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AN INTRODUCTION TO FUZZY SOFT TOPOLOGICAL SPACES

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Abstract

The aim of this study is to define fuzzy soft topology which will be compatible to the fuzzy soft theory and investigate some of its fundamental properties. Firstly, we recall some basic properties of fuzzy soft sets and then we give the definitions of cartesian product of two fuzzy soft sets and projection mappings. Secondly, we introduce fuzzy soft topology and fuzzy soft continuous mapping. Moreover, we induce a fuzzy soft topology after given the definition of a fuzzy soft base. Also, we obtain an initial fuzzy soft topology and give the definition of product fuzzy soft topology. Finally, we prove that the category of fuzzy soft topological spaces **FSTOP** is a topological category over **SET**.

Keywords: fuzzy soft set, fuzzy soft topology, fuzzy soft base, initial fuzzy soft topology, product fuzzy soft topology.

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1. Introduction

Most of the existing mathematical tools for formal modeling, reasoning and computing are crisp, deterministic, and precise in character. But, in real life situation, the problems in economics, engineering, environment, social science, medical science etc. do not always involve crisp data. For this reason, we cannot successfully use the traditional classical methods because of various types of uncertainties presented in these problems. To exceed these uncertainties, some kinds of theories were given like theory of fuzzy sets [21], intuitionistic fuzzy sets [4], rough sets [16],i.e., which we can use as mathematical tools for dealing with uncertainties. But all these theories have their inherent difficulties as what were pointed out by Molodtsov in [15]. The reason for these difficulties is, possibly, the inadequacy

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of the parametrization tool of the theories. Consequently, Molodtsov [15] initiated the concept of soft set theory as a new mathematical tool for dealing with vagueness and uncertainties which is free from the above difficulties.

Applications of Soft Set Theory in other disciplines and real life problems are now catching momentum. Molodtsov [15] successfully applied the soft set theory into several directions, such as smoothness of functions, game theory, Riemann integration, Perron integration, theory of measurement, and so on. Maji et al. [14] gave first practical application of soft sets in decision making problems. They have also introduced the concept of fuzzy soft set, a more generalized concept, which is a combination of fuzzy set and soft set and also studied some of its properties. Ahmad and Kharal [2, 11] also made further contributions to the properties of fuzzy soft sets and fuzzy soft mappings. Soft set and fuzzy soft set theories have a rich potential for applications in several directions, a few of which have been shown by some authors [15, 18].

The algebraic structure of soft set and fuzzy soft set theories dealing with uncertainties has also been studied by some authors. Aktaş and Çağman [3] have introduced the notion of soft groups. Jun [7] applied soft sets to the theory of BCK/BCI-algebras, and introduced the concept of soft BCK/BCI algebras. Jun and Park [8] and Jun et al. [9, 10] reported the applications of soft sets in ideal theory of BCK/BCI-algebras and d-algebras. Feng et al. [6] defined soft semirings and several related notions to establish a connection between soft sets and semirings. Sun et al. [20] presented the definition of soft modules and construct some basic properties using modules and Molodtsov's definition of soft sets. Aygünoğlu and Aygün [5] introduced the concept of fuzzy soft group and in the meantime, discussed some properties and structural characteristic of fuzzy soft group.

In this study, we consider the topological structure of fuzzy soft set theory. First of all, we give the definition of fuzzy soft topology τ which is a mapping from the parameter set E to $[0,1]^{\widetilde{(X,E)}}$ which satisfies the three certain conditions. With respect to this definition the fuzzy soft topology τ is a fuzzy soft set on the family of fuzzy soft sets $\widetilde{(X,E)}$. Also, since the value of a fuzzy soft set f_A under the mapping τ_e gives the degree of openness of the fuzzy soft set with respect to the parameter $e \in E$, τ_e can be thought as a fuzzy soft topology in the sense of Šostak [19]. In this manner, we introduce fuzzy soft cotopology and give the relations between fuzzy soft topology and fuzzy soft cotopology. Then we define fuzzy soft base. Moreover, we induce a fuzzy soft topology by using a fuzzy soft base on the same set. Also, we obtain an initial fuzzy soft topology and then we give the definition of product fuzzy soft topology. Finally, we show that the category of fuzzy soft topological spaces **FSTOP** is a topological category over **SET** with respect to the forgetful functor.

2. Preliminaries

Throughout this paper, X refers to an initial universe, E is the set of all parameters for X, I^X is the set of all fuzzy sets on X (where, $I = [0,1]$) and for $\lambda \in [0,1]$, $\overline{\lambda}(x) = \lambda$, for all $x \in X$.

2.1. Definition. [2, 13] f_A is called a fuzzy soft set on X, where f is a mapping from E into I^X,i.e., $f_e \triangleq f(e)$ is a fuzzy set on X, for each $e \in A$ and $f_e = \overline{0}$, if $e \notin A$, where $\overline{0}$ is zero function on X. f_e, for each $e \in E$, is called an element of the fuzzy soft set f_A.

$\widetilde{(X, E)}$ denotes the collection of all fuzzy soft sets on X and is called a fuzzy soft universe ([13]).

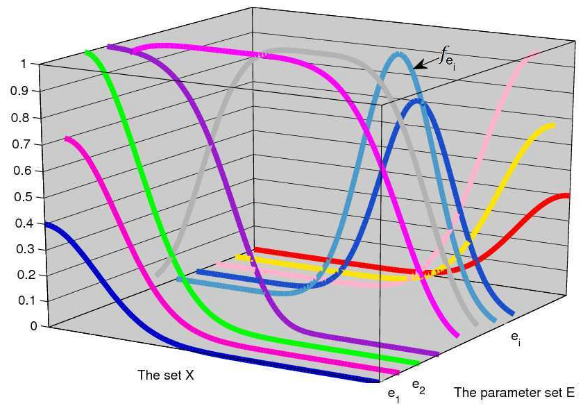

FIGURE 1. A fuzzy soft set f_E

2.2. Definition. [13] For two fuzzy soft sets f_A and g_B on X, we say that f_A is a fuzzy soft subset of g_B and write $f_A \sqsubseteq g_B$ if $f_e \leq g_e$, for each $e \in E$.

2.3. Definition. [13] Two fuzzy soft sets f_A and g_B on X are called equal if $f_A \sqsubseteq g_B$ and $g_B \sqsubseteq f_A$.

2.4. Definition. [13] Union of two fuzzy soft sets f_A and g_B on X is the fuzzy soft set $h_C = f_A \sqcup g_B$, where $C = A \cup B$ and $h_e = f_e \vee g_e$, for each $e \in E$. That is, $h_e = f_e \vee \overline{0} = f_e$ for each $e \in A - B$, $h_e = \overline{0} \vee g_e = g_e$ for each $e \in B - A$ and $h_e = f_e \vee g_e$, for each $e \in A \cap B$.

2.5. Definition. [2, 13] Intersection of two fuzzy soft sets f_A and g_B on X is the fuzzy soft set $h_C = f_A \sqcap g_B$, where $C = A \cap B$ and $h_e = f_e \wedge g_e$, for each $e \in E$.

2.6. Definition. The complement of a fuzzy soft set f_A is denoted by f_A^c, where $f^c : E \longrightarrow I^X$ is a mapping given by $f_e^c = \overline{1} - f_e$, for each $e \in E$.

Clearly $(f_A^c)^c = f_A$.

2.7. Definition. [13] (Null fuzzy soft set) A fuzzy soft set f_E on X is called a null fuzzy soft set and denoted by Φ, if $f_e = \overline{0}$, for each $e \in E$.

2.8. Definition. (Absolute fuzzy soft set) A fuzzy soft set f_E on X is called an absolute fuzzy soft set and denoted by $\widetilde{E}$, if $f_e = \overline{1}$, for each $e \in E$. Clearly $(\widetilde{E})^c = \Phi$ and $\Phi^c = \widetilde{E}$.

2.9. Definition. (λ-absolute fuzzy soft set) A fuzzy soft set f_E on X is called a λ-absolute fuzzy soft set and denoted by $\widetilde{E}^{\lambda}$, if $f_e = \overline{\lambda}$, for each $e \in E$. Clearly, $(\widetilde{E}^{\lambda})^c = \widetilde{E}^{1-\lambda}$.

2.10. Proposition. [2] *Let Δ be an index set and $f_A, g_B, h_C, (f_A)_i \triangleq (f_i)_{A_i}, (g_B)_i \triangleq (g_i)_{B_i} \in \widetilde{(X,E)}$,$\forall i \in \Delta$, then we have the following properties:*

(1) $f_A \sqcap f_A = f_A, \quad f_A \sqcup f_A = f_A$.
(2) $f_A \sqcap g_B = g_B \sqcap f_A, \quad f_A \sqcup g_B = g_B \sqcup f_A$.
(3) $f_A \sqcup (g_B \sqcup h_C) = (f_A \sqcup g_B) \sqcup h_C, \quad f_A \sqcap (g_B \sqcap h_C) = (f_A \sqcap g_B) \sqcap h_C$.
(4) $f_A = f_A \sqcup (f_A \sqcap g_B), \quad f_A = f_A \sqcap (f_A \sqcup g_B)$.
(5) $f_A \sqcap \left(\bigsqcup_{i\in\Delta} (g_B)_i\right) = \bigsqcup_{i\in\Delta} (f_A \sqcap (g_B)_i)$.
(6) $f_A \sqcup \left(\bigsqcap_{i\in\Delta} (g_B)_i\right) = \bigsqcap_{i\in\Delta} (f_A \sqcup (g_B)_i)$.
(7) $\Phi \sqsubseteq f_A \sqsubseteq \widetilde{E}$.
(8) $(f_A^c)^c = f_A$.
(9) $\left(\bigsqcap_{i\in\Delta} (f_A)_i\right)^c = \bigsqcup_{i\in\Delta} (f_A)_i^c$.
(10) $\left(\bigsqcup_{i\in\Delta} (f_A)_i\right)^c = \bigsqcap_{i\in\Delta} (f_A)_i^c$.
(11) If $f_A \sqsubseteq g_B$, then $g_B^c \sqsubseteq f_A^c$.

2.11. Definition. [5, 11] Let $\varphi : X \longrightarrow Y$ and $\psi : E \longrightarrow F$ be two mappings, where E and F are parameter sets for the crisp sets X and Y, respectively. Then the pair φ_ψ is called a fuzzy soft mapping from $\widetilde{(X,E)}$ into $\widetilde{(Y,F)}$ and denoted by $\varphi_\psi : \widetilde{(X,E)} \longrightarrow \widetilde{(Y,F)}$.

2.12. Definition. [5, 11] Let f_A and g_B be two fuzzy soft sets over X and Y, respectively and let φ_ψ be a fuzzy soft mapping from $\widetilde{(X,E)}$ into $\widetilde{(Y,F)}$.

(1) The image of f_A under the fuzzy soft mapping φ_ψ, denoted by $\varphi_\psi(f_A)$, is the fuzzy soft set on Y defined by $\varphi_\psi(f_A) = \varphi(f)_{\psi(A)}$, where

$$\varphi(f)_k(y) = \begin{cases} \bigvee\limits_{x\in\varphi^{-1}(y)} \left(\bigvee\limits_{a\in\psi^{-1}(k)\cap A} f_a(x) \right), & \text{if } \varphi^{-1}(y) \neq \emptyset, \psi^{-1}(k)\cap A \neq \emptyset; \\ 0, & \text{otherwise.} \end{cases},$$

$\forall k \in F$, $\forall y \in Y$.

(2) The pre-image of g_B under the fuzzy soft mapping φ_ψ, denoted by $\varphi_\psi^{-1}(g_B)$, is the fuzzy soft set on X defined by $\varphi_\psi^{-1}(g_B) = \varphi^{-1}(g)_{\psi^{-1}(A)}$, where

$$\varphi^{-1}(g)_a(x) = \begin{cases} g_{\psi(a)}(\varphi(x)), & \text{if } \psi(a) \in B; \\ 0, & \text{otherwise.} \end{cases}, \qquad \forall a \in E,\ \forall x \in X.$$

If φ and ψ is injective (surjective), then φ_ψ is said to be injective (surjective).

2.13. Definition. Let φ_ψ be a fuzzy soft mapping from $\widetilde{(X,E)}$ into $\widetilde{(Y,F)}$ and $\varphi^*_{\psi^*}$ be a fuzzy soft mapping from $\widetilde{(Y,F)}$ into $\widetilde{(Z,K)}$. Then the composition of these mappings from $\widetilde{(X,E)}$ into $\widetilde{(Z,K)}$ is defined as follows: $\varphi^*_{\psi^*} \circ \varphi_\psi \triangleq (\varphi^* \circ \varphi)_{\psi^*\circ\psi}$, where $\psi : E \longrightarrow F$ and $\psi^* : F \longrightarrow K$.

2.14. Proposition. [11] *Let X and Y be two universes $f_A, (f_A)_1, (f_A)_2, (f_A)_i \in \widetilde{(X,E)}$, $g_B, (g_B)_1, (g_B)_2, (g_B)_i \in \widetilde{(Y,F)}$ $\forall i \in \Delta$, where Δ is an index set, and φ_ψ be a fuzzy soft mapping from $\widetilde{(X,E)}$ into $\widetilde{(Y,F)}$.*

(1) *If $(f_A)_1 \sqsubseteq (f_A)_2$, then $\varphi_\psi((f_A)_1) \sqsubseteq \varphi_\psi((f_A)_2)$.*
(2) *If $(g_B)_1 \sqsubseteq (g_B)_2$, then $\varphi_\psi^{-1}((g_B)_1) \sqsubseteq \varphi_\psi^{-1}((g_B)_2)$.*
(3) *$f_A \sqsubseteq \varphi_\psi^{-1}(\varphi_\psi(f_A))$, the equality holds if φ_ψ is injective.*
(4) *$\varphi_\psi\left(\varphi_\psi^{-1}(f_A)\right) \sqsubseteq f_A$, the equality holds if φ_ψ is surjective.*
(5) *$\varphi_\psi\left(\bigsqcup_{i\in\Delta}(f_A)_i\right) = \bigsqcup_{i\in\Delta}\varphi_\psi((f_A)_i)$.*
(6) *$\varphi_\psi\left(\bigsqcap_{i\in\Delta}(f_A)_i\right) \sqsubseteq \bigsqcap_{i\in\Delta}\varphi_\psi((f_A)_i)$, the equality holds if φ_ψ is injective.*
(7) *$\varphi_\psi^{-1}\left(\bigsqcup_{i\in\Delta}(g_B)_i\right) = \bigsqcup_{i\in\Delta}\varphi_\psi^{-1}((g_B)_i)$.*
(8) *$\varphi_\psi^{-1}\left(\bigsqcap_{i\in\Delta}(g_B)_i\right) = \bigsqcap_{i\in\Delta}\varphi_\psi^{-1}((g_B)_i)$.*
(9) *$\varphi_\psi^{-1}(g_B^c) = \left(\varphi_\psi^{-1}(g_B)\right)^c$.*
(10) *$\varphi_\psi^{-1}\left(\widetilde{E}_Y\right) = \widetilde{E}_X, \quad \varphi_\psi^{-1}(\Phi_Y) = \Phi_X$.*
(11) *$\varphi_\psi\left(\widetilde{E}_X\right) = \widetilde{E}_Y$ if φ_ψ is surjective.*
(12) *$\varphi_\psi(\Phi_X) = \Phi_Y$.*

2.15. Definition. (Cartesian product of two fuzzy soft sets) Let X_1 and X_2 be nonempty crisp sets. $f_A \in \widetilde{(X_1,E_1)}$ and $g_B \in \widetilde{(X_2,E_2)}$. The cartesian product $f_A \times g_B$ of f_A and g_B is defined by $(f\times g)_{A\times B}$, where, for each $(e,\mathrm{f}) \in E_1 \times E_2$,
$(f\times g)_{(e,\mathrm{f})}(x,y) = f_e(x) \wedge g_{\mathrm{f}}(y)$, for all $(x,y) \in X \times Y$.

According to this definition the fuzzy soft set $(f\times g)_{A\times B}$ is a fuzzy soft set on $X_1 \times X_2$ and the universal parameter set is $E_1 \times E_2$.

2.16. Definition. Let $(f_A)_1 \times (f_A)_2$ be a fuzzy soft set on $X_1 \times X_2$. The projection mappings $(p_q)_i$, $i \in \{1,2\}$, are defined as follows:

$(p_q)_i((f_A)_1 \times (f_A)_2) = p_i(f_1 \times f_2)_{q_i(A_1\times A_2)} = (f_A)_i$ where $p_i : X_1 \times X_2 \longrightarrow X_i$ and $q_i : E_1 \times E_2 \longrightarrow E_i$ are projection mappings in classical meaning.

3. Fuzzy soft topological spaces

To formulate our program and general ideas more precisely, recall first the concept of fuzzy topological space, that is of a pair (X,τ) where X is a set and $\tau : I^X \longrightarrow I$ is a mapping (satisfying some axioms) which assigns to every fuzzy subset of X the real number, which shows "to what extent" this set is open. According to this idea a fuzzy topology τ is a fuzzy set on I^X. This approach has lead us to define fuzzy soft topology which is compatible to the fuzzy soft theory. By our definition, a fuzzy soft topology is a fuzzy soft set on the set of all fuzzy soft sets $\widetilde{(X,E)}$ which denotes "to what extent" this set is open according to the parameter set.

3.1. Definition. A mapping $\tau : E \longrightarrow [0,1]^{\widetilde{(X,E)}}$ is called a fuzzy soft topology on X if it satisfies the following conditions for each $e \in E$.

(O1) $\tau_e(\Phi) = \tau_e(\widetilde{E}) = 1$.
(O2) $\tau_e(f_A \sqcap g_B) \geq \tau_e(f_A) \wedge \tau_e(g_B), \quad \forall f_A, g_B \in \widetilde{(X,E)}$.

(O3) $\tau_e(\bigsqcup_{i\in\Delta}(f_A)_i) \geq \bigwedge_{i\in\Delta}\tau_e((f_A)_i), \forall (f_A)_i \in \widetilde{(X,E)}, i \in \Delta$.

A fuzzy soft topology is called enriched if it provides that

(O1)$'$ $\tau_e(\widetilde{E}^\lambda) = 1$.

Then the pair (X, τ_E) is called a fuzzy soft topological space. The value $\tau_e(f_A)$ is interpreted as the degree of openness of a fuzzy soft set f_A with respect to parameter $e \in E$.

Let τ_E^1 and τ_E^2 be fuzzy soft topologies on X. We say that τ_E^1 is finer than τ_E^2 (τ_E^2 is coarser than τ_E^1), denoted by $\tau_E^2 \sqsubseteq \tau_E^1$, if $\tau_e^2(f_A) \leq \tau_e^1(f_A)$ for each $e \in E, f_A \in \widetilde{(X,E)}$.

Example Let $\mathcal{T}$ be a fuzzy topology on X in Šostak's sense, that is, $\mathcal{T}$ is a mapping from I^X to I. Take $E = I$ and define $\overline{\mathcal{T}} : E \longrightarrow I^X$ as $\overline{\mathcal{T}}(e) \triangleq \{\mu : \mathcal{T}(\mu) \geq e\}$ which is levelwise fuzzy topology of $\mathcal{T}$ in Chang's sense, for each $e \in I$. However, it is well known that each Chang's fuzzy topology can be considered as Šostak fuzzy topology by using fuzzifying method. Hence, $\mathcal{T}(e)$ satisfies (O1), (O2) and (O3).

According to this definition and by using the decomposition theorem of fuzzy sets [12], if we know the resulting fuzzy soft topology, then we can find the first fuzzy topology. Therefore, we can say that a fuzzy topology can be uniquely represented as a fuzzy soft topology.

3.2. Definition. Let (X, τ) and (Y, τ^*) be fuzzy soft topological spaces. A fuzzy soft mapping φ_ψ from $\widetilde{(X,E)}$ into $\widetilde{(Y,F)}$ is called a fuzzy soft continuous map if $\tau_e(\varphi_\psi^{-1}(g_B)) \geq \tau^*_{\psi(e)}(g_B)$ for all $g_B \in \widetilde{(Y,F)}, e \in E$.

The category of fuzzy soft topological spaces and fuzzy soft continuous mappings is denoted by **FSTOP**.

3.3. Proposition. *Let $\{\tau_k\}_{k\in\Gamma}$ be a family of fuzzy soft topologies on X. Then $\tau = \bigwedge_{k\in\Gamma}\tau_k$ is also a fuzzy soft topology on X, where $\tau_e(f_A) = \bigwedge_{k\in\Gamma}(\tau_k)_e(f_A), \forall e \in E, f_A \in \widetilde{(X,E)}$.*

Proof. It is straightforward and therefore is omitted. □

3.4. Definition. A mapping $\eta : E \longrightarrow [0,1]^{\widetilde{(X,E)}}$ is called a fuzzy soft cotopology on X if it satisfies the following conditions for each $e \in E$:

(C1) $\eta_e(\Phi) = \eta_e(\widetilde{E}) = 1$.

(C2) $\eta_e(f_A \sqcup g_B) \geq \eta_e(f_A) \wedge \eta_e(g_B), \quad \forall f_A, g_B \in \widetilde{(X,E)}$.

(C3) $\eta_e(\bigsqcap_{i\in\Delta}(f_A)_i) \geq \bigwedge_{i\in\Delta}\eta_e((f_A)_i), \forall (f_A)_i \in \widetilde{(X,E)}, i \in \Delta$.

The pair (X, η) is called a fuzzy soft cotopological space.

Let τ be a fuzzy soft topology on X, then the mapping $\eta : E \longrightarrow [0,1]^{\widetilde{(X,E)}}$ defined by $\eta_e(f_A) = \tau_e(f_A^c), \forall e \in E$ is a fuzzy soft cotopology on X. Let η be a fuzzy soft cotopology on X, then the mapping $\tau : E \longrightarrow [0,1]^{\widetilde{(X,E)}}$ defined by $\tau_e(f_A) = \eta_e(f_A^c), \forall e \in E$, is a fuzzy soft topology on X.

3.5. Definition. A mapping $\beta : E \longrightarrow [0,1]^{\widetilde{(X,E)}}$ is called a fuzzy soft base on X if it satisfies the following conditions for each $e \in E$:

(B1) $\beta_e(\Phi) = \beta_e(\widetilde{E}) = 1$.

(B2) $\beta_e(f_A \sqcap g_B) \geq \beta_e(f_A) \wedge \beta_e(g_B), \quad \forall f_A, g_B \in \widetilde{(X,E)}$.

3.6. Theorem. *Let β be a fuzzy soft base on X. Define a map $\tau_\beta : E \longrightarrow [0,1]^{\widetilde{(X,E)}}$ as follows:*

$$(\tau_\beta)_e(f_A) = \bigvee \left\{ \bigwedge_{j\in\Lambda} \beta_e((f_A)_j) \mid f_A = \bigsqcup_{j\in\Lambda} (f_A)_j \right\}, \quad \forall e \in E.$$

Then τ_β is the coarsest fuzzy soft topology on X for which $(\tau_\beta)_e(f_A) \geq \beta_e(f_A)$, for all $e \in E, f_A \in \widetilde{(X,E)}$.

Proof. (O1) It is trivial from the definition of τ_β.

(O2) Let $e \in E$. For all families $\{(f_A)_j \mid f_A = \bigsqcup_{j\in\Lambda}(f_A)_j\}$ and $\{(g_B)_k \mid g_B = \bigsqcup_{k\in\Gamma}(g_B)_k\}$, there exists a family $\{(f_A)_j \sqcap (g_B)_k\}$ such that

$$f_A \sqcap g_B = \left(\bigsqcup_{j\in\Lambda}(f_A)_j\right) \sqcap \left(\bigsqcup_{k\in\Gamma}(g_B)_k\right) = \bigsqcup_{j\in\Lambda, k\in\Gamma} ((f_A)_j \sqcap (g_B)_k).$$

It implies the followings:

$$\begin{aligned}
(\tau_\beta)_e(f_A \sqcap g_B) &\geq \bigwedge_{j\in\Lambda,k\in\Gamma} \beta_e((f_A)_j \sqcap (g_B)_k) \\
&\geq \bigwedge_{j\in\Lambda,k\in\Gamma} (\beta_e((f_A)_j) \wedge \beta_e((g_B)_k)) \\
&\geq (\bigwedge_{j\in\Lambda} \beta_e((f_A)_j)) \wedge (\bigwedge_{k\in\Gamma} \beta_e((g_B)_k)) \\
&\geq (\tau_\beta)_e(f_A) \wedge (\tau_\beta)_e(g_B).
\end{aligned}$$

(O3) Let $e \in E$ and $\wp_i$ be the collection of all index sets K_i such that $\{(f_A)_{i_k} \in \widetilde{(X,E)} \mid (f_A)_i = \bigsqcup_{k\in K_i} (f_A)_{i_k}\}$ with $f_A = \bigsqcup_{i\in\Gamma}(f_A)_i = \bigsqcup_{i\in\Gamma}\bigsqcup_{k\in K_i} (f_A)_{i_k}$. For each $i \in \Gamma$ and each $\Psi \in \Pi_{i\in\Gamma}\wp_i$ with $\Psi(i) = K_i$, we have $(\tau_\beta)_e(f_A) \geq \bigwedge_{i\in\Gamma}(\bigwedge_{k\in K_i} \beta_e((f_A)_{i_k}))$.

Put $a_{i,\Psi_i} = \bigwedge_{k\in K_i} (\beta_e((f_A)_{i_k}))$. Then we have the following:

$$\begin{aligned}
(\tau_\beta)_e(f_A) &\geq \bigvee_{\Psi\in\Pi_{i\in\Gamma}\wp_i} \left(\bigwedge_{i\in\Gamma} a_{i,\Psi(i)}\right) \\
&= \bigwedge_{i\in\Gamma} \left(\bigvee_{M_i\in\wp_i} a_{i,M_i}\right) \\
&= \bigwedge_{i\in\Gamma} \left(\bigvee_{M_i\in\wp_i} \left(\bigwedge_{m\in M_i} (\beta_e((f_A)_{i_m}))\right)\right) \\
&= \bigwedge_{i\in\Gamma} (\tau_\beta)_e((f_A)_i).
\end{aligned}$$

Thus, τ_β is a fuzzy soft topology on X. Let $\tau \sqsupseteq \beta$, then for every $e \in E$ and $f_A = \bigsqcup_{j\in\Lambda} (f_A)_j$, we have

$$\tau_e(f_A) \geq \bigwedge_{j\in\Lambda} \tau_e((f_A)_j) \geq \bigwedge_{j\in\Lambda} \beta_e((f_A)_j).$$

If we take supremum over the family $\{(f_A)_j \in \widetilde{(X,E)} \mid f_A = \bigsqcup_{j\in\Lambda} (f_A)_j\}$, then we obtain that $\tau \sqsupseteq \tau_\beta$. □

3.7. Lemma. *Let τ be a fuzzy soft topology on X and β be a fuzzy soft base on Y. Then a fuzzy soft mapping φ_ψ from $\widetilde{(X,E)}$ into $\widetilde{(Y,F)}$ is fuzzy soft continuous if and only if $\tau_e(\varphi_\psi^{-1}(g_B)) \geq \beta_{\psi(e)}(g_B)$, for each $e \in E, g_B \in \widetilde{(Y,F)}$.*

Proof. $(\Rightarrow)$ Let $\varphi_\psi : (X,\tau) \longrightarrow (Y,\tau_\beta)$ be a fuzzy soft continuous mapping and $g_B \in \widetilde{(Y,F)}$. Then,

$$\tau_e(\varphi_\psi^{-1}(g_B)) \geq (\tau_\beta)_{\psi(e)}(g_B) \geq \beta_{\psi(e)}(g_B).$$

$(\Leftarrow)$ Let $\tau_e(\varphi_\psi^{-1}(g_B)) \geq \beta_{\psi(e)}(g_B)$, for each $g_B \in \widetilde{(Y,F)}$. Let $h_C \in \widetilde{(Y,F)}$. For every family of $\{(h_C)_j \in \widetilde{(Y,F)} \mid h_C = \bigsqcup_{j\in\Gamma} (h_C)_j\}$, we have

$$\begin{aligned}
\tau_e(\varphi_\psi^{-1}(h_C)) &= \tau_e\left(\varphi_\psi^{-1}\left(\bigsqcup_{j\in\Gamma}(h_C)_j\right)\right) \\
&= \tau_e\left(\bigsqcup_{j\in\Gamma}\varphi_\psi^{-1}((h_C)_j)\right) \\
&\geq \bigwedge_{j\in\Gamma}\tau_e(\varphi_\psi^{-1}((h_C)_j)) \\
&\geq \bigwedge_{j\in\Gamma}\beta_{\psi(e)}((h_C)_j).
\end{aligned}$$

If we take supremum over the family of $\{(h_C)_j \in \widetilde{(Y,F)} \mid h_C = \bigsqcup_{j\in\Gamma} (h_C)_j\}$, we obtain
$\tau_e(\varphi_\psi^{-1}(h_C)) \geq (\tau_\beta)_{\psi(e)}(h_C)$. □

3.8. Theorem. *Let $\{(X_i, (\tau_i)_{E_i})\}_{i\in\Gamma}$ be a family of fuzzy soft topological spaces, X be a set, E be a parameter set and for each $i \in \Gamma$, $\varphi_i : X \to X_i$ and $\psi_i : E \to E_i$ be maps. Define $\beta : E \to [0,1]^{\widetilde{(X,E)}}$ on X by:*

$$\beta_e(f_A) = \bigvee\left\{\bigwedge_{j=1}^{n}(\tau_{k_j})_{\psi_{k_j}(e)}((f_A)_{k_j}) \mid f_A = \prod_{j=1}^{n}(\varphi_\psi)_{k_j}^{-1}((f_A)_{k_j})\right\},$$

where $\bigvee$ is taken over all finite subsets $K = \{k_1, k_2, ..., k_n\} \subset \Gamma$. Then,

(1) β is a fuzzy soft base on X.

(2) The fuzzy soft topology τ_β generated by β is the coarsest fuzzy soft topology on X for which all $(\varphi_\psi)_i, i \in \Gamma$ are fuzzy soft continuous maps.

(3) A map $\varphi_\psi : (Y, \delta_F) \to (X, (\tau_\beta)_E)$ is fuzzy soft continuous iff for each $i \in \Gamma$, $(\varphi_\psi)_i \circ \varphi_\psi : (Y, \delta_F) \to (X_i, (\tau_i)_{E_i})$ is a fuzzy soft continuous map.

Proof. (1) (B1) Since $f_A = (\varphi_\psi)_i^{-1}(f_A)$ for each $f_A \in \{\Phi, \widetilde{E}\}$, $\beta_e(\Phi) = \beta_e(\widetilde{E}) = 1$, for each $e \in E$.

(B2) For all finite subsets $K = \{k_1, k_2, ..., k_n\}$ and $J = \{j_1, j_2, ..., j_m\}$ of Γ such that $f_A = \prod_{i=1}^{n} (\varphi_\psi)_{k_i}^{-1}((f_A)_{k_i})$ and $g_B = \prod_{i=1}^{m} (\varphi_\psi)_{j_i}^{-1}((g_B)_{j_i})$, we have $f_A \sqcap g_B = \left(\prod_{i=1}^{n} (\varphi_\psi)_{k_i}^{-1}((f_A)_{k_i})\right) \sqcap \left(\prod_{i=1}^{m} (\varphi_\psi)_{j_i}^{-1}((g_B)_{j_i})\right)$.

Furthermore, we have for each $k \in K \cap J$,

$$(\varphi_\psi)_k^{-1}((f_A)_k) \sqcap (\varphi_\psi)_k^{-1}((g_B)_k) = (\varphi_\psi)_k^{-1}((f_A)_k \sqcap (g_B)_k).$$

Put $f_A \sqcap g_B = \prod_{m_i \in K \cup J} (\varphi_\psi)_{m_i}^{-1}((h_C)_{m_i})$ where

$$(h_C)_{m_i} = \begin{cases} (f_A)_{m_i}, & \text{if } m_i \in K - (K \cap J); \\ (g_B)_{m_i}, & \text{if } m_i \in J - (K \cap J); \\ (f_A)_{m_i} \sqcap (g_B)_{m_i}, & \text{if } m_i \in (K \cap J). \end{cases}$$

So we have

$$\begin{aligned} \beta_e((f_A) \sqcap (g_B)) &\geq \bigwedge_{j \in K \cup J} (\tau_j)_{\psi_j(e)}((h_C)_j) \\ &\geq \left(\bigwedge_{i=1}^{n} (\tau_{k_i})_{\psi_{k_i}(e)}((f_A)_{k_i})\right) \wedge \left(\bigwedge_{i=1}^{m} (\tau_{j_i})_{\psi_{j_i}(e)}((g_B)_{j_i})\right). \end{aligned}$$

If we take supremum over the families $\{f_A = \prod_{i=1}^{n} (\varphi_\psi)_{k_i}^{-1}((f_A)_{k_i})\}$ and $\{g_B = \prod_{i=1}^{m} (\varphi_\psi)_{j_i}^{-1}((g_B)_{j_i})\}$, then we have,

$$\beta_e(f_A \sqcap g_B) \geq \beta_e(f_A) \wedge \beta_e(g_B), \forall e \in E.$$

(2) For each $(f_A)_i \in \widetilde{(X_i, E_i)}$, one family $\{(\varphi_\psi)_i^{-1}((f_A)_i)\}$ and $i \in \Gamma$, we have

$(\tau_\beta)_e((\varphi_\psi)_i^{-1}((f_A)_i)) \geq \beta_e((\varphi_\psi)_i^{-1}((f_A)_i)) \geq (\tau_i)_{\psi_i(e)}((f_A)_i)$, for each $e \in E$.

Therefore, for all $i \in \Gamma$, $(\varphi_\psi)_i : (X, (\tau_\beta)_E) \longrightarrow (X_i, (\tau_i)_{E_i})$ is fuzzy soft continuous.

Let $(\varphi_\psi)_i : (X, \zeta_E) \longrightarrow (X_i, (\tau_i)_{E_i})$ be fuzzy soft continuous, that is, for each $i \in \Gamma$ and $(f_A)_i \in \widetilde{(X_i, E_i)}$, $\zeta_e((\varphi_\psi)_i^{-1}((f_A)_i)) \geq (\tau_i)_{\psi_i(e)}((f_A)_i)$.

For all finite subsets $K = \{k_1, ..., k_n\}$ of Γ such that $f_A = \prod_{i=1}^{n} (\varphi_\psi)_{k_i}^{-1}((f_A)_{k_i})$ we have

$$\zeta_e(f_A) \geq \bigwedge_{i=1}^{n} \zeta_e((\varphi_\psi)_{k_i}^{-1}((f_A)_{k_i})) \geq \bigwedge_{i=1}^{n} (\tau_{k_i})_{\psi_{k_i}(e)}((f_A)_{k_i}).$$

It implies $\zeta_e(f_A) \geq \beta_e(f_A)$, for all $e \in E, f_A \in \widetilde{(X,E)}$. By Theorem 3.6, $\zeta \sqsupseteq \tau_\beta$.

(3) ($\Rightarrow$) Let $\varphi_\psi : (Y, \delta_F) \to (X, (\tau_\beta)_E)$ be fuzzy soft continuous. For each $i \in \Gamma$ and $(f_A)_i \in \widetilde{(X_i, E_i)}$ we have

$$\delta_{\mathrm{f}}((\varphi_i \circ \varphi)^{-1}_{\psi_i \circ \psi}((f_A)_i)) = \delta_{\mathrm{f}}(\varphi_\psi^{-1}((\varphi_\psi)_i^{-1}((f_A)_i))) \geq (\tau_\beta)_{\psi(\mathrm{f})}((\varphi_\psi)_i^{-1}((f_A)_i)) \geq (\tau_i)_{(\psi_i \circ \psi)(\mathrm{f})}((f_A)_i).$$

Hence, $(\varphi_i \circ \varphi)_{\psi_i \circ \psi} : (Y, \delta_F) \to (X_i, (\tau_i)_{E_i})$ is fuzzy soft continuous.

($\Leftarrow$) For all finite subsets $K = \{k_1, ..., k_n\}$ of Γ such that $f_A = \prod_{i=1}^{n} (\varphi_\psi)_{k_i}^{-1}((f_A)_{k_i})$, since
$(\varphi_{k_i} \circ \varphi)_{\psi_{k_i} \circ \psi} : (Y, \delta_F) \to (X_{k_i}, (\tau_{k_i})_{E_{k_i}})$ is fuzzy soft continuous, $\delta_{\mathrm{f}}(\varphi_\psi^{-1}((\varphi_\psi)_{k_i}^{-1}((f_A)_{k_i}))) \geq (\tau_{k_i})_{(\psi_i \circ \psi)(\mathrm{f})}((f_A)_{k_i}), \forall\ \mathrm{f} \in F$.

Hence we have

$$\begin{aligned}
\delta_{\mathrm{f}}(\varphi_\psi^{-1}(f_A)) &= \delta_{\mathrm{f}}(\varphi_\psi^{-1}(\prod_{i=1}^{n} (\varphi_\psi)_{k_i}^{-1}((f_A)_{k_i}))) \\
&= \delta_{\mathrm{f}}(\prod_{i=1}^{n} (\varphi_\psi^{-1}((\varphi_\psi)_{k_i}^{-1}(((f_A)_{k_i}))))) \\
&\geq \bigwedge_{i=1}^{n} \delta_{\mathrm{f}}(\varphi_\psi^{-1}((\varphi_\psi)_{k_i}^{-1}(((f_A)_{k_i})))) \\
&\geq \bigwedge_{i=1}^{n} (\tau_{k_i})_{(\psi_{k_i} \circ \psi)(\mathrm{f})}((f_A)_{k_i}).
\end{aligned}$$

This inequality implies that $\delta_{\mathrm{f}}(\varphi_\psi^{-1}(f_A)) \geq \beta_{\psi(\mathrm{f})}(f_A)$ for each $f_A \in \widetilde{(X,E)}, \mathrm{f} \in F$.

By Lemma 3.7, $\varphi_\psi : (Y, \delta_F) \to (X, (\tau_\beta)_E)$ is fuzzy soft continuous.

Let $\{(X_i, (\tau_i)_{E_i})\}_{i \in \Gamma}$ be a family of fuzzy soft topological spaces, X be a set, E be a parameter set and for each $i \in \Gamma$, $\varphi_i : X \to X_i$ and $\psi_i : E \to E_i$ be maps. The initial fuzzy soft topology τ_β on X is the coarsest fuzzy soft topology on X for which all $(\varphi_\psi)_i, i \in \Gamma$ are fuzzy soft continuous maps. □

3.9. Definition. [1] A category **C** is called a topological category over **SET** with respect to the usual forgetful functor from **C** to **SET** if it satisfies the following conditions:

(TC1) *Existence of initial structures:* For any X, any class J, and any family $((X_j, \xi_j))_{j \in J}$ of **C**-object and any family $(f_j : X \longrightarrow X_j)_{j \in J}$ of maps, there exists a unique **C**-structure ξ on X which is initial with respect to the source $(f_j : X \longrightarrow (X_j, \xi_j))_{j \in J}$, this means that for a **C**-object (Y, η), a map $g : (Y, \eta) \longrightarrow (X, \xi)$ is a **C**-morphism if and only if for all $j \in J$, $f_j \circ g : (Y, \eta) \longrightarrow (X_j, \xi_j)$ is a **C**-morphism.

(TC2) *Fibre smallness:* For any set X, the **C**-fibre of X, i.e., the class of all **C**-structure on X, which we denote **C**(X), is a set.

3.10. Theorem. *The category* **FSTOP** *is a topological category over* **SET** *with respect to the forgetful functor* V : **FSTOP**$\longrightarrow$ **SET** *which is defined by* $V(X, \tau_E) = X$ *and* $V(\varphi_\psi) = \varphi$.

Proof. The proof is straightforward and follows from Theorem 3.8. □

3.11. Definition. Let $\{(X_i,(\tau_i)_{E_i})\}_{i\in\Gamma}$ be a family of fuzzy soft topological spaces, for each $i\in\Gamma$, E_i be parameter sets, $X=\Pi_{i\in\Gamma}X_i$ and $E=\Pi_{i\in\Gamma}E_i$. Let $p_i: X\longrightarrow X_i$ and $q_i: E\longrightarrow E_i$ be projection maps, for all $i\in\Gamma$. The product of fuzzy soft topologies (X,τ_E) with respect to parameter set E is the coarsest fuzzy soft topology on X for which all $(p_q)_i, i\in\Gamma$, are fuzzy soft continuous maps.

4. Conclusion

In this paper, we have considered the topological structure of fuzzy soft set theory. We have given the definition of fuzzy soft topology τ which is a mapping from the parameter set E to $[0,1]^{\widetilde{(X,E)}}$ which satisfy the three certain conditions. Since the value of a fuzzy soft set f_A under the mapping τ_e gives us the degree of openness of the fuzzy soft set with respect to the parameter $e\in E$, τ_e can be thought of as a fuzzy soft topology in the sense of Šostak. In this sense, we have introduced fuzzy soft cotopology and given the relations between fuzzy soft topology and fuzzy soft cotopology. Then we have defined fuzzy soft base and by using a fuzzy soft base we have obtained a fuzzy soft topology on the same set. Also, we have introduced an initial fuzzy soft topology and then we have given the definition of product fuzzy soft topology. Further, we have proved that the category of fuzzy soft topological spaces **FSTOP** is a topological category over **SET** with respect to the forgetful functor.

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