An Introduction to Frames

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Foundations and Trends® in Signal Processing

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PO Box 1024
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www.nowpublishers.com
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Outside North America:
now Publishers Inc.
PO Box 179
2600 AD Delft
The Netherlands
Tel. +31-6-51115274

The preferred citation for this publication is J. Kovačević and A. Chebira, An Introduction to Frames, Foundations and Trends® in Signal Processing, vol 2, no 1, pp 1–94, 2008

ISBN: 978-1-60198-068-7

Foundations and Trends® in Signal Processing
Volume 2 Issue 1, 2008

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Foundations and Trends® in Signal Processing, 2008, Volume 2, 4 issues. ISSN paper version 1932-8346. ISSN online version 1932-8354. Also available as a combined paper and online subscription.

Foundations and Trends® in
Signal Processing
Vol. 2, No. 1 (2008) 1–94

DOI: 10.1561/2000000006

An Introduction to Frames*

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Abstract

This survey gives an introduction to redundant signal representations called frames. These representations have recently emerged as yet another powerful tool in the signal processing toolbox and have become popular through use in numerous applications. Our aim is to familiarize a general audience with the area, while at the same time giving a snapshot of the current state-of-the-art.

* Based on "Life Beyond Bases: The Advent of Frames (Parts I and II)," by Jelena Kovačević and Amina Chebira, © *IEEE Signal Processing Magazine*, 2007.

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1

Introduction

Redundancy is a common tool in our daily lives. We double- and triple-check that we turned off gas and lights, took our keys, money, etc. (at least those worrywarts among us do). When an important date is coming up, we drive our loved ones crazy by confirming "just once more" they are on top of it.

The same idea of removing doubt is present in signal representations. Given a signal, we represent it in another system, typically a basis, where its characteristics are more readily apparent in the transform coefficients. However, these representations are typically nonredundant, and thus corruption or loss of transform coefficients can be serious. In comes redundancy; we build a safety net into our representation so that we can avoid those disasters. The redundant counterpart of a basis is called a *frame*.[1]

It is generally acknowledged[2] that frames were born in 1952 in the work of Duffin and Schaeffer [78]. Despite being over half a century old, frames gained popularity only in the last decade, due mostly to the work of the three wavelet pioneers — Daubechies et al. [67].

[1] No one seems to know why they are called frames, perhaps because of the bounds in (3.8).
[2] At least in the signal processing and harmonic analysis communities.

Frame-like ideas, that is, building redundancy into a signal expansion, can be found in pyramid coding [33], resilience to noise [18, 19, 60, 64, 65, 93, 98, 133], denoising [53, 77, 88, 110, 177], robust transmission [20, 21, 22, 25, 41, 92, 105, 139, 157, 165], CDMA systems [131, 161, 168, 169], multiantenna code design [100, 104], segmentation [69, 124, 162], classification [48, 124, 162], prediction of epileptic seizures [16, 17], restoration and enhancement [113], motion estimation [128], signal reconstruction [6], coding theory [101, 143], operator theory [2], quantum theory and computing [80, 151, 153], and many others.

While frames are often associated with wavelet frames, frames are more general than that. Wavelet frames possess structure; frames are redundant representations that only need to represent signals in a given space with a certain amount of redundancy. The simplest frame, appropriately named Mercedes-Benz, is introduced in Figure 3.2; just have a peek now, we will go into more details later.

Why and where would one use frames? The answer is simple: anywhere where redundancy is a must. The host of the applications mentioned above and discussed later in the survey illustrate that richly.

Now a word about what you are reading: why an introductory survey? The sources on frames are the beautiful book by Daubechies [64], a recent book by Christensen [51] as well as a number of classic papers [39, 63, 99, 103], among others. Although excellent material, none of the above sources offer an introduction to frames geared primarily to engineers and those who just want an introduction into the area. Thus our emphasis; this is a survey, rather than a comprehensive survey of the state of the field. Although we will touch upon a number of applications and theoretical results, we will do so only for the sake of teaching. We will go slowly, whenever possible using the simplest examples. Generalizations will follow naturally. We will be selective and will necessarily give our personal view of frames. We will be rigorous when necessary; however, we will not insist upon it at all times. As often as possible, we will be living in the finite-dimensional world; it is rich enough to give a flavor of the basic concepts. When we do venture into the infinite-dimensional one, we will do so only using filter banks — structured expansions used in many signal processing applications.

This treatment is largely reproduced from two tutorials published in the *IEEE Signal Processing Magazine* [117, 118]. The aim here is to present the material in one piece, with more detail and ease of referencing.

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