbb{R}$.

If the elastic material is *hyperelastic*, there exists a **stored energy function** $\mathcal{W} : \overline{\Omega} \times \mathbb{S}^3 \to \mathbb{R}$ such that

$$\mathcal{R}^{ij}(x, \mathbf{E}) = \frac{\partial \mathcal{W}}{\partial E_{ij}}(x, \mathbf{E}) \text{ for all } (x, \mathbf{E}) = (x, (E_{ij})) \in \overline{\Omega} \times \mathbb{S}^3.$$

In this case, the above variational equations thus become formally equivalent to the equations

$$J'(\boldsymbol{u})\boldsymbol{v} = 0 \text{ for all } \boldsymbol{v} \in \mathbf{W}(\Omega),$$

where the **energy in curvilinear coordinates** $J : \mathbf{W}(\Omega) \to \mathbb{R}$ is defined for all $\boldsymbol{v} \in \mathbf{W}(\Omega)$ by

$$J(\boldsymbol{v}) = \int_\Omega \mathcal{W}(\cdot, (E_{k\ell}(\boldsymbol{v})))\sqrt{g}\,\mathrm{d}x - \Big\{ \int_\Omega f^i v_i \sqrt{g}\,\mathrm{d}x + \int_{\Gamma_1} h^i v_i \sqrt{g}\,\mathrm{d}\Gamma \Big\}.$$

Finding the unknown vector field $\boldsymbol{u}$ in this case thus amounts, at least formally, to finding the *stationary points*, and in particular the *minimizers*, of the energy J over an *ad hoc* space $\mathbf{W}(\Omega)$.

3.6 THE EQUATIONS OF LINEARIZED ELASTICITY IN CURVILINEAR COORDINATES

Formally, the boundary value problem of nonlinear elasticity in curvilinear coordinates (Section 3.5) consists in finding a vector field $\boldsymbol{u} = (u_i)$ such that

$$\boldsymbol{u} \in \mathbf{W}(\Omega) \text{ and } (\boldsymbol{\mathcal{A}}(\boldsymbol{u}), \boldsymbol{\mathcal{B}}(\boldsymbol{u})) = (\boldsymbol{f}, \boldsymbol{h}) \text{ in } \mathbf{F}(\Omega) \times \mathbf{H}(\Gamma_1),$$

where $\mathbf{W}(\Omega)$, $\mathbf{F}(\Omega)$, and $\mathbf{H}(\Gamma_1)$ are spaces of smooth enough vector fields respectively defined on $\overline{\Omega}$ and vanishing on Γ_0, defined in Ω, and defined on Γ_1,

$$\boldsymbol{\mathcal{A}} : \mathbf{W}(\Omega) \to \mathbf{F}(\Omega) \text{ and } \boldsymbol{\mathcal{B}} : \mathbf{W}(\Omega) \to \mathbf{H}(\Gamma_1)$$

are nonlinear operators defined by

$$(\boldsymbol{\mathcal{A}}(\boldsymbol{u}))^i = -(\sigma^{ij} + \sigma^{kj} g^{i\ell} u_{\ell\|k})\|_j \text{ and } (\boldsymbol{\mathcal{B}}(\boldsymbol{u}))^i = (\sigma^{ij} + \sigma^{kj} g^{i\ell} u_{\ell\|k}) n_j,$$

where

$$\sigma^{ij} = \mathcal{R}^{ij}(\cdot, E_{k\ell}(\boldsymbol{u}))$$

and the fields $\boldsymbol{f} = (f^i) \in \mathbf{F}(\Omega)$ and $\boldsymbol{h} = (h^i) \in \mathbf{H}(\Gamma_1)$ are given.

Assume henceforth that the elastic material is *homogeneous* and *isotropic* and that the reference configuration is a *natural state*. This last assumption thus implies that

$$(\boldsymbol{\mathcal{A}}(\mathbf{0}), \boldsymbol{\mathcal{B}}(\mathbf{0})) = (\mathbf{0}, \mathbf{0}),$$

so that $\boldsymbol{u} = \mathbf{0}$ is a particular solution corresponding to $\boldsymbol{f} = \mathbf{0}$ and $\boldsymbol{h} = \mathbf{0}$.

Assume in addition that $\mathbf{W}(\Omega)$, $\mathbf{F}(\Omega)$, and $\mathbf{H}(\Gamma_1)$ are normed vector spaces and that the nonlinear operator $(\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{B}}) : \mathbf{W}(\Omega) \to \mathbf{F}(\Omega) \times \mathbf{H}(\Gamma_1)$ is Fréchet differentiable at the origin $\mathbf{0} \in \mathbf{W}(\Omega)$. Then, *by definition*, the **boundary value problem of linearized elasticity** consists in finding $\boldsymbol{u} = (u_i)$ such that

$$\boldsymbol{u} \in \mathbf{W}(\Omega) \text{ and } (\boldsymbol{\mathcal{A}}'(\mathbf{0})\boldsymbol{u}, \boldsymbol{\mathcal{B}}'(\mathbf{0})\boldsymbol{u}) = (\boldsymbol{f}, \boldsymbol{h}) \text{ in } \mathbf{F}(\Omega) \times \mathbf{H}(\Gamma_1).$$

In other words, this linear boundary value problem is the "*tangent at the origin*" of the nonlinear boundary value problem. As such, its solution can be expected to be a good approximation, at least for "small enough" forces, of the displacement field that satisfies the original nonlinear problem. Indeed, this "raison d'être" can be rigorously justified in *ad hoc* function spaces, but only in some carefully circumscribed situations (see, e.g., Ciarlet [1988, Theorem 6.8-1]). Because it is linear, and thus incomparably simpler to solve than the original nonlinear problem, this linear problem also provides an extraordinarily efficient and versatile means of numerically approximating displacement and stress fields in innumerable elastic structures arising in engineering. For this reason, devising efficient approximation schemes for this problem has pervaded the numerical analysis and computational mechanics literature during the past decades and continues to do so to a very large extent, even to this day.

In order to compute the components of the vector fields $\boldsymbol{\mathcal{A}}'(\mathbf{0})\boldsymbol{u}$ and $\boldsymbol{\mathcal{B}}'(\mathbf{0})\boldsymbol{u}$, it clearly suffices, once derivability is assumed, to compute the terms that are *linear with respect to* $\boldsymbol{v}$ in the differences

$$\boldsymbol{\mathcal{A}}(\boldsymbol{v}) - \boldsymbol{\mathcal{A}}(\mathbf{0}) = \boldsymbol{\mathcal{A}}(\boldsymbol{v}) \text{ and } \boldsymbol{\mathcal{B}}(\boldsymbol{v}) - \boldsymbol{\mathcal{B}}(\mathbf{0}) = \boldsymbol{\mathcal{B}}(\boldsymbol{v}).$$

Doing so shows that *solving the boundary value problem of linearized elasticity consists in finding a vector field* $\boldsymbol{u} = (u_i) : \overline{\Omega} \to \mathbb{R}^3$ *that satisfies*

$$\begin{aligned} -\sigma^{ij}\|_j &= f^i \text{ in } \Omega, \\ u_i &= 0 \text{ on } \Gamma_0, \\ \sigma^{ij} n_j &= h^i \text{ on } \Gamma_1, \\ \sigma^{ij} &= A^{ijk\ell} e_{k\ell}(\boldsymbol{u}) \text{ in } \overline{\Omega}, \end{aligned}$$

where

$$e_{k\ell}(\boldsymbol{u}) := \frac{1}{2}(u_{k\|\ell} + u_{\ell\|k}).$$

Recall that

$$\begin{aligned} \sigma^{ij}\|_j &:= \partial_j \sigma^{ij} + \Gamma^i_{pj}\sigma^{pj} + \Gamma^j_{jq}\sigma^{iq}, \\ A^{ijk\ell} &:= \lambda g^{ij} g^{k\ell} + \mu(g^{ik}g^{j\ell} + g^{i\ell}g^{jk}), \\ v_{\ell\|k} &:= \partial_k v_\ell - \Gamma^p_{ij} v_p. \end{aligned}$$

The functions $e_{k\ell}(\boldsymbol{v}) := \frac{1}{2}(v_{k\|\ell}+v_{\ell\|k})$ are called the **covariant components of the linearized strain tensor**, or the **linearized strains in curvilinear coordinates**, associated with a displacement vector field $v_i\boldsymbol{g}^i$ of the reference configuration $\boldsymbol{\Theta}(\overline{\Omega})$. By definition, they satisfy

$$e_{ij}(\boldsymbol{v}) = [E_{ij}(\boldsymbol{v})]^{\text{lin}},$$

where $[\cdots]^{\text{lin}}$ denotes the linear part with respect to $\boldsymbol{v} = (v_i)$ in the expression $[\cdots]$.

The linearized constitutive equation $\sigma^{ij} = A^{ijk\ell}e_{k\ell}(\boldsymbol{u})$ is called **Hooke's law in curvilinear coordinates** and the functions σ^{ij} that appear in this equation are called the **linearized stresses in curvilinear coordinates**.

The above linear boundary value problem is a **pure displacement problem** if $\Gamma_0 = \Gamma$, a **pure traction problem** if $\Gamma_1 = \Gamma$, and a **displacement-traction problem** if $area\,\Gamma_0 > 0$ and $area\,\Gamma_1 > 0$.

Writing the above boundary value problem as a variational problem naturally relies on the following "linearized version" of Theorem 3.3-1. As before, *ad hoc* smoothness is implicitly assumed throughout.

Theorem 3.6-1. *A symmetric matrix field* $(\sigma^{ij}) : \overline{\Omega} \to \mathbb{S}^3$ *satisfies*

$$-\sigma^{ij}\|_j = f^i \text{ in } \Omega,$$
$$\sigma^{ij} n_j = h^i \text{ on } \Gamma_1,$$

if and only if it satisfies the variational equations

$$\int_\Omega \sigma^{ij} e_{ij}(\boldsymbol{v})\sqrt{g}\,\mathrm{d}x = \int_\Omega f^i v_i \sqrt{g}\,\mathrm{d}x + \int_{\Gamma_1} h^i v_i \sqrt{g}\,\mathrm{d}\Gamma$$

for all $\boldsymbol{v} = (v_i) \in \mathbf{W}(\Omega)$, *where* $\mathbf{W}(\Omega)$ *is a space of smooth enough vector fields that vanish on* Γ_0.

Proof. The proof relies on the integration by part formula

$$\int_\Omega \sigma^{ij} v_{i\|j} \sqrt{g}\,\mathrm{d}x = -\int_\Omega (\sigma^{ij}\|_j) v_i \sqrt{g}\,\mathrm{d}x + \int_\Gamma \sigma^{ij} n_j v_i \sqrt{g}\,\mathrm{d}\Gamma,$$

valid for all vector fields $(v_i) : \overline{\Omega} \to \mathbb{R}^3$, established in the proof of Theorem 3.3-1. ☐

The variational equations derived in Theorem 3.6-1 constitute the **linearized principle of virtual work in curvilinear coordinates**. Together with the assumed symmetry of the matrix field (σ^{ij}), the equations $-\sigma^{ij}\|_j = f^i$ in Ω and $\sigma^{ij} n_j = h^i$ constitute the **linearized equations of equilibrium in curvilinear coordinates.**

Because of Theorem 3.6-1, it is *immediately* seen that *the boundary value problem of linearized elasticity is formally equivalent to finding a vector field* $\boldsymbol{u} \in \mathbf{W}(\Omega)$ *that satisfies the following variational equations*:

$$\int_\Omega A^{ijk\ell} e_{k\ell}(\boldsymbol{u}) e_{ij}(\boldsymbol{v})\sqrt{g}\,\mathrm{d}x = \int_\Omega f^i v_i \sqrt{g}\,\mathrm{d}x + \int_{\Gamma_1} h^i v_i \sqrt{g}\,\mathrm{d}\Gamma$$

for all $\boldsymbol{v} = (v_i) \in \mathbf{W}(\Omega)$.

Thanks to the symmetries $A^{ijk\ell} = A^{k\ell ij}$, finding the solutions to these variational equations also amounts to finding the *stationary points* of the functional $J : \mathbf{W}(\Omega) \to \mathbb{R}$ defined by

$$J(\boldsymbol{v}) = \frac{1}{2}\int_\Omega A^{ijk\ell} e_{k\ell}(\boldsymbol{v}) e_{ij}(\boldsymbol{v})\sqrt{g}\,\mathrm{d}x - \Big\{ \int_\Omega f^i v_i \sqrt{g}\,\mathrm{d}x + \int_{\Gamma_1} h^i v_i \sqrt{g}\,\mathrm{d}\Gamma \Big\}$$

for all $\boldsymbol{v} \in \mathbf{W}(\Omega)$.

Our objective in the next sections is to establish the *existence* and *uniqueness* of the solution to these problems in *ad hoc* function spaces.

More specifically, define the space

$$\mathbf{V}(\Omega) := \{\boldsymbol{v} = (v_i) \in H^1(\Omega; \mathbb{R}^3);\ \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_0\},$$

define the symmetric bilinear form $B : \mathbf{V}(\Omega) \times \mathbf{V}(\Omega) \to \mathbb{R}$ by

$$B(\boldsymbol{u}, \boldsymbol{v}) := \int_\Omega A^{ijk\ell} e_{k\ell}(\boldsymbol{u}) e_{ij}(\boldsymbol{v}) \sqrt{g}\, \mathrm{d}x$$

for all $(\boldsymbol{u}, \boldsymbol{v}) \in \mathbf{V}(\Omega) \times \mathbf{V}(\Omega)$, and define the linear form $L : \mathbf{V}(\Omega) \to \mathbb{R}$ by

$$L(\boldsymbol{v}) := \int_\Omega f^i v_i \sqrt{g}\, \mathrm{d}x - \int_{\Gamma_1} h^i v_i \sqrt{g}\, \mathrm{d}\Gamma$$

for all $\boldsymbol{v} \in \mathbf{V}(\Omega)$. Clearly, both forms B and L are continuous on the space $\mathbf{V}(\Omega)$ if the data are smooth enough.

Our main result will then consist in establishing that, if *area* $\Gamma_0 > 0$, the bilinear form B is $\mathbf{V}(\Omega)$-*elliptic*, i.e., that there exists a constant $\alpha > 0$ such that

$$\alpha \|\boldsymbol{v}\|_{1,\Omega}^2 \leq B(\boldsymbol{v}, \boldsymbol{v}) \text{ for all } \boldsymbol{v} \in \mathbf{V}(\Omega),$$

where $\|\cdot\|_{1,\Omega}$ denotes the norm of the Hilbert space $H^1(\Omega; \mathbb{R}^3)$. This inequality holds, because of a fundamental *Korn inequality in curvilinear coordinates* (Theorem 3.8-3), which asserts the existence of a constant C such that

$$\|\boldsymbol{v}\|_{1,\Omega} \leq C \Big\{ \sum_{i,j} \|e_{ij}(\boldsymbol{v})\|_{0,\Omega}^2 \Big\}^{1/2} \text{ for all } \boldsymbol{v} \in \mathbf{V}(\Omega),$$

in addition to the *uniform positiveness of the elasticity tensor* (Theorem 3.9-1), meaning that there exists a constant C_e such that

$$\sum_{i,j} |t_{ij}|^2 \leq C_e A^{ijk\ell}(x) t_{k\ell} t_{ij}$$

for all $x \in \overline{\Omega}$ and all symmetric matrices (t_{ij}).

The *existence* and *uniqueness* of a solution to the above variational equations then follow from the well-known *Lax-Milgram lemma* and the symmetry of the bilinear form further implies that this solution is also the *unique minimizer of the functional* J over the space $\mathbf{V}(\Omega)$.

3.7 A FUNDAMENTAL LEMMA OF J.L. LIONS

We first review some essential definitions and notations, together with a fundamental *lemma of J.L. Lions* (Theorem 3.7-1). This lemma plays a key rôle in the proof of the Korn inequality in curvilinear coordinates.

Recall that a *domain* Ω in $\mathbb{R}^d$ is an *open, bounded, connected* subset of $\mathbb{R}^d$ with a *Lipschitz-continuous boundary* Γ, the set Ω being *locally on one side of* Γ. As Γ is Lipschitz-continuous, a *measure* $\mathrm{d}\Gamma$ can be defined along Γ and a *unit outer normal vector* $\boldsymbol{\nu} = (\nu_i)_{i=1}^d$ ("unit" means that its Euclidean norm is one) exists $\mathrm{d}\Gamma$-almost everywhere along Γ.

Let Ω be a domain in $\mathbb{R}^d$. For each integer $m \geq 1$, $H^m(\Omega)$ and $H_0^m(\Omega)$ denote the usual *Sobolev spaces*. In particular,

$$H^1(\Omega) := \{v \in L^2(\Omega);\ \partial_i v \in L^2(\Omega),\ 1 \leq i \leq d\},$$
$$H^2(\Omega) := \{v \in H^1(\Omega);\ \partial_{ij} v \in L^2(\Omega),\ 1 \leq i,\ j \leq d\},$$

where $\partial_i v$ and $\partial_{ij} v$ denote partial derivatives in the sense of distributions, and

$$H_0^1(\Omega) := \{v \in H^1(\Omega);\ v = 0 \text{ on } \Gamma\},$$

where the relation $v = 0$ on Γ is to be understood in the sense of trace. Boldface letters denote vector-valued or matrix-valued functions, also called *vector fields* or *matrix fields*, and their associated function spaces. The norm in $L^2(\Omega)$ or $\mathbf{L}^2(\Omega)$ is noted $\|\cdot\|_{0,\Omega}$ and the norm in $H^m(\Omega)$ or $\mathbf{H}^m(\Omega)$, $m \geq 1$, is noted $\|\cdot\|_{m,\Omega}$. In particular then,

$$\|v\|_{0,\Omega} := \Big\{ \int_\Omega |v|^2 \, \mathrm{d}x \Big\}^{1/2} \text{ if } v \in L^2(\Omega),$$

$$\|\boldsymbol{v}\|_{0,\Omega} := \Big\{ \sum_{i=1}^d |v_i|_{0,\Omega}^2 \Big\}^{1/2} \text{ if } \boldsymbol{v} = (v_i)_{i=1}^d \in \mathbf{L}^2(\Omega),$$

$$\|v\|_{1,\Omega} := \Big\{ \|v\|_{0,\Omega}^2 + \sum_{i=1}^d \|\partial_i v\|_{0,\Omega}^2 \Big\}^{1/2} \text{ if } v \in H^1(\Omega),$$

$$\|\boldsymbol{v}\|_{1,\Omega} := \Big\{ \sum_{i=1}^d \|v_i\|_{1,\Omega}^2 \Big\}^{1/2} \text{ if } \boldsymbol{v} = (v_i)_{i=1}^d \in \mathbf{H}^1(\Omega),$$

$$\|v\|_{2,\Omega} = \Big\{ \|v\|_{0,\Omega}^2 + \sum_{i=1}^d \|\partial_i v\|_{0,\Omega}^2 + \sum_{i,j=1}^d \|\partial_{ij} v\|_{0,\Omega}^2 \Big\}^{1/2} \text{ if } v \in H^2(\Omega), \text{ etc.}$$

Detailed treatments of Sobolev spaces are found in Nečas [1967], Lions & Magenes [1968], Dautray & Lions [1984, Chapters 1–3], Adams [1975]. An excellent introduction is given in Brezis [1983].

In this section, we also consider the Sobolev space

$$H^{-1}(\Omega) := \text{dual space of } H_0^1(\Omega).$$

Another possible definition of the space $H_0^1(\Omega)$ being

$$H_0^1(\Omega) = \text{closure of } \mathcal{D}(\Omega) \text{ with respect to } \|\cdot\|_{1,\Omega},$$

where $\mathcal{D}(\Omega)$ denotes the space of infinitely differentiable real-valued functions defined over Ω whose support is a compact subset of Ω, it is clear that

$$v \in L^2(\Omega) \implies v \in H^{-1}(\Omega) \text{ and } \partial_i v \in H^{-1}(\Omega),\ 1 \leq i \leq n,$$

since (the duality between the spaces $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$ is denoted by $\langle\cdot,\cdot\rangle$):

$$|\langle v,\varphi\rangle| = \Big|\int_\Omega v\varphi\,\mathrm{d}x\Big| \le \|v\|_{0,\Omega}\|\varphi\|_{1,\Omega},$$
$$|\langle \partial_i v,\varphi\rangle| = |-\langle v,\partial_i\varphi\rangle| = \Big|-\int_\Omega v\partial_i\varphi\,\mathrm{d}x\Big| \le \|v\|_{0,\Omega}\|\varphi\|_{1,\Omega}$$

for all $\varphi \in \mathcal{D}(\Omega)$. It is remarkable (but also remarkably difficult to prove!) that the converse implication holds:

Theorem 3.7-1. *Let Ω be a domain in $\mathbb{R}^d$ and let v be a distribution on Ω. Then*

$$\{v \in H^{-1}(\Omega) \text{ and } \partial_i v \in H^{-1}(\Omega),\ 1 \le i \le d\} \implies v \in L^2(\Omega).$$

This implication was first proved by J.L. Lions, as stated in Magenes & Stampacchia [1958, p. 320, Note ([27])]; for this reason, it will be henceforth referred to as the **lemma of J.L. Lions**. Its first published proof for domains with smooth boundaries appeared in Duvaut & Lions [1972, p. 111]; another proof was also given by Tartar [1978]. Various extensions to "genuine" domains, i.e., with Lipschitz-continuous boundaries, are given in Bolley & Camus [1976], Geymonat & Suquet [1986], and Borchers & Sohr [1990]; Amrouche & Girault [1994, Proposition 2.10] even proved that the more general implication

$$\{v \in \mathcal{D}'(\Omega) \text{ and } \partial_i v \in H^{m}(\Omega),\ 1 \le i \le n\} \implies v \in H^{m+1}(\Omega)$$

holds for *arbitrary* integers $m \in \mathbb{Z}$.

Note that some minimal regularity of the boundary $\partial\Omega$ is anyway required: Geymonat & Gilardi [1998] have shown that the lemma of J.L. Lions does *not* hold if the open set Ω satisfies only the "segment property" (this means that, for every $x \in \partial\Omega$, there exists an open set U_x containing x and a vector $\boldsymbol{a}_x \neq \mathbf{0}$ such that $(y + t\boldsymbol{a}_x) \in \Omega$ for all $y \in \overline{\Omega} \cap U_x$ and all $0 < t < 1$).

Remark. Although Theorem 3.7-1 shall be referred to as *"the" lemma of J.L. Lions* in this volume, there are other results of his that bear the same name in the literature, such as his "compactness lemmas" (Lions [1961, Proposition 4.1, p. 59] or Lions [1969, Section 5.2, p. 57]) or his "singular perturbation lemma" (Lions [1973, Lemma 5.1, p. 126]). □

3.8 KORN'S INEQUALITIES IN CURVILINEAR COORDINATES

As already noted in Section 3.6, the existence and uniqueness of a solution to the variational equations of three-dimensional linearized elasticity found in Section 3.6 will essentially follow from the *ellipticity of the associated bilinear form.*

To this end, an essential step consists in establishing a *three-dimensional Korn inequality in curvilinear coordinates* (Theorem 3.8-3), due to Ciarlet [1993] (see also Chen & Jost [2002], who have shown that, in fact, such an inequality holds in the more general context of Riemannian geometry). In essence, this inequality asserts that, given a domain $\Omega \subset \mathbb{R}^3$ with boundary Γ and a subset Γ_0 of Γ with *area* $\Gamma_0 > 0$, *the* $\mathbf{L}^2(\Omega)$*-norm of the matrix fields* $(e_{ij}(\boldsymbol{v}))$ *is equivalent to the* $\mathbf{H}^1(\Omega)$*-norm of the vector fields* $\boldsymbol{v}$ *for all vector fields* $\boldsymbol{v} \in \mathbf{H}^1(\Omega)$ *that vanish on* Γ_0 (the ellipticity of the bilinear form also relies on the positive definiteness of the three-dimensional elasticity tensor; cf. Theorem 3.9-1). Recall that the functions $e_{ij}(\boldsymbol{v}) = \frac{1}{2}(v_{i\|j} + v_{j\|i})$ are the covariant components of the linearized strain tensor associated with a displacement field $v_i\boldsymbol{g}^i$ of the reference configuration $\boldsymbol{\Theta}(\overline{\Omega})$.

As a first step towards proving such an inequality, we establish in Theorem 3.8-1 a **Korn's inequality in curvilinear coordinates "without boundary conditions"**. This means that *this inequality is valid for all vector fields* $\boldsymbol{v} = (v_i) \in \mathbf{H}^1(\Omega)$, i.e., that do not satisfy any specific boundary condition on Γ.

This inequality is truly remarkable, since only *six* different combinations of first-order partial derivatives, viz., $\frac{1}{2}(\partial_j v_i + \partial_i v_j)$, occur in its right-hand side, while all *nine* partial derivatives $\partial_i v_j$ occur in its left-hand side! A similarly striking observation applies to part (ii) of the next proof, which entirely rests on the crucial *lemma of J.L. Lions* recalled in the previous section.

Theorem 3.8-1. *Let* Ω *be a domain in* $\mathbb{R}^3$ *and let* $\boldsymbol{\Theta}$ *be a* $\mathcal{C}^2$*-diffeomorphism of* $\overline{\Omega}$ *onto its image* $\boldsymbol{\Theta}(\overline{\Omega}) \subset \mathbf{E}^3$*. Given a vector field* $\boldsymbol{v} = (v_i) \in \mathbf{H}^1(\Omega)$*, let*

$$e_{ij}(\boldsymbol{v}) := \Big\{\frac{1}{2}(\partial_j v_i + \partial_i v_j) - \Gamma_{ij}^p v_p\Big\} \in L^2(\Omega)$$

denote the covariant components of the linearized change of metric tensor associated with the displacement field $v_i\boldsymbol{g}^i$*. Then there exists a constant* $C_0 = C_0(\Omega, \boldsymbol{\Theta})$ *such that*

$$\|\boldsymbol{v}\|_{1,\Omega} \le C_0\Big\{\sum_i \|v_i\|_{0,\Omega}^2 + \sum_{i,j} \|e_{ij}(\boldsymbol{v})\|_{0,\Omega}^2\Big\}^{1/2} \text{ for all } \boldsymbol{v} = (v_i) \in \mathbf{H}^1(\Omega).$$

Proof. The proof is essentially an extension of that given in Duvaut & Lions [1972, p. 110] for proving Korn's inequality without boundary conditions in *Cartesian coordinates.*

(i) *Define the space*

$$\mathbf{W}(\Omega) := \{\boldsymbol{v} = (v_i) \in \mathbf{L}^2(\Omega);\ e_{ij}(\boldsymbol{v}) \in L^2(\Omega)\}.$$

Then, equipped with the norm $\|\cdot\|_{\mathbf{W}(\Omega)}$ *defined by*

$$\|\boldsymbol{v}\|_{\mathbf{W}(\Omega)} := \Big\{\sum_i \|v_i\|_{0,\Omega}^2 + \sum_{i,j} \|e_{ij}(\boldsymbol{v})\|_{0,\Omega}^2\Big\}^{1/2},$$

the space $\mathbf{W}(\Omega)$ *is a Hilbert space.*

The relations "$e_{ij}(\boldsymbol{v}) \in L^2(\Omega)$" appearing in the definition of the space $\mathbf{W}(\Omega)$ are naturally to be understood in the sense of distributions. This means that there exist functions in $L^2(\Omega)$, *denoted* $e_{ij}(\boldsymbol{v})$, such that

$$\int_\Omega e_{ij}(\boldsymbol{v})\varphi \,\mathrm{d}x = -\int_\Omega \Big\{\frac{1}{2}(v_i\partial_j\varphi + v_j\partial_i\varphi) + \Gamma^p_{ij}v_p\varphi\Big\}\,\mathrm{d}x \quad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

Let there be given a Cauchy sequence $(\boldsymbol{v}^k)_{k=1}^\infty$ with elements $\boldsymbol{v}^k = (v_i^k) \in \mathbf{W}(\Omega)$. The definition of the norm $\|\cdot\|_{\mathbf{W}(\Omega)}$ shows that there exist functions $v_i \in L^2(\Omega)$ and $e_{ij} \in L^2(\Omega)$ such that

$$v_i^k \to v_i \text{ in } L^2(\Omega) \text{ and } e_{ij}(\boldsymbol{v}^k) \to e_{ij} \text{ in } L^2(\Omega) \text{ as } k \to \infty,$$

since the space $L^2(\Omega)$ is complete. Given a function $\varphi \in \mathcal{D}(\Omega)$, letting $k \to \infty$ in the relations

$$\int_\Omega e_{ij}(\boldsymbol{v}^k)\varphi \,\mathrm{d}x = -\int_\Omega \Big\{\frac{1}{2}(v_i^k\partial_j\varphi + v_j^k\partial_i\varphi) + \Gamma^p_{ij}v_p^k\varphi\Big\}\,\mathrm{d}x,\ k \geq 1,$$

shows that $e_{ij} = e_{ij}(\boldsymbol{v})$.

(ii) *The spaces* $\mathbf{W}(\Omega)$ *and* $\mathbf{H}^1(\Omega)$ *coincide.*

Clearly, $\mathbf{H}^1(\Omega) \subset \mathbf{W}(\Omega)$. To establish the other inclusion, let $\boldsymbol{v} = (v_i) \in \mathbf{W}(\Omega)$. Then

$$s_{ij}(\boldsymbol{v}) := \frac{1}{2}(\partial_j v_i + \partial_i v_j) = \{e_{ij}(\boldsymbol{v}) + \Gamma^p_{ij}v_p\} \in L^2(\Omega),$$

since $e_{ij}(\boldsymbol{v}) \in L^2(\Omega), \Gamma^p_{ij} \in \mathcal{C}^0(\overline{\Omega})$, and $v_p \in L^2(\Omega)$. We thus have

$$\begin{aligned} \partial_k v_i &\in H^{-1}(\Omega),\\ \partial_j(\partial_k v_i) = \{\partial_j s_{ik}(\boldsymbol{v}) + \partial_k s_{ij}(\boldsymbol{v}) - \partial_i s_{jk}(\boldsymbol{v})\} &\in H^{-1}(\Omega), \end{aligned}$$

since $w \in L^2(\Omega)$ implies $\partial_k w \in H^{-1}(\Omega)$. Hence $\partial_k v_i \in L^2(\Omega)$ by the *lemma of J.L. Lions* (Theorem 3.7-1) and thus $\boldsymbol{v} \in \mathbf{H}^1(\Omega)$.

(iii) *Korn's inequality without boundary conditions.*

The identity mapping ι from the space $\mathbf{H}^1(\Omega)$ equipped with $\|\cdot\|_{1,\Omega}$ into the space $\mathbf{W}(\Omega)$ equipped with $\|\cdot\|_{\mathbf{W}(\Omega)}$ is injective, continuous (there clearly exists a constant c such that $\|\boldsymbol{v}\|_{\mathbf{W}(\Omega)} \leq c\|\boldsymbol{v}\|_{1,\Omega}$ for all $\boldsymbol{v} \in \mathbf{H}^1(\Omega)$), and surjective by (ii). Since both spaces are complete (cf. (i)), the *closed graph theorem* (see, e.g., Brezis [1983, p. 19] for a proof) then shows that the inverse mapping ι^{-1} is also continuous; this continuity is exactly what is expressed by *Korn's inequality without boundary conditions.* □

Our next objective is to dispose of the norms $\|v_i\|_{0,\Omega}$ in the right-hand side of the Korn inequality established in Theorem 3.8-1 when the vector fields $\boldsymbol{v} = (v_i) \in \mathbf{H}^1(\Omega)$ are subjected to the boundary condition $\boldsymbol{v} = \mathbf{0}$ on a subset Γ_0 of the boundary Γ that satisfies *area* $\Gamma_0 > 0$. To this end, we first establish in the next theorem the weaker property that the *semi-norm*

$$\boldsymbol{v} \to \Big\{ \sum_{i,j} \|e_{ij}(\boldsymbol{v})\|^2_{0,\Omega} \Big\}^{1/2}$$

becomes a *norm* for such vector fields.

Part (a) in the next theorem constitutes the **infinitesimal rigid displacement lemma in curvilinear coordinates "without boundary conditions"**, while part (b) constitutes the **infinitesimal rigid displacement lemma in curvilinear coordinates, "with boundary conditions"**.

The adjective "infinitesimal" reminds that if $e_{ij}(\boldsymbol{v}) = 0$ in Ω, i.e., if only the *linearized* part of the full change of metric tensor $E_{ij}(\boldsymbol{v})$ vanishes in Ω (cf. Section 3.6), then the corresponding displacement field $v_i\boldsymbol{g}^i$ is in a specific sense only the *linearized* part of a genuine rigid displacement.

More precisely, let an **infinitesimal rigid displacement** $v_i\boldsymbol{g}^i$ of the set $\boldsymbol{\Theta}(\overline{\Omega})$ be defined as one whose associated vector field $\boldsymbol{v} = (v_i) \in \mathbf{H}^1(\Omega)$ satisfies $e_{ij}(\boldsymbol{v}) = 0$ in Ω. Then one can show that *the space of such infinitesimal rigid displacements coincides with the tangent space at the origin to the manifold of "rigid displacements" of the set* $\boldsymbol{\Theta}(\overline{\Omega})$. Such rigid displacements are defined as those whose associated deformed configuration $(\boldsymbol{\Theta} + v_i\boldsymbol{g}^i)(\overline{\Omega})$ is obtained by means of a "*rigid transformation*" of the reference configuration $\boldsymbol{\Theta}(\overline{\Omega})$ (cf. Section 1.7); for details, see Ciarlet & C. Mardare [2003].

Theorem 3.8-2. *Let Ω be a domain in $\mathbb{R}^3$ and let $\boldsymbol{\Theta}$ be a $\mathcal{C}^2$-diffeomorphism onto its image $\boldsymbol{\Theta}(\overline{\Omega}) \subset \mathbf{E}^3$.*

(a) *Let a vector field $\boldsymbol{v} = (v_i) \in \mathbf{H}^1(\Omega)$ be such that*

$$e_{ij}(\boldsymbol{v}) = 0 \text{ in } \Omega.$$

Then there exist two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ such that the associated vector field $v_i\boldsymbol{g}^i$ is of the form

$$v_i(x)\boldsymbol{g}^i(x) = \boldsymbol{a} + \boldsymbol{b} \wedge \boldsymbol{\Theta}(x) \text{ for all } x \in \overline{\Omega}.$$

(b) *Let Γ_0 be a $d\Gamma$-measurable subset of the boundary $\partial\Omega$ that satisfies* area $\Gamma_0 > 0$, *and let a vector field $\boldsymbol{v} = (v_i) \in \mathbf{H}^1(\Omega)$ be such that*

$$e_{ij}(\boldsymbol{v}) = 0 \text{ in } \Omega \text{ and } \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_0.$$

Then $\boldsymbol{v} = \mathbf{0}$ in Ω.

Proof. An argument similar to that used in part (ii) of the proof of Theorem 3.2-1 shows that

$$\widehat{e}_{ij}(\widehat{\boldsymbol{v}})(\widehat{x}) = \left(e_{k\ell}(\boldsymbol{v})[\boldsymbol{g}^k]_i[\boldsymbol{g}^\ell]_j\right)(x) \text{ for all } \widehat{x} = \boldsymbol{\Theta}(x),\ x \in \Omega,$$

where $\widehat{e}_{ij}(\widehat{\boldsymbol{v}}) = \frac{1}{2}(\widehat{\partial}_j\widehat{v}_i + \widehat{\partial}_i\widehat{v}_j)$ and the vector fields $\widehat{\boldsymbol{v}} = (\widehat{v}_i) \in \mathbf{H}^1(\widehat{\Omega})$ and $\boldsymbol{v} = (v_i) \in \mathbf{H}^1(\Omega)$ are related by

$$\widehat{v}_i(\widehat{x})\widehat{\boldsymbol{e}}^i = v_i(x)\boldsymbol{g}^i(x) \text{ for all } \widehat{x} = \boldsymbol{\Theta}(x),\ x \in \Omega.$$

Hence

$$e_{ij}(\boldsymbol{v}) = 0 \text{ in } \Omega \text{ implies } \widehat{e}_{ij}(\widehat{\boldsymbol{v}}) = 0 \text{ in } \widehat{\Omega},$$

and the identity (actually, the same as in the proof of Theorem 3.8-1)

$$\widehat{\partial}_j(\widehat{\partial}_k\widehat{v}_i) = \widehat{\partial}_j\widehat{e}_{ik}(\widehat{\boldsymbol{v}}) + \widehat{\partial}_k\widehat{e}_{ij}(\widehat{\boldsymbol{v}}) - \widehat{\partial}_i\widehat{e}_{jk}(\widehat{\boldsymbol{v}}) \text{ in } \mathcal{D}'(\widehat{\Omega})$$

further shows that

$$\widehat{e}_{ij}(\widehat{\boldsymbol{v}}) = 0 \text{ in } \widehat{\Omega} \text{ implies } \widehat{\partial}_j(\widehat{\partial}_k\widehat{v}_i) = 0 \text{ in } \mathcal{D}'(\widehat{\Omega}).$$

By a classical result from distribution theory (Schwartz [1966, p. 60]), each function $\widehat{v}_i$ is therefore a polynomial of degree ≤ 1 in the variables $\widehat{x}_j$, since the set $\widehat{\Omega}$ is connected. There thus exist constants a_i and b_{ij} such that

$$\widehat{v}_i(\widehat{x}) = a_i + b_{ij}\widehat{x}_j \text{ for all } \widehat{x} = (\widehat{x}_i) \in \widehat{\Omega}.$$

But $\widehat{e}_{ij}(\widehat{\boldsymbol{v}}) = 0$ also implies that $b_{ij} = -b_{ji}$. Hence there exist two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ such that ($\boldsymbol{o}\widehat{\boldsymbol{x}}$ denotes the column vector with components x_i)

$$\widehat{\boldsymbol{v}}_i(\widehat{x})\widehat{\boldsymbol{e}}^i = \boldsymbol{a} + \boldsymbol{b} \wedge \boldsymbol{o}\widehat{\boldsymbol{x}} \text{ for all } \widehat{x} \in \widehat{\Omega},$$

or equivalently, such that

$$v_i(x)\boldsymbol{g}^i(x) = \boldsymbol{a} + \boldsymbol{b} \wedge \boldsymbol{\Theta}(x) \text{ for all } x \in \Omega.$$

Since the set where such a vector field $\widehat{v}_i\widehat{\boldsymbol{e}}^i$ vanishes is always of zero area unless $\boldsymbol{a} = \boldsymbol{b} = \boldsymbol{0}$ (as is easily proved; see, e.g., Ciarlet [1988, Theorem 6.3-4]), the assumption *area* $\Gamma_0 > 0$ implies that $\widehat{\boldsymbol{v}} = \boldsymbol{0}$. ☐

Remark. Since the fields $\boldsymbol{g}_i$ are of class $\mathcal{C}^1$ on $\overline{\Omega}$ by assumption, the components v_i of a field $\boldsymbol{v} = (v_i) \in \mathbf{H}^1(\Omega)$ satisfying $e_{ij}(\boldsymbol{v}) = 0$ in Ω are thus automatically in $\mathcal{C}^1(\overline{\Omega})$ since $v_i = (v_j\boldsymbol{g}^j)\cdot\boldsymbol{g}_i$. Remarkably, the field $v_i\boldsymbol{g}^i = \boldsymbol{a} + \boldsymbol{b} \wedge \boldsymbol{\Theta}$ inherits in this case even more regularity, as it is of class $\mathcal{C}^2$ on $\overline{\Omega}$! ☐

We are now in a position to prove the announced **Korn inequality in curvilinear coordinates "with boundary conditions"**.

Theorem 3.8-3. *Let Ω be a domain in $\mathbb{R}^3$, let $\boldsymbol{\Theta}$ be a $\mathcal{C}^2$-diffeomorphism of $\overline{\Omega}$ onto its image $\boldsymbol{\Theta}(\overline{\Omega}) \subset \mathbf{E}^3$, let Γ_0 be a* dΓ*-measurable subset of the boundary $\partial\Omega$ that satisfies* area $\Gamma_0 > 0$, *and let the space* $\mathbf{V}(\Omega)$ *be defined by*

$$\mathbf{V}(\Omega) := \{\boldsymbol{v} = (v_i) \in \mathbf{H}^1(\Omega);\ \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_0\}.$$

Then there exists a constant $C = C(\Omega, \Gamma_0, \boldsymbol{\Theta})$ such that

$$\|\boldsymbol{v}\|_{1,\Omega} \leq C\Big\{\sum_{i,j}\|e_{ij}(\boldsymbol{v})\|_{0,\Omega}^2\Big\}^{1/2} \text{ for all } \boldsymbol{v} \in \mathbf{V}(\Omega).$$

Proof. If the announced inequality is false, there exists a sequence $(\boldsymbol{v}^k)_{k=1}^\infty$ of elements $\boldsymbol{v}^k \in \mathbf{V}(\Omega)$ such that

$$\|\boldsymbol{v}^k\|_{1,\Omega} = 1 \text{ for all } k \text{ and } \lim_{k\to\infty} \|e_{ij}(\boldsymbol{v}^k)\|_{0,\Omega} = 0.$$

Since the sequence $(\boldsymbol{v}^k)_{k=1}^\infty$ is bounded in $\mathbf{H}^1(\Omega)$, a subsequence $(\boldsymbol{v}^\ell)_{\ell=1}^\infty$ converges in $\mathbf{L}^2(\Omega)$ by the *Rellich-Kondrašov theorem.* Furthermore, since

$$\lim_{\ell\to\infty} \|e_{ij}(\boldsymbol{v}^\ell)\|_{0,\Omega} = 0,$$

each sequence $(e_{ij}(\boldsymbol{v}^\ell))_{\ell=1}^\infty$ also converges in $L^2(\Omega)$ (to 0, but this information is not used at this stage). The subsequence $(\boldsymbol{v}^\ell)_{\ell=1}^\infty$ is thus a Cauchy sequence with respect to the norm

$$\boldsymbol{v} = (v_i) \to \Big\{ \sum_i \|v_i\|_{0,\Omega}^2 + \sum_{i,j} \|e_{ij}(\boldsymbol{v})\|_{0,\Omega}^2 \Big\}^{1/2},$$

hence with respect to the norm $\|\cdot\|_{1,\Omega}$ by *Korn's inequality without boundary conditions* (Theorem 3.8-1).

The space $\mathbf{V}(\Omega)$ being complete as a closed subspace of $\mathbf{H}^1(\Omega)$, there exists $\boldsymbol{v} \in \mathbf{V}(\Omega)$ such that

$$\lim_{\ell\to\infty} \boldsymbol{v}^\ell = \boldsymbol{v} \text{ in } \mathbf{H}^1(\Omega),$$

and the limit $\boldsymbol{v}$ satisfies $\|e_{ij}(\boldsymbol{v})\|_{0,\Omega} = \lim_{\ell\to\infty} \|e_{ij}(\boldsymbol{v}^\ell)\|_{0,\Omega} = 0$; hence $\boldsymbol{v} = \mathbf{0}$ by Theorem 3.8-2. But this contradicts the relations $\|\boldsymbol{v}^\ell\|_{1,\Omega} = 1$ for all $\ell \geq 1$, and the proof is complete. $\square$

Identifying $\mathbf{E}^3$ with $\mathbb{R}^3$ and letting $\boldsymbol{\Theta}(x) = x$ for all $x \in \overline{\Omega}$ shows that Theorems 3.8-1 to 3.8-3 contain as special cases the Korn inequalities and the infinitesimal rigid displacement lemma in *Cartesian coordinates* (see, e.g., Duvaut & Lions [1972, p. 110]).

Let

$$\mathbf{Rig}(\Omega) := \{\boldsymbol{r} \in \mathbf{H}^1(\Omega);\ e_{ij}(\boldsymbol{r}) = 0 \text{ in } \Omega\}$$

denote the space of vector fields $\boldsymbol{v} = (v_i) \in \mathbf{H}^1(\Omega)$ whose associated displacement fields $v_i\boldsymbol{g}^i$ are infinitesimal rigid displacements of the reference configuration $\boldsymbol{\Theta}(\overline{\Omega})$. We conclude our study of Korn's inequalities in curvilinear coordinates by showing that the Korn inequality "without boundary conditions" (Theorem 3.8-1) is equivalent to yet another **Korn's inequality in curvilinear coordinates, "over the quotient space $\mathbf{H}^1(\Omega)/\,\mathbf{Rig}(\Omega)$"**. As we shall see later (Theorem 3.9-3), this inequality is the basis of the existence theorem for the *pure traction problem.*

In the next theorem, the notation $\dot{\boldsymbol{v}}$ designates the equivalence class of an element $\boldsymbol{v} \in \mathbf{H}^1(\Omega)$ in the quotient space $\mathbf{H}^1(\Omega)/\,\mathbf{Rig}(\Omega)$. In other words,

$$\dot{\boldsymbol{v}} := \{\boldsymbol{w} \in \mathbf{H}^1(\Omega);\ (\boldsymbol{w} - \boldsymbol{v}) \in \mathbf{Rig}(\Omega)\}.$$

Theorem 3.8-4. *Let Ω be a domain in $\mathbb{R}^3$ and let $\boldsymbol{\Theta}$ be a $\mathcal{C}^2$-diffeomorphism of $\overline{\Omega}$ onto its image $\boldsymbol{\Theta}(\overline{\Omega}) \subset \mathbf{E}^3$. Let $\|\cdot\|_{1,\Omega}$ designate the quotient norm $\|\cdot\|_{1,\Omega}$ over the Hilbert space $\mathbf{H}^1(\Omega)/\,\mathbf{Rig}(\Omega)$, defined by*

$$\|\dot{\boldsymbol{v}}\|_{1,\Omega} := \inf_{\boldsymbol{r}\in\mathbf{Rig}(\Omega)} \|\boldsymbol{v}+\boldsymbol{r}\|_{1,\Omega} \text{ for all } \dot{\boldsymbol{v}} \in \mathbf{H}^1(\Omega)/\,\mathbf{Rig}(\Omega).$$

Then there exists a constant $\dot{C} = \dot{C}(\Omega, \boldsymbol{\Theta})$ such that

$$\|\dot{\boldsymbol{v}}\|_{1,\Omega} \le \dot{C}\Big\{\sum_{i,j} \|e_{ij}(\dot{\boldsymbol{v}})\|^2_{0,\Omega}\Big\}^{1/2} \text{ for all } \dot{\boldsymbol{v}} \in \mathbf{H}^1(\Omega)/\,\mathbf{Rig}(\Omega).$$

Moreover, this Korn inequality "over the quotient space $\mathbf{H}^1(\Omega)/\,\mathbf{Rig}(\Omega)$" is equivalent to the Korn inequality "without boundary condition" of Theorem 3.8-1.

Proof. (i) To begin with, we show that *the Korn inequality "without boundary conditions" implies the announced Korn inequality "over the quotient space $\mathbf{H}^1(\Omega)/\,\mathbf{Rig}(\Omega)$".*

By Theorem 3.8-2, a vector field $\boldsymbol{r} = (r_i) \in \mathbf{H}^1(\Omega)$ satisfies $e_{ij}(\boldsymbol{r}) = 0$ in Ω if and only if there exist two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ such that $v_i(x)\boldsymbol{g}^i(x) = \boldsymbol{a} + \boldsymbol{b} \wedge \boldsymbol{\Theta}(x)$ for all $x \in \overline{\Omega}$. This shows that the space $\mathbf{Rig}(\Omega)$ is finite-dimensional, of dimension *six*.

By the Hahn-Banach theorem, there thus exist six continuous linear forms ℓ_α on $\mathbf{H}^1(\Omega)$, $1 \le \alpha \le 6$ with the following property: An element $\boldsymbol{r} \in \mathbf{Rig}(\Omega)$ is equal to $\mathbf{0}$ if and only if $\ell_\alpha(\boldsymbol{r}) = 0$, $1 \le \alpha \le 6$. We then claim that it suffices to establish the existence of a constant D such that

$$\|\boldsymbol{v}\|_{1,\Omega} \le D\Big(\Big\{\sum_{i,j} \|e_{ij}(\boldsymbol{v})\|^2_{0,\Omega}\Big\}^{1/2} + \sum_{\alpha=1}^{6} |\ell_\alpha(\boldsymbol{v})|\Big) \text{ for all } \boldsymbol{v} \in \mathbf{H}^1(\Omega),$$

since this inequality will in turn imply the desired inequality: Given any $\boldsymbol{v} \in \mathbf{H}^1(\Omega)$, let $\boldsymbol{r}(\boldsymbol{v}) \in \mathbf{Rig}(\Omega)$ be such that $\ell_\alpha(\boldsymbol{v} + \boldsymbol{r}(\boldsymbol{v})) = 0$, $1 \le \alpha \le 6$; then

$$\begin{aligned}\|\dot{\boldsymbol{v}}\|_{1,\Omega} &= \inf_{\boldsymbol{r}\in\mathbf{Rig}(\Omega)} \|\boldsymbol{v}+\boldsymbol{r}\|_{1,\Omega} \le \|\boldsymbol{v}+\boldsymbol{r}(\boldsymbol{v})\|_{1,\Omega}\\ &\le D\Big\{\sum_{i,j} \|e_{ij}(\boldsymbol{v})\|^2_{0,\Omega}\Big\}^{1/2} = D\Big\{\sum_{i,j} \|e_{ij}(\dot{\boldsymbol{v}})\|^2_{0,\Omega}\Big\}^{1/2}.\end{aligned}$$

To establish the existence of such a constant D, assume the contrary. Then there exist $\boldsymbol{v}^k \in \mathbf{H}^1(\Omega)$, $k \ge 1$, such that

$$\|\boldsymbol{v}^k\|_{1,\Omega} = 1 \text{ for all } k \ge 1,$$

and

$$\Big(\Big\{\sum_{i,j} \|e_{ij}(\boldsymbol{v}^k)\|^2_{0,\Omega}\Big\}^{1/2} + \sum_{\alpha=1}^{6} |\ell_\alpha(\boldsymbol{v}^k)|\Big) \underset{k\to\infty}{\longrightarrow} 0.$$

By Rellich theorem, there exists a subsequence $(\boldsymbol{v}^\ell)_{\ell=1}^\infty$ that converges in $\mathbf{L}^2(\Omega)$. Since each sequence $(e_{ij}(\boldsymbol{v}^\ell))_{\ell=1}^\infty$ also converges in $\mathbf{L}^2(\Omega)$, the subsequence $(\boldsymbol{v}^\ell)_{\ell=1}^\infty$ is a Cauchy sequence with respect to the norm

$$\boldsymbol{v} \to \Big(\|\boldsymbol{v}\|_{0,\Omega} + \Big\{ \sum_{i,j} \|e_{ij}(\boldsymbol{v})\Big\|_{0,\Omega}^2 \}^{1/2} \Big),$$

hence also with respect to the norm $\|\cdot\|_{1,\Omega}$, by *Korn's inequality "without boundary conditions"* (Theorem 3.8-1). Consequently, there exists $\boldsymbol{v} \in \mathbf{H}^1(\Omega)$ such that $\|\boldsymbol{v}^\ell - \boldsymbol{v}\|_{1,\Omega} \underset{\ell\to\infty}{\longrightarrow} 0$. But then $\boldsymbol{v} = \mathbf{0}$ since $e_{ij}(\boldsymbol{v}) = 0$ and $\ell_\alpha(\boldsymbol{v}) = 0$, $1 \le \alpha \le 6$, in contradiction with the relations $\|\boldsymbol{v}^\ell\|_{1,\Omega} = 1$ for all $\ell \ge 1$.

(ii) We next show that, conversely, *Korn's inequality "in the quotient space* $\mathbf{H}^1(\Omega)/\,\mathbf{Rig}(\Omega)$ *implies Korn's inequality 'without boundary conditions".*

Assume the contrary. Then there exist $\boldsymbol{v}^k \in \mathbf{H}^1(\Omega)$, $k \ge 1$, such that

$$\|\boldsymbol{v}^k\|_{1,\Omega} = 1 \text{ for all } k \ge 1 \text{ and } \big(\|\boldsymbol{v}^k\|_{0,\Omega} + \|\boldsymbol{e}(\boldsymbol{v}^k)\|_{0,\Omega}\big) \underset{k\to\infty}{\longrightarrow} 0.$$

Let $\boldsymbol{r}^k \in \mathbf{Rig}(\Omega)$ denote for each $k \ge 1$ the projection of $\boldsymbol{v}^k$ on $\mathbf{Rig}(\Omega)$ with respect to the inner-product of $\mathbf{H}^1(\Omega)$, which thus satisfies:

$$\|\boldsymbol{v}^k - \boldsymbol{r}^k\|_{1,\Omega} = \inf_{\boldsymbol{r}\in\mathbf{Rig}(\Omega)} \|\boldsymbol{v}^k - \boldsymbol{r}\|_{1,\Omega} \text{ and } \|\boldsymbol{v}^k\|_{1,\Omega}^2 = \|\boldsymbol{v}^k - \boldsymbol{r}^k\|_{1,\Omega}^2 + \|\boldsymbol{r}^k\|_{1,\Omega}^2.$$

The space $\mathbf{Rig}(\Omega)$ being finite-dimensional, the inequalities $\|\boldsymbol{r}^k\|_{1,\Omega} \le 1$ for all $k \ge 1$ imply the existence of a subsequence $(\boldsymbol{r}^\ell)_{\ell=1}^\infty$ that converges in $\mathbf{H}^1(\Omega)$ to an element $\boldsymbol{r} \in \mathbf{Rig}(\Omega)$. Besides, *Korn's inequality in the quotient space* $\mathbf{H}^1(\Omega)/\,\mathbf{Rig}(\Omega)$ implies that $\|\boldsymbol{v}^\ell - \boldsymbol{r}^\ell\|_{1,\Omega} \underset{\ell\to\infty}{\longrightarrow} 0$, so that $\|\boldsymbol{v}^\ell - \boldsymbol{r}\|_{1,\Omega} \underset{\ell\to\infty}{\longrightarrow} 0$. Hence $\|\boldsymbol{v}^\ell - \boldsymbol{r}\|_{0,\Omega} \underset{\ell\to\infty}{\longrightarrow} 0$, which forces $\boldsymbol{r}$ to be $\mathbf{0}$, since $\|\boldsymbol{v}^\ell\|_{0,\Omega} \underset{\ell\to\infty}{\longrightarrow} 0$ on the other hand. We thus reach the conclusion that $\|\boldsymbol{v}^\ell\|_{1,\Omega} \underset{\ell\to\infty}{\longrightarrow} 0$, a contradiction. □

3.9 EXISTENCE AND UNIQUENESS THEOREMS IN LINEARIZED ELASTICITY IN CURVILINEAR COORDINATES

Let the space $\mathbf{V}(\Omega)$ be defined as before by

$$\mathbf{V}(\Omega) = \{\boldsymbol{v} \in \mathbf{H}^1(\Omega);\ \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_0\},$$

where Γ_0 is a subset of the boundary $\partial\Omega$ satisfying $\mathit{area}\,\Gamma_0 > 0$. Our objective consists in showing that the bilinear form $B : \mathbf{V}(\Omega) \times \mathbf{V}(\Omega) \to \mathbb{R}$ defined by

$$B(\boldsymbol{u}, \boldsymbol{v}) := \int_\Omega A^{ijk\ell} e_{k\ell}(\boldsymbol{u}) e_{ij}(\boldsymbol{v}) \sqrt{g}\,\mathrm{d}x$$

for all $(\boldsymbol{u}, \boldsymbol{v}) \in \mathbf{V}(\Omega) \times \mathbf{V}(\Omega)$ is $\mathbf{V}(\Omega)$-elliptic. As a preliminary, we establish the *uniform positive-definiteness of the elasticity tensor* ("uniform" means with respect to points in $\overline{\Omega}$ and to symmetric matrices of order three). Recall that the assumptions made in the next theorem about the Lamé constants λ and μ also reflect experimental evidence.

Theorem 3.9-1. *Let Ω be a domain in $\mathbb{R}^3$ and let $\mathbf{\Theta}$ be a $\mathcal{C}^2$-diffeomorphism of $\overline{\Omega}$ onto $\mathbf{\Theta}(\overline{\Omega}) \subset \mathbf{E}^3$, let the contravariant components $A^{ijk\ell} : \overline{\Omega} \to \mathbb{R}$ of the elasticity tensor be defined by*

$$A^{ijk\ell} = \lambda g^{ij} g^{k\ell} + \mu(g^{ik} g^{j\ell} + g^{i\ell} g^{jk}),$$

and assume that $3\lambda + 2\mu > 0$ and $\mu > 0$. Then there exists a constant $C_e = C_e(\Omega, \mathbf{\Theta}, \lambda, \mu) > 0$ such that

$$\sum_{i,j} |t_{ij}|^2 \leq C_e A^{ijk\ell}(x) t_{k\ell} t_{ij}$$

for all $x \in \overline{\Omega}$ and all symmetric matrices (t_{ij}).

Proof. We recall that $\mathbb{M}^d$ and $\mathbb{S}^d$ respectively designate the set of all real matrices of order d and the set of all real symmetric matrices of order d. The elegant proof of part (ii) given below is due to Cristinel Mardare.

(i) To begin with, we establish a crucial inequality. *Let $d \geq 2$ be an integer and let χ and μ be two constants satisfying $d\chi + 2\mu > 0$ and $\mu > 0$. Then there exists a constant $\alpha = \alpha(d, \chi, \mu) > 0$ such that*

$$\alpha \operatorname{tr}(\mathbf{B}^T \mathbf{B}) \leq \chi (\operatorname{tr} \mathbf{B})^2 + 2\mu \operatorname{tr}(\mathbf{B}^T \mathbf{B}) \text{ for all } \mathbf{B} \in \mathbb{M}^d.$$

If $\chi \geq 0$ and $\mu > 0$, this inequality holds with $\alpha = 2\mu$. It thus remains to consider the case where $-\dfrac{2\mu}{d} < \chi < 0$ and $\mu > 0$. Given any matrix $\mathbf{B} \in \mathbb{M}^d$, define the matrix $\mathbf{C} \in \mathbb{M}^d$ by

$$\mathbf{C} = \mathbb{A}\mathbf{B} := \chi(\operatorname{tr} \mathbf{B})\mathbf{I} + 2\mu \mathbf{B}.$$

The linear mapping $\mathbb{A} : \mathbb{M}^d \to \mathbb{M}^d$ defined in this fashion can be easily inverted if $d\chi + 2\mu \neq 0$ and $\mu \neq 0$, as

$$\mathbf{B} = \mathbb{A}^{-1}\mathbf{C} = -\frac{\chi}{2\mu(d\chi + 2\mu)}(\operatorname{tr} \mathbf{C})\mathbf{I} + \frac{1}{2\mu}\mathbf{C}.$$

Noting that the bilinear mapping

$$(\mathbf{B}, \mathbf{C}) \in \mathbb{M}^d \times \mathbb{M}^d \to \mathbf{B} : \mathbf{C} := \operatorname{tr} \mathbf{B}^T \mathbf{C}$$

defines an inner product over the space $\mathbb{M}^d$, we thus obtain

$$\begin{aligned}\chi(\operatorname{tr} \mathbf{B})^2 + 2\mu \operatorname{tr}(\mathbf{B}^T \mathbf{B}) &= (\mathbb{A}\mathbf{B}) : \mathbf{B} = \mathbf{C} : \mathbb{A}^{-1}\mathbf{C} \\ &= -\frac{\chi}{2\mu(d\chi + 2\mu)}(\operatorname{tr} \mathbf{C})^2 + \frac{1}{2\mu} \operatorname{tr}(\mathbf{C}^T \mathbf{C}) \geq \frac{1}{2\mu}\mathbf{C} : \mathbf{C}\end{aligned}$$

for any $\mathbf{B} = \mathbb{A}^{-1}\mathbf{C} \in \mathbb{M}^d$ if $-\frac{2\mu}{d} < \chi < 0$ and $\mu > 0$. Since there clearly exists a constant $\beta = \beta(d, \chi, \mu) > 0$ such that

$$\mathbf{B} : \mathbf{B} \leq \beta \mathbf{C} : \mathbf{C} \text{ for all } \mathbf{B} = \mathbb{A}^{-1}\mathbf{C} \in \mathbb{M}^d,$$

the announced inequality also holds if $-\frac{2\mu}{d} < \chi < 0$ and $\mu > 0$, with $\alpha = (2\mu\beta)^{-1}$ in this case.

(ii) We next show that, *for any* $x \in \overline{\Omega}$ *and any nonzero symmetric matrix* (t_{ij}),

$$A^{ijk\ell}(x)t_{k\ell}t_{ij} \geq \alpha g^{ik}(x)g^{j\ell}(x)t_{k\ell}t_{ij} > 0,$$

where $\alpha > 0$ *is the constant of* (i) *corresponding to* $d = 3$.

Given any $x \in \overline{\Omega}$ and any symmetric matrix (t_{ij}), let

$$\mathbf{G}(x) := (g^{ij}(x)) \text{ and } \mathbf{T} = (t_{ij}).$$

Then it is easily verified that

$$A^{ijk\ell}(x)t_{k\ell}t_{ij} = \lambda\Big(\operatorname{tr}(\mathbf{G}(x)\mathbf{T})\Big) + 2\mu \operatorname{tr}\Big(\mathbf{G}(x)\mathbf{T}\mathbf{G}(x)\mathbf{T}\Big).$$

In order to render this expression similar to that appearing in the right-hand side of the inequality of (i), let $\mathbf{H}(x) \in \mathbb{S}^3$ be the unique square root of $\mathbf{G}(x) \in \mathbb{S}^3$ (i.e., the unique positive-definite symmetric matrix that satisfies $(\mathbf{H}(x))^2 = \mathbf{G}(x)$; for details about such square roots, see, e.g., Ciarlet [1988, Theorem 3.2-1]), and let

$$\mathbf{B}(x) := \mathbf{H}(x)\mathbf{T}\mathbf{H}(x) \in \mathbb{S}^3.$$

Because $\operatorname{tr}(\mathbf{BC}) = \operatorname{tr}(\mathbf{CB})$ for any $\mathbf{B}, \mathbf{C} \in \mathbb{M}^3$, we may then also write

$$A^{ijk\ell}(x)t_{k\ell}t_{ij} = \lambda\Big(\operatorname{tr}(\mathbf{B}(x))\Big)^2 + 2\mu \operatorname{tr}\Big(\mathbf{B}(x)^T\mathbf{B}(x)\Big).$$

By (i), there thus exists a constant $\alpha > 0$ such that

$$A^{ijk\ell}(x)t_{k\ell}t_{ij} \geq \alpha \operatorname{tr}\Big(\mathbf{B}(x)^T\mathbf{B}(x)\Big).$$

Since $\mathbf{B}(x) = \mathbf{H}(x)\mathbf{T}\mathbf{H}(x) = \mathbf{0}$ only if $\mathbf{T} = \mathbf{0}$, it thus follows that, for any $x \in \overline{\Omega}$ and any *nonzero* symmetric matrix (t_{ij}),

$$\operatorname{tr}\Big(\mathbf{B}(x)^T\mathbf{B}(x)\Big) = \operatorname{tr}\Big(\mathbf{G}(x)\mathbf{T}\mathbf{G}(x)\mathbf{T}\Big) = g^{ik}(x)g^{j\ell}(x)t_{k\ell}t_{ij} > 0.$$

(iii) *Conclusion:* Since the mapping

$$(x, (t_{ij})) \in \mathbf{K} := \overline{\Omega} \times \Big\{(t_{ij}) \in \mathbb{S}^3; \sum_{i,j} |t_{ij}|^2 = 1\Big\} \to g^{ik}(x)g^{j\ell}(x)t_{k\ell}t_{ij}$$

is continuous and its domain of definition is compact, we infer that

$$\beta = \beta(\Omega; \boldsymbol{\Theta}) := \inf_{(x,(t_{ij}))\in \mathbf{K}} g^{ik}(x) g^{j\ell}(x) t_{k\ell} t_{ij} > 0.$$

Hence

$$\beta \sum_{i,j} |t_{ij}|^2 \le g^{ik}(x) g^{j\ell}(x) t_{k\ell} t_{ij},$$

and thus

$$\sum_{i,j} |t_{ij}|^2 \le C_e A^{ijk\ell}(x) t_{k\ell} t_{ij}$$

for all $x \in \overline{\Omega}$ and all symmetric matrices (t_{ij}) with $C_e := (\alpha\beta)^{-1}$. □

Remark. Letting the matrices $\mathbf{B}$ in the inequality of (i) be equal to the identity matrix and to any nonzero matrix with a vanishing trace shows that the inequalities $d\chi + 2\mu > 0$ and $\mu > 0$ are also *necessary* for the validity of this inequality.

With a little further ado, it can likewise be shown that the inequalities $3\lambda + 2\mu > 0$ and $\mu > 0$ necessarily hold if the elasticity tensor is uniformly positive definite. □

Combined with the Korn inequality "with boundary conditions" (Theorem 3.8-3), the positive-definiteness of the elasticity tensor leads to the existence and uniqueness of a **weak solution**, i.e., a solution to the *variational* equations of three-dimensional linearized elasticity in curvilinear coordinates.

Theorem 3.9-2. *Let Ω be a domain in $\mathbb{R}^3$, let Γ_0 be a $\mathrm{d}\Gamma$-measurable subset of $\Gamma = \partial\Omega$ that satisfies $\operatorname{area}\Gamma_0 > 0$, and let $\boldsymbol{\Theta}$ be a $\mathcal{C}^2$-diffeomorphism of $\overline{\Omega}$ onto its image $\boldsymbol{\Theta}(\overline{\Omega}) \subset \mathbf{E}^3$. Finally, let there be given constants λ and μ satisfying $3\lambda + 2\mu > 0$ and $\mu > 0$ and functions $f^i \in L^{6/5}(\Omega)$ and $h^i \in L^{4/3}(\Gamma_1)$, where $\Gamma_1 := \Gamma - \Gamma_0$.*

Then there is one and only one solution $\boldsymbol{u} = (u_i)$ to the variational problem:

$$\boldsymbol{u} \in \mathbf{V}(\Omega) := \{\boldsymbol{v} = (v_i) \in \mathbf{H}^1(\Omega);\ \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_0\},$$
$$\int_\Omega A^{ijk\ell} e_{k\ell}(\boldsymbol{u}) e_{ij}(\boldsymbol{v}) \sqrt{g}\, \mathrm{d}x = \int_\Omega f^i v_i \sqrt{g}\, \mathrm{d}x + \int_{\Gamma_1} h^i v_i \sqrt{g}\, \mathrm{d}\Gamma$$

for all $\boldsymbol{v} = (v_i) \in \mathbf{V}(\Omega)$, where

$$A^{ijk\ell} = \lambda g^{ij} g^{k\ell} + \mu \left(g^{ik} g^{j\ell} + g^{i\ell} g^{jk} \right),$$
$$e_{ij}(\boldsymbol{v}) = \frac{1}{2}(\partial_j v_i + \partial_i v_j) - \Gamma^p_{ij} v_p, \quad \Gamma^p_{ij} = \boldsymbol{g}^p \cdot \partial_i \boldsymbol{g}_j.$$

The field $\boldsymbol{u} \in \mathbf{V}(\Omega)$ is also the unique solution to the minimization problem:

$$J(\boldsymbol{u}) = \inf_{\boldsymbol{v}\in\mathbf{V}(\Omega)} J(\boldsymbol{v}),$$

where

$$J(\boldsymbol{v}) := \frac{1}{2}\int_\Omega A^{ijk\ell} e_{k\ell}(\boldsymbol{v}) e_{ij}(\boldsymbol{v})\sqrt{g}\,\mathrm{d}x - \Big\{\int_\Omega f^i v_i \sqrt{g}\,\mathrm{d}x + \int_{\Gamma_1} h^i v_i \sqrt{g}\,\mathrm{d}\Gamma\Big\}.$$

Proof. As a closed subspace of $\mathbf{H}^1(\Omega)$, the space $\mathbf{V}(\Omega)$ is a Hilbert space. The assumptions made on the mapping $\boldsymbol{\Theta}$ ensure in particular that the functions $A^{ijk\ell}, \Gamma^p_{ij}$, and g are continuous on the compact set $\overline{\Omega}$. Hence the bilinear form

$$B : (\boldsymbol{u}, \boldsymbol{v}) \in \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \longrightarrow \int_\Omega A^{ijk\ell} e_{k\ell}(\boldsymbol{u}) e_{ij}(\boldsymbol{v})\sqrt{g}\,\mathrm{d}x$$

is continuous.

The continuous imbedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ and the continuity of the trace operator $\mathrm{tr} : H^1(\Omega) \to L^4(\Gamma)$ imply that the linear form

$$L : \boldsymbol{v} \in \mathbf{H}^1(\Omega) \longrightarrow \Big\{\int_\Omega f^i v_i \sqrt{g}\,\mathrm{d}x + \int_{\Gamma_1} h^i v_i \sqrt{g}\,\mathrm{d}\Gamma\Big\}$$

is continuous.

Since the symmetric matrix $(g_{ij}(x))$ is positive-definite for all $x \in \overline{\Omega}$, there exists a constant g_0 such that

$$0 < g_0 \le g(x) = \det(g_{ij}(x)) \text{ for all } x \in \overline{\Omega}.$$

Finally, the *Korn inequality "with boundary conditions"* (Theorem 3.8-3) and the *uniform positive-definiteness of the elasticity tensor* (Theorem 3.9-1) together imply that

$$C_e^{-1} C^{-2} \sqrt{g_0} \|\boldsymbol{v}\|^2_{1,\Omega} \le \int_\Omega A^{ijk\ell} e_{k\ell}(\boldsymbol{v}) e_{ij}(\boldsymbol{v})\sqrt{g}\,\mathrm{d}x \text{ for all } \boldsymbol{v} \in \mathbf{V}(\Omega).$$

Hence *the bilinear form B is $\mathbf{V}(\Omega)$-elliptic.*

The bilinear form being also *symmetric* since $A^{ijk\ell} = A^{k\ell ij}$, all the assumptions of the *Lax-Milgram lemma* in its "symmetric" version are satisfied. Therefore, the variational problem has one and only one solution, which may be equivalently characterized as the solution of the minimization problem stated in the theorem (for a proof of the Lax-Milgram lemma in its "symmetric" form used here, see, e.g., Ciarlet [1988, Theorem 6.3-2]). □

An immediate corollary with a more "intrinsic" flavor to Theorem 3.9-2 is the existence and uniqueness of a *displacement field* $u_i \boldsymbol{g}^i$, whose covariant components $u_i \in H^1(\Omega)$ are thus obtained by finding the solution $\boldsymbol{u} = (u_i)$ to the variational problem. Since the vector fields $\boldsymbol{g}^i$ formed by the contravariant bases belong to the space $\mathcal{C}^1(\overline{\Omega})$ by assumption, the displacement field $u_i \boldsymbol{g}^i$ also belongs to the space $\mathbf{H}^1(\Omega)$.

Naturally, the existence and uniqueness result of Theorem 3.9-2 holds *a fortiori* in *Cartesian coordinates* (to see this, identify $\mathbf{E}^3$ with $\mathbb{R}^3$ and let $\boldsymbol{\Theta} = \boldsymbol{id}_{\overline{\Omega}}$).

Remark. Combining the relation

$$\int_{\widehat{\Omega}} \widehat{A}^{ijk\ell} \widehat{e}_{k\ell}(\widehat{\boldsymbol{u}}) e_{ij}(\widehat{\boldsymbol{v}}) \, \mathrm{d}\widehat{x} = \int_{\Omega} A^{ijk\ell} e_{k\ell}(\boldsymbol{u}) e_{ij}(\boldsymbol{v}) \sqrt{g} \, \mathrm{d}x$$

with the three-dimensional Korn inequality in *Cartesian coordinates* (see, e.g., Duvaut & Lions [1972, p. 110]) and with a classical result about composite mappings in Sobolev spaces (see, Nečas [1967, Chapter 2, Lemma 3.2] or Adams [1975, Theorem 3.35]), one can also show directly that the bilinear form

$$B : (\boldsymbol{u}, \boldsymbol{v}) \in \mathbf{V}(\Omega) \times \mathbf{V}(\Omega) \to \int_{\Omega} A^{ijk\ell} e_{k\ell}(\boldsymbol{u}) e_{ij}(\boldsymbol{v}) \sqrt{g} \, \mathrm{d}x$$

is $\mathbf{V}(\Omega)$-elliptic, thus providing another proof to Theorem 3.9-2. □

The above existence and uniqueness result applies to the *linearized pure displacement and displacement-traction problems*, i.e., those that correspond to *area* $\Gamma_0 > 0$.

We now consider the *linearized pure traction problem*, i.e., corresponding to $\Gamma_1 = \Gamma_0$. In this case, we seek a vector field $\boldsymbol{u} = (u_i) \in \mathbf{H}^1(\Omega)$ such that

$$\int_{\Omega} A^{ijk\ell} e_{k\ell}(\boldsymbol{u}) e_{ij}(\boldsymbol{v}) \sqrt{g} \, \mathrm{d}x = \int_{\Omega} f^i v_i \sqrt{g} \, \mathrm{d}x + \int_{\Gamma} h^i v_i \sqrt{g} \, \mathrm{d}\Gamma$$

for all $\boldsymbol{v} \in \mathbf{H}^1(\Omega)$. Clearly, such variational equations can have a solution *only if their right-hand side vanishes for any vector field* $\boldsymbol{r} = (r_i) \in \mathbf{H}^1(\Omega)$ *that satisfies* $e_{ij}(\boldsymbol{r}) = 0$ *in* Ω, since replacing $\boldsymbol{v}$ by $(\boldsymbol{v} + \boldsymbol{r})$ with any such field $\boldsymbol{r}$ does not affect their left-hand side. We now show that this necessary condition is in fact also sufficient for the existence of solutions, thanks in this case to the *Korn inequality "on the quotient space* $\mathbf{H}^1(\Omega)/\,\mathbf{Rig}(\Omega$*"* (Theorem 3.8-4).

Evidently, the *uniqueness* of solutions can then hold only up to the addition of vector fields satisfying $e_{ij}(\boldsymbol{r}) = 0$, which implies that the solution is now sought in the same *quotient space* $\mathbf{H}^1(\Omega)/\,\mathbf{Rig}(\Omega)$.

Theorem 3.9-3. *Let* Ω *be a domain in* $\mathbb{R}^3$ *and let* $\boldsymbol{\Theta}$ *be a* $\mathcal{C}^2$*-diffeomorphism of* $\overline{\Omega}$ *onto its image* $\boldsymbol{\Theta}(\overline{\Omega}) \subset \mathbf{E}^3$. *Let there be given constants* λ *and* μ *satisfying* $3\lambda + 2\mu > 0$ *and* $\mu > 0$ *and functions* $f^i \in L^{6/5}(\Omega)$ *and* $h^i \in L^{4/3}(\Gamma)$. *Define the space*

$$\mathbf{Rig}(\Omega) := \{\boldsymbol{r} \in \mathbf{H}^1(\Omega); e_{ij}(\boldsymbol{r}) = 0 \text{ in } \Omega\},$$

and assume that the functions f^i *and* h^i *are such that*

$$\int_{\Omega} f^i r_i \sqrt{g} \, \mathrm{d}x + \int_{\Gamma} h^i r_i \sqrt{g} \, \mathrm{d}\Gamma = 0 \text{ for all } \boldsymbol{r} = (r_i) \in \mathbf{Rig}(\Omega).$$

Finally, let the functions $A^{ijk\ell}$ *be defined as in* Theorem 3.9-2.

Then there is one and only one solution $\dot{\boldsymbol{u}} \in \mathbf{H}^1(\Omega)/\,\mathbf{Rig}(\Omega)$ *to the variational equations*

$$\int_{\Omega} A^{ijk\ell} e_{k\ell}(\dot{\boldsymbol{u}}) e_{ij}(\dot{\boldsymbol{v}}) \sqrt{g} \, \mathrm{d}x = \int_{\Omega} f^i \dot{v}_i \sqrt{g} \, \mathrm{d}x + \int_{\Gamma} h^i \dot{v}_i \sqrt{g} \, \mathrm{d}\Gamma$$

for all $\dot{\boldsymbol{v}} = (\dot{v}_i) \in \mathbf{H}^1(\Omega)/\,\mathbf{Rig}(\Omega)$.

The equivalence class $\dot{\boldsymbol{u}} \in \mathbf{H}^1(\Omega)/\,\mathbf{Rig}(\Omega)$ *is also the unique solution to the minimization problem*

$$J(\dot{\boldsymbol{u}}) = \inf_{\dot{\boldsymbol{v}} \in \mathbf{H}^1(\Omega)/\,\mathbf{Rig}(\Omega)} J(\dot{\boldsymbol{v}}),$$

where

$$J(\dot{\boldsymbol{v}}) := \frac{1}{2}\int_\Omega A^{ijk\ell} e_{k\ell}(\dot{\boldsymbol{v}}) e_{ij}(\dot{\boldsymbol{v}})\sqrt{g}\,\mathrm{d}x - \Big\{ \int_\Omega f^i \dot{v}_i \sqrt{g}\,\mathrm{d}x + \int_\Gamma h^i \dot{v}_i \sqrt{g}\,\mathrm{d}\Gamma \Big\}.$$

Proof. The proof is analogous to that of Theorem 3.9-2 and for this reason is omitted. ☐

When $\Gamma_0 = \Gamma$ and the boundary Γ is smooth enough, a *regularity result* shows that the weak solution obtained in Theorem 3.9-2 is also a *"classical solution"*, i.e., a solution of the corresponding *pure displacement boundary value problem*, according to the following result.

Theorem 3.9-4. *Let* Ω *be a domain in* $\mathbb{R}^3$ *with a boundary* Γ *of class* $\mathcal{C}^2$ *and let* $\boldsymbol{\Theta}$ *be a* $\mathcal{C}^2$*-diffeomorphism of* $\overline{\Omega}$ *onto its image* $\boldsymbol{\Theta}(\overline{\Omega}) \subset \mathbf{E}^3$.

If $\Gamma_0 = \Gamma$ *and* $\boldsymbol{f} := (f_i) \in \mathbf{L}^p(\Omega)$, $p \geq \frac{6}{5}$, *the weak solution* $\boldsymbol{u} \in \mathbf{V}(\Omega) = \mathbf{H}^1_0(\Omega)$ *found in* Theorem 3.9-2 *is in the space* $\mathbf{W}^{2,p}(\Omega)$ *and satisfies the equations*

$$-A^{ijk\ell} e_{k\ell}(\boldsymbol{u})\|_j = f^i \text{ in } L^p(\Omega).$$

Let $m \geq 1$ *be an integer. If the boundary* Γ *is of class* $\mathcal{C}^{m+2}$, *if* $\boldsymbol{\Theta}$ *is a* $\mathcal{C}^{m+2}$*-diffeomorphism of* $\overline{\Omega}$ *onto its image, and if* $f \in \mathbf{W}^{m,p}(\Omega)$, *the weak solution* $\boldsymbol{u} \in \mathbf{H}^1_0(\Omega)$ *is in the space* $\mathbf{W}^{m+2,p}(\Omega)$.

Proof. We very briefly sketch the main steps of the proof, which is otherwise long and delicate. As in Section 3.6, we let $\boldsymbol{\mathcal{A}}'(\mathbf{0})$ denote the linear operator defined by

$$\boldsymbol{\mathcal{A}}'(0) : \boldsymbol{v} \longrightarrow (-A^{ijk\ell} e_{k\ell}(\boldsymbol{v})\|_j)$$

for any smooth enough vector fields $\boldsymbol{v} = (v_i) : \overline{\Omega} \to \mathbb{R}^3$.

(i) Because the system associated with operator $\boldsymbol{\mathcal{A}}'(0)$ is *strongly elliptic*, the regularity result

$$\boldsymbol{f} \in \mathbf{L}^2(\Omega) \Longrightarrow \boldsymbol{u} \in \mathbf{H}^2(\Omega) \cap \mathbf{H}^1_0(\Omega)$$

holds if the boundary Γ is of class $\mathcal{C}^2$ (Nečas [1967, p. 260]). Hence the announced regularity holds for $m = 0$, $p = 2$.

(ii) Because the linearized pure displacement problem is *uniformly elliptic* and satisfies *the supplementary and complementing conditions*, according to the definitions of Agmon, Douglis & Nirenberg [1964], it follows from Geymonat [1965, Theorem 3.5] that, *considered as acting from the space*

$$\mathbf{V}^p(\Omega) := \{\boldsymbol{v} \in \mathbf{W}^{2,p}(\Omega);\ \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma\}$$

into the space $\mathbf{L}^p(\Omega)$*, the mapping* $\boldsymbol{\mathcal{A}}'(\mathbf{0})$ *has an index* $\operatorname{ind}\boldsymbol{\mathcal{A}}'(\mathbf{0})$ *that is independent of* $p \in]1,\infty[$. Recall that

$$\operatorname{ind}\boldsymbol{\mathcal{A}}'(\mathbf{0}) = \dim \operatorname{Ker}\boldsymbol{\mathcal{A}}'(\mathbf{0}) - \dim \operatorname{Coker}\boldsymbol{\mathcal{A}}'(\mathbf{0}),$$

where $\operatorname{Coker}\boldsymbol{\mathcal{A}}'(\mathbf{0})$ is the quotient space of the space $\mathbf{L}^p(\Omega)$ by the space $\operatorname{Im}\boldsymbol{\mathcal{A}}'(\mathbf{0})$ (the index is well defined only if both spaces $\operatorname{Ker}\boldsymbol{\mathcal{A}}'(\mathbf{0})$ and $\operatorname{Coker}\boldsymbol{\mathcal{A}}'(\mathbf{0})$ are finite-dimensional). In the present case, we know by (i) that $\operatorname{ind}\boldsymbol{\mathcal{A}}'(\mathbf{0}) = 0$ for $p = 2$ since $\boldsymbol{\mathcal{A}}'(\mathbf{0})$ is a bijection in this case ($\operatorname{Ker}\boldsymbol{\mathcal{A}}'(\mathbf{0}) = \{\mathbf{0}\}$ if and only if $\boldsymbol{\mathcal{A}}'(\mathbf{0})$ is injective, and $\operatorname{Coker}\boldsymbol{\mathcal{A}}'(\mathbf{0}) = \{\mathbf{0}\}$ if and only if $\boldsymbol{\mathcal{A}}'(\mathbf{0})$ is surjective).

Since the space $\mathbf{V}^p(\Omega)$ is continuously imbedded in the space $\mathbf{H}^1_0(\Omega)$ for $p \geq \frac{6}{5}$, the mapping $\boldsymbol{\mathcal{A}}'(\mathbf{0}) : \mathbf{V}^p(\Omega) \to \mathbf{L}^p(\Omega)$ is injective for these values of p (if $\boldsymbol{f} \in \mathbf{L}^{6/5}(\Omega)$, the weak solution is unique in the space $\mathbf{H}^1_0(\Omega)$; cf. Theorem 3.9-2); hence $\dim \operatorname{Ker}\boldsymbol{\mathcal{A}}'(\mathbf{0}) = 0$. Since $\operatorname{ind}\boldsymbol{\mathcal{A}}'(\mathbf{0}) = 0$ on the other hand, the mapping $\boldsymbol{\mathcal{A}}'(\mathbf{0})$ is also surjective in this case. Hence the regularity result holds for $m = 0$, $p \geq \frac{6}{5}$.

(iii) The weak solution $\boldsymbol{u} \in \mathbf{W}^{2,p}(\Omega) \cap \mathbf{H}^1_0(\Omega)$ satisfies the variational equations

$$\int_\Omega A^{ijk\ell} e_{k\ell}(\boldsymbol{u}) e_{ij}(\boldsymbol{v}) \sqrt{g}\,\mathrm{d}x = \int_\Omega f^i v_i \sqrt{g}\,\mathrm{d}x \text{ for all } \boldsymbol{v} = (v_i) \in \boldsymbol{\mathcal{D}}(\Omega).$$

Hence we can apply the same integration by parts formula as in Theorem 3.6-1. This gives

$$\int_\Omega A^{ijk\ell} e_{k\ell}(\boldsymbol{u}) e_{ij}(\dot{\boldsymbol{v}}) \sqrt{g}\,\mathrm{d}x = -\int_\Omega (A^{ijk\ell} e_{k\ell}(\boldsymbol{u})) v_i \sqrt{g}\,\mathrm{d}x$$

for all $\boldsymbol{v} = (v_i) \in \boldsymbol{\mathcal{D}}(\Omega)$, and the conclusion follows since $\{\boldsymbol{\mathcal{D}}(\Omega)\}^- = \mathbf{L}^p(\Omega)$.

(iv) Once the regularity result

$$\boldsymbol{f} \in \mathbf{W}^{m,p}(\Omega) \Longrightarrow \boldsymbol{u} \in \mathbf{W}^{m+2,p}(\Omega)$$

has been established for $m = 0$, it follows from Agmon, Douglis & Nirenberg [1964] and Geymonat [1965] that it also holds for higher values of the integer m if the boundary Γ is of class $\mathcal{C}^{m+2}$. $\square$

The regularity results of Theorem 3.9-4 can be extended to *linearized displacement-traction problems*, but *only if the closures of the sets* Γ_0 *and* Γ_1 *do not intersect.* They also apply to *linearized pure traction problems*, provided the functions h^i also possess *ad hoc* regularity. For instance, for $m = 2$, the functions h^i are assumed to belong to the space $W^{1-(1/p),p}(\Gamma)$ (for details about such “trace spaces”, see, e.g., Adams [1975, Chapter 7]).

Chapter 4

APPLICATIONS TO SHELL THEORY

INTRODUCTION

Consider a *nonlinearly elastic shell* with *middle surface* $S = \boldsymbol{\theta}(\overline{\omega})$ and *thickness* $2\varepsilon > 0$, where ω is a domain in $\mathbb{R}^2$ and $\boldsymbol{\theta} : \overline{\omega} \to \mathbf{E}^3$ is a smooth enough injective immersion. The material constituting the shell is *homogeneous* and *isotropic* and the reference configuration is a *natural state*; hence the material is characterized by two *Lamé constants* λ and μ satisfying $3\lambda + 2\mu > 0$ and $\mu > 0$. The shell is subjected to a homogeneous *boundary condition of place* along a portion of its lateral face with $\boldsymbol{\theta}(\gamma_0)$ as its middle curve, where γ_0 is a portion of the boundary $\partial\omega$ that satisfies *length* $\gamma_0 > 0$. Finally, the shell is subjected to *applied body forces* in its interior and to *applied surface forces* on its "upper" and "lower" faces. Let $p^i : \omega \to \mathbb{R}$ denote the contravariant components of the resultant (after integration across the thickness) of these forces, and let

$$a^{\alpha\beta\sigma\tau} = \frac{4\lambda\mu}{\lambda + 2\mu} a^{\alpha\beta} a^{\sigma\tau} + 2\mu(a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma})$$

denote the contravariant components of the *shell elasticity tensor*.

Then *Koiter's equations for a nonlinearly elastic shell*, which are described in detail in Section 4.1, take the following form when they are expressed as a *minimization problem*: The unknown vector field $\boldsymbol{\zeta} = (\zeta_i)$, where the functions $\zeta_i : \overline{\omega} \to \mathbb{R}$ are the covariant components of the displacement field $\zeta_i \boldsymbol{a}^i$ of the middle surface S, should be a *stationary* point (in particular a minimizer) of the functional j defined by

$$\begin{aligned} j(\boldsymbol{\eta}) := &\frac{1}{2} \int_\omega \frac{\varepsilon}{4} a^{\alpha\beta\sigma\tau} (a_{\sigma\tau}(\boldsymbol{\eta}) - a_{\sigma\tau})(a_{\alpha\beta}(\boldsymbol{\eta}) - a_{\alpha\beta}) \sqrt{a} \, \mathrm{d}y \\ &+ \frac{1}{2} \int_\omega \frac{\varepsilon^3}{3} a^{\alpha\beta\sigma\tau} (b_{\sigma\tau}(\boldsymbol{\eta}) - b_{\sigma\tau})(b_{\alpha\beta}(\boldsymbol{\eta}) - b_{\alpha\beta}) \sqrt{a} \, \mathrm{d}y - \int_\omega p^i \eta_i \sqrt{a} \, \mathrm{d}y, \end{aligned}$$

over an appropriate set of vector fields $\boldsymbol{\eta} = (\eta_i)$ satisfying *ad hoc* boundary conditions on γ_0.

For each such field $\boldsymbol{\eta} = (\eta_i)$, the functions $a_{\alpha\beta}(\boldsymbol{\eta})$ and $b_{\alpha\beta}(\boldsymbol{\eta})$ respectively denote the covariant components of the first and second fundamental forms of the deformed surface $(\boldsymbol{\theta} + \eta_i \boldsymbol{a}^i)(\overline{\omega})$, and the functions $\frac{1}{2}(a_{\alpha\beta}(\boldsymbol{\eta}) - a_{\alpha\beta})$ and $(b_{\alpha\beta}(\boldsymbol{\eta}) - b_{\alpha\beta})$ are the covariant components of the *change of metric*, and *change of curvature, tensor fields* associated with the displacement field $\eta_i \boldsymbol{a}^i$ of the middle surface S.

Such equations provide instances of *"two-dimensional" shell equations*. "Two-dimensional" means that such equations are expressed in terms of curvilinear coordinates (those that describe the middle surface of the shell) that vary in a *two-dimensional* domain ω.

The rest of this chapter is then devoted to a mathematical analysis of another set of two-dimensional shell equations, viz., those obtained from the nonlinear Koiter equations by a formal linearization, a procedure detailed in Section 4.2. The resulting *Koiter equations for a linearly elastic shell* take the following *weak*, or *variational*, form, i.e., when they are expressed as a *variational problem*: The unknown $\boldsymbol{\zeta} = (\zeta_i)$, where $\zeta_i \boldsymbol{a}^i$ is now to be interpreted as a "linearized approximation" of the unknown displacement field of the middle surface S, satisfies:

$$\boldsymbol{\zeta} = (\zeta_i) \in \mathbf{V}(\omega) = \{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega);\ \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0\},$$

$$\int_\omega \Big\{\varepsilon a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\boldsymbol{\zeta}) \gamma_{\alpha\beta}(\boldsymbol{\eta}) + \frac{\varepsilon^3}{3} a^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}(\boldsymbol{\zeta}) \rho_{\alpha\beta}(\boldsymbol{\eta})\Big\} \sqrt{a}\,\mathrm{d}y = \int_\omega p^i \eta_i \sqrt{a}\,\mathrm{d}y \text{ for all } \boldsymbol{\eta} = (\eta_i) \in \mathbf{V}(\omega),$$

where, for each $\boldsymbol{\eta} = (\eta_i) \in \mathbf{V}(\omega)$, the functions $\gamma_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega)$ and $\rho_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega)$ are the components of the *linearized change of metric*, and *linearized change of curvature, tensors* associated with a displacement field $\boldsymbol{\eta} = \eta_i \boldsymbol{a}^i$ of S, respectively given by

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \frac{1}{2}(\partial_\beta \widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\alpha + \partial_\alpha \widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\beta)$$
$$\rho_{\alpha\beta}(\boldsymbol{\eta}) = (\partial_{\alpha\beta} \widetilde{\boldsymbol{\eta}} - \Gamma^\sigma_{\alpha\beta} \partial_\sigma \widetilde{\boldsymbol{\eta}}) \cdot \boldsymbol{a}_3.$$

Equivalently, the unknown vector field $\boldsymbol{\zeta} \in \mathbf{V}(\omega)$ minimizes the functional $j : \mathbf{V}(\omega) \to \mathbb{R}$ defined by

$$j(\boldsymbol{\eta}) = \frac{1}{2} \int_\omega \Big\{\varepsilon a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\boldsymbol{\eta}) \gamma_{\alpha\beta}(\boldsymbol{\eta}) + \frac{\varepsilon^3}{3} a^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}(\boldsymbol{\eta}) \rho_{\alpha\beta}(\boldsymbol{\eta})\Big\} \sqrt{a}\,\mathrm{d}y - \int_\omega p^i \eta_i \sqrt{a}\,\mathrm{d}y$$

for all $\boldsymbol{\eta} \in \mathbf{V}(\omega)$.

As shown in Sections 4.3 and 4.4, the *existence* and *uniqueness* of a solution to these equations essentially rely on a fundamental *Korn inequality on a surface* (Theorem 4.3-4), itself a consequence of the same crucial *lemma of J.L. Lions* as in Chapter 3, and on the *uniform positive-definiteness of the shell elasticity tensor*, which holds under the assumptions $3\lambda + 2\mu > 0$ and $\mu > 0$ (Theorem 4.4-1).

The Korn inequality on a surface asserts that, given any subset γ_0 of $\partial\omega$ satisfying *length* $\gamma_0 > 0$, there exists a constant c such that

$$\Big\{ \sum_\alpha \|\eta_\alpha\|^2_{1,\omega} + \|\eta_3\|^2_{2,\omega} \Big\}^{1/2} \le c \Big\{ \sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\boldsymbol{\eta})\|^2_{0,\omega} + \sum_{\alpha,\beta} \|\rho_{\alpha\beta}(\boldsymbol{\eta})\|^2_{0,\omega} \Big\}^{1/2}$$

for all $\boldsymbol{\eta} = (\eta_i) \in \mathbf{V}(\omega)$.

We also derive (Theorem 4.4-4) the *boundary value problem* that is formally equivalent to the above variational equations. This problem takes the form

$$\begin{aligned} m^{\alpha\beta}|_{\alpha\beta} - b^\sigma_\alpha b_{\sigma\beta} m^{\alpha\beta} - b_{\alpha\beta} n^{\alpha\beta} &= p^3 \text{ in } \omega, \\ -(n^{\alpha\beta} + b^\alpha_\sigma m^{\sigma\beta})|_\beta - b^\alpha_\sigma (m^{\sigma\beta}|_\beta) &= p^\alpha \text{ in } \omega, \\ \zeta_i = \partial_\nu \zeta_3 &= 0 \text{ on } \gamma_0, \\ m^{\alpha\beta}\nu_\alpha\nu_\beta &= 0 \text{ on } \gamma_1, \\ (m^{\alpha\beta}|_\alpha)\nu_\beta + \partial_\tau(m^{\alpha\beta}\nu_\alpha\tau_\beta) &= 0 \text{ on } \gamma_1, \\ (n^{\alpha\beta} + 2b^\alpha_\sigma m^{\sigma\beta})\nu_\beta &= 0 \text{ on } \gamma_1, \end{aligned}$$

where $\gamma_1 = \partial\omega - \gamma_0$,

$$n^{\alpha\beta} = \varepsilon a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta}), \quad m^{\alpha\beta} = \frac{\varepsilon^3}{3} a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{\zeta}),$$

and such functions as

$$\begin{aligned} n^{\alpha\beta}|_\beta &= \partial_\beta n^{\alpha\beta} + \Gamma^\alpha_{\beta\sigma} n^{\beta\sigma} + \Gamma^\beta_{\beta\sigma} n^{\alpha\sigma}, \\ m^{\alpha\beta}|_{\alpha\beta} &= \partial_\alpha(m^{\alpha\beta}|_\beta) + \Gamma^\sigma_{\alpha\sigma}(m^{\alpha\beta}|_\beta) \end{aligned}$$

which *naturally* appear in the course of this derivation, provide instances of *first-order*, and *second-order, covariant derivatives of tensor fields defined on a surface*.

This chapter also includes, in Sections 4.1 and 4.5, brief introductions to other nonlinear and linear shell equations that are also *"two-dimensional"*, indicating in particular why Koiter shell equations may be regarded as those of an "all-purpose shell theory". Their choice here was motivated by this observation.

4.1 THE NONLINEAR KOITER SHELL EQUATIONS

To begin with, we briefly recapitulate some important notions already introduced and studied at length in Chapter 2. Note in this respect that we shall extend *without further notice* all the definitions given, or properties studied, on arbitrary *open subsets of* $\mathbb{R}^2$ in Chapter 2 to their analogs on *domains in* $\mathbb{R}^2$ (a similar extension, this time from open subsets to domains in $\mathbb{R}^3$, was carried out in Chapter 3). We recall that a domain U in $\mathbb{R}^d$ is an open, bounded, connected subset of $\mathbb{R}^d$, whose boundary is Lipschitz-continuous, the set U being locally on one side of its boundary.

Greek indices and exponents (except ν in the notation ∂_ν) range in the set $\{1,2\}$, Latin indices and exponents range in the set $\{1,2,3\}$ (save when they are used for indexing sequences), and the summation convention with respect to repeated indices and exponents is systematically used. Let $\mathbf{E}^3$ denote a three-dimensional Euclidean space. The Euclidean scalar product and the exterior product of $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{E}^3$ are noted $\boldsymbol{a} \cdot \boldsymbol{b}$ and $\boldsymbol{a} \wedge \boldsymbol{b}$ and the Euclidean norm of $\boldsymbol{a} \in \mathbf{E}^3$ is noted $|\boldsymbol{a}|$.

Let ω be a domain in $\mathbb{R}^2$. Let $y = (y_\alpha)$ denote a generic point in the set $\overline{\omega}$, and let $\partial_\alpha := \partial/\partial y_\alpha$. Let there be given an *immersion* $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbf{E}^3)$, i.e., a mapping such that the two vectors

$$\boldsymbol{a}_\alpha(y) := \partial_\alpha \boldsymbol{\theta}(y)$$

are linearly independent at all points $y \in \overline{\omega}$. These two vectors thus span the tangent plane to the *surface*

$$S := \boldsymbol{\theta}(\overline{\omega})$$

at the point $\boldsymbol{\theta}(y)$, and the unit vector

$$\boldsymbol{a}_3(y) := \frac{\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)}{|\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)|}$$

is normal to S at the point $\boldsymbol{\theta}(y)$. The three vectors $\boldsymbol{a}_i(y)$ constitute the *covariant basis* at the point $\boldsymbol{\theta}(y)$, while the three vectors $\boldsymbol{a}^i(y)$ defined by the relations

$$\boldsymbol{a}^i(y) \cdot \boldsymbol{a}_j(y) = \delta^i_j,$$

where δ^i_j is the Kronecker symbol, constitute the *contravariant basis* at the point $\boldsymbol{\theta}(y) \in S$ (recall that $\boldsymbol{a}^3(y) = \boldsymbol{a}_3(y)$ and that the vectors $\boldsymbol{a}^\alpha(y)$ are also in the tangent plane to S at $\boldsymbol{\theta}(y)$).

The covariant and contravariant components $a_{\alpha\beta}$ and $a^{\alpha\beta}$ of the *first fundamental form* of S, the *Christoffel symbols* $\Gamma^\sigma_{\alpha\beta}$, and the covariant and mixed components $b_{\alpha\beta}$ and b^β_α of the *second fundamental form* of S are then defined by letting:

$$a_{\alpha\beta} := \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta, \quad a^{\alpha\beta} := \boldsymbol{a}^\alpha \cdot \boldsymbol{a}^\beta, \quad \Gamma^\sigma_{\alpha\beta} := \boldsymbol{a}^\sigma \cdot \partial_\beta \boldsymbol{a}_\alpha,$$
$$b_{\alpha\beta} := \boldsymbol{a}^3 \cdot \partial_\beta \boldsymbol{a}_\alpha, \quad b^\beta_\alpha := a^{\beta\sigma} b_{\sigma\alpha}.$$

The *area* element along S is $\sqrt{a}\,\mathrm{d}y$, where

$$a := \det(a_{\alpha\beta}).$$

Note that $\sqrt{a} = |\boldsymbol{a}_1 \wedge \boldsymbol{a}_2|$.

Let $\Omega := \omega \times]-\varepsilon, \varepsilon[$, let $x = (x_i)$ denote a generic point in the set $\overline{\Omega}$ (hence $x_\alpha = y_\alpha$), and let $\partial_i := \partial/\partial x_i$. Consider an *elastic shell* with *middle surface* $S = \boldsymbol{\theta}(\overline{\omega})$ and *thickness* $2\varepsilon > 0$, i.e., an elastic body whose *reference configuration* is the set $\boldsymbol{\Theta}(\overline{\omega} \times [-\varepsilon, \varepsilon])$, where (cf. Figure 4.1-1)

$$\boldsymbol{\Theta}(y, x_3) := \boldsymbol{\theta}(y) + x_3 \boldsymbol{a}_3(y) \text{ for all } (y, x_3) \in \overline{\omega} \times [-\varepsilon, \varepsilon]\,.$$

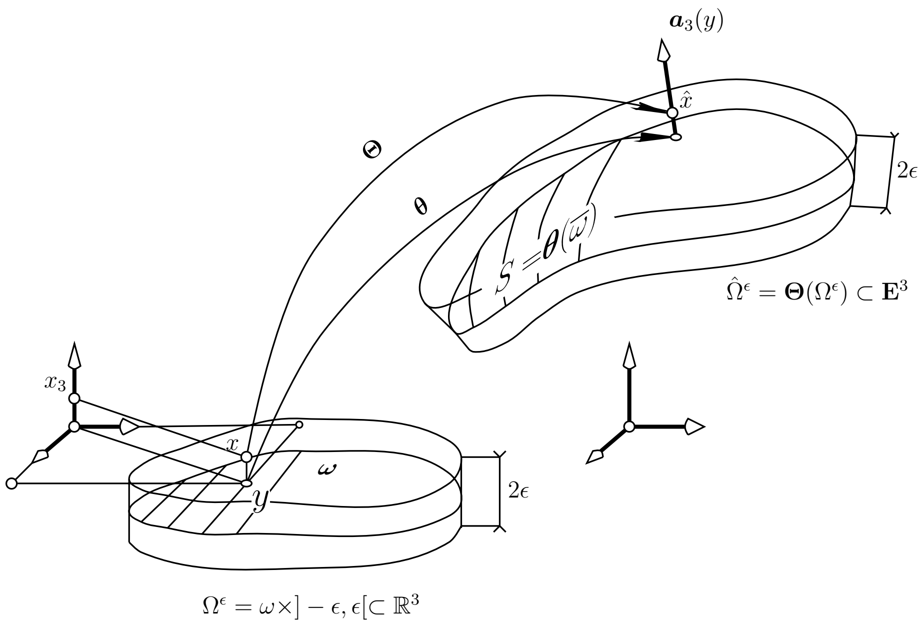

Figure 4.1-1: *The reference configuration of an elastic shell.* Let ω be a domain in $\mathbb{R}^2$, let $\Omega = \omega \times]-\varepsilon, \varepsilon[> 0$, let $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbf{E}^3)$ be an immersion, and let the mapping $\boldsymbol{\Theta} : \overline{\Omega} \to \mathbf{E}^3$ be defined by $\boldsymbol{\Theta}(y, x_3) = \boldsymbol{\theta}(y) + x_3\boldsymbol{a}_3(y)$ for all $(y, x_3) \in \overline{\Omega}$. Then the mapping $\boldsymbol{\Theta}$ is globally injective on $\overline{\Omega}$ if the immersion $\boldsymbol{\theta}$ is globally injective on $\overline{\omega}$ and $\varepsilon > 0$ is small enough (Theorem 4.1-1). In this case, the set $\boldsymbol{\Theta}(\overline{\Omega})$ may be viewed as the reference configuration of an elastic shell with thickness 2ε and middle surface $S = \theta(\overline{\omega})$. The coordinates (y_1, y_2, x_3) of an arbitrary point $x \in \overline{\Omega}$ are then viewed as curvilinear coordinates of the point $\widehat{x} = \boldsymbol{\Theta}(x)$ of the reference configuration of the shell.

Naturally, this definition makes sense physically only if the mapping $\boldsymbol{\Theta}$ is globally injective on the set $\overline{\Omega}$. Following Ciarlet [2000a, Theorem 3.1-1], we now show that this is indeed the case if the immersion $\boldsymbol{\theta}$ is itself globally injective on the set $\overline{\omega}$ and ε is small enough.

Theorem 4.1-1. *Let ω be a domain in $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbf{E}^3)$ be an injective immersion. Then there exists $\varepsilon > 0$ such that the mapping $\boldsymbol{\Theta} : \overline{\Omega} \to \mathbf{E}^3$ defined by*

$$\boldsymbol{\Theta}(y, x_3) := \boldsymbol{\theta}(y) + x_3\boldsymbol{a}_3(y) \text{ for all } (y, x_3) \in \overline{\Omega},$$

where $\Omega := \omega \times]-\varepsilon, \varepsilon[$, is a $\mathcal{C}^2$-diffeomorphism from $\overline{\Omega}$ onto $\boldsymbol{\Theta}(\overline{\Omega})$ and $\det(\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3) > 0$ *in $\overline{\Omega}$, where $\boldsymbol{g}_i := \partial_i \boldsymbol{\Theta}$.*

Proof. The assumed regularity on $\boldsymbol{\theta}$ implies that $\boldsymbol{\Theta} \in \mathcal{C}^2(\overline{\omega} \times [-\varepsilon, \varepsilon]; \mathbf{E}^3)$ for any $\varepsilon > 0$. The relations

$$\boldsymbol{g}_\alpha = \partial_\alpha \boldsymbol{\Theta} = \boldsymbol{a}_\alpha + x_3 \partial_\alpha \boldsymbol{a}_3 \text{ and } \boldsymbol{g}_3 = \partial_3 \boldsymbol{\Theta} = \boldsymbol{a}_3$$

imply that

$$\det(\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3)|_{x_3=0} = \det(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3) > 0 \text{ in } \overline{\omega}.$$

Hence $\det(\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3) > 0$ on $\overline{\omega} \times [-\varepsilon, \varepsilon]$ if $\varepsilon > 0$ is small enough.

Therefore, the *implicit function theorem* can be applied if ε is small enough: It shows that, *locally*, the mapping $\boldsymbol{\Theta}$ is a $\mathcal{C}^2$-diffeomorphism: Given any $y \in \overline{\omega}$, there exist a neighborhood $U(y)$ of y in $\overline{\omega}$ and $\varepsilon(y) > 0$ such that $\boldsymbol{\Theta}$ is a $\mathcal{C}^2$-diffeomorphism from the set $U(y) \times [-\varepsilon(y), \varepsilon(y)]$ onto $\boldsymbol{\Theta}(U(y) \times [-\varepsilon(y), \varepsilon(y)])$. See, e.g., Schwartz [1992, Chapter 3] (the proof of the implicit function theorem, which is almost invariably given for functions defined over open sets, can be easily extended to functions defined over *closures of domains*, such as the sets $\overline{\omega} \times [-\varepsilon, \varepsilon]$; see, e.g., Stein [1970]).

To establish that the mapping $\boldsymbol{\Theta} : \overline{\omega} \times [-\varepsilon, \varepsilon] \to \mathbf{E}^3$ is injective provided $\varepsilon > 0$ is small enough, we proceed by contradiction: If this property is false, there exist $\varepsilon_n > 0, (y^n, x_3^n)$, and $(\widetilde{y}^n, \widetilde{x}_3^n)$, $n \geq 0$, such that

$$\varepsilon_n \to 0 \text{ as } n \to \infty, \quad y^n \in \overline{\omega}, \quad \widetilde{y}^n \in \overline{\omega}, \quad |x_3^n| \leq \varepsilon_n, \quad |\widetilde{x}_3^n| \leq \varepsilon_n,$$
$$(y^n, x_3^n) \neq (\widetilde{y}^n, \widetilde{x}_3^n) \text{ and } \boldsymbol{\Theta}(y^n, x_3^n) = \boldsymbol{\Theta}(\widetilde{y}^n, \widetilde{x}_3^n).$$

Since the set $\overline{\omega}$ is compact, there exist $y \in \overline{\omega}$ and $\widetilde{y} \in \overline{\omega}$, and there exists a subsequence, still indexed by n for convenience, such that

$$y^n \to y, \quad \widetilde{y}^n \to \widetilde{y}, \quad x_3^n \to 0, \quad \widetilde{x}_3^n \to 0 \text{ as } n \to \infty.$$

Hence

$$\boldsymbol{\theta}(y) = \lim_{n\to\infty} \boldsymbol{\Theta}(y^n, x_3^n) = \lim_{n\to\infty} \boldsymbol{\Theta}(\widetilde{y}^n, \widetilde{x}_3^n) = \boldsymbol{\theta}(\widetilde{y}),$$

by the continuity of the mapping $\boldsymbol{\Theta}$ and thus $y = \widetilde{y}$ since the mapping $\boldsymbol{\theta}$ is injective by assumption. But these properties contradict the local injectivity (noted above) of the mapping $\boldsymbol{\Theta}$. Hence there exists $\varepsilon > 0$ such that $\boldsymbol{\Theta}$ is injective on the set $\overline{\Omega} = \overline{\omega} \times [-\varepsilon, \varepsilon]$. $\square$

In what follows, we assume that $\varepsilon > 0$ is small enough so that the conclusions of Theorem 4.1-1 hold. In particular then, $(y_1, y_2, x_3) \in \overline{\Omega}$ constitutes a *bona fide* system of *curvilinear coordinates* for describing the reference configuration $\boldsymbol{\Theta}(\overline{\Omega})$ of the shell.

Let γ_0 be a measurable subset of the boundary $\gamma := \partial\omega$. If *length* $\gamma_0 > 0$, we assume that the shell is subjected to a *homogeneous boundary condition of place* along the portion $\boldsymbol{\Theta}(\gamma_0 \times [-\varepsilon, \varepsilon])$ of its lateral face $\boldsymbol{\Theta}(\gamma \times [-\varepsilon, \varepsilon])$, which means that its displacement field vanishes on the set $\boldsymbol{\Theta}(\gamma_0 \times [-\varepsilon, \varepsilon])$.

The shell is subjected to *applied body forces* in its interior $\boldsymbol{\Theta}(\Omega)$ and to *applied surface forces* on its “upper” and “lower” faces $\boldsymbol{\Theta}(\Gamma_+)$ and $\boldsymbol{\Theta}(\Gamma_-)$, where $\Gamma_\pm := \omega \times \{\pm\varepsilon\}$. The applied forces are given by the contravariant components (i.e., over the covariant bases $\boldsymbol{g}_i = \partial_i\boldsymbol{\Theta}$) $f^i \in L^2(\Omega)$ and $h^i \in L^2(\Gamma_+ \cup \Gamma_-)$ of their densities per unit volume and per unit area, respectively. We then define functions $p^i \in L^2(\omega)$ by letting

$$p^i := \int_{-\varepsilon}^{\varepsilon} f^i x_3 + h^i(\cdot, +\varepsilon) + h^i(\cdot, -\varepsilon).$$

Finally, the elastic material constituting the shell is assumed to be *homogeneous* and *isotropic* and the reference configuration $\boldsymbol{\Theta}(\overline{\Omega})$ of the shell is assumed to be a *natural state*. Hence the material is characterized by two *Lamé constants* λ and μ satisfying $3\lambda + 2\mu > 0$ and $\mu > 0$.

Such a shell, endowed with its "natural" *curvilinear coordinates*, namely the coordinates $(y_1, y_2, x_3) \in \overline{\Omega}$, can thus be modeled as a *three-dimensional problem*. According to Chapter 3, the corresponding *unknowns* are thus the *three covariant components* $u_i : \overline{\Omega} \to \mathbb{R}$ *of the displacement field* $u_i \boldsymbol{g}^i : \overline{\Omega} \to \mathbb{R}$ *of the points of the reference configuration* $\boldsymbol{\Theta}(\overline{\Omega})$, where the vector fields $\boldsymbol{g}^i$ denote the contravariant bases (i.e., defined by the relations $\boldsymbol{g}^i \cdot \boldsymbol{g}_j = \delta^i_j$, where $\boldsymbol{g}_j = \partial_j \boldsymbol{\Theta}$; that these vector fields are well defined also follows from Theorem 4.1-1); cf. Figure 4.1-2. These unknowns then satisfy the *equations of elasticity in curvilinear coordinates*, as described in Sections 3.5 and 3.6.

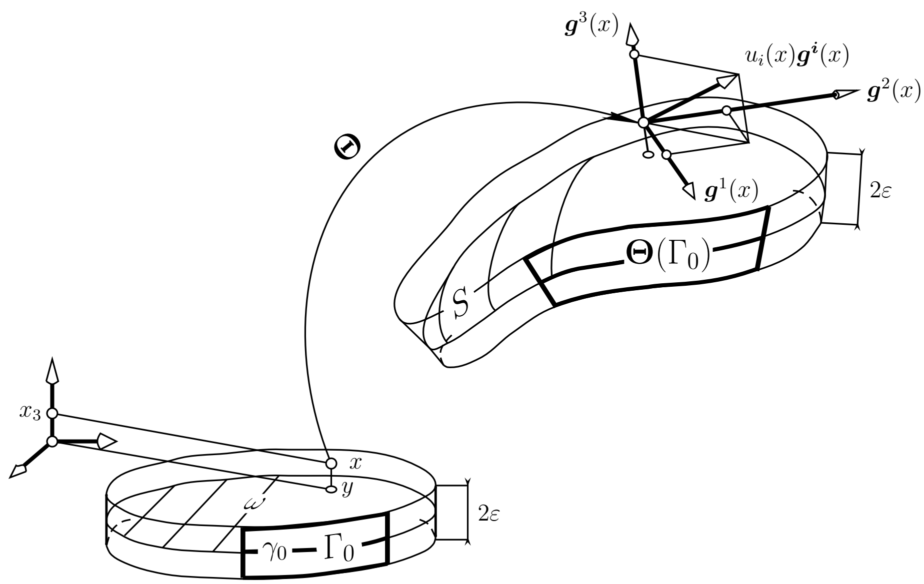

Figure 4.1-2: *An elastic shell modeled as a three-dimensional problem.* Let $\Omega = \omega \times]-\varepsilon, \varepsilon[$. The set $\boldsymbol{\Theta}(\overline{\Omega})$, where $\boldsymbol{\Theta}(y, x_3) = \boldsymbol{\theta}(y) + x_3 \boldsymbol{a}_3(y)$ for all $x = (y, x_3) \in \overline{\Omega}$, is the reference configuration of a shell, with thickness 2ε and middle surface $S = \boldsymbol{\theta}(\overline{\omega})$ (Figure 4.1-1), which is subjected to a boundary condition of place along the portion $\boldsymbol{\Theta}(\Gamma_0)$ of its lateral face (i.e., the displacement vanishes on $\boldsymbol{\Theta}(\Gamma_0)$), where $\Gamma_0 = \gamma_0 \times [-\varepsilon, \varepsilon]$ and $\gamma_0 \subset \gamma = \partial\omega$. The shell is subjected to applied body forces in its interior $\boldsymbol{\Theta}(\Omega)$ and to applied surface forces on its upper and lower faces $\boldsymbol{\Theta}(\Gamma_+)$ and $\boldsymbol{\Theta}(\Gamma_-)$ where $\Gamma_\pm = \overline{\omega} \times \{\pm\varepsilon\}$. Under the influence of these forces, a point $\boldsymbol{\Theta}(x)$ undergoes a displacement $u_i(x)\boldsymbol{g}^i(x)$, where the three vectors $\boldsymbol{g}^i(x)$ form the contravariant basis at the point $\boldsymbol{\Theta}(x)$. The unknowns of the problem are the three covariant components $u_i : \overline{\Omega} \to \mathbb{R}$ of the displacement field $u_i \boldsymbol{g}^i : \overline{\Omega} \to \mathbb{R}^3$ of the points of $\boldsymbol{\Theta}(\overline{\Omega})$, which thus satisfy the boundary conditions $u_i = 0$ on Γ_0. The objective consists in finding *ad hoc* conditions affording the "replacement" of this three-dimensional problem by a "two-dimensional problem posed over the middle surface S" if ε is "small enough"; see Figure 4.1-3.

Note that, for the sake of visual clarity, the thickness is overly exaggerated.

In a "*two-dimensional approach*", the above three-dimensional problem is "replaced" by a presumably much simpler *"two-dimensional" problem*, this time "*posed over the middle surface S of the shell*". This means that the new unknowns should be now the *three covariant components* $\zeta_i : \overline{\omega} \to \mathbb{R}$ *of the displacement field* $\zeta_i \boldsymbol{a}^i : \overline{\omega} \to \mathbf{E}^3$ *of the points of the middle surface* $S = \boldsymbol{\theta}(\overline{\omega})$; cf. Figure 4.1-3.

During the past decades, considerable progress has been made towards a rigorous justification of such a "replacement". The central idea is that of *asymptotic analysis:* It consists in showing that, if the data are of *ad hoc* orders of magnitude, *the three-dimensional displacement vector field* (once properly "scaled") *converges in an appropriate function space as* $\varepsilon \to 0$ *to a "limit" vector field that can be entirely computed by solving a two-dimensional problem.*

In this direction, see Ciarlet [2000a, Part A] for a thorough overview in the *linear* case and the key contributions of Le Dret & Raoult [1996] and Friesecke, James, Mora & Müller [2003] in the *nonlinear* case.

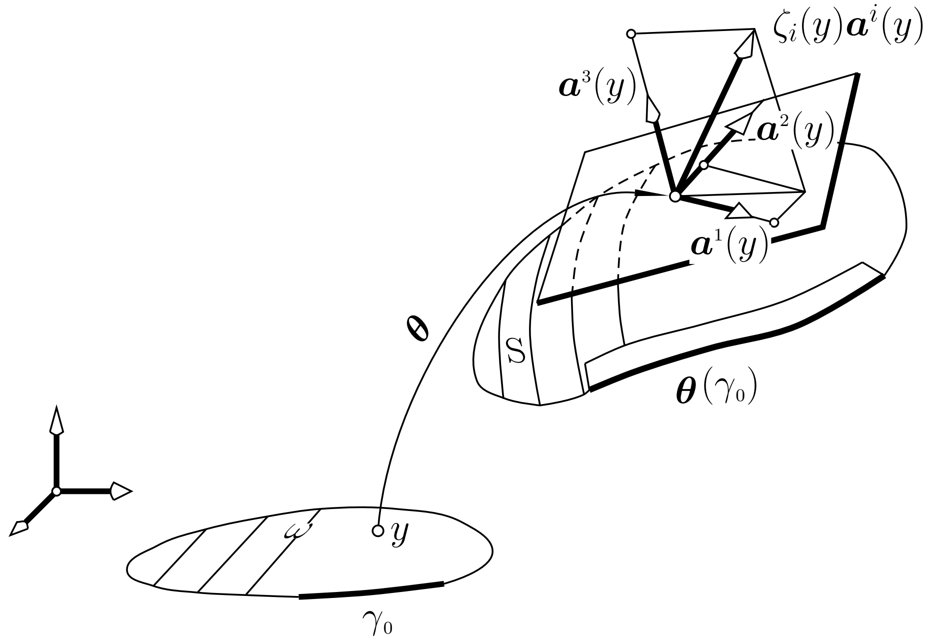

Figure 4.1-3: *An elastic shell modeled as a two-dimensional problem.* For $\varepsilon > 0$ "small enough" and data of *ad hoc* orders of magnitude, the three-dimensional shell problem (Figure 4.1-2) is "replaced" by a "two-dimensional shell problem". This means that the new unknowns are the three covariant components $\zeta_i : \overline{\omega} \to \mathbb{R}$ of the displacement field $\zeta_i \boldsymbol{a}^i : \overline{\omega} \to \mathbb{R}^3$ of the points of the middle surface $S = \boldsymbol{\theta}(\overline{\omega})$. In this process, the "three-dimensional" boundary conditions on Γ_0 need to be replaced by *ad hoc* "two-dimensional" boundary conditions on γ_0. For instance, the "boundary conditions of clamping" $\zeta_i = \partial_\gamma \zeta_3 = 0$ on γ_0 (used in Koiter's linear equations; cf. Section 4.2) mean that the points of, and the tangent spaces to, the deformed and undeformed middle surfaces coincide along the set $\boldsymbol{\theta}(\gamma_0)$.

We now describe the **nonlinear Koiter shell equations**, so named after Koiter [1966], and since then a nonlinear model of choice in computational mechanics (its relation to an asymptotic analysis as $\varepsilon \to 0$ is briefly discussed at the end of this section).

Given an arbitrary displacement field $\eta_i \boldsymbol{a}^i : \overline{\omega} \to \mathbb{R}^3$ of the surface S with smooth enough components $\eta_i : \overline{\omega} \to \mathbb{R}$, define the vector field $\boldsymbol{\eta} := (\eta_i) : \overline{\omega} \to \mathbb{R}^3$ and let

$$a_{\alpha\beta}(\boldsymbol{\eta}) := \boldsymbol{a}_\alpha(\boldsymbol{\eta}) \cdot \boldsymbol{a}_\beta(\boldsymbol{\eta}), \text{ where } \boldsymbol{a}_\alpha(\boldsymbol{\eta}) := \partial_\alpha(\boldsymbol{\theta} + \eta_i \boldsymbol{a}^i),$$

denote the covariant components of the first fundamental form of the *deformed surface* $(\boldsymbol{\theta} + \eta_i \boldsymbol{a}^i)(\overline{\omega})$. Then the functions

$$G_{\alpha\beta}(\boldsymbol{\eta}) := \frac{1}{2}(a_{\alpha\beta}(\boldsymbol{\eta}) - a_{\alpha\beta})$$

denote the **covariant components of the change of metric tensor** associated with the displacement field $\eta_i \boldsymbol{a}^i$ of S.

Remark. An easy computation, which simply relies on the formulas of Gauß and Weingarten (Theorem 2.6-1), shows that

$$G_{\alpha\beta}(\boldsymbol{\eta}) = \frac{1}{2}(\eta_{\alpha\|\beta} + \eta_{\beta\|\alpha} + a^{mn}\eta_{m\|\alpha}\eta_{n\|\beta}),$$

where

$$a^{\alpha 3} = a^{3\alpha} = 0 \text{ and } a^{33} = 1$$

(otherwise the functions $a^{\alpha\beta}$ denote as usual the contravariant components of the first fundamental form of S), and

$$\eta_{\alpha\|\beta} := \partial_\beta \eta_\alpha - \Gamma^\sigma_{\alpha\beta}\eta_\sigma - b_{\alpha\beta}\eta_3 \quad \text{and} \quad \eta_{3\|\beta} := \partial_\beta \eta_3 + b^\sigma_\beta \eta_\sigma.$$

□

If the two vectors $a_\alpha(\boldsymbol{\eta})$ are linearly independent at all points of ω, let

$$b_{\alpha\beta}(\boldsymbol{\eta}) := \frac{1}{\sqrt{a(\boldsymbol{\eta})}} \partial_{\alpha\beta}(\boldsymbol{\theta} + \eta_i \boldsymbol{a}^i) \cdot \{\boldsymbol{a}_1(\boldsymbol{\eta}) \wedge \boldsymbol{a}_2(\boldsymbol{\eta})\},$$

where

$$a(\boldsymbol{\eta}) := \det(a_{\alpha\beta}(\boldsymbol{\eta})),$$

denote the covariant components of the second fundamental form of the *deformed surface* $(\boldsymbol{\theta} + \eta_i \boldsymbol{a}^i)(\overline{\omega})$. Then the functions

$$R_{\alpha\beta}(\boldsymbol{\eta}) := b_{\alpha\beta}(\boldsymbol{\eta}) - b_{\alpha\beta}$$

denote the **covariant components of the change of curvature tensor field** associated with the displacement field $\eta_i \boldsymbol{a}^i$ of S. Note that $\sqrt{a(\boldsymbol{\eta})} = |\boldsymbol{a}_1(\boldsymbol{\eta}) \wedge \boldsymbol{a}_2(\boldsymbol{\eta})|$.

The nonlinear two-dimensional equations proposed by Koiter [1966] for modeling an elastic shell are derived from those of nonlinear three-dimensional elasticity on the basis of two *a priori* assumptions: One assumption, of a *geometrical* nature, is the *Kirchhoff-Love assumption*. It asserts that any point situated on a normal to the middle surface remains on the normal to the deformed middle surface after the deformation has taken place and that, in addition, the distance between such a point and the middle surface remains constant. The other assumption, of a *mechanical* nature, asserts that the state of stress inside the shell is planar and parallel to the middle surface (this second assumption is itself based on delicate *a priori* estimates due to John [1965, 1971]).

Taking these *a priori* assumptions into account, W.T. Koiter then reached the conclusion that the unknown vector field $\boldsymbol{\zeta} = (\zeta_i)$ should be a *stationary point*, in particular a *minimizer*, over a set of smooth enough vector fields $\boldsymbol{\eta} = (\eta_i) : \overline{\omega} \to \mathbb{R}^3$ satisfying *ad hoc* boundary conditions on γ_0, of the functional j defined by (cf. Koiter [1966, eqs. (4.2), (8.1), and (8.3)]):

$$j(\boldsymbol{\eta}) = \frac{1}{2}\int_\omega \Big\{\varepsilon a^{\alpha\beta\sigma\tau} G_{\sigma\tau}(\boldsymbol{\eta}) G_{\alpha\beta}(\boldsymbol{\eta}) + \frac{\varepsilon^3}{3} a^{\alpha\beta\sigma\tau} R_{\sigma\tau}(\boldsymbol{\eta}) R_{\alpha\beta}(\boldsymbol{\eta})\Big\}\sqrt{a}\,\mathrm{d}y$$
$$- \int_\omega p^i \eta_i \sqrt{a}\,\mathrm{d}y,$$

where the functions

$$a^{\alpha\beta\sigma\tau} := \frac{4\lambda\mu}{\lambda + 2\mu} a^{\alpha\beta} a^{\sigma\tau} + 2\mu(a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma})$$

denote the **contravariant components of the shell elasticity tensor**. The above functional j is called **Koiter's energy for a nonlinear elastic shell**.

Remark. The specific form of the functions $a^{\alpha\beta\sigma\tau}$ can be fully justified, in both the *linear* and *nonlinear* cases, by means of an asymptotic analysis of the solution of the three-dimensional equations as the thickness 2ε approaches zero; see Ciarlet [2000a], Le Dret & Raoult [1996] and Friesecke, James, Mora & Müller [2003]. □

The *stored energy function* w_K found in Koiter's energy j is thus defined by

$$w_K(\boldsymbol{\eta}) = \frac{\varepsilon}{2} a^{\alpha\beta\sigma\tau} G_{\sigma\tau}(\boldsymbol{\eta}) G_{\alpha\beta}(\boldsymbol{\eta}) + \frac{\varepsilon^3}{6} a^{\alpha\beta\sigma\tau} R_{\sigma\tau}(\boldsymbol{\eta}) R_{\alpha\beta}(\boldsymbol{\eta})$$

for *ad hoc* vector fields $\boldsymbol{\eta}$. This expression is the sum of the *"membrane" part*

$$w_M(\boldsymbol{\eta}) = \frac{\varepsilon}{2} a^{\alpha\beta\sigma\tau} G_{\sigma\tau}(\boldsymbol{\eta}) G_{\alpha\beta}(\boldsymbol{\eta})$$

and of the *"flexural" part*

$$w_F(\boldsymbol{\eta}) = \frac{\varepsilon^3}{6} a^{\alpha\beta\sigma\tau} R_{\sigma\tau}(\boldsymbol{\eta}) R_{\alpha\beta}(\boldsymbol{\eta}).$$

As hinted at earlier, the long-standing question of how to rigorously identify and justify the nonlinear two-dimensional equations of elastic shells from three-dimensional elasticity was finally settled in two key contributions, one by Le Dret & Raoult [1996] and one by Friesecke, James, Mora & Müller [2003], who respectively justified the equations of a *nonlinearly elastic membrane shell* and those of a *nonlinearly elastic flexural shell* by means of Γ-*convergence theory* (a nonlinearly elastic shell is a membrane shell if there are no nonzero admissible displacements of its middle surface S that preserve the metric of S; it is a flexural shell otherwise).

The stored energy function $w_M^\sharp$ of a nonlinearly elastic membrane shell is an *ad hoc* quasiconvex envelope, which turns out to be only a function of the covariant components $a_{\alpha\beta}(\boldsymbol{\eta})$ of the first fundamental form of the unknown deformed middle surface (the notion of quasiconvexity, which plays a central role in the calculus of variations, is due to Morrey [1952]; an excellent introduction to this notion is provided in Dacorogna [1989, Chapter 5]). The function $w_M^\sharp$ reduces to the above "membrane" part w_M in Koiter's stored energy function w_K only for a restricted class of displacement fields $\eta_i \boldsymbol{a}^i$ of the middle surface. By contrast, the stored energy function of a nonlinearly elastic flexural shell is always equal to the above "flexural" part w_F in Koiter's stored energy function w_K.

Remark. Interestingly, a *formal* asymptotic analysis of the three-dimensional equations is only capable of delivering the above "restricted" expression $w_M(\boldsymbol{\eta})$, but otherwise fails to provide the general expression, i.e., valid for all types of displacements, found by Le Dret & Raoult [1996]. By contrast, the same formal approach yields the correct expression $w_F(\boldsymbol{\eta})$. For details, see Miara [1998], Lods & Miara [1998], and Ciarlet [2000a, Part B]. □

Another closely related set of nonlinear shell equations "of Koiter's type" has been proposed by Ciarlet [2000b]. In these equations, the denominator $\sqrt{a(\boldsymbol{\eta})}$ that appears in the functions $R_{\alpha\beta}(\boldsymbol{\eta}) = b_{\alpha\beta}(\boldsymbol{\eta}) - b_{\alpha\beta}$ is simply replaced by $\sqrt{a}$, thereby avoiding the possibility of a vanishing denominator in the expression $w_K(\boldsymbol{\eta})$. Then Ciarlet & Roquefort [2001] have shown that the leading term of a formal asymptotic expansion of a solution to this two-dimensional model, with the thickness 2ε as the "small" parameter, coincides with that found by a formal asymptotic analysis of the three-dimensional equations. This result thus raises hopes that a rigorous justification, again by means of Γ-convergence theory, of either types of nonlinear Koiter's models might be possible.

4.2 THE LINEAR KOITER SHELL EQUATIONS

Consider the Koiter energy j for a nonlinearly elastic shell, defined by (cf. Section 4.1)

$$j(\boldsymbol{\eta}) = \frac{1}{2}\int_\omega \Big\{\varepsilon a^{\alpha\beta\sigma\tau} G_{\sigma\tau}(\boldsymbol{\eta}) G_{\alpha\beta}(\boldsymbol{\eta}) + \frac{\varepsilon^3}{3} a^{\alpha\beta\sigma\tau} R_{\sigma\tau}(\boldsymbol{\eta}) R_{\alpha\beta}(\boldsymbol{\eta})\Big\}\sqrt{a}\,\mathrm{d}y$$
$$- \int_\omega p^i \eta_i \sqrt{a}\,\mathrm{d}y,$$

for smooth enough vector fields $\boldsymbol{\eta} = (\eta_i) : \overline{\omega} \to \mathbb{R}^3$. One of its virtues is that the integrands of the first two integrals are *quadratic* expressions in terms of the covariant components $G_{\alpha\beta}(\boldsymbol{\eta})$ and $R_{\alpha\beta}(\boldsymbol{\eta})$ of the change of metric, and change of curvature, tensors associated with a displacement field $\eta_i \boldsymbol{a}^i$ of the middle surface $S = \boldsymbol{\theta}(\overline{\omega})$ of the shell. In order to obtain the energy corresponding to the *linear* equations of Koiter [1970], it thus suffices to replace the covariant components

$$G_{\alpha\beta}(\boldsymbol{\eta}) = \frac{1}{2}(a_{\alpha\beta}(\boldsymbol{\eta}) - a_{\alpha\beta}) \text{ and } R_{\alpha\beta}(\boldsymbol{\eta}) = b_{\alpha\beta}(\boldsymbol{\eta}) - b_{\alpha\beta},$$

of these tensors by their *linear parts with respect to* $\boldsymbol{\eta} = (\eta_i)$, respectively denoted $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta}(\boldsymbol{\eta})$ below. Accordingly, our first task consists in finding explicit expressions of such linearized tensors. To begin with, we compute the components $\gamma_{\alpha\beta}(\boldsymbol{\eta})$.

A word of caution. *The vector fields*

$$\boldsymbol{\eta} = (\eta_i) \text{ and } \widetilde{\boldsymbol{\eta}} := \eta_i \boldsymbol{a}^i,$$

which are both defined on $\overline{\omega}$, *must be carefully distinguished!* While the latter has an *intrinsic character*, the former has not; it only provides a means of recovering the field $\widetilde{\boldsymbol{\eta}}$ *via* its covariant components η_i. □

Theorem 4.2-1. *Let* ω *be a domain in* $\mathbb{R}^2$ *and let* $\boldsymbol{\theta} \in \mathcal{C}^2(\overline{\omega}; \mathbf{E}^3)$ *be an immersion. Given a displacement field* $\widetilde{\boldsymbol{\eta}} := \eta_i \boldsymbol{a}^i$ *of the surface* $S = \boldsymbol{\theta}(\overline{\omega})$ *with smooth enough covariant components* $\eta_i : \overline{\omega} \to \mathbb{R}$, *let the function* $\gamma_{\alpha\beta}(\boldsymbol{\eta}) : \overline{\omega} \to \mathbb{R}$ *be defined by*

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) := \frac{1}{2}\left[a_{\alpha\beta}(\boldsymbol{\eta}) - a_{\alpha\beta}\right]^{\text{lin}},$$

where $a_{\alpha\beta}$ *and* $a_{\alpha\beta}(\boldsymbol{\eta})$ *are the covariant components of the first fundamental form of the surfaces* $\boldsymbol{\theta}(\overline{\omega})$ *and* $(\boldsymbol{\theta} + \eta_i \boldsymbol{a}^i)(\overline{\omega})$, *and* $[\cdots]^{\text{lin}}$ *denotes the linear part with respect to* $\boldsymbol{\eta} = (\eta_i)$ *in the expression* $[\cdots]$. *Then*

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \frac{1}{2}\left(\partial_\beta \widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\alpha + \partial_\alpha \widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\beta\right) = \gamma_{\beta\alpha}(\boldsymbol{\eta})$$
$$= \frac{1}{2}(\eta_{\alpha|\beta} + \eta_{\beta|\alpha}) - b_{\alpha\beta}\eta_3$$
$$= \frac{1}{2}(\partial_\beta \eta_\alpha + \partial_\alpha \eta_\beta) - \Gamma^\sigma_{\alpha\beta}\eta_\sigma - b_{\alpha\beta}\eta_3,$$

where the covariant derivatives $\eta_{\alpha|\beta}$ *are defined by* $\eta_{\alpha|\beta} = \partial_\beta \eta_\alpha - \Gamma^\sigma_{\alpha\beta}\eta_\sigma$ (Theorem 2.6-1). *In particular then,*

$$\eta_\alpha \in H^1(\omega) \text{ and } \eta_3 \in L^2(\omega) \Rightarrow \gamma_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega).$$

Proof. The covariant components $a_{\alpha\beta}(\boldsymbol{\eta})$ of the metric tensor of the surface $(\boldsymbol{\theta} + \eta_i \boldsymbol{a}^i)(\overline{\omega})$ are by definition given by

$$a_{\alpha\beta}(\boldsymbol{\eta}) = \partial_\alpha(\boldsymbol{\theta} + \widetilde{\boldsymbol{\eta}}) \cdot \partial_\beta(\boldsymbol{\theta} + \widetilde{\boldsymbol{\eta}}).$$

Note that *both* surfaces $\boldsymbol{\theta}(\overline{\omega})$ and $(\boldsymbol{\theta} + \eta_i \boldsymbol{a}^i)(\overline{\omega})$ are thus equipped with the *same* curvilinear coordinates y_α. The relations

$$\partial_\alpha(\boldsymbol{\theta} + \widetilde{\boldsymbol{\eta}}) = \boldsymbol{a}_\alpha + \partial_\alpha \widetilde{\boldsymbol{\eta}}$$

then show that

$$\begin{aligned} a_{\alpha\beta}(\boldsymbol{\eta}) &= (\boldsymbol{a}_\alpha + \partial_\alpha \widetilde{\boldsymbol{\eta}}) \cdot (\boldsymbol{a}_\beta + \partial_\beta \widetilde{\boldsymbol{\eta}}) \\ &= a_{\alpha\beta} + \partial_\beta \widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\alpha + \partial_\alpha \widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\beta + \partial_\alpha \widetilde{\boldsymbol{\eta}} \cdot \partial_\beta \widetilde{\boldsymbol{\eta}}, \end{aligned}$$

hence that

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \frac{1}{2}[a_{\alpha\beta}(\boldsymbol{\eta}) - a_{\alpha\beta}]^{\text{lin}} = \frac{1}{2}(\partial_\beta \widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\alpha + \partial_\alpha \widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\beta).$$

The other expressions of $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ immediately follow from the expression of $\partial_\alpha \widetilde{\boldsymbol{\eta}} = \partial_\alpha(\eta_i \boldsymbol{a}^i)$ given in Theorem 2.6-1. $\square$

The functions $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ are called the **covariant components of the linearized change of metric tensor** associated with a displacement $\eta_i \boldsymbol{a}^i$ of the surface S.

We next compute the components $\rho_{\alpha\beta}(\boldsymbol{\eta})$.

Theorem 4.2-2. *Let* ω *be a domain in* $\mathbb{R}^2$ *and let* $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbf{E}^3)$ *be an immersion. Given a displacement field* $\widetilde{\boldsymbol{\eta}} := \eta_i \boldsymbol{a}^i$ *of the surface* $S = \boldsymbol{\theta}(\overline{\omega})$ *with smooth enough and "small enough" covariant components* $\eta_i : \overline{\omega} \to \mathbb{R}$, *let the functions* $\rho_{\alpha\beta}(\boldsymbol{\eta}) : \overline{\omega} \to \mathbb{R}$ *be defined by*

$$\rho_{\alpha\beta}(\boldsymbol{\eta}) := [b_{\alpha\beta}(\boldsymbol{\eta}) - b_{\alpha\beta}]^{\text{lin}},$$

where $b_{\alpha\beta}$ *and* $b_{\alpha\beta}(\boldsymbol{\eta})$ *are the covariant components of the second fundamental form of the surfaces* $\boldsymbol{\theta}(\overline{\omega})$ *and* $(\boldsymbol{\theta} + \eta_i \boldsymbol{a}^i)(\overline{\omega})$, *and* $[\cdots]^{\text{lin}}$ *denotes the linear part with respect to* $\boldsymbol{\eta} = (\eta_i)$ *in the expression* $[\cdots]$. *Then*

$$\begin{aligned} \rho_{\alpha\beta}(\boldsymbol{\eta}) &= (\partial_{\alpha\beta} \widetilde{\boldsymbol{\eta}} - \Gamma^\sigma_{\alpha\beta} \partial_\sigma \widetilde{\boldsymbol{\eta}}) \cdot \boldsymbol{a}_3 = \rho_{\beta\alpha}(\boldsymbol{\eta}) \\ &= \eta_{3|\alpha\beta} - b^\sigma_\alpha b_{\sigma\beta} \eta_3 + b^\sigma_\alpha \eta_{\sigma|\beta} + b^\tau_\beta \eta_{\tau|\alpha} + b^\tau_\beta|_\alpha \eta_\tau \\ &= \partial_{\alpha\beta}\eta_3 - \Gamma^\sigma_{\alpha\beta}\partial_\sigma \eta_3 - b^\sigma_\alpha b_{\sigma\beta}\eta_3 \\ &\quad + b^\sigma_\alpha(\partial_\beta \eta_\sigma - \Gamma^\tau_{\beta\sigma}\eta_\tau) + b^\tau_\beta(\partial_\alpha \eta_\tau - \Gamma^\sigma_{\alpha\tau}\eta_\sigma) \\ &\quad + (\partial_\alpha b^\tau_\beta + \Gamma^\tau_{\alpha\sigma} b^\sigma_\beta - \Gamma^\sigma_{\alpha\beta} b^\tau_\sigma)\eta_\tau, \end{aligned}$$

where the covariant derivatives $\eta_{\alpha|\beta}$ *are defined by* $\eta_{\alpha|\beta} = \partial_\beta \eta_\alpha - \Gamma^\sigma_{\alpha\beta}\eta_\sigma$ (Theorem 2.6-1) *and*

$$\eta_{3|\alpha\beta} := \partial_{\alpha\beta}\eta_3 - \Gamma^\sigma_{\alpha\beta}\partial_\sigma\eta_3 \text{ and } b^\tau_\beta|_\alpha := \partial_\alpha b^\tau_\beta + \Gamma^\tau_{\alpha\sigma}b^\sigma_\beta - \Gamma^\sigma_{\alpha\beta}b^\tau_\sigma.$$

In particular then,

$$\eta_\alpha \in H^1(\omega) \text{ and } \eta_3 \in H^2(\omega) \Rightarrow \rho_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega).$$

The functions $b^\tau_\beta|_\alpha$ *satisfy the symmetry relations*

$$b^\tau_\beta|_\alpha = b^\tau_\alpha|_\beta.$$

Proof. For convenience, the proof is divided into five parts. In parts (i) and (ii), we establish elementary relations satisfied by the vectors $\boldsymbol{a}_i$ and $\boldsymbol{a}^i$ of the covariant and contravariant bases along S.

(i) *The two vectors* $\boldsymbol{a}_\alpha = \partial_\alpha\boldsymbol{\theta}$ *satisfy* $|\boldsymbol{a}_1 \wedge \boldsymbol{a}_2| = \sqrt{a}$, *where* $a = \det(a_{\alpha\beta})$.

Let $\mathbf{A}$ denote the matrix of order three with $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ as its column vectors. Consequently,

$$\det \mathbf{A} = (\boldsymbol{a}_1 \wedge \boldsymbol{a}_2)\cdot\boldsymbol{a}_3 = (\boldsymbol{a}_1 \wedge \boldsymbol{a}_2)\cdot\frac{\boldsymbol{a}_1 \wedge \boldsymbol{a}_2}{|\boldsymbol{a}_1 \wedge \boldsymbol{a}_2|} = |\boldsymbol{a}_1 \wedge \boldsymbol{a}_2|.$$

Besides,

$$(\det \mathbf{A})^2 = \det(\mathbf{A}^T\mathbf{A}) = \det(a_{\alpha\beta)} = a,$$

since $\boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta = a_{\alpha\beta}$ and $\boldsymbol{a}_\alpha \cdot \boldsymbol{a}_3 = \delta_{\alpha 3}$. Hence $|\boldsymbol{a}_1 \wedge \boldsymbol{a}_2| = \sqrt{a}$.

(ii) *The vectors* $\boldsymbol{a}_i$ *and* $\boldsymbol{a}^\alpha$ *are related by* $\boldsymbol{a}_1 \wedge \boldsymbol{a}_3 = -\sqrt{a}\boldsymbol{a}^2$ *and* $\boldsymbol{a}_3 \wedge \boldsymbol{a}_2 = -\sqrt{a}\boldsymbol{a}^1$.

To prove that two vectors $\boldsymbol{c}$ and $\boldsymbol{d}$ coincide, it suffices to prove that $\boldsymbol{c}\cdot\boldsymbol{a}_i = \boldsymbol{d}\cdot\boldsymbol{a}_i$ for $i \in \{1,2,3\}$. In the present case,

$$(\boldsymbol{a}_1 \wedge \boldsymbol{a}_3)\cdot\boldsymbol{a}_1 = 0 \text{ and } (\boldsymbol{a}_1 \wedge \boldsymbol{a}_3)\cdot\boldsymbol{a}_3 = 0,$$
$$(\boldsymbol{a}_1 \wedge \boldsymbol{a}_3)\cdot\boldsymbol{a}_2 = -(\boldsymbol{a}_1 \wedge \boldsymbol{a}_2)\cdot\boldsymbol{a}_3 = -\sqrt{a},$$

since $\sqrt{a}\boldsymbol{a}_3 = \boldsymbol{a}_1 \wedge \boldsymbol{a}_2$ by (i), on the one hand; on the other hand,

$$-\sqrt{a}\boldsymbol{a}^2\cdot\boldsymbol{a}_1 = -\sqrt{a}\boldsymbol{a}^2\cdot\boldsymbol{a}_3 = 0 \text{ and } -\sqrt{a}\boldsymbol{a}^2\cdot\boldsymbol{a}_2 = -\sqrt{a},$$

since $\boldsymbol{a}^i \cdot \boldsymbol{a}_j = \delta^i_j$. Hence $\boldsymbol{a}_1 \wedge \boldsymbol{a}_3 = -\sqrt{a}\boldsymbol{a}^2$. The other relation is similarly established.

(iii) *The covariant components* $b_{\alpha\beta}(\boldsymbol{\eta})$ *satisfy*

$$b_{\alpha\beta}(\boldsymbol{\eta}) = b_{\alpha\beta} + (\partial_{\alpha\beta}\widetilde{\boldsymbol{\eta}} - \Gamma^\sigma_{\alpha\beta}\partial_\sigma\widetilde{\boldsymbol{\eta}})\cdot\boldsymbol{a}_3 + \text{ h.o.t.},$$

where "h.o.t." *stands for "higher-order terms"*, i.e., terms of order higher than linear with respect to $\boldsymbol{\eta} = (\eta_i)$. Consequently,

$$\rho_{\alpha\beta}(\boldsymbol{\eta}) := [b_{\alpha\beta}(\boldsymbol{\eta}) - b_{\alpha\beta}]^{\text{lin}} = \left(\partial_{\alpha\beta}\widetilde{\boldsymbol{\eta}} - \Gamma^\sigma_{\alpha\beta}\partial_\sigma\widetilde{\boldsymbol{\eta}}\right)\cdot\boldsymbol{a}_3 = \rho_{\beta\alpha}(\boldsymbol{\eta}).$$

Since the vectors $\boldsymbol{a}_\alpha = \partial_\alpha \boldsymbol{\theta}$ are linearly independent in $\overline{\omega}$ and the fields $\boldsymbol{\eta} = (\eta_i)$ are smooth enough by assumption, *the vectors* $\partial_\alpha(\boldsymbol{\theta} + \eta_i \boldsymbol{a}^i)$ *are also linearly independent in* $\overline{\omega}$ *provided the fields* $\boldsymbol{\eta}$ *are "small enough"*, e.g., with respect to the norm of the space $\mathcal{C}^1(\overline{\omega}; \mathbb{R}^3)$. The following computations are therefore licit as they apply to a linearization around $\boldsymbol{\eta} = \mathbf{0}$.

Let

$$\boldsymbol{a}_\alpha(\boldsymbol{\eta}) := \partial_\alpha(\boldsymbol{\theta} + \widetilde{\boldsymbol{\eta}}) = \boldsymbol{a}_\alpha + \partial_\alpha \widetilde{\boldsymbol{\eta}} \text{ and } \boldsymbol{a}_3(\boldsymbol{\eta}) := \frac{\boldsymbol{a}_1(\boldsymbol{\eta}) \wedge \boldsymbol{a}_2(\boldsymbol{\eta})}{\sqrt{a(\boldsymbol{\eta})}},$$

where

$$a(\boldsymbol{\eta}) := \det(a_{\alpha\beta}(\boldsymbol{\eta})) \text{ and } a_{\alpha\beta}(\boldsymbol{\eta}) := a_\alpha(\boldsymbol{\eta}) \cdot \boldsymbol{a}_\beta(\eta).$$

Then

$$\begin{aligned}
b_{\alpha\beta}(\boldsymbol{\eta}) &= \partial_\alpha \boldsymbol{a}_\beta(\boldsymbol{\eta}) \cdot \boldsymbol{a}_3(\boldsymbol{\eta}) \\
&= \frac{1}{\sqrt{a(\boldsymbol{\eta})}}(\partial_\alpha \boldsymbol{a}_\beta + \partial_{\alpha\beta}\widetilde{\boldsymbol{\eta}}) \cdot (\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 + \boldsymbol{a}_1 \wedge \partial_2 \widetilde{\boldsymbol{\eta}} + \partial_1 \widetilde{\boldsymbol{\eta}} \wedge \boldsymbol{a}_2 + \text{ h.o.t.}) \\
&= \frac{1}{\sqrt{a(\boldsymbol{\eta})}} \left\{ \sqrt{a}(b_{\alpha\beta} + \partial_{\alpha\beta}\widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_3) \right\} \\
&\quad + \frac{1}{\sqrt{a(\boldsymbol{\eta})}} \left\{ (\Gamma^\sigma_{\alpha\beta} \boldsymbol{a}_\sigma + b_{\alpha\beta} \boldsymbol{a}_3) \cdot (\boldsymbol{a}_1 \wedge \partial_2 \widetilde{\boldsymbol{\eta}} + \partial_1 \widetilde{\boldsymbol{\eta}} \wedge \boldsymbol{a}_2) + \text{ h.o.t. } \right\},
\end{aligned}$$

since $b_{\alpha\beta} = \partial_\alpha \boldsymbol{a}_\beta \cdot \boldsymbol{a}_3$ and $\partial_\alpha \boldsymbol{a}_\beta = \Gamma^\sigma_{\alpha\beta} \boldsymbol{a}_\sigma + b_{\alpha\beta} \boldsymbol{a}_3$ by the formula of Gauß (Theorem 2.6-1 (a)). Next,

$$\begin{aligned}
&(\Gamma^\sigma_{\alpha\beta} \boldsymbol{a}_\sigma + b_{\alpha\beta} \boldsymbol{a}_3) \cdot (\boldsymbol{a}_1 \wedge \partial_2 \widetilde{\boldsymbol{\eta}}) \\
&\quad = \Gamma^2_{\alpha\beta} \boldsymbol{a}_2 \cdot (\boldsymbol{a}_1 \wedge \partial_2 \widetilde{\boldsymbol{\eta}}) - b_{\alpha\beta} \partial_2 \widetilde{\boldsymbol{\eta}} \cdot (\boldsymbol{a}_1 \wedge \boldsymbol{a}_3) \\
&\quad = \sqrt{a} \left(-\Gamma^2_{\alpha\beta} \partial_2 \widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_3 + b_{\alpha\beta} \partial_2 \widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}^2 \right),
\end{aligned}$$

since, by (ii), $\boldsymbol{a}_2 \cdot (\boldsymbol{a}_1 \wedge \partial_2 \widetilde{\boldsymbol{\eta}}) = -\partial_2 \widetilde{\boldsymbol{\eta}} \cdot (\boldsymbol{a}_1 \wedge \boldsymbol{a}_2) = -\sqrt{a} \partial_2 \widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_3$ and $\boldsymbol{a}_1 \wedge \boldsymbol{a}_3 = -\sqrt{a} \boldsymbol{a}^2$; likewise,

$$(\Gamma^\sigma_{\alpha\beta} \boldsymbol{a}_\sigma + b_{\alpha\beta} \boldsymbol{a}_3) \cdot (\partial_1 \widetilde{\boldsymbol{\eta}} \wedge \boldsymbol{a}_2) = \sqrt{a}(-\Gamma^1_{\alpha\beta} \partial_1 \widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_3 + b_{\alpha\beta} \partial_1 \boldsymbol{\eta} \cdot \boldsymbol{a}^1).$$

Consequently,

$$b_{\alpha\beta}(\boldsymbol{\eta}) = \sqrt{\frac{a}{a(\boldsymbol{\eta})}} \left\{ b_{\alpha\beta}(1 + \partial_\sigma \tilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}^\sigma) + (\partial_{\alpha\beta} \widetilde{\boldsymbol{\eta}} - \Gamma^\sigma_{\alpha\beta} \partial_\sigma \widetilde{\boldsymbol{\eta}}) \cdot \boldsymbol{a}_3 + \text{ h.o.t.} \right\}.$$

There remains to find the linear term with respect to $\boldsymbol{\eta} = (\eta_i)$ in the expansion $\dfrac{1}{\sqrt{a(\boldsymbol{\eta})}} = \dfrac{1}{\sqrt{a}}(1 + \cdots)$. To this end, we note that

$$\det(\mathbf{A} + \mathbf{H}) = (\det \mathbf{A})(1 + \operatorname{tr} \mathbf{A}^{-1}\mathbf{H} + o(\mathbf{H})),$$

with $\mathbf{A} := (a_{\alpha\beta})$ and $\mathbf{A} + \mathbf{H} := (a_{\alpha\beta}(\boldsymbol{\eta}))$. Hence

$$\mathbf{H} = (\partial_\beta \widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\alpha + \partial_\alpha \widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\beta + \text{ h.o.t.}),$$

since $[a_{\alpha\beta}(\boldsymbol{\eta}) - a_{\alpha\beta}]^{\text{lin}} = \partial_\beta \widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\alpha + \partial_\alpha \widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\beta$ (Theorem 4.2-1). Therefore,

$$a(\boldsymbol{\eta}) = \det(a_{\alpha\beta}(\boldsymbol{\eta})) = \det(a_{\alpha\beta})(1 + 2\partial_\alpha \widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}^\alpha + \text{ h.o.t.}),$$

since $\mathbf{A}^{-1} = (a^{\alpha\beta})$; consequently,

$$\frac{1}{\sqrt{a(\boldsymbol{\eta})}} = \frac{1}{\sqrt{a}}(1 - \partial_\alpha \widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}^\alpha + \text{ h.o.t.}).$$

Noting that there are no linear terms with respect to $\boldsymbol{\eta} = (\eta_i)$ in the product $(1 - \partial_\alpha \widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}^\alpha)(1 + \partial_\sigma \widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}^\sigma)$, we find the announced expansion, viz.,

$$b_{\alpha\beta}(\boldsymbol{\eta}) = b_{\alpha\beta} + (\partial_{\alpha\beta} \widetilde{\boldsymbol{\eta}} - \Gamma^\sigma_{\alpha\beta} \partial_\sigma \widetilde{\boldsymbol{\eta}}) \cdot \boldsymbol{a}_3 + \text{ h.o.t.}$$

(iv) *The components $\rho_{\alpha\beta}(\boldsymbol{\eta})$ can be also written as*

$$\rho_{\alpha\beta}(\boldsymbol{\eta}) = \eta_{3|\alpha\beta} - b^\sigma_\alpha b_{\sigma\beta} \eta_3 + b^\sigma_\alpha \eta_{\sigma|\beta} + b^\tau_\beta \eta_{\tau|\alpha} + b^\tau_\beta|_\alpha \eta_\tau,$$

where the functions $\eta_{3|\alpha\beta}$ and $b^\tau_\beta|_\alpha$ are defined as in the statement of the theorem.

By Theorem 2.6-1 (b),

$$\partial_\sigma \widetilde{\boldsymbol{\eta}} = (\partial_\sigma \eta_\beta - \Gamma^\tau_{\sigma\beta} \eta_\tau - b_{\sigma\beta} \eta_3) \boldsymbol{a}^\beta + (\partial_\sigma \eta_3 + b^\tau_\sigma \eta_\tau) \boldsymbol{a}^3.$$

Hence

$$-\Gamma^\sigma_{\alpha\beta} \partial_\sigma \widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_3 = -\Gamma^\sigma_{\alpha\beta}(\partial_\sigma \eta_3 + b^\tau_\sigma \eta_\tau),$$

since $\boldsymbol{a}^i \cdot \boldsymbol{a}_3 = \delta^i_3$. Again by Theorem 2.6-1 (b),

$$\begin{aligned}
\partial_{\alpha\beta} \widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_3 &= \partial_\alpha \big\{ (\partial_\beta \eta_\sigma - \Gamma^\tau_{\beta\sigma} \eta_\tau - b_{\beta\sigma} \eta_3) \boldsymbol{a}^\sigma \\
&\quad + (\partial_\beta \eta_3 + b^\tau_\beta \eta_\tau) \boldsymbol{a}^3 \big\} \cdot \boldsymbol{a}_3 \\
&= (\partial_\beta \eta_\sigma - \Gamma^\tau_{\beta\sigma} \eta_\tau - b_{\beta\sigma} \eta_3) \partial_\alpha \boldsymbol{a}^\sigma \cdot \boldsymbol{a}_3 \\
&\quad + (\partial_{\alpha\beta} \eta_3 + (\partial_\alpha b^\tau_\beta) \eta_\tau + b^\tau_\beta \partial_\alpha \eta_\tau) \boldsymbol{a}^3 \cdot \boldsymbol{a}_3 \\
&\quad + (\partial_\beta \eta_3 + b^\tau_\beta \eta_\tau) \partial_\alpha \boldsymbol{a}^3 \cdot \boldsymbol{a}_3 \\
&= b^\sigma_\alpha (\partial_\beta \eta_\sigma - \Gamma^\tau_{\beta\sigma} \eta_\tau) - b^\sigma_\alpha b_{\sigma\beta} \eta_3 + \partial_{\alpha\beta} \eta_3 \\
&\quad + (\partial_\alpha b^\tau_\beta) \eta_\tau + b^\tau_\beta \partial_\alpha \eta_\tau,
\end{aligned}$$

since

$$\partial_\alpha \boldsymbol{a}^\sigma \cdot \boldsymbol{a}_3 = (-\Gamma^\sigma_{\alpha\tau} \boldsymbol{a}^\tau + b^\sigma_\alpha \boldsymbol{a}^3) \cdot \boldsymbol{a}_3 = b^\sigma_\alpha,$$
$$\partial_\alpha \boldsymbol{a}^3 \cdot \boldsymbol{a}_3 = -b_{\alpha\sigma} \boldsymbol{a}^\sigma \cdot \boldsymbol{a}_3 = 0,$$

by the formulas of Gauß and Weingarten (Theorem 2.6-1 (a)). We thus obtain

$$\begin{aligned}\rho_{\alpha\beta}(\boldsymbol{\eta}) &= (\partial_{\alpha\beta}\widetilde{\boldsymbol{\eta}} - \Gamma^{\sigma}_{\alpha\beta}\partial_\sigma\widetilde{\boldsymbol{\eta}})\cdot\boldsymbol{a}_3\\ &= b^\sigma_\alpha(\partial_\beta\eta_\sigma - \Gamma^\tau_{\beta\sigma}\eta_\tau) - b^\sigma_\alpha b_{\sigma\beta}\eta_3 + \partial_{\alpha\beta}\eta_3 + (\partial_\alpha b^\tau_\beta)\eta_\tau + b^\tau_\beta\partial_\alpha\eta_\tau\\ &\quad -\Gamma^\sigma_{\alpha\beta}(\partial_\sigma\eta_3 + b^\tau_\sigma\eta_\tau).\end{aligned}$$

While this relation seemingly involves only the covariant derivatives $\eta_{3|\alpha\beta}$ and $\eta_{\sigma|\beta}$, it may be easily rewritten so as to involve in addition the functions $\eta_{\tau|\alpha}$ and $b^\tau_{\beta|\alpha}$. The stratagem simply consists in using the relation $\Gamma^\tau_{\alpha\sigma}b^\sigma_\beta\eta_\tau - \Gamma^\sigma_{\alpha\tau}b^\tau_\beta\eta_\sigma = 0$! This gives

$$\begin{aligned}\rho_{\alpha\beta}(\boldsymbol{\eta}) &= (\partial_{\alpha\beta}\eta_3 - \Gamma^\sigma_{\alpha\beta}\partial_\tau\eta_3) - b^\sigma_\alpha b_{\sigma\beta}\eta_3\\ &\quad + b^\sigma_\alpha(\partial_\beta\eta_\sigma - \Gamma^\tau_{\beta\sigma}\eta_\tau) + b^\tau_\beta(\partial_\alpha\eta_\tau - \Gamma^\sigma_{\alpha\tau}\eta_\tau)\\ &\quad + (\partial_\alpha b^\tau_\beta + \Gamma^\tau_{\alpha\sigma}b^\sigma_\beta b^\sigma_\beta - \Gamma^\sigma_{\alpha\beta}b^\tau_\sigma)\eta_\tau.\end{aligned}$$

(v) *The functions $b^\tau_\beta|_\alpha$ are symmetric with respect to the indices α and β.* Again, because of the formulas of Gauß and Weingarten, we can write

$$\begin{aligned}\boldsymbol{0} &= \partial_{\alpha\beta}\boldsymbol{a}^\tau - \partial_{\beta\alpha}\boldsymbol{a}^\tau = \partial_\alpha\left(-\Gamma^\tau_{\beta\sigma}\boldsymbol{a}^\sigma + b^\tau_\beta\boldsymbol{a}^3\right) - \partial_\beta\left(-\Gamma^\tau_{\alpha\sigma}\boldsymbol{a}^\sigma + b^\tau_\alpha\boldsymbol{a}^3\right)\\ &= -(\partial_\alpha\Gamma^\tau_{\beta\sigma})\boldsymbol{a}^\sigma + \Gamma^\tau_{\beta\sigma}\Gamma^\sigma_{\alpha\nu}\boldsymbol{a}^\nu - \Gamma^\tau_{\beta\sigma}b^\sigma_\alpha\boldsymbol{a}^3 + (\partial_\alpha b^\tau_\beta)\boldsymbol{a}^3 - b^\tau_\beta b_{\alpha\sigma}\boldsymbol{a}^\sigma\\ &\quad + (\partial_\beta\Gamma^\tau_{\alpha\sigma})\boldsymbol{a}^\sigma - \Gamma^\tau_{\alpha\sigma}\Gamma^\sigma_{\beta\mu}\boldsymbol{a}^\mu + \Gamma^\tau_{\alpha\sigma}b^\sigma_\beta\boldsymbol{a}^3 - (\partial_\beta b^\tau_\alpha)\boldsymbol{a}^3 + b^\tau_\alpha b_{\beta\sigma}\boldsymbol{a}^\sigma.\end{aligned}$$

Consequently,

$$0 = (\partial_{\alpha\beta}\boldsymbol{a}^\tau - \partial_{\beta\alpha}\boldsymbol{a}^\tau)\cdot\boldsymbol{a}^3 = \partial_\alpha b^\tau_\beta - \partial_\beta b^\tau_\alpha + \Gamma^\tau_{\alpha\sigma}b^\sigma_\beta - \Gamma^\tau_{\beta\sigma}b^\sigma_\alpha,$$

on the one hand. On the other hand, we immediately infer from the definition of the functions $b^\tau_\beta|_\alpha$ that we also have

$$b^\tau_\beta|_\alpha - b^\tau_\alpha|_\beta = \partial_\alpha b^\tau_\beta - \partial_\beta b^\tau_\alpha + \Gamma^\tau_{\alpha\sigma}b^\sigma_\beta - \Gamma^\tau_{\beta\sigma}b^\sigma_\alpha,$$

and thus the proof is complete. □

The functions $\rho_{\alpha\beta}(\boldsymbol{\eta})$ are called the **covariant components of the linearized change of curvature tensor** associated with a displacement $\eta_i\boldsymbol{a}^i$ of the surface S. The functions

$$\eta_{3|\alpha\beta} = \partial_{\alpha\beta}\eta_3 - \Gamma^\sigma_{\alpha\beta}\partial_\sigma\eta_3 \text{ and } b^\tau_\beta|_\alpha = \partial_\alpha b^\tau_\beta + \Gamma^\tau_{\alpha\sigma}b^\sigma_\beta - \Gamma^\sigma_{\alpha\beta}b^\tau_\sigma$$

respectively represent a **second-order covariant derivative of the vector field** $\eta_i\boldsymbol{a}^i$ and a **first-order covariant derivative of the second fundamental form** of S, defined here by means of its *mixed* components b^τ_β.

Remarks. (1) Covariant derivatives $b_{\alpha\beta|\sigma}$ can be likewise defined. More specifically, each function

$$b_{\alpha\beta|\sigma} := \partial_\sigma b_{\alpha\beta} - \Gamma^\tau_{\alpha\sigma}b_{\tau\beta} - \Gamma^\tau_{\beta\sigma}b_{\alpha\tau}$$

represents a *first-order covariant derivatives of the second fundamental form*, defined here by means of its *covariant* components $b_{\alpha\beta}$. By a proof analogous to that given in Theorem 4.2-2 for establishing the symmetry relations $b^\tau_\beta|_\alpha = b^\tau_\alpha|_\beta$, one can then show that these covariant derivatives likewise satisfy the symmetry relations

$$b_{\alpha\beta|\sigma} = b_{\alpha\sigma|\beta},$$

which are themselves equivalent to the relations

$$\partial_\sigma b_{\alpha\beta} - \partial_\beta b_{\alpha\sigma} + \Gamma^\tau_{\alpha\beta} b_{\tau\sigma} - \Gamma^\tau_{\alpha\sigma} b_{\tau\beta} = 0,$$

i.e., the familiar Codazzi-Mainardi equations (Theorem 2.7-1)!

(2) The functions $c_{\alpha\beta} := b^\sigma_\alpha b_{\sigma\beta} = c_{\beta\alpha}$ appearing in the expression of $\rho_{\alpha\beta}(\boldsymbol{\eta})$ are the covariant components of the *third fundamental form* of S. For details, see, e.g., Stoker [1969, p. 98] or Klingenberg [1973, p. 48].

(3) The functions $b_{\alpha\beta}(\boldsymbol{\eta})$ are not always well defined (in order that they be, the vectors $a_\alpha(\boldsymbol{\eta})$ must be linearly independent in $\overline{\omega}$), but the functions $\rho_{\alpha\beta}(\boldsymbol{\eta})$ are *always* well defined.

(4) The symmetry $\rho_{\alpha\beta}(\boldsymbol{\eta}) = \rho_{\beta\alpha}(\boldsymbol{\eta})$ follows immediately by inspection of the expression $\rho_{\alpha\beta}(\boldsymbol{\eta}) = (\partial_{\alpha\beta}\widetilde{\boldsymbol{\eta}} - \Gamma^\sigma_{\alpha\beta}\partial_\sigma\widetilde{\boldsymbol{\eta}}) \cdot \boldsymbol{a}_3$ found there. By contrast, deriving the same symmetry from the other expression of $\rho_{\alpha\beta}(\boldsymbol{\eta})$ requires proving first that the covariant derivatives $b^\sigma_\beta|_\alpha$ are themselves symmetric with respect to the indices α and β (cf. part (v) of the proof of Theorem 4.2-1). □

While the expression of the components $\rho_{\alpha\beta}(\boldsymbol{\eta})$ in terms of the covariant components η_i of the displacement field is fairly complicated but well known (see, e.g., Koiter [1970]), that in terms of $\widetilde{\boldsymbol{\eta}} = \eta_i \boldsymbol{a}^i$ is *remarkably simple* but seems to have been mostly ignored, although it already appeared in Bamberger [1981]. Together with the expression of the components $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ in terms of $\widetilde{\boldsymbol{\eta}}$ (Theorem 4.2-1), this simpler expression was efficiently put to use by Blouza & Le Dret [1999], who showed that their principal merit is to afford the definition of the components $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta}(\boldsymbol{\eta})$ under substantially weaker regularity assumptions on the mapping $\boldsymbol{\theta}$.

More specifically, we were led to assume that $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbf{E}^3)$ in Theorem 4.2-1 in order to insure that $\rho_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega)$ if $\boldsymbol{\eta} \in H^1(\omega) \times H^1(\omega) \times H^2(\omega)$. The culprits responsible for this regularity are the functions $b^\tau_\beta|_\alpha$ appearing in the functions $\rho_{\alpha\beta}(\boldsymbol{\eta})$. Otherwise Blouza & Le Dret [1999] have shown how this regularity assumption on $\boldsymbol{\theta}$ can be weakened if only the expressions of $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta}(\boldsymbol{\eta})$ in terms of the field $\widetilde{\boldsymbol{\eta}}$ are considered. We shall return to such aspects in Section 4.3.

We are now in a position to describe the **linear Koiter shell equations**. Let γ_0 be a measurable subset of $\gamma = \partial\omega$ that satisfies *length*$\gamma_0 > 0$, let ∂_ν denote the outer normal derivative operator along $\partial\omega$, and let the space $\mathbf{V}(\omega)$ be defined by

$$\mathbf{V}(\omega) := \big\{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega); \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0\big\}.$$

Then the unknown vector field $\boldsymbol{\zeta} = (\zeta_i) : \overline{\omega} \to \mathbb{R}^3$, where the functions ζ_i are the covariant components of the displacement field $\zeta_i \boldsymbol{a}^i$ of the middle surface

$S = \boldsymbol{\theta}(\overline{\omega})$ of the shell, should be a *stationary point over the space* $\mathbf{V}(\omega)$ of the **functional** j defined by

$$j(\boldsymbol{\eta}) = \frac{1}{2}\int_\omega \Big\{\varepsilon a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\eta})\gamma_{\alpha\beta}(\boldsymbol{\eta}) + \frac{\varepsilon^3}{3}a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{\eta})\rho_{\alpha\beta}(\boldsymbol{\eta})\Big\}\sqrt{a}\,\mathrm{d}y$$
$$- \int_\omega p^i\eta_i\sqrt{a}\,\mathrm{d}y$$

for all $\boldsymbol{\eta} \in \mathbf{V}(\omega)$. This functional j is called **Koiter's energy for a linearly elastic shell**.

Equivalently, the vector field $\boldsymbol{\zeta} \in \mathbf{V}(\omega)$ should satisfy the *variational equations*

$$\int_\omega \Big\{\varepsilon a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta})\gamma_{\alpha\beta}(\boldsymbol{\eta}) + \frac{\varepsilon^3}{3}a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{\zeta})\rho_{\alpha\beta}(\boldsymbol{\eta})\Big\}\sqrt{a}\,\mathrm{d}y$$
$$= \int_\omega p^i\eta_i\sqrt{a}\,\mathrm{d}y \text{ for all } \boldsymbol{\eta} = (\eta_i) \in \mathbf{V}(\omega).$$

We recall that the functions

$$a^{\alpha\beta\sigma\tau} := \frac{4\lambda\mu}{\lambda+2\mu}a^{\alpha\beta}a^{\sigma\tau} + 2\mu(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma})$$

denote the *contravariant components of the shell elasticity tensor* (λ and μ are the Lamé constants of the elastic material constituting the shell), $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta}(\boldsymbol{\eta})$ denote the covariant components of the *linearized change of metric*, and *change of curvature, tensors* associated with a displacement field $\eta_i\boldsymbol{a}^i$ of S, and the given functions $p^i \in L^2(\omega)$ account for the applied forces. Finally, the boundary conditions $\eta_i = \partial_\nu\eta_3 = 0$ on γ_0 express that the shell is *clamped* along the portion $\boldsymbol{\theta}(\gamma_0)$ of its middle surface (see Figure 4.1-3).

The choice of the function spaces $H^1(\omega)$ and $H^2(\omega)$ for the tangential components η_α and normal components η_3 of the displacement fields $\eta_i\boldsymbol{a}^i$ is guided by the natural requirement that the functions $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta}(\boldsymbol{\eta})$ be both in $L^2(\omega)$, so that the energy is in turn well defined for $\boldsymbol{\eta} = (\eta_i) \in \mathbf{V}(\omega)$. Otherwise these choices can be weakened to accommodate shells whose middle surfaces have little regularity (cf. Section 4.3).

Remark. A justification of Koiter's linear equations (by means of an asymptotic analysis of the "three-dimensional" equations as $\varepsilon \to 0$) is provided in Section 4.5. □

Our objective in the next sections is to study the existence and uniqueness of the solution to the above variational equations. To this end, we shall establish (Theorem 4.4-1) that, under the assumptions $3\lambda + 2\mu > 0$ and $\mu > 0$, there exists a constant $c_e > 0$ such that

$$\sum_{\alpha,\beta} |t_{\alpha\beta}|^2 \leq c_e a^{\alpha\beta\sigma\tau}(y)t_{\sigma\tau}t_{\alpha\beta}$$

for all $y \in \overline{\omega}$ and all symmetric matrices $(t_{\alpha\beta})$. When *length* $\gamma_0 > 0$, the existence and uniqueness of a solution to this variational problem by means of the *Lax-Milgram lemma* will then be a consequence of the existence of a constant c such that

$$\Big\{ \sum_\alpha \|\eta_\alpha\|^2_{1,\omega} + \|\eta_3\|^2_{2,\omega} \Big\}^{1/2} \\ \leq c \Big\{ \sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\boldsymbol{\eta})\|^2_{0,\omega} + \sum_{\alpha,\beta} \|\rho_{\alpha\beta}(\boldsymbol{\eta})\|^2_{0,\omega} \Big\}^{1/2} \text{ for all } \boldsymbol{\eta} \in \mathbf{V}(\omega).$$

The objective of the next section precisely consists in showing that such a fundamental *"Korn's inequality on a surface" indeed holds* (Theorem 4.3-4).

4.3 KORN'S INEQUALITIES ON A SURFACE

In Section 3.8, we established three-dimensional Korn inequalities, first "without boundary conditions" (Theorem 3.8-1), then "with boundary conditions" (Theorem 3.8-3), the latter depending on an "infinitesimal rigid displacement lemma" (Theorem 3.8-2). Both Korn inequalities involved the covariant components $e_{ij}(\boldsymbol{v})$ of the *three-dimensional linearized change of metric tensor*.

But while this tensor is the only one that is attached to a displacement $v_i \boldsymbol{g}^i$ of the three-dimensional set $\boldsymbol{\Theta}(\overline{\Omega})$ in $\mathbf{E}^3$, we saw in the previous section that *two* tensors, the linearized change of metric and the linearized change of curvature tensors, are attached to a displacement field $\eta_i \boldsymbol{a}^i$ of a *surface* in $\mathbf{E}^3$.

It is thus natural to seek to likewise establish *Korn's inequalities "on a surface"*, first *without boundary conditions* (Theorem 4.3-1), then *with boundary conditions* (Theorem 4.3-4), the latter again depending on an *infinitesimal rigid displacement lemma on a surface* (Theorem 4.3-3). As expected, such inequalities will now involve the covariant components $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta}(\boldsymbol{\eta})$ of the *linearized change of metric tensor* and *linearized change of curvature tensor* defined in the previous section.

The infinitesimal rigid displacement lemma and the Korn inequality "with boundary conditions" were first established by Bernadou & Ciarlet [1976]. A simpler proof, which we follow here, was then proposed by Ciarlet & Miara [1992] (see also Bernadou, Ciarlet & Miara [1994]). Its first stage consists in establishing a **Korn's inequality on a surface, "without boundary conditions"**, again as a consequence of the same *lemma of J.L. Lions* as in dimension three (cf. Theorem 3.8-1). Note that such an inequality holds as well in the more general context of Riemannian geometry; cf. Chen & Jost [2002].

Recall that the notations $\|\cdot\|_{0,\omega}$ and $\|\cdot\|_{m,\omega}$ respectively designate the norms in $L^2(\omega)$ and $H^m(\omega)$, $m \geq 1$; cf. Section 3.6.

Theorem 4.3-1. *Let ω be a domain in $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbf{E}^3)$ be an injective immersion. Given $\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega)$, let*

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) := \Big\{\frac{1}{2}(\partial_\beta \widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\alpha + \partial_\alpha \widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\beta)\Big\} \in L^2(\omega),$$
$$\rho_{\alpha\beta}(\boldsymbol{\eta}) := \Big\{(\partial_{\alpha\beta} \widetilde{\boldsymbol{\eta}} - \Gamma^\sigma_{\alpha\beta} \partial_\sigma \widetilde{\boldsymbol{\eta}}) \cdot \boldsymbol{a}_3\Big\} \in L^2(\omega)$$

denote the covariant components of the linearized change of metric, and linearized change of curvature, tensors associated with the displacement field $\widetilde{\boldsymbol{\eta}} := \eta_i \boldsymbol{a}^i$ of the surface $S = \boldsymbol{\theta}(\overline{\omega})$. Then there exists a constant $c_0 = c_0(\omega, \boldsymbol{\theta})$ such that

$$\Big\{\sum_\alpha \|\eta_\alpha\|^2_{1,\omega} + \|\eta_3\|^2_{2,\omega}\Big\}^{1/2} \leq c_0 \Big\{\sum_\alpha \|\eta_\alpha\|^2_{0,\omega} + \|\eta_3\|^2_{1,\omega} + \sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\boldsymbol{\eta})\|^2_{0,\omega} + \sum_{\alpha,\beta} \|\rho_{\alpha\beta}(\boldsymbol{\eta})\|^2_{0,\omega}\Big\}^{1/2}$$

for all $\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega)$.

Proof. The "fully explicit" expressions of the functions $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta}(\boldsymbol{\eta})$, as found in Theorems 4.2-1 and 4.2-2, are used in this proof, simply because they are more convenient for its purposes.

(i) *Define the space*

$$\mathbf{W}(\omega) := \big\{\boldsymbol{\eta} = (\eta_i) \in L^2(\omega) \times L^2(\omega) \times H^1(\omega);\ \gamma_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega),\ \rho_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega)\big\}.$$

Then, equipped with the norm $\|\cdot\|_{\mathbf{W}(\omega)}$ defined by

$$\|\boldsymbol{\eta}\|_{\mathbf{W}(\omega)} := \Big\{\sum_\alpha \|\eta_\alpha\|^2_{0,\omega} + \|\eta_3\|^2_{1,\omega} + \sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\boldsymbol{\eta})\|^2_{0,\omega} + \sum_{\alpha,\beta} \|\rho_{\alpha\beta}(\boldsymbol{\eta})\|^2_{0,\omega}\Big\}^{1/2},$$

the space $\mathbf{W}(\omega)$ is a Hilbert space.

The relations "$\gamma_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega)$" and "$\rho_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega)$" appearing in the definition of the space $\mathbf{W}(\omega)$ are to be understood in the sense of distributions. They mean that a vector field $\boldsymbol{\eta} \in L^2(\omega) \times L^2(\omega) \times H^1(\omega)$ belongs to $\mathbf{W}(\omega)$ if there exist functions in $L^2(\omega)$, *denoted* $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta}(\boldsymbol{\eta})$, such that for all $\varphi \in \mathcal{D}(\omega)$,

$$\int_\omega \gamma_{\alpha\beta}(\boldsymbol{\eta})\varphi \,\mathrm{d}y = -\int_\omega \Big\{\frac{1}{2}(\eta_\beta \partial_\alpha \varphi + \eta_\alpha \partial_\beta \varphi) + \Gamma^\sigma_{\alpha\beta} \eta_\sigma \varphi + b_{\alpha\beta} \eta_3 \varphi\Big\} \mathrm{d}y,$$

$$\begin{aligned}\int_\omega \rho_{\alpha\beta}(\boldsymbol{\eta})\varphi \,\mathrm{d}y = -\int_\omega \big\{&\partial_\alpha \eta_3 \partial_\beta \varphi + \Gamma^\sigma_{\alpha\beta} \partial_\sigma \eta_3 \varphi + b^\sigma_\alpha b_{\sigma\beta} \eta_3 \varphi \\ &+ \eta_\sigma \partial_\beta (b^\sigma_\alpha \varphi) + b^\sigma_\alpha \Gamma^\tau_{\beta\sigma} \eta_\tau \varphi \\ &+ \eta_\tau \partial_\alpha (b^\tau_\beta \varphi) + b^\tau_\beta \Gamma^\sigma_{\alpha\tau} \eta_\sigma \varphi \\ &- \big(\partial_\alpha b^\tau_\beta + \Gamma^\tau_{\alpha\sigma} b^\sigma_\beta - \Gamma^\sigma_{\alpha\beta} b^\tau_\sigma\big) \eta_\tau \varphi\big\} \,\mathrm{d}y.\end{aligned}$$

Let there be given a Cauchy sequence $(\boldsymbol{\eta}^k)_{k=1}^{\infty}$ with elements $\boldsymbol{\eta}^k = (\eta_i^k) \in \mathbf{W}(\omega)$. The definition of the norm $\|\cdot\|_{\mathbf{W}(\omega)}$ shows that there exist $\eta_\alpha \in L^2(\omega)$, $\eta_3 \in H^1(\omega)$, $\gamma_{\alpha\beta} \in L^2(\omega)$, and $\rho_{\alpha\beta} \in L^2(\omega)$ such that

$$\eta_\alpha^k \to \eta_\alpha \text{ in } L^2(\omega), \quad \eta_3^k \to \eta_3 \text{ in } H^1(\omega),$$
$$\gamma_{\alpha\beta}(\boldsymbol{\eta}^k) \to \gamma_{\alpha\beta} \text{ in } L^2(\omega), \quad \rho_{\alpha\beta}(\boldsymbol{\eta}^k) \to \rho_{\alpha\beta} \text{ in } L^2(\omega)$$

as $k \to \infty$. Given a function $\varphi \in \mathcal{D}(\omega)$, letting $k \to \infty$ in the relations $\int_\omega \gamma_{\alpha\beta}(\boldsymbol{\eta}^k)\varphi \,\mathrm{d}\omega = \dots$ and $\int_\omega \rho_{\alpha\beta}(\boldsymbol{\eta}^k)\varphi \,\mathrm{d}\omega = \dots$ then shows that $\gamma_{\alpha\beta} = \gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta} = \rho_{\alpha\beta}(\boldsymbol{\eta})$.

(ii) *The spaces* $\mathbf{W}(\omega)$ *and* $H^1(\omega) \times H^1(\omega) \times H^2(\omega)$ *coincide.*

Clearly, $H^1(\omega) \times H^1(\omega) \times H^2(\omega) \subset \mathbf{W}(\omega)$. To prove the other inclusion, let $\boldsymbol{\eta} = (\eta_i) \in \mathbf{W}(\omega)$. The relations

$$s_{\alpha\beta}(\boldsymbol{\eta}) := \frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha) = \gamma_{\alpha\beta}(\boldsymbol{\eta}) + \Gamma_{\alpha\beta}^\sigma \eta_\sigma + b_{\alpha\beta}\eta_3$$

then imply that $e_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega)$ since the functions $\Gamma_{\alpha\beta}^\sigma$ and $b_{\alpha\beta}$ are continuous on $\overline{\omega}$. Therefore,

$$\partial_\sigma \eta_\alpha \in H^{-1}(\omega),$$
$$\partial_\beta(\partial_\sigma \eta_\alpha) = \{\partial_\beta s_{\alpha\sigma}(\boldsymbol{\eta}) + \partial_\sigma s_{\alpha\beta}(\boldsymbol{\eta}) - \partial_\alpha s_{\beta\sigma}(\boldsymbol{\eta})\} \in H^{-1}(\omega),$$

since $\chi \in L^2(\omega)$ implies $\partial_\sigma \chi \in H^{-1}(\omega)$. Hence $\partial_\sigma \eta_\alpha \in L^2(\omega)$ by the *lemma of J.L. Lions* (Theorem 3.7-1) and thus $\eta_\alpha \in H^1(\omega)$.

The definition of the functions $\rho_{\alpha\beta}(\boldsymbol{\eta})$, the continuity over $\overline{\omega}$ of the functions $\Gamma_{\alpha\beta}^\sigma, b_{\sigma\beta}, b_\alpha^\sigma$, and $\partial_\alpha b_\beta^\tau$, and the relations $\rho_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega)$ then imply that $\partial_{\alpha\beta}\eta_3 \in L^2(\omega)$, hence that $\eta_3 \in H^2(\omega)$.

(iii) *Korn's inequality without boundary conditions.*

The identity mapping ι from the space $H^1(\omega)\times H^1(\omega)\times H^2(\omega)$ equipped with its product norm $\boldsymbol{\eta} = (\eta_i) \to \{\sum_\alpha \|\eta_\alpha\|_{1,\omega}^2 + \|\eta_3\|_{2,\omega}^2\}^{1/2}$ into the space $\mathbf{W}(\omega)$ equipped with $\|\cdot\|_{\mathbf{W}(\omega)}$ is injective, continuous, and surjective by (ii). Since both spaces are complete (cf. (i)), the *closed graph theorem* then shows that the inverse mapping ι^{-1} is also continuous or equivalently, that the inequality of Korn's type without boundary conditions holds. □

In order to establish a Korn's inequality "with boundary conditions", we have to identify classes of boundary conditions to be imposed on the fields $\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega)$ in order that we can "get rid" of the norms $\|\eta_\alpha\|_{0,\omega}$ and $\|\eta_3\|_{1,\omega}$ in the right-hand side of the above inequality, i.e., situations where the *semi-norm*

$$\boldsymbol{\eta} = (\eta_i) \to \Big\{\sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\boldsymbol{\eta})\|_{0,\omega}^2 + \sum_{\alpha,\beta} \|\rho_{\alpha\beta}(\boldsymbol{\eta})\|_{0,\omega}^2\Big\}^{1/2}$$

becomes a *norm*, which should be in addition *equivalent* to the product norm.

To this end, the first step consists in establishing in Theorem 4.3-3 an *infinitesimal rigid displacement lemma*, which provides in particular one instance of boundary conditions implying that this semi-norm becomes a norm.

The proof of this lemma relies on the preliminary observation, quite worthwhile *per se*, that a vector field $\eta_i \boldsymbol{a}^i$ on a surface may be "canonically" extended to a three-dimensional vector field $v_i \boldsymbol{g}^i$ in such a way that all the components $e_{ij}(\boldsymbol{v})$ of the associated three-dimensional linearized change of metric tensor have remarkable expressions in terms of the components $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta}(\boldsymbol{\eta})$ of the linearized change of metric and curvature tensors of the surface vector field.

Theorem 4.3-2. *Let ω be a domain in $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbf{E}^3)$ be an injective immersion. By* Theorem 4.1-1, *there exists $\varepsilon > 0$ such that the mapping $\boldsymbol{\Theta}$ defined by*

$$\boldsymbol{\Theta}(y, x_3) := \boldsymbol{\theta}(y) + x_3 \boldsymbol{a}_3(y) \text{ for all } (y, x_3) \in \overline{\Omega},$$

where $\Omega := \omega \times \left]-\varepsilon, \varepsilon\right[$, is a $\mathcal{C}^2$-diffeomorphism from $\overline{\Omega}$ onto $\boldsymbol{\Theta}(\overline{\Omega})$ and thus the three vectors $\boldsymbol{g}_i := \partial_i \boldsymbol{\Theta}$ are linearly independent at all points of $\overline{\Omega}$.

With any vector field $\eta_i \boldsymbol{a}^i$ with covariant components η_α in $H^1(\omega)$ and η_3 in $H^2(\omega)$, let there be associated the vector field $v_i \boldsymbol{g}^i$ defined on $\overline{\Omega}$ by

$$v_i(y, x_3)\boldsymbol{g}^i(y, x_3) = \eta_i(y)\boldsymbol{a}^i(y) - x_3(\partial_\alpha \eta_3 + b_\alpha^\sigma \eta_\sigma)(y)\boldsymbol{a}^\alpha(y)$$

for all $(y, x_3) \in \overline{\Omega}$, where the vectors $\boldsymbol{g}^i$ are defined by $\boldsymbol{g}^i \cdot \boldsymbol{g}_j = \delta_j^i$.

Then the covariant components v_i of the vector field $v_i \boldsymbol{g}^i$ are in $H^1(\Omega)$ and the covariant components $e_{ij}(\boldsymbol{v}) \in L^2(\Omega)$ of the associated linearized change of metric tensor (Section 3.6) *are given by*

$$\begin{aligned} e_{\alpha\beta}(\boldsymbol{v}) &= \gamma_{\alpha\beta}(\boldsymbol{\eta}) - x_3 \rho_{\alpha\beta}(\boldsymbol{\eta}) \\ &\quad + \frac{x_3^2}{2}\left\{ b_\alpha^\sigma \rho_{\beta\sigma}(\boldsymbol{\eta}) + b_\beta^\tau \rho_{\alpha\tau}(\boldsymbol{\eta}) - 2 b_\alpha^\sigma b_\beta^\tau \gamma_{\sigma\tau}(\boldsymbol{\eta}) \right\}, \\ e_{i3}(\boldsymbol{v}) &= 0. \end{aligned}$$

Proof. As in the above expressions of the functions $e_{\alpha\beta}(\boldsymbol{v})$, the dependence on x_3 is explicit, but the dependence with respect to $y \in \overline{\omega}$ is omitted, throughout the proof. The explicit expressions of the functions $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta}(\boldsymbol{\eta})$ in terms of the functions η_i (Theorems 4.2-1 and 4.2-2) are used in this proof.

(i) *Given functions $\eta_\alpha, \mathcal{X}_\alpha \in H^1(\omega)$ and $\eta_3 \in H^2(\omega)$, let the vector field $v_i \boldsymbol{g}^i$ be defined on $\overline{\Omega}$ by*

$$v_i \boldsymbol{g}^i = \eta_i \boldsymbol{a}^i + x_3 \mathcal{X}_\alpha \boldsymbol{a}^\alpha.$$

Then the functions v_i *are in* $H^1(\Omega)$ *and the covariant components* $e_{i\|j}(\boldsymbol{v})$ *of the linearized change of metric tensor associated with the field* $v_i\boldsymbol{g}^i$ *are given by*

$$
\begin{aligned}
e_{\alpha\beta}(\boldsymbol{v}) \;&=\; \Big\{\frac{1}{2}(\eta_{\alpha|\beta}+\eta_{\beta|\alpha})-b_{\alpha\beta}\eta_3\Big\}\\
&\quad+\frac{x_3}{2}\big\{\mathcal{X}_{\alpha|\beta}+\mathcal{X}_{\beta|\alpha}-b_\alpha^\sigma(\eta_{\sigma|\beta}-b_{\beta\sigma}\eta_3)-b_\beta^\tau(\eta_{\tau|\alpha}-b_{\alpha\tau}\eta_3)\big\}\\
&\quad+\frac{x_3^2}{2}\big\{-b_\alpha^\sigma\mathcal{X}_{\sigma|\beta}-b_\beta^\tau\mathcal{X}_{\tau|\alpha}\big\},\\
e_{\alpha 3}(\boldsymbol{v}) \;&=\; \frac{1}{2}(\mathcal{X}_\alpha+\partial_\alpha\eta_3+b_\alpha^\sigma\eta_\sigma),\\
e_{33}(\boldsymbol{v}) \;&=\; 0,
\end{aligned}
$$

where $\eta_{\alpha|\beta}=\partial_\beta\eta_\alpha-\Gamma_{\alpha\beta}^\sigma\eta_\sigma$ *and* $\mathcal{X}_{\alpha|\beta}=\partial_\beta\mathcal{X}_\alpha-\Gamma_{\alpha\beta}^\sigma\mathcal{X}_\sigma$ *designate the covariant derivatives of the fields* $\eta_i\boldsymbol{a}^i$ *and* $\mathcal{X}_i\boldsymbol{a}^i$ *with* $\mathcal{X}_3=0$ (Section 2.6).

Since

$$\partial_\alpha\boldsymbol{a}_3=-b_\alpha^\sigma\boldsymbol{a}_\sigma$$

by the second formula of Weingarten (Theorem 2.6-1), *the vectors of the covariant basis associated with the mapping* $\boldsymbol{\Theta}=\boldsymbol{\theta}+x_3\boldsymbol{a}_3$ *are given by*

$$\boldsymbol{g}_\alpha=\boldsymbol{a}_\alpha-x_3b_\alpha^\sigma\boldsymbol{a}_\sigma \text{ and } \boldsymbol{g}_3=\boldsymbol{a}_3.$$

The assumed regularities of the functions η_i and $\mathcal{X}_\alpha$ imply that

$$v_i=(v_j\boldsymbol{g}^j)\cdot\boldsymbol{g}_i=(\eta_j\boldsymbol{a}^j+x_3\mathcal{X}_\alpha\boldsymbol{a}^\alpha)\cdot\boldsymbol{g}_i\in H^1(\Omega)$$

since $\boldsymbol{g}_i\in\mathbf{C}^1(\overline{\Omega})$. The announced expressions for the functions $e_{ij}(\boldsymbol{v})$ are obtained by simple computations, based on the relations $v_{i\|j}=\{\partial_j(v_k\boldsymbol{g}^k)\}\cdot\boldsymbol{g}_i$ (Theorem 1.4-1) and $e_{ij}(\boldsymbol{v})=\frac{1}{2}(v_{i\|j}+v_{j\|i})$.

(ii) *When* $\mathcal{X}_\alpha=-(\partial_\alpha\eta_3+b_\alpha^\sigma\eta_\sigma)$, *the functions* $e_{ij}(\boldsymbol{v})$ *in* (i) *take the expressions announced in the statement of the theorem.*

We first note that $\mathcal{X}_\alpha\in H^1(\omega)$ (since $b_\alpha^\sigma\in\mathcal{C}^1(\overline{\omega})$) and that $e_{\alpha3}(\boldsymbol{v})=0$ when $\mathcal{X}_\alpha=-(\partial_\alpha\eta_3+b_\alpha^\sigma\eta_\sigma)$. It thus remains to find the explicit forms of the functions $e_{\alpha\beta}(\boldsymbol{v})$ in this case. Replacing the functions $\mathcal{X}_\alpha$ by their expressions and using the symmetry relations $b_\alpha^\sigma|_\beta=b_\beta^\sigma|_\alpha$ (Theorem 4.2-2), we find that

$$
\begin{aligned}
&\frac{1}{2}\big\{\mathcal{X}_{\alpha|\beta}+\mathcal{X}_{\beta|\alpha}-b_\alpha^\sigma(\eta_{\sigma|\beta}-b_{\beta\sigma}\eta_3)-b_\beta^\tau(\eta_{\tau|\alpha}-b_{\alpha\tau}\eta_3)\big\}\\
&\qquad=-\eta_{3|\alpha\beta}-b_\alpha^\sigma\eta_{\sigma|\beta}-b_\beta^\tau\eta_{\tau|\alpha}-b_{\beta|\alpha}^\tau\eta_\tau+b_\alpha^\sigma b_{\sigma\beta}\eta_3,
\end{aligned}
$$

i.e., the factor of x_3 in $e_{\alpha\beta}(\boldsymbol{v})$ is equal to $-\rho_{\alpha\beta}(\boldsymbol{\eta})$. Finally,

$$
\begin{aligned}
&-b_\alpha^\sigma\mathcal{X}_{\sigma|\beta}-b_\beta^\tau\mathcal{X}_{\tau|\alpha}\\
&\quad=b_\alpha^\sigma\big(\eta_{3|\beta\sigma}+b_{\sigma|\beta}^\tau\eta_\tau+b_\sigma^\tau\eta_{\tau|\beta}\big)+b_\beta^\tau\big(\eta_{3|\alpha\tau}+b_{\tau|\alpha}^\sigma\eta_\sigma+b_\tau^\sigma\eta_{\sigma|\alpha}\big)\\
&\quad=b_\alpha^\sigma\big(\rho_{\beta\sigma}(\boldsymbol{\eta})-b_\beta^\tau\eta_{\tau|\sigma}+b_\beta^\tau b_{\tau\sigma}\eta_3\big)+b_\beta^\tau\big(\rho_{\alpha\tau}(\boldsymbol{\eta})-b_\alpha^\sigma\eta_{\sigma|\tau}+b_\alpha^\sigma b_{\sigma\tau}\eta_3\big)\\
&\quad=b_\alpha^\sigma\rho_{\beta\sigma}(\boldsymbol{\eta})+b_\beta^\tau\rho_{\alpha\tau}(\boldsymbol{\eta})-2b_\alpha^\sigma b_\beta^\tau\gamma_{\sigma\tau}(\boldsymbol{\eta}),
\end{aligned}
$$

i.e., the factor of $\frac{x_3^2}{2}$ in $e_{\alpha\beta}(\boldsymbol{v})$ is that announced in the theorem. □

As shown by Destuynder [1985, Theorem 3.1] (see also Ciarlet & S. Mardare [2001]), the mapping

$$\mathbf{F} : (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega) \to (v_i) \in \mathbf{H}^1(\Omega)$$

defined in Theorem 4.3-2 is in fact an *isomorphism* from the Hilbert space $H^1(\omega) \times H^1(\omega) \times H^2(\omega)$ onto the Hilbert space

$$\mathbf{V}_{KL}(\Omega) = \{\boldsymbol{v} \in \mathbf{H}^1(\Omega); e_{i3}(\boldsymbol{v}) = 0 \text{ in } \Omega\}.$$

This identification of the image $\operatorname{Im}\mathbf{F}$ as the space $\mathbf{V}_{KL}(\Omega)$ has an interest *per se* in *linearized shell theory.* This result shows that, inside an *elastic shell*, the *Kirchhoff-Love displacement fields*, i.e. those displacement fields $v_i\boldsymbol{g}^i$ that satisfy the relations $e_{i3}(\boldsymbol{v}) = 0$ in Ω, are of the form

$$v_i\boldsymbol{g}^i = \eta_i\boldsymbol{a}^i - x_3(\partial_\alpha\eta_3 + b_\alpha^\sigma\eta_\sigma)\boldsymbol{a}^\alpha \text{ with } \eta_\alpha \in H^1(\omega) \text{ and } \eta_3 \in H^2(\omega),$$

and *vice versa.* This identification thus constitutes an extension of the well-known identification of *Kirchhoff-Love displacement fields* inside an *elastic plate* (cf. Ciarlet & Destuynder [1979] and also Theorem 1.4-4 of Ciarlet [1997]).

We next establish an *infinitesimal rigid displacement lemma "on a surface"*. The adjective "infinitesimal" reminds that only the *linearized* parts $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta}(\boldsymbol{\eta})$ of the "full" change of metric and curvature tensors $\frac{1}{2}(a_{\alpha\beta}(\boldsymbol{\eta}) - a_{\alpha\beta})$ and $(b_{\alpha\beta}(\boldsymbol{\eta}) - b_{\alpha\beta})$ are required to vanish in ω. Thanks to Theorem 4.3-2, this lemma becomes a simple consequence of the "three-dimensional" infinitesimal rigid displacement lemma in curvilinear coordinates (Theorem 3.8-2), to which it should be profitably compared.

This lemma is due to Bernadou & Ciarlet [1976, Theorems 5.1-1 and 5.2-1], who gave a more direct, but less "transparent", proof (see also Bernadou [1994, Part 1, Lemma 5.1.4]).

Part (a) in the next theorem is an **infinitesimal rigid displacement lemma on a surface, "without boundary conditions"**, while part (b) is an **infinitesimal rigid displacement lemma on a surface, "with boundary conditions"**.

Theorem 4.3-3. *Let there be given a domain ω in $\mathbb{R}^2$ and an injective immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbf{E}^3)$.*

(a) *Let $\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega)$ be such that*

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \rho_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega.$$

Then there exist two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ such that

$$\eta_i(y)\boldsymbol{a}^i(y) = \boldsymbol{a} + \boldsymbol{b} \wedge \boldsymbol{\theta}(y) \text{ for all } y \in \overline{\omega}.$$

(b) *Let γ_0 be a $d\gamma$-measurable subset of $\gamma = \partial\omega$ that satisfies* length $\gamma_0 > 0$ *and let a vector field $\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega)$ be such that*

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \rho_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega \text{ and } \eta_i = \partial_\nu\eta_3 = 0 \text{ on } \gamma_0.$$

Then $\boldsymbol{\eta} = \mathbf{0}$ in ω.

Proof. Let the set $\Omega = \omega \times]-\varepsilon, \varepsilon[$ and the vector field $\boldsymbol{v} = (v_i) \in \mathbf{H}^1(\Omega)$ be defined as in Theorem 4.3-2. By this theorem,

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \rho_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega \quad \text{implies that} \quad e_{i\|j}(\boldsymbol{v}) = 0 \text{ in } \Omega.$$

Therefore, by Theorem 3.8-2 (a), there exist two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ such that

$$v_i(y, x_3)\boldsymbol{g}^i(y, x_3) = \boldsymbol{a} + \boldsymbol{b} \wedge \{\boldsymbol{\theta}(y) + x_3\boldsymbol{a}_3(y)\} \text{ for all } (y, x_3) \in \overline{\Omega}.$$

Hence

$$\eta_i(y)\boldsymbol{a}^i(y) = v_i(y, x_3)\boldsymbol{g}^i(y, x_3)|_{x_3=0} = \boldsymbol{a} + \boldsymbol{b} \wedge \boldsymbol{\theta}(y) \text{ for all } y \in \overline{\omega},$$

and part (a) is established.

Let $\gamma_0 \subset \gamma$ be such that *length* $\gamma_0 > 0$. If $\eta_i = \partial_\nu \eta_3 = 0$ on γ_0, the functions $(\partial_\alpha \eta_3 + b_\alpha^\sigma \eta_\sigma)$ vanish on γ_0, since $\eta_3 = \partial_\nu \eta_3 = 0$ on γ_0 implies $\partial_\alpha \eta_3 = 0$ on γ_0. Theorem 4.3-2 then shows that

$$v_i = (v_j \boldsymbol{g}^j) \cdot \boldsymbol{g}_i = (\eta_j \boldsymbol{a}^j + x_3 \mathcal{X}_\alpha \boldsymbol{a}^\alpha) \cdot \boldsymbol{g}_i = 0 \text{ on } \Gamma_0 := \gamma_0 \times [-\varepsilon, \varepsilon].$$

Since *area* $\Gamma_0 > 0$, Theorem 3.8-2 (b) implies that $\boldsymbol{v} = \boldsymbol{0}$ in $\overline{\Omega}$, hence that $\boldsymbol{\eta} = \boldsymbol{0}$ on $\overline{\omega}$. $\square$

Remark. If a field $\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega)$ satisfies $\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \rho_{\alpha\beta}(\boldsymbol{\eta}) = 0$ in ω, its three components η_i are automatically in $\mathcal{C}^2(\overline{\omega})$ since $\eta_i = (\eta_j \boldsymbol{a}^j) \cdot \boldsymbol{a}_i$ and the fields $\boldsymbol{a}_i$ are of class $\mathcal{C}^2$ on $\overline{\omega}$. Remarkably, the field $\eta_i \boldsymbol{a}^i = \boldsymbol{a} + \boldsymbol{b} \wedge \boldsymbol{\theta}$ inherits in this case even *more* regularity, as it is of class $\mathcal{C}^3$ on $\overline{\omega}$. $\square$

An **infinitesimal rigid displacement** $\eta_i \boldsymbol{a}^i$ **of the surface** $S = \boldsymbol{\theta}(\overline{\omega})$ is defined as one whose associated vector field $\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega)$ satisfies $\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \rho_{\alpha\beta}(\boldsymbol{\eta}) = 0$ in ω. The vector fields $\boldsymbol{\eta}$ associated with such an infinitesimal rigid displacement thus span the vector space

$$\mathbf{Rig}(\omega) := \{\boldsymbol{\eta} \in H^1(\omega) \times H^1(\omega) \times H^2(\omega);\ \gamma_{\alpha\beta}(\boldsymbol{\eta}) = \rho_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega\},$$

which, because of Theorem 4.3-3, is also given by

$$\mathbf{Rig}(\omega) = \{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega); \\ \eta_i \boldsymbol{a}^i = \boldsymbol{a} + \boldsymbol{b} \wedge \boldsymbol{\theta} \text{ for some } \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3\}.$$

This relation shows in particular that *the infinitesimal rigid displacements of the surface S span a vector space of dimension six.* Furthermore, Ciarlet & C. Mardare [2004a] have shown that *this vector space is precisely the tangent space at the origin to the manifold* (also of dimension six) *formed by the rigid displacements of the surface* $S = \boldsymbol{\theta}(\overline{\omega})$, i.e., those whose associated deformed surface $(\boldsymbol{\theta} + \eta_i \boldsymbol{a}^i)(\overline{\omega})$ is obtained by means of a rigid deformation of the surface $\boldsymbol{\theta}(\omega)$ (see Section 2.9). In other words, this result shows that an infinitesimal rigid displacement of S is indeed the "linearized part of a genuine rigid displacement of S", thereby fully justifying the use of the adjective "infinitesimal".

We are now in a position to prove the announced **Korn's inequality on a surface, "with boundary conditions"**. This inequality plays a *fundamental rôle in the analysis of linearly elastic shells*, in particular for establishing the existence and uniqueness of the solution to the *linear Koiter equations* (see Theorem 4.4-2).

This inequality was first proved by Bernadou & Ciarlet [1976]. It was later given other proofs by Ciarlet & Miara [1992] and Bernadou, Ciarlet & Miara [1994]; then by Akian [2003] and Ciarlet & S. Mardare [2001], who showed that it can be directly derived from the three-dimensional Korn inequality in curvilinear coordinates, "with boundary conditions" (Theorem 3.8-3), for *ad hoc* choices of set Ω, mapping $\boldsymbol{\Theta}$, and vector fields $\boldsymbol{v}$ (this idea goes back to Destuynder [1985]); then by Blouza & Le Dret [1999], who showed that it still holds under a less stringent smoothness assumption on the mapping $\boldsymbol{\theta}$. We follow here the proof of Bernadou, Ciarlet & Miara [1994].

Theorem 4.3-4. *Let ω be a domain in $\mathbb{R}^2$, let $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbf{E}^3)$ be an injective immersion, let γ_0 be a $d\gamma$-measurable subset of $\gamma = \partial\omega$ that satisfies length $\gamma_0 > 0$, and let the space $\mathbf{V}(\omega)$ be defined as:*

$$\mathbf{V}(\omega) := \{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega);\ \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0\}.$$

Given $\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega)$, let

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) := \Big\{\frac{1}{2}(\partial_\beta \widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\alpha + \partial_\alpha \widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\beta\Big\} \in L^2(\omega),$$

$$\rho_{\alpha\beta}(\boldsymbol{\eta}) := \Big\{(\partial_{\alpha\beta} \widetilde{\boldsymbol{\eta}} - \Gamma^\sigma_{\alpha\beta} \partial_\sigma \widetilde{\boldsymbol{\eta}}) \cdot \boldsymbol{a}_3\Big\} \in L^2(\omega)$$

denote the covariant components of the linearized change of metric and linearized change of curvature tensors associated with the displacement field $\widetilde{\boldsymbol{\eta}} := \eta_i \boldsymbol{a}^i$ of the surface $S = \boldsymbol{\theta}(\overline{\omega})$. Then there exists a constant $c = c(\omega, \gamma_0, \boldsymbol{\theta})$ such that

$$\Big\{\sum_\alpha \|\eta_\alpha\|^2_{1,\omega} + \|\eta_3\|^2_{2,\omega}\Big\}^{1/2} \le c\Big\{\sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\boldsymbol{\eta})\|^2_{0,\omega} + \sum_{\alpha,\beta} \|\rho_{\alpha\beta}(\boldsymbol{\eta})\|^2_{0,\omega}\Big\}^{1/2}$$

for all $\boldsymbol{\eta} = (\eta_i) \in \mathbf{V}(\omega)$.

Proof. Let

$$\|\boldsymbol{\eta}\|_{H^1(\omega)\times H^1(\omega)\times H^2(\omega)} := \Big\{\sum_\alpha \|\eta_\alpha\|^2_{1,\omega} + \|\eta_3\|^2_{2,\omega}\Big\}^{1/2}.$$

If the announced inequality is false, there exists a sequence $(\boldsymbol{\eta}^k)_{k=1}^\infty$ of vector fields $\boldsymbol{\eta}^k \in \mathbf{V}(\omega)$ such that

$$\|\boldsymbol{\eta}^k\|_{H^1(\omega)\times H^1(\omega)\times H^2(\omega)} = 1 \text{ for all } k,$$

$$\lim_{k\to\infty} \Big\{\sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\boldsymbol{\eta}^k)\|^2_{0,\omega} + \sum_{\alpha,\beta} \|\rho_{\alpha\beta}(\boldsymbol{\eta}^k)\|^2_{0,\omega}\Big\}^{1/2} = 0.$$

Since the sequence $(\boldsymbol{\eta}^k)_{k=1}^{\infty}$ is bounded in $H^1(\omega) \times H^1(\omega) \times H^2(\omega)$, a subsequence $(\boldsymbol{\eta}^\ell)_{\ell=1}^{\infty}$ converges in $L^2(\omega) \times L^2(\omega) \times H^1(\omega)$ by the *Rellich-Kondrašov theorem*. Furthermore, each sequence $(\gamma_{\alpha\beta}(\boldsymbol{\eta}^\ell))_{\ell=1}^{\infty}$ and $(\rho_{\alpha\beta}(\boldsymbol{\eta}^\ell))_{\ell=1}^{\infty}$ also converges in $L^2(\omega)$ (to 0, but this information is not used at this stage) since

$$\lim_{\ell\to\infty} \Big\{ \sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\boldsymbol{\eta}^\ell)\|_{0,\omega}^2 + \sum_{\alpha,\beta} \|\rho_{\alpha\beta}(\boldsymbol{\eta}^\ell)\|_{0,\omega}^2 \Big\}^{1/2} = 0.$$

The subsequence $(\boldsymbol{\eta}^\ell)_{\ell=1}^{\infty}$ is thus a Cauchy sequence with respect to the norm

$$\boldsymbol{\eta} \to \Big\{ \sum_{\alpha} \|\eta_\alpha\|_{0,\omega}^2 + \|\eta_3\|_{1,\omega}^2 + \sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\boldsymbol{\eta})\|_{0,\omega}^2 + \sum_{\alpha,\beta} |\rho_{\alpha\beta}(\boldsymbol{\eta})\|_{0,\omega}^2 \Big\}^{1/2},$$

hence with respect to the norm $\|\cdot\|_{H^1(\omega)\times H^1(\omega)\times H^2(\omega)}$ by *Korn's inequality without boundary conditions* (Theorem 4.3-1).

The space $\mathbf{V}(\omega)$ being complete as a closed subspace of the space $H^1(\omega) \times H^1(\omega) \times H^2(\omega)$, there exists $\boldsymbol{\eta} \in \mathbf{V}(\omega)$ such that

$$\boldsymbol{\eta}^\ell \to \boldsymbol{\eta} \text{ in } H^1(\omega) \times H^1(\omega) \times H^2(\omega),$$

and the limit $\boldsymbol{\eta}$ satisfies

$$\|\gamma_{\alpha\beta}(\boldsymbol{\eta})\|_{0,\omega} = \lim_{\ell\to\infty} \|\gamma_{\alpha\beta}(\boldsymbol{\eta}^\ell)\|_{0,\omega} = 0,$$
$$\|\rho_{\alpha\beta}(\boldsymbol{\eta})\|_{0,\omega} = \lim_{\ell\to\infty} \|\rho_{\alpha\beta}(\boldsymbol{\eta}^\ell)\|_{0,\omega} = 0.$$

Hence $\boldsymbol{\eta} = \mathbf{0}$ by Theorem 4.3-3. But this last relation contradicts the relations $\|\boldsymbol{\eta}^\ell\|_{H^1(\omega)\times H^1(\omega)\times H^2(\omega)} = 1$ for all $\ell \geq 1$, and the proof is complete. □

If the mapping $\boldsymbol{\theta}$ is of the form $\boldsymbol{\theta}(y_1, y_2) = (y_1, y_2, 0)$ for all $(y_1, y_2) \in \overline{\omega}$, the inequality of Theorem 4.3-4 reduces to two distinct inequalities (obtained by letting first $\eta_\alpha = 0$, then $\eta_3 = 0$):

$$\|\eta_3\|_{2,\omega} \leq c\Big\{ \sum_{\alpha,\beta} \|\partial_{\alpha\beta}\eta_3\|_{0,\omega}^2 \Big\}^{1/2}$$

for all $\eta_3 \in H^2(\omega)$ satisfying $\eta_3 = \partial_\nu \eta_3 = 0$ on γ_0, and

$$\Big\{ \sum_{\alpha} \|\eta_\alpha\|_{1,\omega}^2 \Big\}^{1/2} \leq c\Big\{ \sum_{\alpha} \Big\| \frac{1}{2}(\partial_\beta\eta_\alpha + \partial_\alpha\eta_\beta) \Big\|_{0,\omega}^2 \Big\}^{1/2}$$

for all $\eta_\alpha \in \mathbf{H}^1(\omega)$ satisfying $\eta_\alpha = 0$ on γ_0. The first inequality is a well-known property of Sobolev spaces. The second inequality is the *two-dimensional Korn inequality in Cartesian coordinates*. Both play a central rôle in the *existence theory* for *linear two-dimensional plate equations* (see, e.g., Ciarlet [1997, Theorems 1.5-1 and 1.5-2]).

As shown by Blouza & Le Dret [1999], Le Dret [2004], and Anicic, Le Dret & Raoult [2005], *the regularity assumptions made on the mapping $\boldsymbol{\theta}$ and on* *the field* $\boldsymbol{\eta} = (\eta_i)$ *in both the infinitesimal rigid displacement lemma and the Korn inequality on a surface of* Theorems 4.3-3 *and* 4.3-4 *can be substantially weakened.*

This improvement relies on the observation that the "fully explicit" expressions of the covariant components of the linearized change of metric and change of curvature tensors that have been used in the proofs of these theorems, viz.,

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \frac{1}{2}(\partial_\beta\eta_\alpha + \partial_\alpha\eta_\beta) - \Gamma^\sigma_{\alpha\beta}\eta_\sigma - b_{\alpha\beta}\eta_3$$

and

$$\begin{aligned}\rho_{\alpha\beta}(\boldsymbol{\eta}) = \partial_{\alpha\beta}\eta_3 &- \Gamma^\sigma_{\alpha\beta}\partial_\sigma\eta_3 - b^\sigma_\alpha b_{\sigma\beta}\eta_3 \\ &+ b^\sigma_\alpha(\partial_\beta\eta_\sigma - \Gamma^\tau_{\beta\sigma}\eta_\tau) + b^\tau_\beta(\partial_\alpha\eta_\tau - \Gamma^\sigma_{\alpha\tau}\eta_\sigma) \\ &+ (\partial_\alpha b^\tau_\beta + \Gamma^\tau_{\alpha\sigma}b^\sigma_\beta - \Gamma^\sigma_{\alpha\beta}b^\tau_\sigma)\eta_\tau,\end{aligned}$$

can be advantageously replaced by expressions such that

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \frac{1}{2}(\partial_\beta\widetilde{\boldsymbol{\eta}}\cdot\boldsymbol{a}_\alpha + \partial_\alpha\widetilde{\boldsymbol{\eta}}\cdot\boldsymbol{a}_\beta)$$

and

$$\rho_{\alpha\beta}(\boldsymbol{\eta}) = (\partial_{\alpha\beta}\widetilde{\boldsymbol{\eta}} - \Gamma^\sigma_{\alpha\beta}\partial_\sigma\widetilde{\boldsymbol{\eta}})\cdot\boldsymbol{a}_3,$$

or

$$\rho_{\alpha\beta}(\widetilde{\boldsymbol{\eta}}) = \boldsymbol{a}_\alpha\cdot\partial_\beta\big\{(\partial_\sigma\widetilde{\boldsymbol{\eta}}\cdot\boldsymbol{a}_3)\boldsymbol{a}^\sigma\big\} - \partial_\alpha\widetilde{\boldsymbol{\eta}}\cdot\partial_\beta\boldsymbol{a}_3,$$

in terms of the field $\widetilde{\boldsymbol{\eta}} := \eta_i\boldsymbol{a}^i$. Note in passing that this last expression no longer involves Christoffel symbols.

The interest of such expressions is that they still define *bona fide* distributions under significantly weaker smoothness assumptions than those of Theorem 4.3-4, viz., $\boldsymbol{\theta}\in\mathcal{C}^3(\overline{\omega};\mathbf{E}^3)$ and $\boldsymbol{\eta} = (\eta_i)\in H^1(\omega)\times H^1(\omega)\times H^2(\omega)$. For instance, it is easily verified that $\gamma_{\alpha\beta}(\boldsymbol{\eta})\in L^2(\omega)$ and $\rho_{\alpha\beta}(\boldsymbol{\eta})\in H^{-1}(\omega)$ if $\boldsymbol{\theta}\in W^{2,\infty}(\omega;\mathbf{E}^3)$ and $\widetilde{\boldsymbol{\eta}}\in\mathbf{H}^1(\omega)$; or that $\gamma_{\alpha\beta}(\boldsymbol{\eta})\in L^2(\omega)$ and $\rho_{\alpha\beta}(\boldsymbol{\eta})\in L^2(\omega)$ if $\boldsymbol{\theta}\in W^{2,\infty}(\omega;\mathbf{E}^3)$ and $\widetilde{\boldsymbol{\eta}}\in\mathbf{H}^1(\Omega)$ and $\partial_{\alpha\beta}\widetilde{\boldsymbol{\eta}}\cdot\boldsymbol{a}_3\in L^2(\omega)$.

This approach clearly widens the class of shells that can be modeled by Koiter's linear equations, since discontinuities in the second derivatives of the mapping $\boldsymbol{\theta}$ are allowed, provided these derivatives stay in $L^\infty(\omega)$. For instance, it affords the consideration of a shell whose middle surface is composed of a portion of a plane and a portion of a circular cylinder meeting along a segment and having a common tangent plane along this segment.

We continue our study of Korn's inequalities on a surface by showing that the Korn inequality "without boundary conditions" (Theorem 4.3-1) is equivalent to yet another **Korn's inequality on a surface**, **"over the quotient space**

$H^1(\omega) \times H^1(\omega) \times H^2(\omega)/\,\mathbf{Rig}(\omega)$". As we shall see, this inequality is the key to the existence theory for the *pure traction problem* for a shell modeled by the linear Koiter equations (cf. Theorem 4.4-3).

We recall that the vector space

$$\mathbf{Rig}(\omega) = \{\boldsymbol{\eta} \in H^1(\omega) \times H^1(\omega) \times H^2(\omega);\ \gamma_{\alpha\beta}(\boldsymbol{\eta}) = \rho_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega\}$$

denotes the space of vector fields $\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega)$ whose associated displacement fields $\eta_i \boldsymbol{a}^i$ constitute the *infinitesimal rigid displacements of the surface S* and that *the space* $\mathbf{Rig}(\omega)$ *is of dimension six* (Theorem 4.3-3).

In the next theorem, the notation $\dot{\boldsymbol{\eta}}$ designates the equivalence class of an element $\boldsymbol{\eta} \in H^1(\omega) \times H^1(\omega) \times H^2(\omega)$ in the quotient space $H^1(\omega) \times H^1(\omega) \times H^2(\omega)/\,\mathbf{Rig}(\omega)$. In other words,

$$\dot{\boldsymbol{\eta}} := \{\boldsymbol{\zeta} \in H^1(\omega) \times H^1(\omega) \times H^2(\omega); (\boldsymbol{\zeta} - \boldsymbol{\eta}) \in \mathbf{Rig}(\omega)\}.$$

Theorem 4.3-5. *Let ω be a domain in $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbf{E}^3)$ be an injective immersion. Define the quotient space*

$$\dot{\mathbf{V}}(\omega) := (H^1(\omega) \times H^1(\omega) \times H^2(\omega))/\,\mathbf{Rig}(\omega),$$

which is a Hilbert space, equipped with the quotient norm $\|\cdot\|_{\dot{\mathbf{V}}(\Omega)}$ defined by

$$\|\dot{\boldsymbol{\eta}}\|_{\dot{\mathbf{V}}(\omega)} := \inf_{\boldsymbol{\xi} \in \mathbf{Rig}(\omega)} \|\boldsymbol{\eta} + \boldsymbol{\xi}\|_{H^1(\omega) \times H^1(\omega) \times H^2(\omega)} \textit{ for all } \dot{\boldsymbol{\eta}} \in \mathbf{V}(\omega).$$

Then there exists a constant $\dot{c} = \dot{c}(\omega, \boldsymbol{\theta})$ such that

$$\|\dot{\boldsymbol{\eta}}\|_{\dot{\mathbf{V}}(\omega)} \le \dot{c} \Big\{ \sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\dot{\boldsymbol{\eta}})\|^2_{0,\omega} + \sum_{\alpha,\beta} \|\rho_{\alpha\beta}(\dot{\boldsymbol{\eta}})\|^2_{0,\omega} \Big\}^{1/2} \text{ for all } \dot{\boldsymbol{\eta}} \in \dot{\mathbf{V}}(\omega).$$

Moreover, this Korn inequality "over the quotient space $\dot{\mathbf{V}}(\omega)$" is equivalent to the Korn inequality "without boundary condition" of Theorem 4.3-1.

Proof. To begin with, we observe that, thanks to the definition of the space $\mathbf{Rig}(\omega)$, the functions $\gamma_{\alpha\beta}(\dot{\boldsymbol{\eta}}) \in L^2(\omega)$ and $\rho_{\alpha\beta}(\dot{\boldsymbol{\eta}}) \in L^2(\omega)$ are unambiguously defined, viz., as $\gamma_{\alpha\beta}(\dot{\boldsymbol{\eta}}) = \gamma_{\alpha\beta}(\boldsymbol{\zeta})$ and $\rho_{\alpha\beta}(\dot{\boldsymbol{\eta}}) = \rho_{\alpha\beta}(\boldsymbol{\zeta})$ for any $\boldsymbol{\zeta} \in \dot{\boldsymbol{\eta}}$. *In this proof*, we let

$$\mathbf{V}(\omega) := H^1(\omega) \times H^1(\omega) \times H^2(\omega) \text{ and } \|\boldsymbol{\eta}\|_{\mathbf{V}(\omega)} = \Big\{ \sum_{\alpha} \|\eta_\alpha\|^2_{1,\omega} + \|\eta_3\|^2_{2,\omega} \Big\}^{1/2}$$

for the sake of notational conciseness.

(i) We first show that *the Korn inequality "without boundary conditions"* (Theorem 4.3-1) *implies the announced Korn inequality "over the quotient space $\dot{\mathbf{V}}(\omega)$"*.

By the Hahn-Banach theorem, there exist six continuous linear forms ℓ_α on the space $\mathbf{V}(\omega)$, $1 \le \alpha \le 6$, with the following property: A vector field $\boldsymbol{\xi} \in \mathbf{Rig}(\omega)$ is equal to $\mathbf{0}$ if and only if $\ell_\alpha(\boldsymbol{\xi}) = 0$, $1 \le \alpha \le 6$. It thus suffices to show that there exists a constant $\dot{c}$ such that

$$\|\boldsymbol{\eta}\|_{\mathbf{V}(\omega)} \le \dot{c}\Big(\Big\{\sum_{\alpha,\beta}\|\gamma_{\alpha\beta}(\boldsymbol{\eta})\|_{0,\omega}^2 + \sum_{\alpha,\beta}\|\rho_{\alpha\beta}(\boldsymbol{\eta})\|_{0,\omega}^2\Big\}^{1/2} + \sum_{\alpha=1}^{6}|\ell_\alpha(\boldsymbol{\eta})|\Big)$$

for all $\boldsymbol{\eta} \in \mathbf{V}(\omega)$. For, given any $\boldsymbol{\eta} \in \mathbf{V}(\omega)$, let $\boldsymbol{\xi}(\boldsymbol{\eta}) \in \mathbf{Rig}(\omega)$ be defined by the relations $\ell_\alpha(\boldsymbol{\eta} + \boldsymbol{\xi}(\boldsymbol{\eta})) = 0$, $1 \le \alpha \le 6$. The above inequality then implies that, for all $\dot{\boldsymbol{\eta}} \in \dot{\mathbf{V}}(\omega)$,

$$\|\dot{\boldsymbol{\eta}}\|_{\dot{\mathbf{V}}(\omega)} \le \|\boldsymbol{\eta} + \boldsymbol{\xi}(\boldsymbol{\eta})\|_{\mathbf{V}(\omega)} \le \dot{c}\Big\{\sum_{\alpha,\beta}\|\gamma_{\alpha\beta}(\boldsymbol{\eta})\|_{0,\omega}^2 + \sum_{\alpha,\beta}\|\rho_{\alpha\beta}(\boldsymbol{\eta})\|_{0,\omega}^2\Big\}^{1/2}.$$

Assume that there does not exist such a constant $\dot{c}$. Then there exist $\boldsymbol{\eta}^k \in \mathbf{V}(\omega)$, $k \ge 1$, such that $\|\boldsymbol{\eta}^k\|_{\mathbf{V}(\omega)} = 1$ for all $k \ge 1$,

$$\Big\{\sum_{\alpha,\beta}\|\gamma_{\alpha\beta}(\boldsymbol{\eta}^k)\|_{0,\omega}^2 + \sum_{\alpha,\beta}\|\rho_{\alpha\beta}(\boldsymbol{\eta}^k)\|_{0,\omega}^2\Big\}^{1/2} + \sum_{\alpha=1}^{6}|\ell_\alpha(\boldsymbol{\eta}^k)|\Big) \underset{k\to\infty}{\longrightarrow} 0.$$

By Rellich theorem, there thus exists a subsequence $(\boldsymbol{\eta}^\ell)_{\ell=1}^\infty$ that converges in the space $L^2(\omega) \times L^2(\omega) \times H^1(\omega)$ on the one hand; on the other hand, each subsequence $(\boldsymbol{\gamma}_{\alpha\beta}(\boldsymbol{\eta}^\ell))_{\ell=1}^\infty$ and $(\boldsymbol{\rho}_{\alpha\beta}(\boldsymbol{\eta}^\ell))_{\ell=1}^\infty$ converges in the space $L^2(\omega)$. Therefore, the subsequence $(\boldsymbol{\eta}^\ell)_{\ell=1}^\infty$ is a Cauchy sequence with respect to the norm

$$\boldsymbol{\eta} = (\eta_i) \in \mathbf{V}(\omega) \to \Big\{\sum_{\alpha}\|\eta_\alpha\|_{0,\omega}^2 + \|\eta_3\|_{1,\omega}^2 + \sum_{\alpha,\beta}\|\gamma_{\alpha\beta}(\boldsymbol{\eta})\|_{0,\omega}^2 + \sum_{\alpha,\beta}\|\rho_{\alpha\beta}(\boldsymbol{\eta})\|_{0,\omega}^2\Big\}^{1/2},$$

hence also with respect to the norm $\|\cdot\|_{\mathbf{V}(\omega)}$ by *Korn's inequality without boundary conditions.*

Consequently, there exists $\boldsymbol{\eta} \in \mathbf{V}(\omega)$ such that $\|\boldsymbol{\eta}^\ell - \boldsymbol{\eta}\|_{\mathbf{V}(\omega)} \underset{\ell\to\infty}{\longrightarrow} 0$. But then $\boldsymbol{\eta} = \mathbf{0}$, since $\ell_\alpha(\boldsymbol{\eta}) = 0$ and $\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \rho_{\alpha\beta}(\boldsymbol{\eta}) = \mathbf{0}$ in ω, in contradiction with the relations $\|\boldsymbol{\eta}^\ell\|_{\mathbf{V}(\omega)} = 1$ for all $\ell \ge 1$.

(ii) We next show that, conversely, the *Korn inequality "over the quotient space* $\dot{\mathbf{V}}(\omega)$*"* implies the *Korn inequality "without boundary condition"* of Theorem 4.3-1.

Assume that this Korn inequality does not hold. Then there exist $\boldsymbol{\eta}^k = (\eta_i^k) \in \mathbf{V}(\omega)$, $k \ge 1$, such that

$$\|\boldsymbol{\eta}^k\|_{\mathbf{V}(\omega)} = 1 \text{ for all } k \ge 1,$$

$$\Big(\Big\{\sum_{\alpha}\|\eta_\alpha^k\|_{0,\omega}^2 + \|\eta_3^k\|_{1,\omega}^2 + \sum_{\alpha,\beta}\|\gamma_{\alpha\beta}(\boldsymbol{\eta}^k)\|_{0,\omega}^2 + \sum_{\alpha,\beta}\|\rho_{\alpha\beta}(\boldsymbol{\eta}^k)\|_{0,\omega}^2\Big\}^{1/2}\Big) \underset{k\to\infty}{\longrightarrow} 0.$$

Let $\boldsymbol{\zeta}^k \in \mathbf{Rig}(\omega)$ denote for each $k \geq 1$ the projection of $\boldsymbol{\eta}^k$ on $\mathbf{Rig}(\omega)$ with respect to the inner-product of the space $\mathbf{V}(\omega)$. This projection thus satisfies

$$\|\boldsymbol{\eta}^k - \boldsymbol{\zeta}^k\|_{\mathbf{V}(\omega)} = \inf_{\boldsymbol{\xi}\in\mathbf{Rig}(\omega)} \|\boldsymbol{\eta}^k + \boldsymbol{\xi}\|_{\mathbf{V}(\omega)} = \|\dot{\boldsymbol{\eta}}^k\|_{\dot{\mathbf{V}}(\omega)},$$

$$\|\boldsymbol{\eta}^k\|_{\mathbf{V}(\omega)} = \|\boldsymbol{\eta}^k - \boldsymbol{\zeta}^k\|_{\mathbf{V}(\omega)} + \|\boldsymbol{\zeta}^k\|_{\mathbf{V}(\omega)}.$$

The space $\mathbf{Rig}(\omega)$ being finite-dimensional, the inequalities $\|\boldsymbol{\zeta}^k\|_{\mathbf{V}(\omega)} \leq 1$ for all $k \geq 1$ imply the existence of a subsequence $(\boldsymbol{\zeta}^\ell)_{\ell=1}^\infty$ that converges in the space $\mathbf{V}(\omega)$ to an element $\boldsymbol{\zeta} = (\zeta_i) \in \mathbf{Rig}(\omega)$. Besides, Korn's inequality in the quotient space $\dot{\mathbf{V}}(\omega)$ obtained in part (i) implies that

$$\|\boldsymbol{\eta}^\ell - \boldsymbol{\zeta}^\ell\|_{\mathbf{V}(\omega)} = \|\dot{\boldsymbol{\eta}}^\ell\|_{\dot{\mathbf{V}}(\omega)} \underset{\ell\to\infty}{\longrightarrow} 0,$$

since

$$\Big\{ \sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\boldsymbol{\eta}^\ell)\|_{0,\omega}^2 + \sum_{\alpha,\beta} \|\rho_{\alpha\beta}(\boldsymbol{\eta}^\ell)\|_{0,\omega}^2 \Big\}^{1/2} \underset{\ell\to\infty}{\longrightarrow} 0.$$

Consequently,

$$\|\boldsymbol{\eta}^\ell - \boldsymbol{\zeta}\|_{\mathbf{V}(\omega)} \underset{\ell\to\infty}{\longrightarrow} 0.$$

Hence $\{\sum_\alpha \|\eta_\alpha^\ell - \zeta_\alpha^\ell\|_{0,\omega}^2 + \|\eta_3^\ell\|_{1,\omega}^2\}^{1/2} \underset{\ell\to\infty}{\longrightarrow} 0$ *a fortiori*, which shows that $\boldsymbol{\zeta} = \mathbf{0}$ since $\{\sum_\alpha \|\eta_\alpha^\ell\|_{0,\omega}^2 + \|\eta_3^\ell\|_{1,\omega}^2\}^{1/2} \underset{\ell\to\infty}{\longrightarrow} 0$ on the other hand. We have thus reached the conclusion that $\|\boldsymbol{\eta}^\ell\|_{\mathbf{V}(\omega)} \underset{\ell\to\infty}{\longrightarrow} 0$, a contradiction. □

The various Korn inequalities established or mentioned so far, which apply to "general" surfaces (i.e., without any restrictions bearing on their geometry save some regularity assumptions), all involve *both* the linearized change of metric, and the linearized change of curvature, tensors.

It is remarkable that, *for specific geometries and boundary conditions, a Korn inequality can be established that only involves the linearized change of metric tensors.* More specifically, Ciarlet & Lods [1996a] and Ciarlet & Sanchez-Palencia [1996] have established the following **Korn inequality "on an elliptic surface"**:

Let ω be a domain in $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^{2,1}(\overline{\omega}; \mathbf{E}^3)$ be an injective immersion with the property that *the surface* $S = \boldsymbol{\theta}(\overline{\omega})$ is **elliptic**, in the sense that all its points are elliptic (this means that the Gaussian curvature is > 0 everywhere on S; cf. Section 2.5). Then *there exists a constant* $c_M = c_M(\omega, \boldsymbol{\theta}) > 0$ *such that*

$$\Big\{ \sum_\alpha \|\eta_\alpha\|_{1,\omega}^2 + \|\eta_3\|_{0,\omega}^2 \Big\}^{1/2} \leq c_M \Big\{ \sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\boldsymbol{\eta})\|_{0,\omega}^2 \Big\}^{1/2}$$

for all $\boldsymbol{\eta} = (\eta_i) \in H_0^1(\omega) \times H_0^1(\omega) \times L^2(\omega)$.

Remarks. (1) The norm $\|\eta_3\|_{2,\omega}$ appearing in the left-hand side of the Korn inequality on a "general" surface (Theorem 4.3-4) is now replaced by the norm

$\|\eta_3\|_{0,\omega}$. This replacement reflects that it is enough that $\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times L^2(\omega)$ in order that $\gamma_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega)$. As a result, no boundary condition can be imposed on η_3.

(2) The Korn inequality on an elliptic surface was first established by Destuynder [1985, Theorem 6.1 and 6.5], under the additional assumptions that the surface S can be covered by a single system of lines of curvature (Section 2.5) and that the $\mathcal{C}^0(\overline{\omega})$-norms of the corresponding Christoffel symbols are small enough. □

Only compact surfaces defined by a *single* injective immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega})$ have been considered so far. By contrast, a *compact surface S "without boundary"* (such as an ellipsoid or a torus) is defined by means of a *finite number $I \geq 2$ of injective immersions $\boldsymbol{\theta}_i \in \mathcal{C}^3(\overline{\omega}_i)$*, $1 \leq i \leq I$, where the sets ω_i are domains in $\mathbb{R}^2$, in such a way that $S = \bigcup_{i \in I} \boldsymbol{\theta}_i(\omega_i)$. As shown by S. Mardare [2003a], the Korn inequality "without boundary conditions" (Theorem 4.3-1) and the Korn inequality "on the quotient space $H^1(\omega) \times H^1(\omega) \times H^2(\omega)/\,\mathbf{Rig}(\omega)$" (Theorem 4.3-5) can be both extended to such surfaces without boundary.

Likewise, Slicaru [1998] has shown that the above Korn inequality "on an elliptic surface" can be extended to *elliptic surfaces without boundary.*

4.4 EXISTENCE AND UNIQUENESS THEOREMS FOR THE LINEAR KOITER SHELL EQUATIONS; COVARIANT DERIVATIVES OF A TENSOR FIELD DEFINED ON A SURFACE

Let the space $\mathbf{V}(\omega)$ be defined by

$$\mathbf{V}(\omega) := \{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega);\ \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0\},$$

where γ_0 is a $d\gamma$-measurable subset of $\gamma := \partial\omega$ that satisfies *length*$\gamma_0 > 0$. Our primary objective consists in showing that the bilinear form $B : \mathbf{V}(\omega) \times \mathbf{V}(\omega) \to \mathbb{R}$ defined by

$$B(\boldsymbol{\zeta}, \boldsymbol{\eta}) := \int_\omega \Big\{ \varepsilon a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\boldsymbol{\zeta}) \gamma_{\alpha\beta}(\boldsymbol{\eta}) + \frac{\varepsilon^3}{3} a^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}(\boldsymbol{\zeta}) \rho_{\alpha\beta}(\boldsymbol{\eta}) \Big\} \sqrt{a}\, dy$$

for all $(\boldsymbol{\zeta}, \boldsymbol{\eta}) \in \mathbf{V}(\omega) \times \mathbf{V}(\omega)$ is $\mathbf{V}(\omega)$-elliptic.

As a preliminary, we establish the *uniform positive-definiteness of the elasticity tensor of the shell*, given here by means of its *contravariant components* $a^{\alpha\beta\sigma\tau}$. Note that the assumptions on the Lamé constants are the same as in three-dimensional elasticity (Theorem 3.9-1).

Theorem 4.4-1. *Let ω be a domain in $\mathbb{R}^2$, let $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbf{E}^3)$ be an injective immersion, let $a^{\alpha\beta}$ denote the contravariant components of the metric tensor of the surface $\boldsymbol{\theta}(\overline{\omega})$, let the contravariant components of the two-dimensional elasticity tensor of the shell be given by*

$$a^{\alpha\beta\sigma\tau} = \frac{4\lambda\mu}{\lambda + 2\mu} a^{\alpha\beta} a^{\sigma\tau} + 2\mu (a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma}),$$

and assume that $3\lambda + 2\mu > 0$ *and* $\mu > 0$. *Then there exists a constant* $c_e = c_e(\omega, \boldsymbol{\theta}, \lambda, \mu) > 0$ *such that*

$$\sum_{\alpha,\beta} |t_{\alpha\beta}|^2 \leq c_e a^{\alpha\beta\sigma\tau}(y) t_{\sigma\tau} t_{\alpha\beta}$$

for all $y \in \overline{\omega}$ *and all symmetric matrices* $(t_{\alpha\beta})$.

Proof. This proof is similar to that of Theorem 3.9-1 and for this reason, is only sketched. Given any $y \in \overline{\omega}$ and any symmetric matrix $(t_{\alpha\beta})$, let

$$\mathbf{A}(y) = (a^{\alpha\beta}(y)) \text{ and } \mathbf{T} = (t_{\alpha\beta}),$$

let $\mathbf{K}(y) \in \mathbb{S}^2$ be the unique square root of $\mathbf{A}(y)$, and let

$$\mathbf{B}(y) := \mathbf{K}(y)\mathbf{T}\mathbf{K}(y) \in \mathbb{S}^2.$$

Then

$$\frac{1}{2} a^{\alpha\beta\sigma\tau}(y) t_{\sigma\tau} t_{\alpha\beta} = \chi \Big(\operatorname{tr} \mathbf{B}(y) \Big)^2 + 2\mu \operatorname{tr} \Big(\mathbf{B}(y)^T \mathbf{B}(y) \Big) \text{ with } \chi := \frac{2\lambda\mu}{\lambda + 2\mu}.$$

By the inequality established in part (i) of the proof of Theorem 3.9-1 (with $d = 2$ in this case), there thus exists a constant $\alpha(\lambda, \mu) > 0$ such that

$$\frac{1}{2} a^{\alpha\beta\sigma\tau}(y) t_{\sigma\tau} t_{\alpha\beta} \geq \alpha \operatorname{tr} \Big(\mathbf{B}(y)^T \mathbf{B}(y) \Big)$$

if $\chi + \mu > 0$ and $\mu > 0$, or equivalently, if $3\lambda + 2\mu > 0$ and $\mu > 0$. The proof is then concluded as the proof of Theorem 3.9-1. □

Combined with Korn's inequality "with boundary conditions" (Theorem 4.3-4), the positive definiteness of the elasticity tensor leads to the existence of a *weak* solution, i.e., a solution to the variational equations of the *linear Koiter shell equations.*

Theorem 4.4-2. *Let* ω *be a domain in* $\mathbb{R}^2$, *let* γ_0 *be a subset of* $\gamma = \partial\omega$ *with* length $\gamma_0 > 0$, *and let* $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbf{E}^3)$ *be an injective immersion. Finally, let there be given constants* λ *and* μ *that satisfy* $3\lambda + 2\mu > 0$ *and* $\mu > 0$, *and functions* $p^\alpha \in L^r(\omega)$ *for some* $r > 1$ *and* $p^3 \in L^1(\omega)$.

Then there is one and only one solution $\boldsymbol{\zeta} = (\zeta_i)$ *to the variational problem:*

$$\boldsymbol{\zeta} \in \mathbf{V}(\omega) = \{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega);\ \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0\},$$

$$\int_\omega \Big\{ \varepsilon a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\boldsymbol{\zeta}) \gamma_{\alpha\beta}(\boldsymbol{\eta}) + \frac{\varepsilon^3}{3} a^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}(\boldsymbol{\zeta}) \rho_{\alpha\beta}(\boldsymbol{\eta}) \Big\} \sqrt{a}\, \mathrm{d}y$$
$$= \int_\omega p^i \eta_i \sqrt{a}\, \mathrm{d}y \text{ for all } \boldsymbol{\eta} = (\eta_i) \in \mathbf{V}(\omega),$$

where

$$a^{\alpha\beta\sigma\tau} = \frac{4\lambda\mu}{\lambda + 2\mu} a^{\alpha\beta} a^{\sigma\tau} + 2\mu(a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma}),$$

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \frac{1}{2}(\partial_\beta \widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\alpha + \partial_\alpha \widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\beta) \text{ and } \rho_{\alpha\beta}(\boldsymbol{\eta}) = (\partial_{\alpha\beta} \widetilde{\boldsymbol{\eta}} - \Gamma^\sigma_{\alpha\beta} \partial_\sigma \widetilde{\boldsymbol{\eta}}) \cdot \boldsymbol{a}_3,$$

where $\widetilde{\boldsymbol{\eta}} := \eta_i \boldsymbol{a}^i$.

The field $\boldsymbol{\zeta} \in \mathbf{V}(\omega)$ *is also the unique solution to the minimization problem:*

$$j(\boldsymbol{\zeta}) = \inf_{\boldsymbol{\eta} \in \mathbf{V}(\omega)} j(\boldsymbol{\eta}),$$

where

$$j(\boldsymbol{\eta}) := \frac{1}{2} \int_\omega \Big\{ \varepsilon a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\boldsymbol{\eta}) \gamma_{\alpha\beta}(\boldsymbol{\eta}) + \frac{\varepsilon^3}{3} a^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}(\boldsymbol{\eta}) \rho_{\alpha\beta}(\boldsymbol{\eta}) \Big\} \sqrt{a}\, \mathrm{d}y$$
$$- \int_\omega p^i \eta_i \sqrt{a}\, \mathrm{d}y.$$

Proof. As a closed subspace of $H^1(\omega) \times H^1(\omega) \times H^2(\omega)$, the space $\mathbf{V}(\omega)$ is a Hilbert space. The assumptions made on the mapping $\boldsymbol{\theta}$ ensure in particular that the vector fields $\boldsymbol{a}_i$ and $\boldsymbol{a}^i$ belong to $\mathcal{C}^2(\overline{\omega}; \mathbb{R}^3)$ and that the functions $a^{\alpha\beta\sigma\tau}$, $\Gamma^\sigma_{\alpha\beta}$, and a are continuous on the compact set $\overline{\omega}$. Hence the bilinear form defined by the left-hand side of the variational equations is *continuous over the space* $H^1(\omega) \times H^1(\omega) \times H^2(\omega)$.

The continuous embeddings of the space $H^1(\omega)$ into the space $L^s(\omega)$ for any $s \geq 1$ and of the space $H^2(\omega)$ into the space $\mathcal{C}^1(\overline{\omega})$ show that the linear form defined by the right-hand side is continuous over the same space.

Since the symmetric matrix $(a_{\alpha\beta}(y))$ is positive-definite for all $y \in \overline{\omega}$, there exists a_0 such that $a(y) \geq a_0 > 0$ for all $y \in \overline{\omega}$.

Finally, the Korn inequality "with boundary conditions" (Theorem 4.3-4) and the uniform positive definiteness of the elasticity tensor of the shell (Theorem 4.4-1) together imply that

$$\min\Big\{\varepsilon, \frac{\varepsilon^3}{3}\Big\} c_e^{-1} c^{-2} \sqrt{a_0} \Big(\sum_\alpha \|\eta_\alpha\|^2_{1,\omega} + \|\eta_3\|^2_{2,\omega} \Big)$$
$$\leq \int_\omega \Big\{ \varepsilon a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\boldsymbol{\eta}) \gamma_{\alpha\beta}(\boldsymbol{\eta}) + \frac{\varepsilon^3}{3} a^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}(\boldsymbol{\eta}) \rho_{\alpha\beta}(\boldsymbol{\eta}) \Big\} \sqrt{a}\, \mathrm{d}y$$

for all $\boldsymbol{\eta} = (\eta_i) \in \mathbf{V}(\omega)$. Hence *the bilinear form* B *is* $\mathbf{V}(\omega)$*-elliptic.*

The *Lax-Milgram* lemma then shows that the variational equations have one and only one solution. Since the bilinear form is symmetric, this solution is also the unique solution of the minimization problem stated in the theorem. □

The above existence and uniqueness result applies to linearized *pure displacement* and *displacement-traction* problems, i.e., those that correspond to *length* $\gamma_0 > 0$.

We next consider the linearized *pure traction problem*, i.e., corresponding to the case where the set γ_0 is empty. In this case, we seek a vector field $\boldsymbol{\zeta} = (\zeta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega)$ that satisfies

$$\int_\omega \Big\{\varepsilon a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta})\gamma_{\alpha\beta}(\boldsymbol{\eta}) + \frac{\varepsilon^3}{3}a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{\zeta})\rho_{\alpha\beta}(\boldsymbol{\eta})\Big\}\sqrt{a}\,\mathrm{d}y$$
$$= \int_\omega p^i\eta_i\sqrt{a}\,\mathrm{d}y \text{ for all } \boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega).$$

Clearly, such variational equations can have a solution *only if their right-hand side vanishes for any vector field* $\boldsymbol{\xi} = (\xi_i) \in \mathbf{Rig}(\omega)$, where

$$\mathbf{Rig}(\omega) := \big\{\boldsymbol{\xi} \in H^1(\omega) \times H^1(\omega) \times H^2(\omega);\ \gamma_{\alpha\beta}(\boldsymbol{\xi}) = \rho_{\alpha\beta}(\boldsymbol{\xi}) = 0 \text{ in } \omega\big\},$$

since replacing $\boldsymbol{\eta}$ by $(\boldsymbol{\eta} + \boldsymbol{\xi})$ for any $\boldsymbol{\xi} \in \mathbf{Rig}(\omega)$ does not affect their left-hand side.

We now show that this necessary condition is in fact also sufficient for the existence of solutions, because in this case of the *Korn inequality "over the quotient space* $H^1(\omega) \times H^1(\omega) \times H^2(\omega)/\,\mathbf{Rig}(\omega)$*"* (Theorem 4.3-5). Evidently, the existence of solutions can then hold only up to the addition of vector fields $\boldsymbol{\xi} \in \mathbf{Rig}(\omega)$. This means that the solution is naturally sought in this quotient space.

Theorem 4.4-3. *Let* ω *be a domain in* $\mathbb{R}^2$ *and let* $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbf{E}^3)$ *be an injective immersion. Let there be given constants* λ *and* μ *that satisfy* $3\lambda + 2\mu > 0$ *and* $\mu > 0$ *and functions* $p^\alpha \in L^r(\omega)$ *for some* $r > 1$ *and* $p^3 \in L^1(\omega)$*. Define the quotient space*

$$\dot{\mathbf{V}}(\omega) := H^1(\omega) \times H^1(\omega) \times H^2(\omega)/\,\mathbf{Rig}(\omega),$$

and assume that the functions p^i *satisfy*

$$\int_\omega p^i\xi_i\sqrt{a}\,\mathrm{d}\omega = 0 \text{ for all } \boldsymbol{\xi} \in \mathbf{Rig}(\omega).$$

Finally, let the functions $a^{\alpha\beta\sigma\tau}$ *be defined as in* Theorem 4.4-1*.*

Then there is one and only one solution $\dot{\boldsymbol{\zeta}} \in \dot{\mathbf{V}}(\omega)$ *to the variational equations*

$$\int_\omega \Big\{\varepsilon a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\dot{\boldsymbol{\zeta}})\gamma_{\alpha\beta}(\dot{\boldsymbol{\eta}}) + \frac{\varepsilon^3}{3}a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\dot{\boldsymbol{\zeta}})\rho_{\alpha\beta}(\dot{\boldsymbol{\eta}})\Big\}\sqrt{a}\,\mathrm{d}y$$
$$= \int_\omega p^i\dot{\eta}_i\sqrt{a}\,\mathrm{d}y \text{ for all } \dot{\boldsymbol{\eta}} = (\eta_i) \in \dot{\mathbf{V}}(\omega).$$

The equivalence class $\dot{\boldsymbol{\zeta}} \in \dot{\mathbf{V}}(\omega)$ *is also the unique solution to the minimization problem*

$$j(\dot{\boldsymbol{\zeta}}) = \inf_{\dot{\boldsymbol{\eta}} \in \dot{\mathbf{V}}(\omega)} j(\dot{\boldsymbol{\eta}}),$$

where

$$j(\dot{\boldsymbol{\eta}}) := \frac{1}{2}\int_\omega \Big\{\varepsilon a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\dot{\boldsymbol{\eta}})\gamma_{\alpha\beta}(\dot{\boldsymbol{\eta}}) + \frac{\varepsilon^3}{3}a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\dot{\boldsymbol{\eta}})\rho_{\alpha\beta}(\dot{\boldsymbol{\eta}})\Big\}\sqrt{a}\,\mathrm{d}y$$
$$- \int_\omega p^i\dot{\eta}_i\sqrt{a}\,\mathrm{d}y.$$

Proof. The proof is analogous to that of Theorem 4.4-2, the Korn inequality of Theorem 4.3-4 being now replaced by that of Theorem 4.3-5. □

An immediate consequence of Theorems 4.4-2 and 4.4-3 is the existence and uniqueness (up to infinitesimal rigid displacements when the set γ_0 is empty) of a *displacement field* $\zeta_i\boldsymbol{a}^i$ *of the middle surface* S, whose components $\zeta_\alpha \in H^1(\omega)$ and $\zeta_3 \in H^2(\omega)$ are thus obtained by finding the solution $\boldsymbol{\zeta} = (\zeta_i)$ to the variational equations of either theorem. Since the vector fields $\boldsymbol{a}^i$ formed by the covariant bases belong to the space $\mathcal{C}^2(\overline{\omega};\mathbb{R}^3)$ by assumption, the vector fields $\zeta_\alpha\boldsymbol{a}^\alpha$ and $\zeta_3\boldsymbol{a}^3$ belong respectively to the spaces $\mathbf{H}^1(\omega)$ and $\mathbf{H}^2(\omega)$.

We next derive the *boundary value problem* that is, at least formally, equivalent to the variational equations of the Theorems 4.4-2 or 4.4-3, the latter corresponding to the case where the set γ_0 is empty. In what follows, $\gamma_1 := \gamma - \gamma_0$, (ν_α) is the unit outer normal vector along γ, $\tau_1 := -\nu_2$, $\tau_2 := \nu_1$, and $\partial_\tau\chi := \tau_\alpha\partial_\alpha\chi$ denotes the tangential derivative of χ in the direction of the vector (τ_α).

Theorem 4.4-4. *Let ω be a domain in $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega};\mathbf{E}^3)$ be an injective immersion. Assume that the boundary γ of ω and the functions p^i are smooth enough. If the solution $\boldsymbol{\zeta} = (\zeta_i)$ to the variational equations found in either* Theorem 4.4-2 *or* Theorem 4.4-3 *is smooth enough, then $\boldsymbol{\zeta}$ is also a solution to the following boundary value problem:*

$$\begin{aligned}
m^{\alpha\beta}|_{\alpha\beta} - b_\alpha^\sigma b_{\sigma\beta}m^{\alpha\beta} - b_{\alpha\beta}n^{\alpha\beta} &= p^3 \text{ in } \omega,\\
-(n^{\alpha\beta} + b_\sigma^\alpha m^{\sigma\beta})|_\beta - b_\sigma^\alpha(m^{\sigma\beta}|_\beta) &= p^\alpha \text{ in } \omega,\\
\zeta_i = \partial_\nu\zeta_3 &= 0 \text{ on } \gamma_0,\\
m^{\alpha\beta}\nu_\alpha\nu_\beta &= 0 \text{ on } \gamma_1,\\
(m^{\alpha\beta}|_\alpha)\nu_\beta + \partial_\tau(m^{\alpha\beta}\nu_\alpha\tau_\beta) &= 0 \text{ on } \gamma_1,\\
(n^{\alpha\beta} + 2b_\sigma^\alpha m^{\sigma\beta})\nu_\beta &= 0 \text{ on } \gamma_1,
\end{aligned}$$

where

$$n^{\alpha\beta} := \varepsilon a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta}) \text{ and } m^{\alpha\beta} := \frac{\varepsilon^3}{3}a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{\zeta}),$$

and, for an arbitrary tensor field with smooth enough covariant components $t^{\alpha\beta} : \overline{\omega} \to \mathbb{R}$,

$$\begin{aligned}
t^{\alpha\beta}|_\beta &:= \partial_\beta t^{\alpha\beta} + \Gamma_{\beta\sigma}^\alpha t^{\beta\sigma} + \Gamma_{\beta\sigma}^\beta t^{\alpha\sigma},\\
t^{\alpha\beta}|_{\alpha\beta} &:= \partial_\alpha(t^{\alpha\beta}|_\beta) + \Gamma_{\alpha\sigma}^\sigma(t^{\alpha\beta}|_\beta).
\end{aligned}$$

Proof. For simplicity, we give the proof only in the case where $\gamma_0 = \gamma$, i.e., when the space $\mathbf{V}(\omega)$ of Theorem 4.4-2 reduces to

$$\mathbf{V}(\omega) = H_0^1(\omega) \times H_0^1(\omega) \times H_0^2(\omega).$$

The extension to the case where *length* $\gamma_1 > 0$ is straightforward.

In what follows, we assume that the solution $\boldsymbol{\zeta}$ is "smooth enough" in the sense that $n^{\alpha\beta} \in H^1(\omega)$ and $m^{\alpha\beta} \in H^2(\omega)$.

(i) We first establish the relations

$$\partial_\alpha \sqrt{a} = \sqrt{a}\Gamma^\sigma_{\sigma\alpha}.$$

Let $\mathbf{A}$ denote the matrix of order three with $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ as its column vectors, so that $\sqrt{a} = \det \mathbf{A}$ (see part (i) of the proof of Theorem 4.2-2). Consequently,

$$\begin{aligned}\partial_\alpha \sqrt{a} &= \det(\partial_\alpha \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3) + \det(\boldsymbol{a}_1, \partial_\alpha \boldsymbol{a}_2, \boldsymbol{a}_3) + \det(\boldsymbol{a}_1, \boldsymbol{a}_2, \partial_\alpha \boldsymbol{a}_3)\\ &= (\Gamma^1_{1\alpha} + \Gamma^2_{2\alpha} + \Gamma^3_{3\alpha}) \det(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3) = \sqrt{a}\Gamma^\sigma_{\sigma\alpha}\end{aligned}$$

since $\partial_\alpha \boldsymbol{a}_\beta = \Gamma^\sigma_{\beta\alpha} \boldsymbol{a}_\sigma + b_{\alpha\beta} \boldsymbol{a}_3$ (Theorem 2.6-1).

(ii) Using the *Green formula in Sobolev spaces* (see, e.g., Nečas [1967]) and assuming that the functions $n^{\alpha\beta} = n^{\beta\alpha}$ are in $H^1(\omega)$, we first transform the first integral appearing in the left-hand side of the variational equations. This gives, for all $\boldsymbol{\eta} = (\eta_i) \in H_0^1(\omega) \times H_0^1(\omega) \times L^2(\omega)$, hence *a fortiori* for all $\boldsymbol{\eta} = (\eta_i) \in H_0^1(\omega) \times H_0^1(\omega) \times H_0^2(\omega)$,

$$\begin{aligned}&\int_\omega a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\boldsymbol{\zeta}) \gamma_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a}\, \mathrm{d}y = \int_\omega n^{\alpha\beta} \gamma_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a}\, \mathrm{d}y\\ &= \int_\omega \sqrt{a} n^{\alpha\beta} \Big(\frac{1}{2} (\partial_\beta \eta_\alpha + \partial_\alpha \eta_\beta) - \Gamma^\sigma_{\alpha\beta} \eta_\sigma - b_{\alpha\beta} \eta_3 \Big)\, \mathrm{d}y\\ &= \int_\omega \sqrt{a} n^{\alpha\beta} \partial_\beta \eta_\alpha\, \mathrm{d}y - \int_\omega \sqrt{a} n^{\alpha\beta} \Gamma^\sigma_{\alpha\beta} \eta_\sigma\, \mathrm{d}y - \int_\omega \sqrt{a} n^{\alpha\beta} b_{\alpha\beta} \eta_3\, \mathrm{d}y\\ &= -\int_\omega \partial_\beta (\sqrt{a} n^{\alpha\beta}) \eta_\alpha\, \mathrm{d}y - \int_\omega \sqrt{a} n^{\alpha\beta} \Gamma^\sigma_{\alpha\beta} \eta_\sigma\, \mathrm{d}y - \int_\omega \sqrt{a} n^{\alpha\beta} b_{\alpha\beta} \eta_3\, \mathrm{d}y\\ &= -\int_\omega \sqrt{a} (\partial_\beta n^{\alpha\beta} + \Gamma^\alpha_{\tau\beta} n^{\tau\beta} + \Gamma^\beta_{\beta\tau} n^{\alpha\tau}) \eta_\alpha\, \mathrm{d}y - \int_\omega \sqrt{a} n^{\alpha\beta} b_{\alpha\beta} \eta_3\, \mathrm{d}y\\ &= -\int_\omega \sqrt{a} \Big\{ (n^{\alpha\beta}|_\beta) \eta_\alpha + b_{\alpha\beta} n^{\alpha\beta} \eta_3 \Big\}\, \mathrm{d}y.\end{aligned}$$

(iii) We then likewise transform the second integral appearing in the left-hand side of the variational equations, viz.,

$$\frac{1}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{\zeta})\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y = \int_\omega m^{\alpha\beta}\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y$$
$$= \int_\omega \sqrt{a}\,m^{\alpha\beta}\partial_{\alpha\beta}\eta_3\,\mathrm{d}y$$
$$+\int_\omega \sqrt{a}\,m^{\alpha\beta}(2b^\sigma_\alpha\partial_\beta\eta_\sigma - \Gamma^\sigma_{\alpha\beta}\partial_\sigma\eta_3)\,\mathrm{d}y$$
$$+\int_\omega \sqrt{a}\,m^{\alpha\beta}(-2b^\tau_\beta\Gamma^\sigma_{\alpha\tau}\eta_\sigma + b^\sigma_\beta|_\alpha\eta_\sigma - b^\sigma_\alpha b_{\sigma\beta}\eta_3)\,\mathrm{d}y,$$

for all $\boldsymbol{\eta} = (\eta_i) \in H^1_0(\omega)\times H^1_0(\omega)\times H^2_0(\omega)$. Using the symmetry $m^{\alpha\beta} = m^{\beta\alpha}$, the relation $\partial_\beta\sqrt{a} = \sqrt{a}\,\Gamma^\sigma_{\beta\sigma}$ (cf. part (i)), and the same Green formula as in part (ii), we obtain

$$\int_\omega m^{\alpha\beta}\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y = -\int_\omega \sqrt{a}(\partial_\beta m^{\alpha\beta} + \Gamma^\sigma_{\beta\sigma}m^{\alpha\beta} + \Gamma^\alpha_{\sigma\beta}m^{\sigma\beta})\partial_\alpha\eta_3\,\mathrm{d}y$$
$$+2\int_\omega \sqrt{a}m^{\alpha\beta}b^\sigma_\alpha\partial_\beta\eta_\sigma\,\mathrm{d}y$$
$$+\int_\omega \sqrt{a}m^{\alpha\beta}(-2b^\tau_\beta\Gamma^\sigma_{\alpha\tau}\eta_\sigma + b^\sigma_\beta|_\alpha\eta_\sigma - b^\sigma_\alpha b_{\sigma\beta}\eta_3)\,\mathrm{d}y.$$

The same Green formula further shows that

$$-\int_\omega \sqrt{a}(\partial_\beta m^{\alpha\beta} + \Gamma^\sigma_{\beta\sigma}m^{\alpha\beta} + \Gamma^\alpha_{\sigma\beta}m^{\sigma\beta})\partial_\alpha\eta_3\,\mathrm{d}y$$
$$= -\int_\omega \sqrt{a}(m^{\alpha\beta}|_\beta)\partial_\alpha\eta_3\,\mathrm{d}y = \int_\omega \partial_\alpha(\sqrt{a}\,m^{\alpha\beta}|_\beta)\eta_3\,\mathrm{d}y$$
$$= \int_\omega \sqrt{a}(m^{\alpha\beta}|_{\alpha\beta})\eta_3\,\mathrm{d}y,$$
$$2\int_\omega \sqrt{a}\,m^{\alpha\beta}b^\sigma_\alpha\partial_\beta\eta_\sigma\,\mathrm{d}y = -2\int_\omega \sqrt{a}\big\{\partial_\beta(b^\sigma_\alpha m^{\alpha\beta}) + \Gamma^\tau_{\beta\tau}b^\sigma_\alpha m^{\alpha\beta}\big\}\eta_\sigma\,\mathrm{d}y.$$

Consequently,

$$\int_\omega m^{\alpha\beta}\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y = \int_\omega \sqrt{a}\big\{-2(b^\alpha_\sigma m^{\sigma\beta})|_\beta + (b^\alpha_\beta|_\sigma)m^{\sigma\beta}\big\}\eta_\alpha\,\mathrm{d}y$$
$$+\int_\omega \sqrt{a}\big\{m^{\alpha\beta}|_{\alpha\beta} - b^\sigma_\alpha b_{\sigma\beta}m^{\alpha\beta}\big\}\eta_3\,\mathrm{d}y.$$

Using in this relation the easily verified formula

$$(b^\alpha_\sigma m^{\sigma\beta})|_\beta = (b^\alpha_\beta|_\sigma)m^{\sigma\beta} + b^\alpha_\sigma(m^{\sigma\beta}|_\beta)$$

and the symmetry relations $b^\alpha_\beta|_\sigma = b^\alpha_\sigma|_\beta$ (Theorem 4.2-2), we finally obtain

$$\int_\omega m^{\alpha\beta}\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y = -\int_\omega \sqrt{a}\big\{(b^\alpha_\sigma m^{\sigma\beta})|_\beta + b^\alpha_\sigma(m^{\sigma\beta}|_\beta)\big\}\eta_\alpha\,\mathrm{d}y$$
$$-\int_\omega \sqrt{a}\big\{b^\sigma_\alpha b_{\sigma\beta}m^{\alpha\beta} - m^{\alpha\beta}|_{\alpha\beta}\big\}\eta_3\,\mathrm{d}y.$$

(iv) By parts (ii) and (iii), the variational equations

$$\int_\omega \Big\{a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta})\gamma_{\alpha\beta}(\boldsymbol{\eta}) + \frac{1}{3}a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{\zeta})\rho_{\alpha\beta}(\boldsymbol{\eta}) - p^i\eta_i\Big\}\sqrt{a}\,\mathrm{d}y = 0$$

imply that

$$\int_\omega \sqrt{a}\big\{(n^{\alpha\beta} + b^\alpha_\sigma m^{\sigma\beta})|_\beta + b^\alpha_\sigma(m^{\sigma\beta}|_\beta) + p^\alpha\big\}\eta_\alpha\,\mathrm{d}y$$
$$+\int_\omega \sqrt{a}\big\{b_{\alpha\beta}n^{\alpha\beta} + b^\sigma_\alpha b_{\sigma\beta}m^{\alpha\beta} - m^{\alpha\beta}|_{\alpha\beta} + p^3\big\}\eta_3\,\mathrm{d}y = 0$$

for all $(\eta_i) \in H^1_0(\omega)\times H^1_0(\omega)\times H^2_0(\omega)$. The announced partial differential equations are thus satisfied in ω. □

The functions
$$n^{\alpha\beta} = \varepsilon a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta})$$
are the **contravariant components of the linearized stress resultant tensor field** inside the shell, and the functions

$$m^{\alpha\beta} = \frac{\varepsilon^3}{3}a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{\zeta})$$

are the **contravariant components of the linearized stress couple**, or **linearized bending moment, tensor field** inside the shell.

The functions

$$t^{\alpha\beta}|_\beta = \partial_\beta t^{\alpha\beta} + \Gamma^\alpha_{\beta\sigma}t^{\beta\sigma} + \Gamma^\beta_{\beta\sigma}t^{\alpha\sigma},$$
$$t^{\alpha\beta}|_{\alpha\beta} = \partial_\alpha(t^{\alpha\beta}|_\beta) + \Gamma^\sigma_{\alpha\sigma}(t^{\alpha\beta}|_\beta),$$

which have naturally appeared in the course of the proof of Theorem 4.4-4, constitute examples of **first-order**, and **second order, covariant derivatives of a tensor field defined on a surface**, here by means of its contravariant components $t^{\alpha\beta} : \overline{\omega} \to \mathbb{R}$.

Finally, we state a *regularity result* that provides an instance where the *weak solution*, viz., the solution of the variational equations, is also a *classical solution*, viz., a solution of the associated boundary value problem. The proof of this result, which is due to Alexandrescu [1994], is long and delicate and for this reason is only briefly sketched here (as expected, it follows the same pattern as in the three-dimensional case, considered in Theorem 3.9-4).

Theorem 4.4-5. *Let ω be a domain in $\mathbb{R}^2$ with boundary γ and let $\boldsymbol{\theta} : \overline{\omega} \to \mathbf{E}^3$ be an injective immersion. Assume that, for some integer $m \geq 0$ and some real number $q > 1$, γ is of class $\mathcal{C}^{m+4}$, $\boldsymbol{\theta} \in \mathcal{C}^{m+4}(\overline{\omega}; \mathbf{E}^3)$, $p^\alpha \in W^{m+1,q}(\omega)$, and $p^3 \in W^{m,q}(\omega)$. Finally, assume that $\gamma_0 = \gamma$. Then the weak solution found in* Theorem 4.4-2 *satisfies*

$$\boldsymbol{\zeta} = (\zeta_i) \in W^{m+3,q}(\omega) \times W^{m+3,q}(\omega) \times W^{m+4,q}(\omega).$$

Sketch of the proof. To begin with, assume that the boundary γ is of class $\mathcal{C}^4$ and the mapping $\boldsymbol{\theta}$ belongs to the space $\mathcal{C}^4(\overline{\omega}; \mathbf{E}^3)$.

One first verifies that the linear system of partial differential equations found in Theorem 4.4-4 (which is of the third order with respect to the unknowns ζ_α and of the fourth order with respect to the unknown ζ_3) is *uniformly elliptic* and satisfies the *supplementing condition on L* and the *complementing boundary conditions*, in the sense of Agmon, Douglis & Nirenberg [1964].

One then verifies that the same system is also *strongly elliptic* in the sense of Nečas [1967, p. 185]. A regularity result of Nečas [1967, Lemma 3.2, p. 260] then shows that the weak solution $\boldsymbol{\zeta} = (\zeta_i)$ found in Theorem 4.4-2, which belongs to the space $H_0^1(\omega) \times H_0^1(\omega) \times H_0^2(\omega)$ since $\gamma_0 = \gamma$ by assumption, satisfies

$$\boldsymbol{\zeta} = (\zeta_i) \in H^3(\omega) \times H^3(\omega) \times H^4(\omega)$$

if $p^\alpha \in H^1(\omega)$ and $p^3 \in L^2(\omega)$.

A result of Geymonat [1965, Theorem 3.5] about the *index* of the associated linear operator then implies that

$$\boldsymbol{\zeta}^\varepsilon = (\zeta_i^\varepsilon) \in W^{3,q}(\omega) \times W^{3,q}(\omega) \times W^{4,q}(\omega)$$

if $p^\alpha \in W^{1,q}(\omega)$ and $p^3 \in L^q(\omega)$ for some $q > 1$.

Assume finally that, for some integer $m \geq 1$ and some real number $q > 1$, γ is of class $\mathcal{C}^{m+4}$ and $\boldsymbol{\theta} \in \mathcal{C}^{m+4}(\overline{\omega}; \mathbf{E}^3)$. Then a regularity result of Agmon, Douglis & Nirenberg [1964] implies that

$$\boldsymbol{\zeta} = (\zeta_i) \in W^{m+3,q}(\omega) \times W^{m+3,q}(\omega) \times W^{m+4,q}(\omega).$$

if $p^\alpha \in W^{m+1,q}(\omega)$ and $p^3 \in W^{m,q}(\omega)$. □

4.5 A BRIEF REVIEW OF LINEAR SHELL THEORIES

In order to put the *linear Koiter shell equations* in their proper perspective, we briefly review the genesis of those *two-dimensional linear shell theories* that can be found, and rigorously justified, as the outcome of an *asymptotic analysis of the equations of three-dimensional linearized elasticity as $\varepsilon \to 0$.*

The asymptotic analysis of elastic shells has been a subject of considerable attention during the past decades. After the landmark attempt of Goldenveizer [1963], a major step for linearly elastic shells was achieved by Destuynder [1980] in his Doctoral Dissertation, where a convergence theorem for *"membrane shells"* was "almost proved". Another major step was achieved by Sanchez-Palencia [1990], who clearly delineated the kinds of geometries of the middle surface and boundary conditions that yield either two-dimensional *membrane*, or two-dimensional *flexural*, equations when the method of formal asymptotic expansions is applied to the variational equations of three-dimensional linearized elasticity (see also Caillerie & Sanchez-Palencia [1995] and Miara & Sanchez-Palencia [1996]).

Then Ciarlet & Lods [1996a,b] and Ciarlet, Lods & Miara [1996] carried out an *asymptotic analysis of linearly elastic shells that covers all possible cases*: Under three distinct sets of assumptions on the geometry of the middle surface, on the boundary conditions, and on the order of magnitude of the applied forces, they established *convergence theorems* in H^1, in L^2, or in *ad hoc* completion spaces, that justify either the linear two-dimensional equations of a *"membrane shell"*, or those of a *"generalized membrane shell"*, or those of a *"flexural shell"*.

More specifically, consider a *family of linearly elastic shells of thickness* 2ε that satisfy the following *assumptions*: All the shells have the *same* middle surface $S = \theta(\overline{\omega}) \subset \mathbf{E}^3$, where ω is a domain in $\mathbb{R}^2$ with boundary γ, and $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbf{E}^3)$. Their reference configurations are thus of the form $\boldsymbol{\Theta}(\overline{\Omega}^\varepsilon)$, $\varepsilon > 0$, where

$$\Omega^\varepsilon := \omega \times \left]-\varepsilon, \varepsilon\right[,$$

and the mapping $\boldsymbol{\Theta}$ is defined by

$$\boldsymbol{\Theta}(y, x_3^\varepsilon) := \boldsymbol{\theta}(y) + x_3^\varepsilon \boldsymbol{a}_3(y) \text{ for all } (y, x_3^\varepsilon).$$

All the shells in the family are made with the *same homogeneous isotropic elastic material* and that *their reference configurations are natural states.* Their elastic material is thus characterized by two Lamé constants λ and μ satisfying $3\lambda + 2\mu > 0$ and $\mu > 0$.

The shells are subjected to *body forces* and that the corresponding *applied body force density is* $O(\varepsilon^p)$ *with respect to* ε, for some *ad hoc* power p (which will be specified later). This means that, for each $\varepsilon > 0$, the contravariant components $f^{i,\varepsilon} \in L^2(\Omega^\varepsilon)$ of the body force density are of the form

$$f^{i,\varepsilon}(y, \varepsilon x_3) = \varepsilon^p f^i(y, x_3) \text{ for all } (y, x_3) \in \Omega := \omega \times \left]-1, 1\right[,$$

and the functions $f^i \in L^2(\Omega)$ are *independent* of ε (surface forces acting on the "upper" and "lower" faces of the shell could be as well taken into account but will not be considered here, for simplicity of exposition). Let then the functions $p^{i,\varepsilon} \in L^2(\omega)$ be defined for each $\varepsilon > 0$ by

$$p^{i,\varepsilon} := \int_{-\varepsilon}^{\varepsilon} f^{i,\varepsilon} \, dx_3^\varepsilon.$$

Then, for each $\varepsilon > 0$, the associated *equations of linearized three-dimensional elasticity in curvilinear coordinates*, viz., $(y_1, y_2, x_3^\varepsilon) \in \overline{\Omega}^\varepsilon$, have one and only one solution (Theorem 3.9-2) $\boldsymbol{u}^\varepsilon = (u_i^\varepsilon) \in \mathbf{H}^1(\Omega^\varepsilon)$, where the functions u_i^ε are the covariant components of the displacement field of the reference configuration $\boldsymbol{\Theta}(\overline{\Omega}^\varepsilon)$.

Finally, each shell is subjected to a *boundary condition of place* on the portion $\boldsymbol{\Theta}(\gamma_0 \times [-\varepsilon, \varepsilon])$ of its lateral face, where γ_0 is a *fixed* portion of γ, with *length* $\gamma_0 > 0$.

Incidentally, such particular instances of sets Ω^ε and mappings $\boldsymbol{\Theta}$ provide a fundamental motivation for studying the equations of linear elasticity in *curvilinear coordinates*, since they constitute a most natural point of departure of any asymptotic analysis of shells.

Let

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \frac{1}{2}(\partial_\beta \widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\alpha + \partial_\alpha \widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\beta) \text{ and } \rho_{\alpha\beta}(\boldsymbol{\eta}) = (\partial_{\alpha\beta}\widetilde{\boldsymbol{\eta}} - \Gamma^\sigma_{\alpha\beta}\partial_\sigma \widetilde{\boldsymbol{\eta}}) \cdot \boldsymbol{a}_3,$$

where $\widetilde{\boldsymbol{\eta}} = \eta_i \boldsymbol{a}^i$, denote as usual the covariant components of the *linearized change of metric*, and *linearized change of curvature, tensors.*

In Ciarlet, Lods & Miara [1996] it is first assumed that *the space of linearized inextensional displacements* (introduced by Sanchez-Palencia [1989a])

$$\mathbf{V}_F(\omega) := \{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega); \\ \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_{\alpha\beta}(\eta) = 0 \text{ in } \omega\}$$

contains non-zero functions. This assumption is in fact one in disguise about the geometry of the surface S and on the set γ_0. For instance, it is satisfied if S is a portion of a cylinder and $\boldsymbol{\theta}(\gamma_0)$ is contained in one or two generatrices of S, or if S is contained in a plane, in which case the shells are *plates.*

Under this assumption Ciarlet, Lods & Miara [1996] showed that, *if the applied body force density is* $O(\varepsilon^2)$ *with respect to* ε*, then*

$$\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u_i^\varepsilon \, \mathrm{d}x_3^\varepsilon \to \zeta_i \text{ in } H^1(\omega) \text{ as } \varepsilon \to 0,$$

where the limit vector field $\boldsymbol{\zeta} := (\zeta_i)$ *belongs to the space* $\mathbf{V}_F(\omega)$ *and satisfies the* **equations of a linearly elastic "flexural shell"** , viz.,

$$\frac{\varepsilon^3}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{\zeta})\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y = \int_\omega p^{i,\varepsilon}\eta_i\sqrt{a}\,\mathrm{d}y$$

for all $\boldsymbol{\eta} = (\eta_i) \in \mathbf{V}_F(\omega)$. Observe in passing that the limit $\boldsymbol{\zeta}$ is indeed *independent of* ε, since both sides of these variational equations are of the same order (viz., ε^3), because of the assumptions made on the applied forces.

Equivalently, the vector field $\boldsymbol{\zeta}$ satisfies the following *constrained minimization problem*:

$$\boldsymbol{\zeta} \in \mathbf{V}_F(\omega) \text{ and } j_F^\varepsilon(\boldsymbol{\zeta}) = \inf j_F^\varepsilon(\boldsymbol{\eta}),$$

where

$$j_F^\varepsilon(\boldsymbol{\eta}) := \frac{1}{2}\int_\omega \frac{\varepsilon^3}{3} a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{\eta})\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y - \int_\omega p^{i,\varepsilon}\eta_i\sqrt{a}\,\mathrm{d}y$$

for all $\boldsymbol{\eta} = (\eta_i) \in \mathbf{V}_F(\omega)$, where the functions

$$a^{\alpha\beta\sigma\tau} = \frac{4\lambda\mu}{(\lambda+2\mu)}a^{\alpha\beta}a^{\sigma\tau} + 2\mu(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma})$$

are precisely the familiar contravariant components of the *shell elasticity tensor.*

If $\mathbf{V}_F(\omega) \neq \{\mathbf{0}\}$, *the two-dimensional equations of a linearly elastic "flexural shell" are therefore justified.*

If $\mathbf{V}_F(\omega) = \{\mathbf{0}\}$, the above convergence result still applies. However, the only information it provides is that $\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u_i^\varepsilon \,\mathrm{d}x_3^\varepsilon \to 0$ in $H^1(\omega)$ as $\varepsilon \to 0$. Hence a more refined asymptotic analysis is needed in this case.

A first instance of such a refinement was given by Ciarlet & Lods [1996a], where it was assumed that $\gamma_0 = \gamma$ and that the surface S is *elliptic*, in the sense that its Gaussian curvature is > 0 everywhere. As shown in Ciarlet & Lods [1996a] and Ciarlet & Sanchez-Palencia [1996], these two conditions, together with *ad hoc* regularity assumptions, indeed imply that $\mathbf{V}_F(\omega) = \{\mathbf{0}\}$.

In this case, Ciarlet & Lods [1996b] showed that, *if the applied body force density is* $O(1)$ *with respect to* ε, *then*

$$\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u_\alpha^\varepsilon\,\mathrm{d}x_3^\varepsilon \to \zeta_\alpha \text{ in } H^1(\omega) \text{ and } \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u_3^\varepsilon\,\mathrm{d}x_3^\varepsilon \to \zeta_3 \text{ in } L^2(\omega) \text{ as } \varepsilon \to 0,$$

where the limit vector field $\boldsymbol{\zeta} := (\zeta_i)$ *belongs to the space*

$$\mathbf{V}_M(\omega) := H_0^1(\omega) \times H_0^1(\omega) \times L^2(\omega),$$

and solves the **equations of a linearly elastic "membrane shell"**, viz.,

$$\int_\omega \varepsilon a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta}^\varepsilon)\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y = \int_\omega p^{i,\varepsilon}\eta_i\sqrt{a}\,\mathrm{d}y$$

for all $\boldsymbol{\eta} = (\eta_i) \in \mathbf{V}_M(\omega)$, where the functions $a^{\alpha\beta\sigma\tau}, \gamma_{\alpha\beta}(\boldsymbol{\eta}), a$, and $p^{i,\varepsilon}$ have the same meanings as above. *If* $\gamma_0 = \gamma$ *and* S *is elliptic, the two-dimensional equations of a linearly elastic "membrane shell" are therefore justified.* Observe that the limit $\boldsymbol{\zeta}$ is again *independent of* ε, since both sides of these variational equations are of the same order (viz., ε), because of the assumptions made on the applied forces.

Equivalently, the field $\boldsymbol{\zeta}$ satisfies the following *unconstrained minimization problem*:

$$\boldsymbol{\zeta} \in \mathbf{V}_M(\omega) \text{ and } j_M^\varepsilon(\boldsymbol{\zeta}) = \inf_{\boldsymbol{\eta}\in\mathbf{V}_M(\omega)} j_M^\varepsilon(\boldsymbol{\eta}),$$

where

$$j_M^\varepsilon(\boldsymbol{\eta}) := \frac{1}{2}\int_\omega \varepsilon a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\eta})\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y - \int_\omega p^{i,\varepsilon}\eta_i\sqrt{a}\,\mathrm{d}y.$$

Finally, Ciarlet & Lods [1996d] studied *all the "remaining"* cases where $\mathbf{V}_F(\omega) = \{\mathbf{0}\}$, e.g., when S is elliptic but *length* $\gamma_0 <$ *length* γ, or when S is for instance a portion of a hyperboloid of revolution, etc. To give a flavor of their results, consider the *important special case where the semi-norm*

$$|\cdot|_\omega^M : \boldsymbol{\eta} = (\eta_i) \to |\boldsymbol{\eta}|_\omega^M = \Big\{ \sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\boldsymbol{\eta})\|_{0,\omega}^2 \Big\}^{1/2}$$

becomes a norm over the space

$$\mathbf{W}(\omega) := \{\boldsymbol{\eta} \in \mathbf{H}^1(\omega);\ \boldsymbol{\eta} = \mathbf{0} \text{ on } \gamma_0\}.$$

In this case, Ciarlet & Lods [1996d] showed that, *if the applied body forces are "admissible" in a specific sense* (but a bit too technical to be described here), *and if their density is again $O(1)$ with respect to ε, then*

$$\frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \boldsymbol{u}^\varepsilon \, \mathrm{d}x_3^\varepsilon \longrightarrow \boldsymbol{\zeta} \text{ in } \mathbf{V}_M^\sharp(\omega) \text{ as } \varepsilon \to 0,$$

where

$$\mathbf{V}_M^\sharp(\omega) := \text{ completion of } \mathbf{W}(\omega) \text{ with respect to } |\cdot|_\omega^M .$$

Furthermore, the limit field $\boldsymbol{\zeta} \in \mathbf{V}_M^\sharp(\omega)$ solves "limit" variational equations of the form

$$\varepsilon B_M^\sharp(\boldsymbol{\zeta}^\varepsilon, \boldsymbol{\eta}) = L_M^{\sharp,\varepsilon}(\boldsymbol{\eta}) \text{ for all } \boldsymbol{\eta} \in \mathbf{V}_M^\sharp(\omega),$$

where $B_M^\sharp$ is the unique extension to $\mathbf{V}_M^\sharp(\omega)$ of the bilinear form B_M defined by

$$B_M(\boldsymbol{\zeta}, \boldsymbol{\eta}) := \frac{1}{2} \int_\omega a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\boldsymbol{\zeta}) \gamma_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a} \, \mathrm{d}y \text{ for all } \boldsymbol{\zeta}, \boldsymbol{\eta} \in \mathbf{W}(\omega),$$

i.e., εB_M is the bilinear form found above for a linearly elastic "membrane shell", and $L_M^{\sharp,\varepsilon} : \mathbf{V}_M^\sharp(\omega) \to \mathbf{R}$ is an *ad hoc* linear form, determined by the behavior as $\varepsilon \to 0$ of the admissible body forces.

In the "last" remaining case, where $\mathbf{V}_F(\omega) = \{\mathbf{0}\}$ but $|\cdot|_\omega^M$ is not a norm over the space $\mathbf{W}(\omega)$, a similar convergence result can be established, but only in the completion $\dot{\mathbf{V}}_M^\sharp(\omega)$ with respect of $|\cdot|_\omega^M$ of the *quotient* space $\mathbf{W}(\omega)/\mathbf{W}_0(\omega)$, where $\mathbf{W}_0(\omega) = \{\boldsymbol{\eta} \in \mathbf{W}(\omega);\ \gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega\}$.

Either one of the above variational problems corresponding to the "remaining" cases where $\mathbf{V}_F = \{\mathbf{0}\}$ constitute the **equations of a linearly elastic "generalized" membrane shell,** *whose two-dimensional equations are therefore justified.*

The proofs of the above convergence results are long and technically difficult. Suffice it to say here that they crucially hinge on the *Korn inequality "with boundary conditions"* (Theorem 4.3-4) and on the *Korn inequality "on an elliptic surface"* mentioned at the end of Section 4.3.

Combining these convergences with earlier results of Destuynder [1985] and Sanchez-Palencia [1989a,b, 1992] (see also Sanchez-Hubert & Sanchez-Palencia [1997]), Ciarlet & Lods [1996b,c] have also justified as follows the *linear Koiter shell equations* studied in Sections 4.2 to 4.4, *again in all possible cases.*

Let $\boldsymbol{\zeta}_K^\varepsilon$ denote for each $\varepsilon > 0$ the unique solution (Theorem 4.4-2) to the *linear Koiter shell equations*, viz., the vector field that satisfies

$$\boldsymbol{\zeta}_K^\varepsilon \in \mathbf{V}(\omega) = \{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega);\, \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0\},$$

$$\int_\omega \Big\{\varepsilon a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\boldsymbol{\zeta}_K^\varepsilon)\gamma_{\alpha\beta}(\boldsymbol{\eta}) + \frac{\varepsilon^3}{3} a^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}(\boldsymbol{\zeta}_K^\varepsilon)\rho_{\alpha\beta}(\boldsymbol{\eta})\Big\}\sqrt{a}\,\mathrm{d}y = \int_\omega p^{i,\varepsilon}\eta_i\sqrt{a}\,\mathrm{d}y \text{ for all } \boldsymbol{\eta} = (\eta_i) \in \mathbf{V}(\omega),$$

or equivalently, the unique solution to the *minimization problem*

$$\boldsymbol{\zeta}_K^\varepsilon \in \mathbf{V}(\omega) \text{ and } j(\boldsymbol{\zeta}_K^\varepsilon) = \inf_{\boldsymbol{\eta}\in\mathbf{V}(\omega)} j(\boldsymbol{\eta})$$

where

$$j(\boldsymbol{\eta}) = \frac{1}{2}\int_\omega \Big\{\varepsilon a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\eta})\gamma_{\alpha\beta}(\boldsymbol{\eta}) + \frac{\varepsilon^3}{3}a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{\eta})\rho_{\alpha\beta}(\boldsymbol{\eta})\Big\}\sqrt{a}\,\mathrm{d}y - \int_\omega p^i\eta_i\sqrt{a}\,\mathrm{d}y.$$

Observe in passing that, *for a linearly elastic shell, the stored energy function found in Koiter's energy, viz.,*

$$\boldsymbol{\eta} \longrightarrow \Big\{\frac{\varepsilon}{2}a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\eta})\gamma_{\alpha\beta}(\boldsymbol{\eta}) + \frac{\varepsilon^3}{6}a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{\eta})\rho_{\alpha\beta}(\boldsymbol{\eta})\Big\}$$

is thus exactly the sum of the stored energy function of a linearly elastic "membrane shell" and of that of a linearly elastic "flexural shell" (a similar, *albeit* less satisfactory, observation holds for a nonlinearly elastic shell, cf. Section 4.1).

Then, *for each category of linearly elastic shells* (membrane, generalized membrane, or flexural), *the vector fields* $\boldsymbol{\zeta}_K^\varepsilon$ *and* $\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} \boldsymbol{u}^\varepsilon\,\mathrm{d}x_3^\varepsilon$, where $\boldsymbol{u}^\varepsilon = (u_i^\varepsilon)$ denotes the solution of the three-dimensional problem, *have exactly the same asymptotic behavior as* $\varepsilon \to 0$*, in precisely the same function spaces that were found in the asymptotic analysis of the three-dimensional solution.*

It is all the more remarkable that *Koiter's equations can be fully justified for all types of shells*, since it is clear that Koiter's equations *cannot* be recovered as the outcome of an asymptotic analysis of the three-dimensional equations, the two-dimensional equations of linearly elastic, membrane, generalized membrane, or flexural, shells exhausting all such possible outcomes!

So, *even though Koiter's linear model is not a limit model, it is in a sense the "best" two-dimensional one for linearly elastic shells*!

One can thus only marvel at the insight that led W.T. Koiter to conceive the “right” equations, whose versatility is indeed remarkable, out of purely mechanical and geometrical intuitions!

We refer to Ciarlet [2000a] for a detailed analysis of the asymptotic analysis of linearly elastic shells, for a detailed description and analysis of other linear shell models, such as those of Naghdi, Budiansky and Sanders, Novozilov, etc., and for an extensive list of references.

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