AN INTRODUCTION TO O-MINIMAL GEOMETRY

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November 1999

Preface

These notes have served as a basis for a course in Pisa in Spring 1999. A parallel course on the construction of o-minimal structures was given by A. Macintyre.

The content of these notes owes a great deal to the excellent book by L. van den Dries [vD]. Some interesting topics contained in this book are not included here, such as the Vapnik-Chervonenkis property.

Part of the material which does not come from [vD] is taken from the paper [Co1]. This includes the sections on the choice of good coordinates and the triangulation of functions in Chapter 4 and Chapter 5. The latter chapter contains the results on triviality in families of sets or functions which were the main aim of this course.

The last chapter on smoothness was intended to establish property "D$\mathcal{C}^k$ - all k" which played a crucial role in the course of Macintyre (it can be easily deduced from the results in [vDMi]). It is also the occasion to give a few results on tubular neighborhoods.

I am pleased to thank Francesca Acquistapace, Fabrizio Broglia and all colleagues of the Dipartimento di Matematica for the invitation to give this course in Pisa and their friendly hospitality. Also many thanks to Antonio Diaz-Cano, Pietro Di Martino, Jesus Escribano and Federico Ponchio for reading these notes and correcting mistakes.

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Chapter 1

O-minimal Structures

1.1 Introduction

The main feature of o-minimal structures is that there are no "monsters" in such structures. Let us take an example of the pathological behaviour that is ruled out. Let $\Gamma \subset \mathbb{R}^2$ be the graph of the function $x \mapsto \sin(1/x)$ for $x > 0$, and let $\operatorname{clos}(\Gamma)$ be the closure of Γ in $\mathbb{R}^2$. This set $\operatorname{clos}(\Gamma)$ is the union of Γ and the closed segment joining the two points $(0, -1)$ and $(0, 1)$. We have

$$\dim(\operatorname{clos}(\Gamma) \setminus \Gamma) = \dim \Gamma = 1.$$

Observe also that $\operatorname{clos}(\Gamma)$ is connected, but not arcwise connected: there is no continuous path inside $\operatorname{clos}(\Gamma)$ joining the origin with a point of Γ. The o-minimal structures will allow to develop a "tame topology" in which such bad things cannot happen. The model for o-minimal structures is the class of semialgebraic sets.

A *semialgebraic subset* of $\mathbb{R}^n$ is a subset defined by a boolean combination of polynomial equations and inequalities with real coefficients. In other words, the semialgebraic subsets of $\mathbb{R}^n$ form the smallest class $\mathcal{SA}_n$ of subsets of $\mathbb{R}^n$ such that:

1. If $P \in \mathbb{R}[X_1, \dots, X_n]$, then $\{x \in \mathbb{R}^n ;\ P(x) = 0\} \in \mathcal{SA}_n$ and $\{x \in \mathbb{R}^n ;\ P(x) > 0\} \in \mathcal{SA}_n$.
2. If $A \in \mathcal{SA}_n$ and $B \in \mathcal{SA}_n$, then $A \cup B$, $A \cap B$ and $\mathbb{R}^n \setminus A$ are in $\mathcal{SA}_n$.

On the one hand the class of semialgebraic sets is stable under many constructions (such as taking projections, closure, connected components…), and on the other hand the topology of semialgebraic sets is very simple, without pathological behaviour. The o-minimal structures may be seen as an axiomatic treatment of semialgebraic geometry. An o-minimal structure (expanding the field of reals) is the data, for every positive integer n, of a subset $\mathcal{S}_n$ of the powerset of $\mathbb{R}^n$, satisfying certain axioms. There are axioms which allow to perform many constructions inside the structure, and an "o-minimality axiom" which guarantee the tameness of the topology.

One can distinguish two kinds of activities in the study of o-minimal structures. The first one is to develop the geometry of o-minimal structures from the axioms. In this activity one tries to follow the semialgebraic model. The second activity is to discover new interesting classes statisfying the axioms of o-minimal structures. This activity is more innovative, and the progress made in this direction (starting with the proof by Wilkie that the field of reals with the exponential function defines an o-minimal structure) justifies the study of o-minimal structures. The subject of this course is the first activity (geometry of o-minimal structures), while the course of A. Macintyre is concerned with the construction of o-minimal structures.

1.2 Semialgebraic Sets

The semialgebraic subsets of $\mathbb{R}^n$ were defined above. We denote by $\mathcal{SA}_n$ the set of all semialgebraic subsets of $\mathbb{R}^n$. Some stability properties of the class of semialgebraic sets follow immediately from the definition.

1. All algebraic subsets of $\mathbb{R}^n$ are in $\mathcal{SA}_n$. Recall that an algebraic subset is a subset defined by a finite number of polynomial equations
$$P_1(x_1, \ldots, x_n) = \ldots = P_k(x_1, \ldots, x_n) = 0 \ .$$
2. $\mathcal{SA}_n$ is stable under the boolean operations, i.e. finite unions and intersections and taking complement. In other words, $\mathcal{SA}_n$ is a Boolean subalgebra of the powerset $\mathcal{P}(\mathbb{R}^n)$.
3. The cartesian product of semialgebraic sets is semialgebraic. If $A \in \mathcal{SA}_n$ and $B \in \mathcal{SA}_p$, then $A \times B \in \mathcal{SA}_{n+p}$.

Sets are not sufficient, we need also maps. Let $A \subset \mathbb{R}^n$ be a semialgebraic set. A map $f : A \to \mathbb{R}^p$ is said to be semialgebraic if its graph $\Gamma(f) \subset \mathbb{R}^n \times \mathbb{R}^p = \mathbb{R}^{n+p}$ is semialgebraic. For instance, the polynomial maps are semialgebraic. The function $x \mapsto \sqrt{1-x^2}$ for $|x| \leq 1$ is semialgebraic.

The most important stability property of semialgebraic sets is known as "Tarski-Seidenberg theorem". This central result in semialgebraic geometry is not obvious from the definition.

4. Denote by $p : \mathbb{R}^{n+1} \to \mathbb{R}^n$ the projection on the first n coordinates. Let A be a semialgebraic subset of $\mathbb{R}^{n+1}$. Then the projection $p(A)$ is semialgebraic.

The Tarski-Seidenberg theorem has many consequences. For instance, it implies that the composition of two semialgebraic maps is semialgebraic. We shall say more on the consequences of the stability under projection in the context of o-minimal structures.

The semialgebraic subsets of the line are very simple to describe. Indeed, a semialgebraic subset A of $\mathbb{R}$ is described by a boolean combination of sign conditions (< 0, $= 0$ or > 0) on polynomials in one variable. Consider the finitely many roots of all polynomials appearing in the definition of A. The signs of the polynomials are constant on the intervals delimited by these roots. Hence, such an interval is either disjoint from or contained in A. We obtain the following description.

5. The elements of $\mathcal{SA}_1$ are the finite unions of points and open intervals.

We cannot hope for such a simple description of semialgebraic subsets of $\mathbb{R}^n$, $n > 1$. However, we have that every semialgebraic set has a finite partition into semialgebraic subsets homeomorphic to open boxes (i.e. cartesian product of open intervals). This is a consequence of the so-called "cylindrical algebraic decomposition" (cad), which is the main tool in the study of semialgebraic sets. Actually, the Tarki-Seidenberg theorem can be proved by using cad. A cad of $\mathbb{R}^n$ is a partition of $\mathbb{R}^n$ into finitely many semialgebraic subsets (which are called the cells of the cad), satisfying certain properties. We define the cad of $\mathbb{R}^n$ by induction on n.

- A cad of $\mathbb{R}$ is a subdivision by finitely many points $a_1 < \ldots < a_\ell$. The cells are the singletons $\{a_i\}$ and the open intervals delimited by these points.
- For $n > 1$, a cad of $\mathbb{R}^n$ is given by a cad of $\mathbb{R}^{n-1}$ and, for each cell C of $\mathbb{R}^{n-1}$, continuous semialgebraic functions

$$\zeta_{C,1} < \ldots < \zeta_{C,\ell_C} : C \to \mathbb{R} \ .$$

The cells of $\mathbb{R}^n$ are the graphs of the functions $\zeta_{C,i}$ and the bands in the cylinder $C \times \mathbb{R} \subset \mathbb{R}^n$ delimited by these graphs. For $i = 0, \ldots, \ell_C$, the band $(\zeta_{C,i}, \zeta_{C,i+1})$ is

$$(\zeta_{C,i}, \zeta_{C,i+1}) = \{(x', x_n) \in \mathbb{R}^n \ ; \ x' \in C \ \text{ and } \ \zeta_{C,i}(x') < x_n < \zeta_{C,i+1}(x')\} \ ,$$

where we set $\zeta_{C,0} = -\infty$ and $\zeta_{C,\ell_C+1} = +\infty$.

Observe that every cell of a cad is homeomorphic to an open box. This is proved by induction on n, since a graph of $\zeta_{C,i}$ is homeomorphic to C and a band $(\zeta_{C,i}, \zeta_{C,i+1})$ is homeomorphic to $C \times (0,1)$.

Given a finite list $(P_1, \ldots, P_k)$ of polynomials in $\mathbb{R}[X_1, \ldots, X_n]$, a subset A of $\mathbb{R}^n$ is said to be $(P_1, \ldots, P_k)$-invariant if the sign (< 0, $= 0$ or > 0) of each P_i is constant on A. The main result concerning cad is the following.

Theorem 1.1 *Given a finite list $(P_1, \ldots, P_k)$ of polynomials in $\mathbb{R}[X_1, \ldots, X_n]$, there is a cad of $\mathbb{R}^n$ such that each cell is $(P_1, \ldots, P_k)$-invariant.*

It follows from Theorem 1.1 that, for every semialgebraic subset A of $\mathbb{R}^n$, there is a cad such that A is a union of cells (such a cad is called adapted to A). Indeed, A is defined by a boolean combination of sign conditions on a finite list of polynomials $(P_1, \ldots, P_k)$, and it suffices to take a cad such that each cell is $(P_1, \ldots, P_k)$-invariant.

We illustrate Theorem 1.1 with the example of a cad of $\mathbb{R}^3$ such that each cell is $(X_1^2 + X_2^2 + X_3^2 - 1)$-invariant (a cad adapted to the unit sphere). Such a cad is shown on Figure 1.1.

Exercise 1.2 How many cells of $\mathbb{R}^3$ are there in this cad? Is it possible to have a cad of $\mathbb{R}^3$, adapted to the unit sphere, with less cells?

We refer to [Co2] for an introduction to semialgebraic geometry.

Semialgebraic geometry can also be developed over an arbitrary real closed field, instead of the field of reals. A real closed field R is an ordered field satisfying one of the equivalent conditions:

- Every positive element is a square and every polynomial in $R[X]$ with odd degree has a root in R.

- For every polynomial $F \in R[X]$ and all a, b in R such that $a < b$ and $F(a)F(b) < 0$, there exists $c \in R$, $a < c < b$, such that $F(c) = 0$.

- $R[\sqrt{-1}] = R[X]/(X^2 + 1)$ is an algebraically closed field.

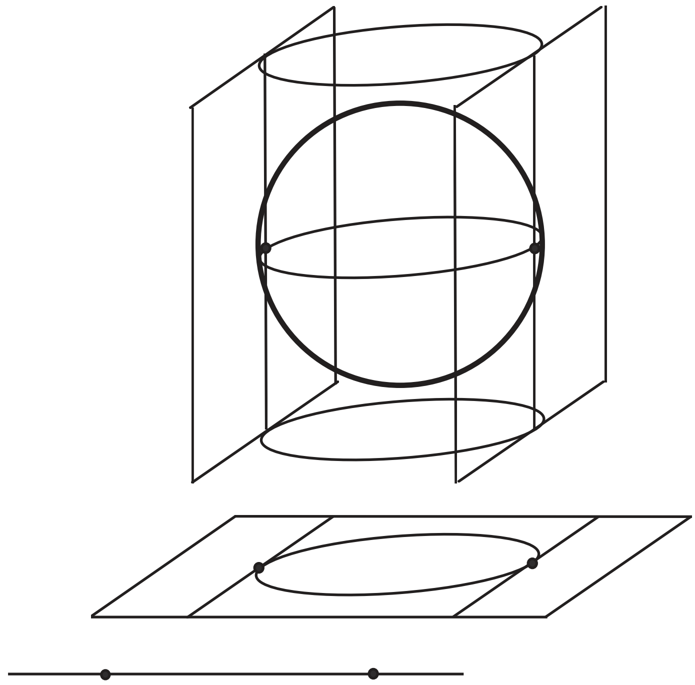


Figure 1.1: A cad adapted to the sphere

A semialgebraic subset of R^n is defined as for $\mathbb{R}^n$. The properties of semialgebraic sets that we have recalled in this section also hold for semialgebraic subsets of R^n. We refer to [BCR] for the study of semialgebraic geometry over an arbitrary real closed field.

1.3 Definition of an O-minimal Structure

We shall now define the o-minimal structures expanding a real closed fied R. The fact that we consider a situation more general than the field of reals will be important only in Chapter 5. Otherwise, the reader may take $R = \mathbb{R}$.

An *interval in* R will always be an open interval (a, b) (for $a < b$) or $(a, +\infty)$ or $(-\infty, b)$. We insist that an interval always has endpoints in $R \cup \{-\infty, +\infty\}$. For instance, if R is the field of real algebraic numbers (which is the smallest real closed field), the set of x in R such that $0 < x < \pi$ ($\pi = 3.14\ldots$) is not an interval, because there is no right endpoint in R. Occasionally, we shall also use the notation $[a, b]$ for closed segments in R.

The field R has a topology for which the intervals form a basis. Affine spaces R^n are endowed with the product topology. The *open boxes*, i.e. the cartesian products of open intervals $(a_1, b_1) \times \cdots \times (a_n, b_n)$ form a basis for this topology. The polynomials are continuous for this topology.

Exercise 1.3 Prove that the polynomials are continuous.

When dealing with the topology, one should take into account that a real closed field is generally not locally connected nor locally compact (think of real algebraic numbers).

Definition 1.4 *A* structure expanding the real closed field R *is a collection* $\mathcal{S} = (\mathcal{S}^n)_{n\in\mathbb{N}}$*, where each* $\mathcal{S}^n$ *is a set of subsets of the affine space* R^n*, satisfying the following axioms:*

1. *All algebraic subsets of* R^n *are in* S_n*.*
2. *For every* n*,* S_n *is a Boolean subalgebra of the powerset of* R^n*.*
3. *If* $A \in S_m$ *and* $B \in S_n$*, then* $A \times B \in S_{m+n}$*.*
4. *If* $p : R^{n+1} \to R^n$ *is the projection on the first* n *coordinates and* $A \in S_{n+1}$*, then* $p(A) \in S_n$*.*

The elements of S_n *are called the* definable subsets of R^n*. The structure* $\mathcal{S}$ *is said to be* o-minimal *if, moreover, it satisfies:*

5. *The elements of* S_1 *are precisely the finite unions of points and intervals.*

In the following, we shall always work in a o-minimal structure expanding a real closed field R*.*

Definition 1.5 *A map* $f : A \to R^p$ *(where* $A \subset R^n$*) is called* definable *if its graph is a definable subset of* $R^n \times R^p$*. (Applying* p *times property 4, we deduce that* A *is definable).*

Proposition 1.6 *The image of a definable set by a definable map is definable.*

Proof. Let $f : A \to R^p$ be definable, where $A \subset R^n$, and let B be a definable subset of A. Denote by $\Gamma_f = \{(x, f(x))\ ;\ x \in A\} \subset R^{n+p}$ the graph of f. Let Δ be the algebraic (in fact, linear) subset of R^{p+n+p} consisting of those $(z, x, y) \in R^p \times R^n \times R^p$ such that $z = y$. Then $C = \Delta \cap (R^p \times \Gamma) \cap (R^p \times B \times R^p)$ is a definable subset of R^{p+n+p}. Let $p_{p+n+p,p} : R^{p+n+p} \to R^p$ be the projection on the first p coordinates. We have $p_{p+n+p,p}(C) = f(B)$, and, applying $n + p$ times property 4, we deduce that $f(B)$ is definable. $\square$

Observe that every polynomial map is a definable map, since its graph is an algebraic set.

Exercise 1.7 Every semialgebraic subset of R^n is definable (Hint: the set defined by $P(x_1, \ldots, x_n) > 0$ is the projection of the algebraic set with equation $x_{n+1}^2 P(x_1, \ldots, x_n) - 1 = 0$). Hence, the collection of $\mathcal{SA}_n$ is the smallest o-minimal structure expanding R.

Exercise 1.8 Show that every nonempty definable subset of R has a least upper bound in $R \cup \{+\infty\}$.

Exercise 1.9 Assume that $\mathcal{S}$ is an o-minimal structure expanding an ordered field R (same definition as above). Show that R is real closed. (Hint: one can use the second equivalent condition for a field to be real closed, the continuity of polynomials and property 5 of o-minimal structures.)

Exercise 1.10 Let $f = (f_1, \ldots, f_p)$ be a map from $A \subset R^n$ into R^p. Show that f is definable if and only if each of its coordinate functions $f_1, \ldots, f_p$ is definable

Exercise 1.11 Show that the composition of two definable maps is definable. Show that the definable functions $A \to R$ form an R-algebra.

Proposition 1.12 *The closure and the interior of a definable subset of R^n are definable.*

Proof. It is sufficient to prove the assertion concerning the closure. The case of the interior follows by taking complement. Let A be a definable subset of R^n. The closure of A is

$$\operatorname{clos}(A) = \left\{ x \in R^n \ ; \ \forall \varepsilon \in R, \ \varepsilon > 0 \Rightarrow \exists y \in R^n, \ y \in A \ \text{ and } \ \sum_{i=1}^{n} (x_i - y_i)^2 < \varepsilon^2 \right\} .$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. The closure of A can also be described as

$$\operatorname{clos}(A) = R^n \setminus \left(p_{n+1,n} \left(R^{n+1} \setminus p_{2n+1,n+1}(B) \right) \right) ,$$

where

$$B = (R^n \times R \times A) \cap \left\{ (x, \varepsilon, y) \in R^n \times R \times R^n \ ; \ \sum_{i=1}^{n} (x_i - y_i)^2 < \varepsilon^2 \right\} ,$$

$p_{n+1,n}(x,\varepsilon) = x$ and $p_{2n+1,n+1}(x,\varepsilon,y) = (x,\varepsilon)$. Then observe that B is definable. □

The example above shows that it is usually boring to write down projections in order to show that a subset is definable. We are more used to write down formulas. Let us make precise what is meant by a *first-order formula* (of the language of the o-minimal structure). A first-order formula is constructed according to the following rules.

1. If $P \in R[X_1,\dots,X_n]$, then $P(x_1,\dots,x_n) = 0$ and $P(x_1,\dots,x_n) > 0$ are first-order formulas.
2. If A is a definable subset of R^n, then $x \in A$ (where $x = (x_1,\dots,x_n)$) is a first-order formula.
3. If $\Phi(x_1,\dots,x_n)$ and $\Psi(x_1,\dots,x_n)$ are first-order formulas, then "Φ and Ψ", "Φ or Ψ", "not Φ" , $\Phi \Rightarrow \Psi$ are first order formulas.
4. If $\Phi(y,x)$ is a first-order formula (where $y = (y_1,\dots,y_p)$ and $x = (x_1,\dots,x_p)$) and A is a definable subset of R^n, then $\exists x \in A\ \Phi(y,x)$ and $\forall x \in A\ \Phi(y,x)$ are first-order formulas.

Theorem 1.13 *If $\Phi(x_1,\dots,x_n)$ is a first-order formula, the set of $(x_1,\dots,x_n)$ in R^n which satisfy $\Phi(x_1,\dots,x_n)$ is definable.*

Proof. By induction on the construction of formulas. Rule 1 produces semi-algebraic sets, which are definable. Rule 2 obviously produces definable sets. Rule 3 works because $\mathcal{SA}_n$ is closed under boolean operation. Rule 4 reflects the fact that the projection of a definable set is definable. Indeed, if $B = \{(y,x) \in R^{p+n}\ ;\ \Phi(y,x)\}$ is definable and $p_{p+n,p} : R^{p+n} \to R^p$ denotes the projection on the first p coordinates, we have

$$\begin{aligned}
\{y \in R^p\ ;\ \exists x \in A\ \Phi(y,x)\} &= p_{p+n,p}((R^p \times A) \cap B)\ , \\
\{y \in R^p\ ;\ \forall x \in A\ \Phi(y,x)\} &= R^p \setminus p_{p+n,p}((R^p \times A) \cap (R^{p+n} \setminus B))\ ,
\end{aligned}$$

which shows that both sets are definable. □

One should pay attention to the fact that the quantified variables have to range over definable sets. For instance,

$$\{(x,y) \in \mathbb{R}^2\ ;\ \exists n \in \mathbb{N}\ y = nx\}$$

is not definable (why ?).

We refer the reader to [Pr] for notions of model theory.

Exercise 1.14 Let $f : A \to R$ be a definable function. Show that $\{x \in A\ ;\ f(x) > 0\}$ is definable. Hence, we can accept inequalities involving definable functions in formulas defining definable sets.

Exercise 1.15 Let A be a non empty definable subset of R^n. For $x \in R^n$, define $\mathrm{dist}(x, A)$ as the greatest lower bound of the set of $\|y - x\| = \sqrt{\sum_{i=1}^n (y_i - x_i)^2}$ for $y \in A$. Show that $\mathrm{dist}(x, A)$ is well-defined and that $x \mapsto \mathrm{dist}(x, A)$ is a continuous definable function on R^n.

Exercise 1.16 Let $f : A \to R$ be a definable function and $a \in \mathrm{clos}(A)$. For $\varepsilon > 0$, define

$$m(\varepsilon) = \inf\{f(x)\ ;\ x \in A \ \text{ and } \ \|x - a\| < \varepsilon\} \in R \cup \{-\infty\}\ .$$

Show that m is a definable function (this means that $m^{-1}(-\infty)$ is definable and $m|_{m^{-1}(R)}$ is definable). Show that $\liminf_{x \to a} f(x)$ exists in $R \cup \{-\infty, +\infty\}$.

Chapter 2

Cell Decomposition

In this chapter we prove the cell decomposition for definable sets, which generalizes the cylindrical algebraic decomposition of semialgebraic sets. This result is the most important for the study of o-minimal geometry. The proofs are rather technical. The main results come from [PS, KPS], and we follow [vD] rather closely.

2.1 Monotonicity Theorem

Theorem 2.1 (Monotonicity Theorem) *Let $f : (a, b) \to R$ be a definable function. There exists a finite subdivision $a = a_0 < a_1 < \ldots < a_k = b$ such that, on each interval (a_i, a_{i+1}), f is continuous and either constant or strictly monotone.*

The key of the proof of the Monotonicity Theorem is the following Lemma.

Lemma 2.2 *Let $f : (a, b) \to R$ be a definable function. There exists a subinterval of (a, b) on which f is constant, or there exists a subinterval on which f is strictly monotone and continuous.*

Proof. Suppose that there is no subinterval of (a, b) on which f is constant.

First step: there exists a subinterval on which f is injective. It follows from the assumption that, for all y in R, the definable set $f^{-1}(y)$ is finite: otherwise, $f^{-1}(y)$ would contain an interval, on which $f = y$. Hence the definable set $f((a, b))$ is infinite and contains an interval J. The function $g : J \to (a, b)$ defined by $g(y) = \min(f^{-1}(y))$ is definable and satisfies $f \circ g = \mathrm{Id}_J$. Since g is injective, $g(J)$ is infinite and contains a subinterval I of (a, b). We have $g \circ f|_I = \mathrm{Id}_I$, and f is injective on I.

Second step: there exists a subinterval on which f is strictly monotone. We know that f is injective on a subinterval I of (a, b). Take $x \in I$. The sets

$$\begin{array}{rcl} I_+ & = & \{y \in I \ ; \ f(y) > f(x)\} \\ I_- & = & \{y \in I \ ; \ f(y) < f(x)\} \end{array}$$

form a definable partition of $I \setminus \{x\}$. Therefore, there is $\varepsilon > 0$ such that $(x-\varepsilon, x)$ (resp. $(x, x+\varepsilon)$) is contained in I_+ or in I_-. There are four possibilities $\Phi_{+,+}(x)$, $\Phi_{+,-}(x)$, $\Phi_{-,+}(x)$ and $\Phi_{-,-}(x)$ which give a definable partition of I. For instance, $\Phi_{+,-}(x)$ is

$$\exists \varepsilon \ \ \forall y \in I \ \ \Big((x - \varepsilon < y < x \Rightarrow f(y) > f(x)) \ \text{ and } \\ (x < y < x + \varepsilon \Rightarrow f(y) < f(x)) \Big) \ .$$

We claim that $\Phi_{+,+}$ and $\Phi_{-,-}$ are finite. It is sufficient to prove the claim for $\Phi_{+,+}$ (for $\Phi_{-,-}$, replace f with $-f$). If $\Phi_{+,+}$ is not finite, it holds on a subinterval of I, which we still call I. Set

$$B = \{x \in I \ ; \ \forall y \in I \quad y > x \Rightarrow f(y) > f(x)\} \ .$$

If B contains an interval, then f is strictly increasing on this interval, which contradicts $\Phi_{+,+}$. Hence, B is finite. Replacing I with $(\max(B), +\infty) \cap I$ if $B \neq \emptyset$, we can assume

$$(*) \qquad \forall x \in I \ \exists y \in I \ (y > x \ \text{ and } \ f(y) < f(x)) \ .$$

Take $x \in I$. The definable set $C_x = \{y \in I \ ; \ y > x \ \text{ and } \ f(y) < f(x)\}$ is nonempty. If C_x were finite, its maximum would be an element of I contradicting the property (*). Therefore, C_x contains an interval. Let z be the greatest lower bound of the interior of C_x. We have $z > x$ because $f > f(x)$ on some interval $(x, x + \varepsilon)$. We have also $f > f(x)$ on some interval $(z - \varepsilon, z)$ and $f < f(x)$ on some interval $(z, z+\varepsilon)$. Hence, the following formula $\Psi_{+,-}(z)$ holds:

$$\exists \varepsilon > 0 \ \forall t \in I \ \forall u \in I \ (z - \varepsilon < t < z < u < z + \varepsilon \Rightarrow f(t) > f(u)) \ .$$

We have shown

$$(**) \qquad \forall x \in I \ \exists z \in I \ (x < z \ \text{ and } \ \Psi_{+,-}(z)) \ .$$

The definable set of elements of I satisfying $\Psi_{+,-}$ is not empty and it is not finite: otherwise, its maximum would contradict (**). It follows that, replacing

I with a smaller subinterval, we can assume that $\Psi_{+,-}$ holds on I. Consider the definable function $h : -I \to R$ defined by $h(x) = f(-x)$. The property $\Phi_{+,+}$ for h holds on $-I$. Hence, by the preceding argument, there is a subinterval of $-I$ on which $\Psi_{+,-}$ for h holds. This means that there is a subinterval of I on which $\Psi_{-,+}$ for f holds (exchange left and right). On this subinterval we have both $\Psi_{+,-}$ and $\Psi_{-,+}$, which are contradictory. This contradiction proves the claim.

Since $\Phi_{+,+}$ and $\Phi_{-,-}$ are finite, replacing I with a smaller subinterval, we can assume that $\Phi_{-,+}$ holds on I or $\Phi_{+,-}$ holds on I. Say $\Phi_{-,+}$ holds on I. Then f is strictly increasing on I. Indeed, for all $x \in I$, the definable set $\{y \in I \; ; \; y > x \text{ and } f > f(x) \text{ on } (x,y)\}$ is nonempty and its least upper bound is necessarily the right endpoint of I (Otherwise, this l.u.b. would not satisfy $\Phi_{-,+}$). Similarly, if $\Phi_{+,-}$ holds on I, f is strictly decreasing on I.

Last step: there is a subinterval on which f is strictly monotone and continuous. Recall that f is strictly monotone on I. The definable set $f(I)$ is infinite and contains an interval J. The inverse image $f^{-1}(J)$ is the interval $(\inf(f^{-1}(J)), \sup(f^{-1}(J)))$. Replacing I with this interval, we can assume that f is a strictly monotone bijection from the interval I onto the interval J. Since the inverse image of a subinterval of J is a subinterval of I, f is continuous. $\square$

Proof of the Monotonicity Theorem. Let $X_=$ (resp. $X_\nearrow$, resp. $X_\searrow$) be the definable set of $x \in (a,b)$ such that f is constant (resp. continuous and strictly increasing, resp. continuous and strictly decreasing) on an interval containing x. Then

$$(a,b) \setminus (X_= \cup X_\nearrow \cup X_\searrow)$$

is finite: otherwise, it would contain an interval, and we get a contradiction by applying Lemma 2.2 to this interval. Hence, there is a subdivision $a = a_0 < a_1 < \ldots < a_k = b$ such that each (a_i, a_{i+1}) is contained in $X_=$ or in $X_\nearrow$ or in $X_\searrow$. Clearly, f is continuous on each (a_i, a_{i+1}). Take x in (a_i, a_{i+1}). If (a_i, a_{i+1}) is contained in $X_=$ (resp. $X_\nearrow$, resp. $X_\searrow$), let D_x be the set of $y \in (a_i, a_{i+1})$ such that $x < y$ and $f = f(x)$ (resp. $f > f(x)$, resp. $f < f(x)$) on (x,y). The definable set D_x is nonempty and its least upper bound is necessarily a_{i+1}. It follows that f is constant or strictly monotone on (a_i, a_{i+1}). $\square$

Exercise 2.3 Let $f : (a,b) \to R$ be a definable function. Then $\lim_{x \to b_-} f(x)$ and $\lim_{x \to a_+} f(x)$ exist in $R \cup \{-\infty, +\infty\}$.

2.2 Uniform Finiteness, Cell Decomposition and Piecewise Continuity

The notion of a *cylindrical definable cell decomposition (cdcd) of* R^n is a generalization of cylindrical algebraic cell decomposition. We define the cdcd by induction on n

Definition 2.4 *A cdcd of* R^n *is a finite partition of* R^n *into definable sets* $(C_i)_i \in I$ *satisfying certain properties explained below. The* C_i *are called the* cells *of the cdcd.*

$n = 1$**:** *A cdcd of* R *is given by a finite subdivision* $a_1 < \ldots < a_\ell$ *of* R*. The cells of* R *are the singletons* $\{a_i\}$*,* $0 < i \leq \ell$*, and the intervals* (a_i, a_{i+1})*,* $0 \leq i \leq \ell$*, where* $a_0 = -\infty$ *and* $a_{\ell+1} = +\infty$*.*

$n > 1$**:** *A cdcd of* R^n *is given by a cdcd of* R^{n-1} *and, for each cell* D *of* R^{n-1}*, continuous definable functions*

$$\zeta_{D,1} < \ldots < \zeta_{D,\ell(D)} : D \longrightarrow R \ .$$

The cells of R^n *are the* graphs

$$\{(x, \zeta_{D,i}(x)) \ ; \ x \in D\} \ , \qquad 0 < i \leq \ell(D) \ ,$$

and the bands

$$(\zeta_{D,i}, \zeta_{D,i+1}) = \{(x, y) \ ; \ x \in D \ \text{ and } \ \zeta_{D,i}(x) < y < \zeta_{D,i+1}(x)\}$$

for $0 \leq i \leq \ell(D)$*, where* $\zeta_{D,0} = -\infty$ *and* $\zeta_{D,\ell(D)+1} = +\infty$*.*

Note that the fact that a cell of a cdcd is definable follows immediately from the definition and the axioms of o-minimal structure. Note also that if $p_{n,m} : R^n \to R^m$, $m < n$, is the projection on the first m coordinates, the images by p of the cells of a cdcd of R^n are the cells of a cdcd of R^m.

We define by induction the *dimension of a cell*. The dimension of a singleton is 0 and the dimension of an interval is 1. If C is a cell of R^n, its dimension is $\dim(p_{n,n-1}(C))$ if C is a graph and $\dim(p_{n,n-1}(C)) + 1$ if C is a band.

Proposition 2.5 *For each cell* C *of a cdcd of* R^n*, there is a definable homeomorphism* $\theta_C : C \to R^{\dim(C)}$*.*

Proof. Let $D = p_{n,n-1}(C)$ and assume $\theta_D : D \to R^{\dim(D)}$ is already defined. Let $(x, y) \in C$, where $x \in D$. We define $\theta_C(x, y)$ as $\theta_D(x)$ if C is a graph, and as

$$\left(\theta_D(x)\ ,\ \frac{1}{\zeta_{D,i}(x) - y} + y + \frac{1}{\zeta_{D,i+1}(x) - y}\right)$$

if C is the band $(\zeta_{D,i}, \zeta_{D,i+1})$ (fractions where infinite functions appear in the denominator have to be omitted). ☐

Exercise 2.6 Prove (by induction on n) that a cell is open in R^n if and only if its dimension is n. Prove that the union of all cells of dimension n is dense in R^n (hint: show by induction on n that every open box in R^n meets a cell of dimension n).

We have already said that the notion of connectedness does not behave well if $R \neq \mathbb{R}$. It has to be replaced by the notion of definable connectedness.

Definition 2.7 *A definable set A is said to be* definably connected *if, for all disjoint definable open subsets U and V of A such that $A = U \cup V$, one has $A = U$ or $A = V$. A definable set A is said to be* definably arcwise connected *if, for all points a and b in A, there is a definable continuous map $\gamma : [0, 1] \to A$ such that $\gamma(0) = a$ and $\gamma(1) = b$.*

Exercise 2.8 Prove the following facts: The segment $[0, 1]$ is definably connected. Definably arcwise connected implies definably connected. Every box $(0, 1)^d$ is definably connected. Every cell of a cdcd is definably connected.

We now state the three main results of this section. In the following, we denote by $\sharp S$ the number of elements of a finite set S.

Theorem 2.9 (Uniform Finiteness UF_n**)** *Let A be a definable subset of R^n such that, for every $x \in R^{n-1}$, the set*

$$A_x = \{y \in R\ ;\ (x, y) \in A\}$$

is finite. Then there exists $k \in \mathbf{N}$ such that $\sharp A_x \leq k$ for every $x \in R^{n-1}$.

Theorem 2.10 (Cell Decomposition CDCD_n**)** *Let $A_1, \ldots, A_k$ be definable subsets of R^n. There is a cdcd of R^n such that each A_i is a union of cells.*

A cdcd of R^n satisfying the property of the theorem will be called *adapted to* $A_1, \ldots, A_k$.

Exercise 2.11 Prove the following consequence of CDCD_n: Let X be a definable subset of R^n such that $\mathrm{clos}(X)$ has nonempty interior. Then X has nonempty interior. (Hint: use Exercise 2.6 for a convenient cdcd of R^n.)

Theorem 2.12 (Piecewise Continuity PC_n**)** *Let A be a definable subset of R^n and $f : A \to R$ a definable function. There is a cdcd of R^n adapted to A such that, for every cell C contained in A, $f|_C$ is continuous.*

Exercise 2.13 Prove the following consequence of PC_n: Let X be an open definable subset of R^n and $f : X \to R$ a definable function. Then there is an open box $B \subset X$ such that $f|_B$ is continuous (hint: use Exercise 2.6).

We prove the three theorems simultaneously by induction on n. For $n = 1$, UF_1 is trivial (R^0 is just one point), CDCD_1 follows immediately from the o-minimality, and PC_1 is a consequence of the Monotonicity Theorem 2.1. In the following we assume $n > 1$ and UF_m, CDCD_m and PC_m hold for all integers m such that $0 < m < n$.

Proof of UF_n. We can assume that, for every $x \in R^{n-1}$, the set A_x is contained in $(-1, 1)$. Let $\mu : R \to (-1, 1)$ be the semialgebraic homeomorphism defined by $\mu(y) = y/\sqrt{1 + y^2}$. We can replace A with its image by $(x, y) \mapsto (x, \mu(y))$, which satisfies the assumption.

For $x \in R^{n-1}$ and $i = 1, 2, \ldots$, define $f_i(x)$ to be the i-th element of A_x, if it exists. Note that the function f_i is definable. Call $a \in R^{n-1}$ *good* if $f_1, \ldots, f_{\sharp(A_a)}$ are defined and continuous on an open box containing a and a does not belong to the closure of the domain of $f_{\sharp(A_a)+1}$. In other words, a is good if and only if there is an open box $B \ni a$ in R^{n-1}, such that $(B \times R) \cap A$ is the union of graphs of continuous definable functions

$$\zeta_1 < \ldots < \zeta_{\sharp(A_a)} : B \to (-1, 1) \qquad (\zeta_i = f_i|B) .$$

Call $a \in R^{n-1}$ *bad* if it is not good.

First step: the set of good points is definable. Let a be a point of R^{n-1}. If b belongs to $[-1, 1]$, we say that $(a, b) \in R^n$ *is normal* if there is an open box $C = B \times (c, d) \subset R^n$ containing (a, b) such that $A \cap C$ is either empty or the graph of a continuous definable function $B \to (c, d)$.

Clearly, if a is good, (a, b) is normal for every $b \in [-1, 1]$. Now assume that a is bad. Let f_ℓ be the first function f_i such that a is in the closure of the domain of f_i and there is no open box containing a on which f_i is defined and continuous. We set $\beta(a) = \liminf_{x \to a} f_\ell(x) \in [-1, 1]$. We claim that $(a, \beta(a))$ is not normal. Suppose $(a, \beta(a))$ is normal. There is an open box $B \times (c, d)$ containing $(a, \beta(a))$ whose intersection with A is the graph of a continuous function $g : B \to (c, d)$. We can assume that $f_\ell(x) > c$ for all $x \in B$ such that $f_\ell(x)$ is defined. If $\ell > 1$ and $\beta(a) = f_{\ell-1}(a)$, we would deduce $g = f_{\ell-1}|_B$ since B is definably connected. We would have $f_\ell(x) \geq d$ for all $x \in B$ such that $f_\ell(x)$ is defined, which contradicts $\beta(a) < d$. Hence, we can assume $\ell = 1$ or $f_{\ell-1} < c$ on B. It follows that $g = f_\ell|_B$, which contradicts the definition of ℓ. Therefore the claim is proved.

We have shown that $a \in R^{n-1}$ is good if and only if, for all $b \in [-1, 1]$, (a, b) is normal. From this we deduce easily that the set of good points is definable.

Second step: the set of good points is dense. Otherwise, there is an open box $B \subset R^{n-1}$ contained in the set of bad points. Consider the definable function $\beta : B \to [-1, 1]$ defined as in Step 1. By PC_{n-1}, we can assume that β is continuous (see Exercise 2.13). For $x \in B$, we define $\beta_-(x)$ (resp. $\beta_+(x)$) to be the maximum (resp. minimum) of the $y \in A_x$ such that $y < \beta(x)$ (resp. $y > \beta(x)$), if such y exist. Using PC_{n-1} and shrinking the box B, we can assume that β_- (resp. β_+) either is nowhere defined on B or is continuous on B. Then the set of $(x, y) \in A \cap (B \times R)$ such that $y \neq \beta(x)$ is open and closed in $A \cap (B \times R)$. Shrinking further the box B, we can assume that the graph of $\beta|_B$ is either disjoint from A or contained in A. The first case contradicts the definition of β. In the second case, $(x, \beta(x))$ would be normal for every $x \in B$, which contradicts what was proved in the first step. We have proved that the set of good points is dense.

Third step. By CDCD_{n-1}, there is a cdcd of R^{n-1} adapted to the set of good points. Let C be a cell of dimension $n-1$ of R^{n-1}. Since good points are dense, every $x \in C$ is good. Take $a \in C$. The set of $x \in C$ such that $\sharp(A_x) = \sharp(A_a)$ is definable, open and closed in C. By definable connectedness of C, it is equal to C. If D is a cell of R^{n-1} of smaller dimension, we can use the definable homeomorphism of $\theta_D : D \to R^{\dim(D)}$ and the assumption that $\mathrm{UF}_{\dim(D)}$ holds to prove that $\sharp(A_x)$ is uniformly bounded for $x \in D$. Since there are finitely many cells, the proof of UF_n is completed. $\square$

Proof of CDCD_n. Let A be the set of $(x, y) \in R^{n-1} \times R$ such that y belongs to the boundary of one of $A_{1,x}, \dots, A_{k,x}$ (the boundary of S in R is $\mathrm{clos}(S) \setminus \mathrm{int}(S)$). Clearly A is definable and satisfies the assumptions of UF_n. Hence

$\sharp(A_x)$ has a maximum ℓ for $x \in R^{n-1}$, and A is the union of the graphs of functions $f_1, \ldots, f_\ell$ defined at the beginning of the proof of UF_n. We define the type of x in R^{n-1} to consist of the following data:

- $\sharp(A_x)$,
- the sets of $j \in \{1, \ldots, k\}$ such that $f_i(x) \in A_{j,x}$ for $i = 1, \ldots, \sharp(A_x)$,
- the sets of $j \in \{1, \ldots, k\}$ such that $(f_i(x), f_{i+1}(x)) \subset A_{j,x}$, for $i = 0, \ldots, \sharp(A_x)$ (where $f_0(x) = -\infty$ and $f_{\sharp(A_x)+1}(x) = +\infty$).

Since there are finitely many possible types and the set of points in R^{n-1} with a given type is definable, we deduce from CDCD_{n-1} that there is a cdcd of R^{n-1} such that two points in the same cell of R^{n-1} have the same type. Moreover, using PC_{n-1}, we can assume that the cdcd of R^{n-1} is such that, for each cell C and $i = 1, \ldots, \ell$, either f_i is defined nowhere on C or f_i is defined and continuous on C. The cdcd of R^{n-1} we have obtained, together with the restrictions of the functions f_i to the cells of this cdcd, define a cdcd of R^n adapted to $A_1, \ldots, A_k$. $\square$

Proof of PC_n.

First step. Assume that A is an open box $B \times (a, b) \subset R^{n-1} \times R$. We claim that there is an open box $A' \subset A$ such that $f|_{A'}$ is continuous.

For every $x \in B$, let $\lambda(x)$ be the least upper bound of the set of $y \in (a, b)$ such that $f(x, \cdot)$ is continuous and monotone on (a, y). The Monotonicity Theorem 2.1 implies $\lambda(x) > a$ for all $x \in B$. The function λ is definable. Applying PC_{n-1} to λ and replacing B with a smaller open subbox, we can assume that λ is continuous (see Exercise 2.13). Replacing again B with a smaller subbox, we can assume that there is $c > a$ such that $\lambda > c$. Replacing b with c, we can assume that, for every $x \in B$, $f(x, \cdot)$ is continuous and monotone on (a, b).

Now consider the set C of points $(x, y) \in B \times (a, b)$ such that $f(\cdot, y)$ is continuous at x. The set C is definable. It follows from PC_{n-1} that, for every $y \in (a, b)$, the set of x such that $f(\cdot, y)$ is continuous at x is dense in B. Hence, C is dense in A. Applying CDCD_n, we deduce that C contains an open subbox of A. Replacing A with this smaller subbox, we can assume that, for every $y \in (a, b)$, $f(\cdot, y)$ is continuous.

So it suffices to consider the case where $f(x, \cdot)$ is continuous and monotone on (a, b) for every $x \in B$ and $f(\cdot, y)$ is continuous on B for every $y \in (a, b)$. In this situation, f is continuous on $B \times (a, b)$. Indeed, take $(x^0, y^0) \in B \times (a, b)$ and I an interval containing $f(x^0, y^0)$. By continuity of $f(x^0, \cdot)$, we find $y^1 <$

$y^0 < y^2$ such that $f(x^0, y^i) \in I$ for $i = 1, 2$. By continuity of $f(\cdot, y^i)$, we find an open box $B' \ni x^0$ in B such that $f(B' \times \{y^i\}) \subset I$ for $i = 1, 2$. It follows from the monotonicity of $f(x, \cdot)$ that $f(B' \times (y^1, y^2))$ is contained in I. This proves the continuity of f and completes the proof of the claim.

Second step. Now we can prove PC_n. Consider the set D of points of A where f is continuous. The set D is definable. By CDCD_n, there is a cdcd of R^n adapted to A and D. If E is an open cell contained in A, the first step shows that $E \cap D$ is nonempty, therefore $E \subset D$ and f is continuous on E. If F is a cell of dimension $d < n$ contained in A, there is a definable homeomorphism $\theta_F : F \to R^d$. Composing f with θ_F^{-1} and applying PC_d, we obtain a finite partition of F into definable subsets F_i such that $f|_{F_i}$ is continuous. Hence, there is a finite partition of A into definable subsets $A_1, \ldots, A_k$ such that $f|_{A_i}$ is continuous for $i = 1, \ldots, k$. Using CDCD_n, we obtain a cdcd of R^n adapted to $A_1, \ldots, A_k$. So we can assume that the A_i are cells of a cdcd of R^n. $\square$

Chapter 3

Connected Components and Dimension

3.1 Curve Selection

We begin with a useful result. It says that, if a formula "$\forall x \in X\ \exists y \in Y\ (x, y) \in Z$" holds, where X, Y and Z are definable sets, then y can be choosed as a *definable* function of $x \in X$.

Theorem 3.1 (Definable Choice) *Let A be a definable subset of $R^m \times R^n$. Denote by $p : R^m \times R^n \to R^m$ the projection on the first m coordinates. There is a definable function $f : p(A) \to R^n$ such that, for every $x \in p(A)$, $(x, f(x))$ belongs to A.*

Proof. It is sufficient to consider the case $n = 1$. The general case follows by decomposing the projection $R^{m+n} \to R^m$ as $R^{m+n} \to R^{m+n-1} \to \ldots \to R^{m+1} \to R^m$.

Make a cdcd of R^{m+1} adapted to A. The projection $p(A)$ is the union of the images by p of cells contained in A. Hence, we can assume that A is a cell of R^{m+1}, and consequently $p(A)$ is a cell of R^m. If A is the graph of $\zeta_i : p(A) \to R$, we take $f = \zeta_i$. If A is a band (ζ_i, ζ_{i+1}), where both functions are finite, we take $f = \frac{1}{2}(\zeta_i + \zeta_{i+1})$. If for instance ζ_i is finite and $\zeta_{i+1} = +\infty$, we take $f = \zeta_i + 1$. $\square$

We introduce a notation. Given $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in R^n, we set $\|x - y\| = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$. Given $r > 0$ in R, we define the open and closed balls

$$\begin{aligned} B(x,r) &= \{z \in R^n \ ; \ \|z-x\| < r\} \ , \\ \overline{B}(x,r) &= \{z \in R^n \ ; \ \|z-x\| \le r\} \ . \end{aligned}$$

The open balls $B(x,r)$ form a basis of open neighborhoods of x.

Theorem 3.2 (Curve Selection Lemma) *Let A be a definable subset of R^n, b a point in* $\mathrm{clos}(A)$. *There is a continuous definable map $\gamma : [0,1) \to R^n$ such that $\gamma(0) = b$ and $\gamma((0,1)) \subset A$.*

Proof. Set $X = \{(t,x) \in R \times R^n \ ; \ x \in A$ and $\|x-b\| < t\}$. Let $p : R \times R^n \to R$ be the projection on the first coordinate. Since $b \in \mathrm{clos}(A)$, we have $p(X) = \{t \in R \ ; \ t > 0\}$. Applying Definable Choice 3.1 and the Monotonicity Theorem 2.1, we find $\varepsilon > 0$ and a continuous definable map $\delta : (0,\varepsilon) \to A$ such that $\|\delta(t) - b\| < t$. Clearly, δ extends continuously to $\overline{\delta} : [0,\varepsilon) \to R^n$ with $\overline{\delta}(0) = b$, and we define $\gamma : [0,1) \to R^n$ by $\gamma(t) = \overline{\delta}(t\varepsilon)$. □

The curve selection lemma replaces the use of sequences in many situations.

Exercise 3.3 Show that a definable function $f : A \to R$ is continuous if and only if, for every continuous definable $\gamma : [0,1) \to A$, $\lim_{t \to 0_+} f(\gamma(t)) = f(\gamma(0))$.

We have seen that compactness is a problem for a general real closed field. However, the following result shows that we retain the good properties of compactness if we deal with definable objects.

Theorem 3.4 *Let A be a definable subset of R^n. The following properties are equivalent.*

1. *A is closed and bounded.*

2. *Every definable continuous map $(0,1) \to A$ extends by continuity to a map $[0,1) \to A$.*

3. *For every definable continuous function $f : A \to R$, $f(A)$ is closed and bounded.*

Proof. $1 \Rightarrow 2$. A definable continuous map $(0,1) \to A$ extends by continuity to a map $[0,1) \to R^n$: every coordinate of the map has a limit as $t \to 0_+$ (cf. Exercise 2.3), and this limit is in R since A is bounded. Since A is closed, the value of the extension at 0 belongs to A.

$2 \Rightarrow 3$. Suppose that $f(A)$ is not bounded. Set

$$X = \{(t,x) \in R \times R^n \ ; \ x \in A \ \text{ and } \ t\,|f(x)| = 1\} \ .$$

Then the projection of X on the first coordinate contains some interval $(0,\varepsilon)$. Using Definable Choice 3.1 and the Monotonicity Theorem 2.1, and rescaling the interval of definition, we can assume that there is a continuous map $\delta : (0,1) \to A$ such that $\lim_{x\to 0_+} |f(\delta(t))| = +\infty$. This implies that δ cannot be extended continuously to a map $[0,1) \to A$, which contradicts 2. Hence, $f(A)$ is bounded. Now let b belong to $\mathrm{clos}(f(A))$. Set

$$Y = \{(t,x) \in R \times R^n \ ; \ x \in A \ \text{ and } \ |b - f(x)| < t\} \ .$$

The same argument as above shows that there is a continuous map $\gamma : (0,1) \to A$ such that $\lim_{t\to 0_+} f(\gamma(t)) = b$. By 2, γ extends continuously to a map $[0,1) \to A$ and its value at 0 is an element a of A such that $f(a) = b$. This shows that $f(A)$ is closed.

$3 \Rightarrow 1$. Since the image of A by each coordinate function is bounded, A is bounded. Let b belong to $\mathrm{clos}(A)$. Since the image of A by $x \mapsto \|x - b\|$ is closed, it contains 0. Therefore b belongs to A, which shows that A is closed. □

Exercise 3.5 Show that a definable continuous function on a closed and bounded definable set is uniformly continuous.

Corollary 3.6 *Let A be a closed and bounded definable set. If B is a definable set definably homeomorphic to A, then B is also closed and bounded.*

The preceding corollary shows that the property of being closed and bounded, for a definable set, is intrisic (in the sense that it does not depend on an imbedding in affine space). The property of being locally closed is also intrisic, in the same sense. Recall that a subset of R^n is said to be locally closed if it is open in its closure.

Proposition 3.7 *1) A definable set $A \subset R^n$ is locally closed if and only if every point $x \in A$ has a basis of closed and bounded definable neighborhoods in A.*

2) If A is a locally closed definable set, and B, a definable set definably homeomorphic to A, then B is locally closed.

Proof. 1) First assume that A is locally closed. Take $x \in A$. There is $r_0 > 0$ such that the intersection of the ball $B(x, r_0) \subset R^n$ with $\operatorname{clos}(A)$ is contained in A. Then the intersections of $\operatorname{clos}(A)$ with the closed balls $\overline{B}(x, r)$, for $0 < r < r_0$, form a basis of closed and bounded definable neighborhoods of x in A.

Conversely, assume that every point $x \in A$ has a closed neighborhood N in A. Take $s > 0$ such that $B(x, s) \cap A \subset N$. Since $B(x, s) \cap (R^n \setminus N)$ is open and disjoint from A, we have $B(x, s) \cap \operatorname{clos}(A) \subset N \subset A$. This shows that A is locally closed.

2) This follows from 1) and Corollary 3.6. $\square$

Exercise 3.8 Show:

1) A cell of a cdcd is locally closed. (Use 3.7, statement 2.)

2) Every definable set is the union of finitely many locally closed definable sets.

3.2 Connected Components

Theorem 3.9 *Let A be a definable subset of R^n. There is a partition of A into finitely many definable subsets $A_1, \dots, A_k$ such that each A_i is nonempty, open and closed in A, and definably arcwise connected. Such a partition is unique. The $A_1, \dots, A_k$ are called the* definable connected components of A.

Proof. Make a cdcd of R^n adapted to A. We say that a cell C is adjacent to a cell D if $C \cap \operatorname{clos}(D) \neq \emptyset$, and we denote this fact by $C \prec D$.

We claim that, if $C \prec D$, every $x \in C$ can be joined to every $y \in D$ by a definable continuous path in $C \cup D$. Take $c \in C \cap \operatorname{clos}(D)$. By the Curve Selection Lemma, there is a continuous definable $\gamma : [0, 1) \to C \cup D$ such that $\gamma(0) = c$ and $\gamma((0, 1)) \subset D$. Set $d = \gamma(1/2)$. The points c and d are joined by a continuous definable path in $C \cup D$. Since every cell is definably homeomorphic to an affine space, every point $x \in C$ can be joined to c by a definable continuous path in C and every point $y \in D$ can be joined to d by a definable continuous path in D. This proves the claim.

Let $\sim$ be the smallest equivalence relation on the set of cells contained in A containing the adjacency relation: we have $C \sim D$ if and only if there is a chain $C = C_0 \prec C_1 \succ C_2 \prec \ldots \succ C_{2k} = D$ where all cells C_i are contained in A (note that we can have $C_i = C_{i+1}$). Let $\mathcal{E}_1, \dots, \mathcal{E}_k$ be the equivalence classes for $\sim$, and let A_i be the union of all cells in $\mathcal{E}_i$. The A_i form a finite partition of A into definable sets. The claim proved above and the definition of $\sim$ imply that each A_i is definably arcwise connected. If a cell $C \subset A$ has a nonempty intersection with $\mathrm{clos}(A_i)$, it is adjacent to a cell in $\mathcal{E}_i$, which implies $C \subset A_i$. Hence, each A_i is closed in A. Since the A_i form a finite partition of A, they are also open in A.

Assume that $A = B_1 \cup \ldots \cup B_\ell$ is another partition with the same properties. We have $A_i = \bigcup_{j=1}^{\ell} (A_i \cap B_j)$, and each $(A_i \cap B_j)$ is definable, open and closed in A_i. Since A_i is definably connected, there is exactly one j such that $A_i \subset B_j$. For the same reason, for every j there is exactly one i such that $B_j \subset A_i$. This shows that the two partitions coincide up to order. □

Remark that it follows from the proof that the number of definable connected components of A is not greater that the number of cells contained in A for any cdcd of R^n adapted to A.

Corollary 3.10 *A definably connected definable set is definably arcwise connected.*

Consider the case where $R = \mathbb{R}$, i.e. the o-minimal structure expands the field of real numbers. It is clear that a connected definable set is definably connected. It is also clear that a definably arcwise connected definable set is arcwise connected. Hence, a definable set is connected if and only it is definably connected, and the definable connected components of a definable set are its usual connected components. It follows that a definable set has finitely many connected components, which are definable.

We shall prove now that there is a uniform bound for the number of connected components in a definable family of subsets of R^n. First we make precise this notion of definable family. Let A be a definable subset of $R^m \times R^n$. For $x \in R^m$, set $A_x = \{y \in R^n \ ; \ (x, y) \in A\}$. We call the family $(A_x)_{x \in R^m}$ a *definable family of subsets of R^n parametrized by R^m.*

The next result allows to regard a cdcd of $R^m \times R^n$ as a “definable family of cdcd of R^n”. Assume that a cdcd of $R^m \times R^n$ is given. Let $p_{m+n,m} : R^m \times R^n \to R^m$ be the projection on the first m coordinates. Recall that the $p_{m+n,m}(C)$, for all cells C of $R^m \times R^n$, are the cells of a cdcd of R^m.

Proposition 3.11 *Let B be a cell of R^m, $a \in B$. For all cells $C \subset R^m \times R^n$ such that $p_{m+n,m}(C) = B$, we set $C_a = \{y \in R^n \ ; \ (a, y) \in C\}$. Then these C_a are the cells of a cdcd of R^n. The dimension of the cell C_a is equal to* $\dim(C) - \dim(B)$.

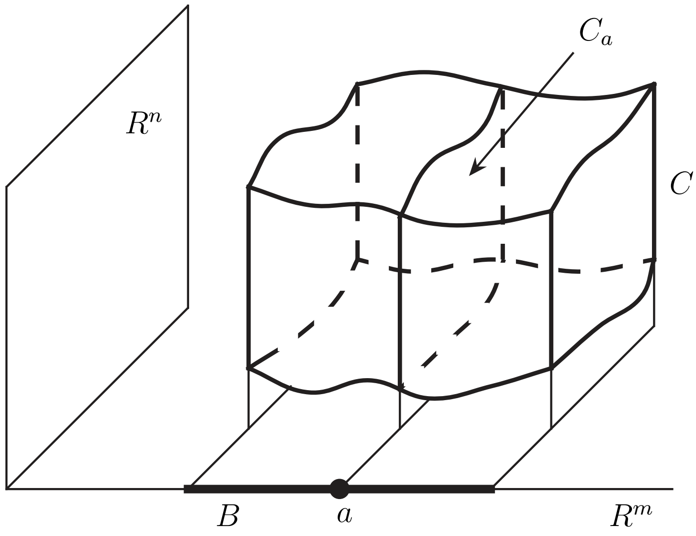


Figure 3.1: The slice C_a of a cell C.

Proof. We proceed by induction on n. The case $n = 0$ is trivial. Now assume that $n > 0$ and the proposition is proved for $n-1$. The cdcd of $R^m \times R^n$ induces by the projection $p_{m+n,m+n-1} : R^m \times R^n \to R^m \times R^{n-1}$ a cdcd of $R^m \times R^{n-1}$. For each cell D of $R^m \times R^{n-1}$, there are continuous definable functions $\zeta_{D,1} < \ldots < \zeta_{D,\ell(D)} : D \to R$, and the cells C of $R^m \times R^n$ such that $p_{m+n,m+n-1}(C) = D$ are graphs of $\zeta_{D,i}$ or bands $(\zeta_{D,i}, \zeta_{D,i+1})$. By the inductive assumption, the D_a, for all cells D of $R^m \times R^{n-1}$ such that $p_{m+n-1,m}(D) = B$, are the cells of a cdcd of R^{n-1} and $\dim(D_a) = \dim(D) - \dim(B)$. For such cells D, define $(\zeta_{D,i})_a : D_a \to R$ by $(\zeta_{D,i})_a(y) = \zeta_{D,i}(a, y)$. Now $C \subset R^m \times R^n$ is the graph of $\zeta_{D,i}$ (resp. the band $(\zeta_{D,i}, \zeta_{D,i+1})$) if and only if $C_a \subset R^n$ is the graph of $(\zeta_{D,i})_a$ (resp. the band $((\zeta_{D,i})_a, (\zeta_{D,i+1})_a)$). Hence the C_a, for all cells C such that $B = p_{m+n,m}(C)$, form a cdcd of R^n. If C is a graph over $D = p_{m+n,m+n-1}(C)$, $\dim(C_a) = \dim(D_a) = \dim(D) - \dim(B) = \dim(C) - \dim(B)$. If C is a band, $\dim(C_a) = \dim(D_a) + 1 = \dim(D) - \dim(B) + 1 = \dim(C) - \dim(B)$. $\square$

Theorem 3.12 *Let A be a definable subset of $R^m \times R^n$. There is $\beta \in \mathbb{N}$ such that, for every $x \in R^m$ the number of definable connected components of A_x is not greater than β.*

Proof. We choose a cdcd of $R^m \times R^n$ adapted to A. We adopt the notation of Proposition 3.11 and its proof. Take $x \in R^m$ and let B be the cell of

R^m containing x. The collection of all C_x, for C cell of $R^m \times R^n$ such that $p_{m+n,m}(C) = B$, is a cdcd of R^n adapted to A_x. Hence, the number of definable connected components of A_x is not greater that the number of cells of $R^m \times R^n$ contained in A. $\square$

Exercise 3.13 The aim of this exercise is to prove the following fact:

($\sharp$) Let A be a definable subset of R^n. There exists $\beta(A) \in \mathbb{N}$ such that, for every affine subspace L of R^n, the number of definable connected components of $L \cap A$ is not greater than $\beta(A)$.

1) (The Grassmannian of affine subspaces of R^n as an algebraic subset of R^{n^2+n}.) Let M be a $n \times n$ matrix with coefficients in R, which we identify with an element of R^{n^2}, and let v be a vector in R^n. We associate to $g = (M, v) \in R^{n^2} \times R^n$ the affine subspace $L_g = \{x \in R^n \ ; \ Mx = v\}$ of R^n. Show that the mapping $g \mapsto L_g$ restricted to the algebraic subset

$$G = \{g = (M, v) \in R^{n^2} \times R^n \ ; \ M^2 = M \ \text{ and } \ {}^tM = M \ \text{ and } \ Mv = v\}$$

is a bijection onto the set of all affine subspaces of R^n.
2) Let A be a definable subset of R^n. Construct a definable set $\mathcal{A} \subset G \times R^n$ such that, for all $g \in G$, $\mathcal{A}_g = A \cap L_g$ (where $\mathcal{A}_g$ is defined as $\{x \in R^n \ ; \ (g, x) \in \mathcal{A}\}$).
3) Prove ($\sharp$).

3.3 Dimension

We have already defined the dimension of a cell of a cdcd. Now let A be a definable subset of R^n. Take a cdcd of R^n adapted to A. A "naive" definition of the dimension of A is the maximum of the dimension of the cells contained in A. But this definition is not intrinsic. We have to prove that the dimension so defined does not depend on the choice of a cdcd adapted to A. We shall rather introduce an intrisic definition of dimension, and show that it coincides with the "naive" one

Definition 3.14 *The dimension of a definable set A is the sup of d such that there exists a injective definable map from R^d to A. By convention, the dimension of the empty set is $-\infty$.*

Remark that it is not obvious for the moment that the dimension is always $< +\infty$. It is also not clear that this definition of dimension agrees with the one already given for cells. Both facts will follow from the next lemma.

Lemma 3.15 *Let A be a definable subset of R^n with nonempty interior. Let $f : A \to R^n$ be an injective definable map. Then $f(A)$ has nonempty interior.*

Proof. We prove the lemma by induction on n. If $n = 1$, A is infinite, hence $f(A) \subset R$ is infinite and contains an interval. Assume that $n > 1$ and the lemma is proved for all $m < n$. Using Piecewise Continuity 2.12, we can assume moreover that f is continuous. Take a cdcd of R^n adapted to $f(A)$. If $f(A)$ has empty interior, it contains no open cell. Hence $f(A)$ is the union of nonopen cells $C_1, \dots, C_k$ and, for $i = 1, \dots, k$, there is a definable homeomorphism $C_i \to R^{m_i}$ with $m_i < n$. Take a cdcd of R^n adapted to the $f^{-1}(C_i)$. Since $A = \bigcup_{i=1}^k f^{-1}(C_i)$ has nonempty interior, one of the $f^{-1}(C_i)$, say $f^{-1}(C_1)$, must contain an open cell B. The restriction of f to B gives an injective continuous definable map $B \to C_1$. Since B is definably homeomorphic to R^n and C_1 definably homeomorphic to R^m with $m < n$, we obtain an injective continuous definable map $g : R^n \to R^m$. Set $a = (0, \dots, 0) \in R^{n-m}$ and consider the mapping $g_a : R^m \to R^m$ defined by $g_a(x) = g(a, x)$. We can apply the inductive assumption to g_a. It implies that $g_a(R^m)$ has nonempty interior in R^m. Take a point $c = g_a(b)$ in the interior of $g_a(R^m)$. Since g is continuous we can find $x \in R^{n-m}$, $x \neq a$ and close to a, such that $g(x, b) \in g_a(R^m)$. There is $y \in R^m$ such that $g(x, b) = g_a(y) = g(a, y)$, which contradicts the fact that g is injective. Hence, $f(A)$ has nonempty interior. □

Corollary 3.16 *The dimension of R^d (according to Definition 3.14) is d. The dimension of a cell, as defined in Section 2.2, agrees with its dimension according to Definition 3.14.*

Proof. There is no injective definable map from R^e to R^d if $e > d$. Otherwise, the composition of such a map with the embedding of R^d in $R^e = R^d \times R^{e-d}$ as $R^d \times \{0\}$ would contradict Lemma 3.15. This shows the first part of the corollary. The second part follows, using the fact that the dimension according to 3.14 is invariant by definable bijection. □

Proposition 3.17 1. *If $A \subset B$ are definable sets,* $\dim A \leq \dim B$.

2. *If A and $f : A \to R^n$ are definable,* $\dim(f(A)) \leq \dim(A)$. *If moreover f is injective,* $\dim(f(A)) = \dim(A)$.

3. *If A and B are definable subsets of R^n,* $\dim(A \cup B) = \max(\dim A, \dim B)$.

4. *Let $A \subset R^n$ be definable and take a cdcd of R^n adapted to A. Then the dimension of A is the maximum of the dimension of the cells contained in A.*

5. *If A and B are definable sets,* $\dim(A \times B) = \dim A + \dim B$.

Proof. 1 is clear from the definition.

2. The second part is obvious since dimension is invariant by definable bijection. If f is definable, we get by Definable Choice a definable mapping $g : f(A) \to A$ such that $f \circ g = \mathrm{Id}_{f(A)}$. Hence, g is injective and $\dim(f(A)) = \dim(g(f(A))) \leq \dim(A)$.

3. The inequality $\dim(A \cup B) \geq \max(\dim A, \dim B)$ follows from 1. Now let $f : R^d \to A \cup B$ be a definable injective map. Taking a cdcd of R^d adapted to $f^{-1}(A)$ and $f^{-1}(B)$, we see that $f^{-1}(A)$ or $f^{-1}(B)$ contains a cell of dimension d. Since f is injective, we have $\dim A \geq d$ or $\dim B \geq d$. This proves the reverse inequality $\dim(A \cup B) \leq \max(\dim A, \dim B)$.

4 is an immediate consequence of 3.

5. By 3, it is sufficient to consider the case where A and B are cells. Since $A \times B$ is definably homeomorphic to $R^{\dim A} \times R^{\dim B}$, the assertion in this case follows from Corollary 3.16. □

Now we study the variation of dimension in a definable family. Recall that a definable subset A of $R^m \times R^n$ can be considered as a definable family $(A_x)_{x \in R^m}$ of subsets of R^n, where $A_x = \{y \in R^n \ ; \ (x, y) \in A\}$.

Theorem 3.18 *Let A be a definable subset of $R^m \times R^n$. For $d \in \mathbb{N} \cup \{-\infty\}$, set $X_d = \{x \in R^m \ ; \ \dim(A_x) = d\}$. Then X_d is a definable subset of R^m, and* $\dim(A \cap (X_d \times R^n)) = \dim(X_d) + d$.

Proof. We take a cdcd of $R^m \times R^n$ adapted to A and use Proposition 3.11. Let B be a cell of R^m. For every x in B, A_x is the union of the cells $C_x \subset R^n$, for all cells $C \subset R^m \times R^n$ contained in A such that $p_{m+n,m}(C) = B$. Moreover, $\dim(C_x) = \dim C - \dim B$. It follows that, for every $x \in B$, we have $\dim A_x = \dim(A \cap (B \times R^n)) - \dim B$. Hence, each X_d is the union of some cells B of R^m. This implies that X_d is definable. Since $\dim(A \cap (B \times R^n)) = \dim B + d$ for a cell $B \subset X_d$, we have $\dim(A \cap (X_d \times R^n)) = \dim X_d + d$. □

Exercise 3.19 Let A and B be subsets of $R^m \times R^n$, with A nonempty. Assume that, for every $x \in R^m$, $\dim(B_x) < \dim(A_x)$ or B_x is empty. Prove that $\dim B < \dim A$.

Exercise 3.20 Let $f : A \to R^m$ be a definable map. Prove that the set Y_d of $x \in R^m$ such that $\dim(f^{-1}(x)) = d$ is definable. Prove that $\dim(Y_d) = \dim(f^{-1}(Y_d)) - d$.

We finish this section on dimension by showing that the dimension behaves well with respect to closure. The following lemma will be useful.

Lemma 3.21 *Let A be a definable subset of $R^m \times R^p$. Let M be the set of $x \in R^m$ such that the closure of A_x in R^p is different from $(\mathrm{clos}(A))_x$. Then M is definable and $\dim M < m$. In particular, if $m = 1$, M is finite.*

Proof. Note that we have always $\mathrm{clos}(A_x) \subset (\mathrm{clos}(A))_x$, since $(\mathrm{clos}(A))_x$ is closed and contains A_x.

We leave the verification of the definability of M as an exercise. Suppose that $\dim(M) = m$. Then M contains an open cell C of a cdcd of R^m. For every $x \in C$, we can find a box $B = \prod_{i=1}^p (a_i, b_i)$ such that $B \cap (\mathrm{clos}(A))_x \neq \emptyset$ and $B \cap A_x = \emptyset$. By Definable Choice 3.1 and Piecewise Continuity 2.12, we can assume that all a_i and b_i are definable continuous functions of $x \in C$. Let U be the set of $(x, y_1, \ldots, y_p) \in C \times R^p$ such that $a_i(x) < y_i < b_i(x)$ for $i = 1, \ldots, p$. The set U is open in $R^m \times R^p$, disjoint from A and has nonempty intersection with $\mathrm{clos}(A)$. This is impossible. Hence, $\dim M < m$. $\square$

Theorem 3.22 *Let A be a nonempty definable subset of R^n. Then*

$$\dim(\mathrm{clos}(A) \setminus A) < \dim(A) .$$

It follows from the theorem that $\dim(\mathrm{clos}(A)) = \dim A$.

Proof. We proceed by induction on n. The theorem is obvious for $n = 1$. We assume that $n > 1$ and the theorem is proved for $n - 1$. We denote by $\xi_1, \ldots, \xi_n$ the coordinate functions on R^n. For $i = 1, \ldots, n$, let $\mathrm{clos}_i(A)$ be the set of $x \in R^n$ such that x belongs to the closure of the intersection of A with the hyperplane $\xi_i^{-1}(\xi_i(x))$.

First step. We claim that $\mathrm{clos}(A) \setminus A$ has dimension not greater than the maximum of 0 and $\dim(\mathrm{clos}_i(A) \setminus A)$ for $i = 1, \ldots, n$. We have $\mathrm{clos}_i(A) \subset \mathrm{clos}(A)$. Applying Lemma 3.21 after a permutation of the coordinates which puts the i-th coordinate in the first place, we obtain that the difference $\mathrm{clos}(A) \setminus \mathrm{clos}_i(A)$ is contained in finitely many hyperplanes $\xi_i^{-1}(a_{i,j})$, for $j = 1, \ldots, \ell(i)$. Hence, $\mathrm{clos}(A) \setminus \bigcup_{i=1}^n \mathrm{clos}_i A$ is contained in the finite set consisting of the $\prod_{i=1}^n \ell(i)$ points $(a_{1,j_1}, \ldots, a_{n,j_n}) \in R^n$. The claim follows.

Second step. We claim that $\dim(\mathrm{clos}_i(A) \setminus A) < \dim A$ for $i = 1, \ldots, n$. Take $a \in R$. Since the hyperplane $\xi_1^{-1}(a)$ has dimension $n-1$, the inductive assumption implies that $\mathrm{clos}(A \cap \xi_1^{-1}(a)) \setminus (A \cap \xi_1^{-1}(a))$ is empty or has dimension strictly smaller than the dimension of $A \cap \xi_1^{-1}(a)$; note that the first set is $(\mathrm{clos}_1(A) \setminus A) \cap \xi_1^{-1}(a)$. Hence, applying the result of Exercise 3.19, we obtain $\dim(\mathrm{clos}_1(A) \setminus A) < \dim A$. Permuting the coordinates, we prove the claim.

Now we can complete the proof of the theorem. Steps 1 and 2 imply that $\dim(\mathrm{clos}(A) \setminus A) \leq 0$ or $\dim(\mathrm{clos}(A) \setminus A) < \dim A$. If $\dim A = 0$, A is closed. Hence, for every nonempty A, $\dim(\mathrm{clos}(A) \setminus A) < \dim A$. $\square$

Chapter 4

Definable Triangulation

In this chapter we show that the topology of definable sets can be entirely encoded in finite terms. This will be done by means of a triangulation.

4.1 Good Coordinates

As we have seen in the last chapter, the cdcd is a very powerful tool. But it does not give sufficient control on the relative disposition of the cells, when they are not contained in the same cylinder. In particular, one cannot, in general, reconstruct the topology of a definable set from its decomposition into cells of an adapted cdcd. The main difficulty is that we have no control on how a definable continuous function $\zeta : C \to R$ on a cell C behaves as one approaches the boundary of C. The function ζ, even if it is bounded, may not extend to a continuous function on $\mathrm{clos}(C)$. For instance, the definable continuous function ζ defined on the set of $(x, y) \in R^2$ such that $x > 0$ by $\zeta(x, y) = 2xy/(x^2 + y^2)$ does not extend continuously to $(0, 0)$. All points $(0, 0, z)$ with $-1 \leq z \leq 1$ belong to the closure Γ of the graph of ζ. The definable set $\Gamma \subset R^3$ has dimension 2, but the restriction to Γ of the projection $R^3 \to R^2$ on the first two coordinates is not finite-to-one (see Figure 4.1).

In this section, we consider the following problem: given a definable subset G of R^n of dimension $< n$, can we make a polynomial change of coordinates in R^n such that the restriction to G of the projection on the first $n - 1$ new coordinates becomes finite-to-one? In other words, we want a polynomial automorphism $u : R^n \to R^n$ such that $p_{n,n-1}|_{u(G)} : u(G) \to R^{n-1}$ is finite-to-one. We look for u of the form $u(x', x_n) = (x' - v'(x_n), x_n)$, where $x' = (x_1, \ldots, x_{n-1})$ and $v' : R \to R^{n-1}$ is a polynomial map. The condition that

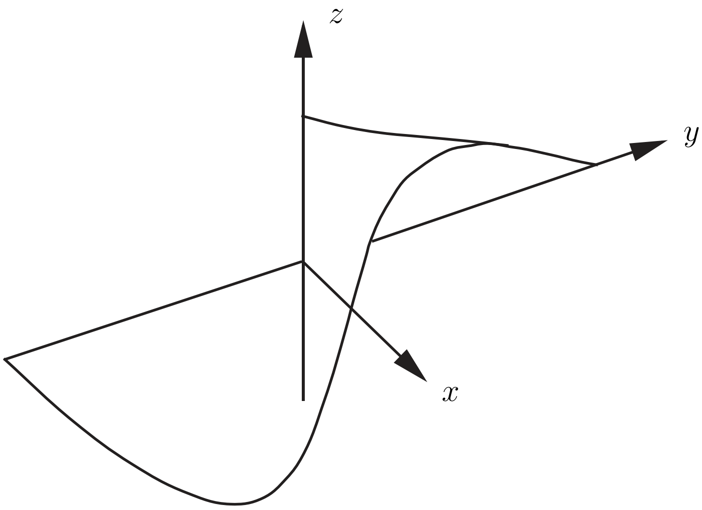


Figure 4.1: $\Gamma \subset R^3$ near the origin

$p_{n,n-1}|_{u(G)}$ is finite-to-one reads:

$$(*) \qquad \begin{cases} \text{for all } y' \in R^{n-1}, \text{ the set of } x_n \in R \\ \text{such that } (y' + v'(x_n), x_n) \in G \text{ is finite.} \end{cases}$$

We introduce the definable set

$$W = \{(y', x', x_n) \in R^{n-1} \times R^{n-1} \times R^n \ ; \ (y' + x', x_n) \in G\} \ ,$$

and we set, for $y' \in R^{n-1}$, $W_{y'} = \{x \in R^n \ ; \ (y', x) \in W\}$. Note that, for all $y' \in R^{n-1}$, $W_{y'} = G - (y', 0)$ has dimension $< n$. We can translate the condition $(*)$ as follows:

$$\text{for all } y' \in R^{n-1}, \text{ the set of } x_n \in R \text{ such that } (v'(x_n), x_n) \in W_{y'} \text{ is finite.}$$

Hence, our problem will be solved by the following lemma. In the statement of the lemma, we exchange x_1 and x_n; it will make the proof by induction easier to write.

Lemma 4.1 *Let $W \subset R^m \times R^n$ ($n \geq 2$) be a definable set. For $s \in R^m$, define $W_s = \{y \in R^n \ ; \ (s, y) \in W\}$. Assume that, for all s in R^m, $\dim(W_s) < n$. Then there exist a polynomial map $v' : R \to R^{n-1}$ of degree not greater than m such that, for all s in R^m, the set of $x_1 \in R$ such that $(x_1, v'(x_1)) \in W_s$ is finite.*

Proof. We proceed by induction on n. We begin with $n = 2$. Let V be a definable subset of dimension 1 of R^2. A cdcd of R^2 adapted to V decomposes V as the disjoint union of finitely many points, vertical open intervals and graphs of definable continuous functions $\zeta_i : I_i \to R$, where I_i is an interval. Consider such a function ζ_i. For $a = (a_0, a_1, \ldots, a_m) \in R^{m+1}$, define $f_a : R \to R$ by $f_a(t) = \sum_{i=0}^m a_i t^i$. The set

$$\Big\{x_1 \in I_i \ ; \ \exists a \in R^{m+1} \ \exists \epsilon \in R \\ \Big(\epsilon > 0 \text{ and } \forall y \in I_i \ (|y - x_1| < \epsilon \Rightarrow f_a(y) = \zeta_i(y))\Big)\Big\}$$

is a definable open subset of R. It has finitely many definable connected components which are open intervals. If U is one of its definable connected components, there is a unique a such that $f_a|_U = \zeta_i|_U$. Hence, there are finitely many $a \in R^{m+1}$ such that the set of $x_1 \in R$ such that $(x_1, f_a(x_1)) \in V$ is infinite. From this we deduce that, for all $s \in R^m$, there are finitely many $a \in R^{m+1}$ such that the set $B_{a,s} = \{x_1 \in R \ ; \ (x_1, f_a(x_1)) \in W_s\}$ is infinite. Therefore the definable set of (a, s) such that $B_{a,s}$ is infinite has dimension at most m. Then also the set of $a \in R^{m+1}$, such that there is $s \in R^m$ such that $B_{a,s}$ is infinite, has dimension at most m. Hence, there exists a polynomial f_a of degree not greater than m such that, for all $s \in R^m$, the set $B_{a,s}$ is finite.

Given $n > 2$, assume the lemma proved for $n - 1$. Let Z be the definable set of $(s, x_1, u) \in R^m \times R \times R^{n-2}$ such that $\{x_n \in R \ ; \ (s, x_1, u, x_n) \in W\}$ is infinite. For all $s \in R^m$, the set Z_s has dimension at most $n-2$ because W_s has dimension at most $n - 1$. Therefore, we can apply the inductive assumption to obtain a polynomial map $g : R \to R^{n-2}$ of degree at most m such that, for all $s \in R^m$, the set of $x_1 \in R$ such that $(x_1, g(x_1)) \in Z_s$ is finite. Consider the definable subset of $R^m \times R^2$

$$M = \{(s, x_1, x_n) \in R^m \times R^2 \ ; \ (s, x_1, g(x_1), x_n) \in W\} \ .$$

For all $s \in R^m$, the set M_s has dimension at most 1. Therefore, by the argument above, there is a polynomial f of degree at most m such that for all $s \in R^m$ the set of $x_1 \in R$ such that $(x_1, f(x_1)) \in M_s$ is finite. Set $v' = (g, f)$, and the proof is complete. $\square$

The lemma actually solves more than our initial problem.

Proposition 4.2 *Let G be a definable subset of $R^q \times R^n$. Assume that, for every t in R^q, the dimension of $G_t = \{x \in R^n \ ; \ (t, x) \in G\}$ is $< n$. Let*

$p_{q+n,q+n-1} : R^q \times R^n \to R^q \times R^{n-1}$ *be the projection on the first* $q+n-1$ *coordinates. Then there is a polynomial automorphism* u *of* R^n *such that*

$$\forall (t,x') \in R^q \times R^{n-1} \quad p^{-1}_{q+n,q+n-1}(t,x') \cap (R^q \times u(G_t)) \quad \textit{is finite} .$$

Proof. Set

$$\begin{aligned} W &= \{(t,y',x',x_n) \in R^q \times R^{n-1} \times R^{n-1} \times R \ ; \ (t, y'+x', x_n) \in G\} \\ W_{(t,y')} &= \{(x',x_n) \in R^{n-1} \times R \ ; \ (t,y',x',x_n) \in W\} . \end{aligned}$$

For all (t,y') in $R^q \times R^{n-1}$, the dimension of $W_{(t,y')} = G_t - (y',0)$ is $< n$. By Lemma 4.1 (exchanging x_1 and x_n), there is a polynomial mapping $v' : R \to R^{n-1}$ such that the set of x_n in R such that $(v'(x_n), x_n) \in W_{(t,y')}$ is finite. Now set $u(x',x_n) = (x' - v'(x_n), x_n)$. Then u is a polynomial automorphism of R^n which satisfies the condition of the proposition. □

The case $q = 0$ is our initial problem. In fact, this problem can be solved with u a linear automorphism (see [vD], Theorem 4.2). The case $q = 1$ will be useful for the triangulation of functions. The point of having a finite-to-one projection is explained by the following proposition.

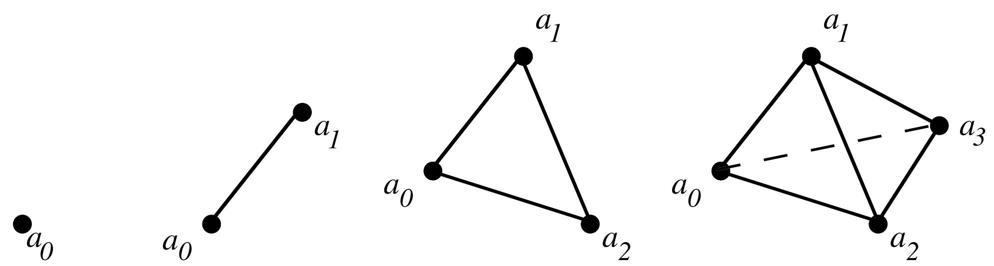
Proposition 4.3 *Let* F *be a closed and bounded definable subset of* R^n*, such that* $p_{n,n-1}|_F$ *is finite-to-one. Let* $X \subset p_{n,n-1}(F)$ *be a definable subset of* R^{n-1} *such that every* $x' \in \operatorname{clos}(X)$ *has a basis of neighborhoods* U *such that* $U \cap X$ *is definably connected (this is the case, for instance, if* X *is convex). Then every continuous definable function* $\zeta : X \to R$ *whose graph is contained in* F *extends continuously to* $\operatorname{clos}(X)$.

Proof. Let $\Gamma \subset F$ be the graph of ζ. Its closure is bounded, and $p_{n,n-1}(\operatorname{clos}(\Gamma)) = \operatorname{clos}(X)$. Take $x' \in \operatorname{clos}(X)$. The set $\operatorname{clos}(\Gamma) \cap p^{-1}_{n,n-1}(x')$ is nonempty and finite. Take $a \in R$ and $\delta > 0$ such that $(\{x'\} \times (a-\delta, a+\delta)) \cap \operatorname{clos}(\Gamma) = (x', a)$. For every ε such that $0 < \varepsilon < \delta$, there is neighborhood U of x' such that $U \cap X$ is definably connected, $U \cap \zeta^{-1}(a-\varepsilon, a+\varepsilon) \neq \emptyset$, and $\zeta(y') \neq a \pm \varepsilon$ for every $y' \in X \cap U$. It follows that $\zeta(U \cap X) \subset (a-\varepsilon, a+\varepsilon)$. This proves that ζ extends continuously to x', with value a. □

4.2 Simplicial Complex

We recall some definitions concerning simplicial complexes that we shall need. Let $a_0, \ldots, a_d$ be points of R^n which are affine independent (i.e. not contained in an affine subspace of dimension $d-1$). The *d-simplex* with vertices $a_0, \ldots, a_d$ is

$$[a_0, \ldots, a_d] =$$
$$\{x \in R^n \ ; \ \exists \lambda_0, \ldots, \lambda_d \in [0,1] \quad \sum_{i=0}^{d} \lambda_i = 1 \ \text{ and } \ x = \lambda_0 a_0 + \ldots + \lambda_d a_d\} \ .$$



0-simplex 1-simplex 2-simplex 3-simplex

Figure 4.2: Simplices

The corresponding *open simplex* is

$$\{x \in R^n \ ; \ \exists \lambda_0, \ldots, \lambda_d \in (0,1] \ \sum_{i=0}^{d} \lambda_i = 1 \ \text{ and } \ x = \lambda_0 a_0 + \cdots + \lambda_d a_d\}$$

We shall denote by σ the open simplex corresponding to the simplex $\overline{\sigma}$. A *face* of the simplex $\overline{\sigma} = [a_0, \ldots, a_d]$ is a simplex $\overline{\tau} = [b_0, \ldots, b_e]$ such that

$$\{b_0, \ldots, b_e\} \subset \{a_0, \ldots, a_d\} \ .$$

A *finite simplicial complex* in R^n is a finite collection $K = \{\overline{\sigma}_1, \ldots, \overline{\sigma}_p\}$ of simplices $\overline{\sigma}_i \subset R^n$ such that, for every $\overline{\sigma}_i, \overline{\sigma}_j \in K$, the intersection $\overline{\sigma}_i \cap \overline{\sigma}_j$ is a common face of $\overline{\sigma}_i$ and $\overline{\sigma}_j$ (see Figure 4.3).

We set $|K| = \bigcup_{\overline{\sigma}_i \in K} \overline{\sigma}_i$; this is a semialgebraic subset of R^n. A *polyhedron* in R^n is a subset P of R^n, such that there exists a finite simplicial complex K in R^n with $P = |K|$. Such a K will be called a *simplicial decomposition* of P. Note that a polyhedron is a closed and bounded definable set. In the following, it will be convenient to agree that if a simplex $\overline{\sigma}$ belongs to a finite simplicial complex K, then all faces of $\overline{\sigma}$ also belong to K. With this convention, $|K|$ is the disjoint union of all open simplices σ for $\overline{\sigma} \in K$.

We shall also use the notion of a *cone*. Let B be a polyhedron in R^n, and $a \in (R^n \setminus B)$ such that every half-line from a intersects B in at most one point (i.e., for every $x \in B$, $[a,x] \cap B = \{x\}$). The cone with base B and vertex a is the polyhedron

$$a * B = \{tx + (1-t)a \; ; \; x \in B \;\text{ and }\; t \in [0,1]\} \, .$$

Given a simplicial decomposition of B, we obtain a simplicial decomposition of $a * B$ by taking all $a * \overline{\sigma}$, for $\overline{\sigma}$ a simplex of the simplicial decomposition of B

4.3 Triangulation of Definable Sets

Theorem 4.4 *Let A be a closed and bounded definable subset of R^n and let B_i, $i = 1, \ldots, k$, be definable subsets of A. Then there exist a finite simplicial complex K with vertices in $\mathbb{Q}^n$ and a definable homeomorphism $\Phi : |K| \to A$ such that each B_i is a union of images by Φ of open simplices of K.*

Proof. We proceed by induction on n. The case of $n = 1$ is obvious. We can subdivide R with finitely many points $x_1 < \ldots < x_p$ such that A and the B_i are unions of points x_i and intervals (x_j, x_{j+1}). Then we choose a definable homeomorphism $\tau : R \to R$ such that $\tau(x_i) = i$ for $i = 1, \ldots, p$. We take for Φ the restriction of τ^{-1} to $\tau(A)$.

Now assume that $n > 1$ and the theorem is proved for $n - 1$. Since every definable set is a finite union of locally closed definable sets (cf. Exercise 3.8), we may assume that the B_i are locally closed. Then we can replace B_i with its closure $\mathrm{clos}(B_i)$ and the difference $\mathrm{clos}(B_i) \setminus B_i$. Hence, we can assume that all B_i are closed. Let F_0 be the boundary $A \cap \mathrm{clos}(R^n \setminus A)$ of A and F_i the boundary of B_i, for $i = 1, \ldots, k$. Set $F = \bigcup_{i=0}^k F_i$. Then F is a closed and bounded definable set of dimension $< n$. Let $p = p_{n,n-1} : R^n \to R^{n-1}$ be the

Figure 4.3: Simplicial complex

projection on the first $n-1$ coordinates. By Proposition 4.2, we can assume that, for all $x' \in R^{n-1}$, $p^{-1}(x') \cap F$ is finite.

We make a cdcd of R^n adapted to $F_0, F_1, , \ldots, F_k$. We get a finite partition of $p(A) = p(F_0)$ into definably connected definable subsets X_λ of R^{n-1}, and definable continuous functions

$$\zeta_{\lambda,1} < \ldots < \zeta_{\lambda,m(\lambda)} : X_\lambda \longrightarrow R \,,$$

such that every graph of $\zeta_{\lambda,\mu}$ is contained in some F_i and every F_i is a union of graphs of $\zeta_{\lambda,\mu}$. Since every band $(\zeta_{\lambda,\mu}, \zeta_{\lambda,\mu+1})$ is definably connected and disjoint from the boundaries F_i, it is contained in or disjoint from each one of $A, B_1, \ldots, B_k$.

Applying the inductive assumption, we obtain a simplicial complex L with vertices in $\mathbb{Q}^{n-1}$ and a definable homeomorphism $\Psi : |L| \to p(A)$ and such that all X_λ are the images by Ψ of unions of open simplices of L. Replacing A with $\{(x', x_n) \in |L| \times R \; ; \; (\Psi(x'), x_n) \in A\}$, we can assume that Ψ is the identity. Moreover, we can partition the X_λ and assume that they are the open simplices σ_λ of L (the open simplices are no longer cells of a cdcd). We denote by $(C_\alpha)_{\alpha=1,\ldots,\ell}$ the collection of all graphs of $\zeta_{\lambda,\mu} : \sigma_\lambda \to R$ and all bands $(\zeta_{\lambda,\mu}, \zeta_{\lambda,\mu+1}) \subset \sigma_\lambda \times R$ which are contained in A. By Proposition 4.3, since F is closed and bounded and $p|_F : F \to R^{n-1}$ is finite-to-one, every $\zeta_{\lambda,\mu} : \sigma_\lambda \to R$ can be continuously extended to the closed simplex $\overline{\sigma}_\lambda$. We denote the extension by $\overline{\zeta}_{\lambda,\mu}$. The graph of this extension is contained in F. It follows that, if $\sigma_{\lambda'}$ is contained in $\overline{\sigma}_\lambda$, the restriction of $\overline{\zeta}_{\lambda,\mu}$ to $\sigma_{\lambda'}$ coincides with some $\zeta_{\lambda',\mu'}$. We denote by $\overline{C}_\alpha$ the closure of C_α. It is either the graph of some $\overline{\zeta}_{\lambda,\mu}$ or some closed band $[\overline{\zeta}_{\lambda,\mu}, \overline{\zeta}_{\lambda,\mu+1}] \subset \overline{\sigma}_\lambda \times R$ (obvious notation). The set $\partial C_\alpha = \overline{C}_\alpha \setminus C_\alpha$ is a union of C_β with $\dim(C_\beta) < \dim(C_\alpha)$.

For every simplex $\overline{\sigma}_\lambda$ of L, let $b(\sigma_\lambda) \in \mathbb{Q}^{n-1}$ be its barycenter. If C_α is the graph of $\zeta_{\lambda,\mu} : \sigma_\lambda \to R$, we set $b_\alpha = (b(\sigma_\lambda), \mu) \in \mathbb{Q}^n$. If C_α is the band $(\zeta_{\lambda,\mu}, \zeta_{\lambda,\mu+1})$, we set $b_\alpha = (b(\sigma_\lambda), \mu + \frac{1}{2}) \in \mathbb{Q}^n$. Now we associate to each $\overline{C}_\alpha$ a polyhedron $\overline{D}_\alpha$, by induction on $\dim(\overline{C}_\alpha)$. Moreover, we give a simplicial decomposition of $\overline{D}_\alpha$. If $\overline{C}_\alpha$ is a point, we set $\overline{D}_\alpha = \{b_\alpha\}$. If $\dim(\overline{C_\alpha}) > 0$, then $\overline{D_\alpha}$ is the cone with vertex b_α and base the union ∂D_α of all $\overline{D}_\beta$ such that $C_\beta \subset \partial C_\alpha$. We decompose $\overline{D_\alpha}$ by taking the cones with vertex b_α over all simplices of the simplicial decompositions of $\overline{D}_\beta \subset \partial D_\alpha$. We obtain in this way a finite simplicial complex K such that $|K| = \bigcup_{\alpha=1}^{\ell} \overline{D}_\alpha$. The simplices of K are all simplices $[b_{\alpha_0}, b_{\alpha_1}, \ldots, b_{\alpha_q}]$ such that $C_{\alpha_{i-1}} \subset \partial C_{\alpha_i}$ for $i = 1, \ldots, q$. The complex K has all its vertices in $\mathbb{Q}^n$. Note that, by construction, $p(\overline{D}_\alpha) = p(\overline{C}_\alpha)$. We construct a definable homeomorphism $\theta_\alpha : \overline{D}_\alpha \to \overline{C}_\alpha$ such that $p \circ \theta_\alpha = p|_{\overline{D}_\alpha}$. If $\overline{C}_\alpha$ is the graph of $\overline{\zeta}_{\lambda,\mu}$, we must have $\theta_\alpha(x', x_n) = (x', \overline{\zeta}_{\lambda,\mu}(x'))$.

If $\overline{C}_\alpha$ is the closed band $[\overline{\zeta}_{\lambda,\mu}, \overline{\zeta}_{\lambda,\mu+1}]$, we choose θ_α so that, for every $x' \in \overline{\sigma}_\lambda$, it carries the segment $(\{x'\} \times R) \cap \overline{D}_\alpha$ onto the segment $\{x'\} \times [\overline{\zeta}_{\lambda,\mu}(x'), \overline{\zeta}_{\lambda,\mu+1}(x')]$ in an affine way (note that $(\{x'\} \times R) \cap \overline{D}_\alpha$ is reduced to a point if and only if $\overline{\zeta}_{\lambda,\mu}(x') = \overline{\zeta}_{\lambda,\mu+1}(x')$). The homeomorphism θ_α carries ∂D_α onto ∂C_α.

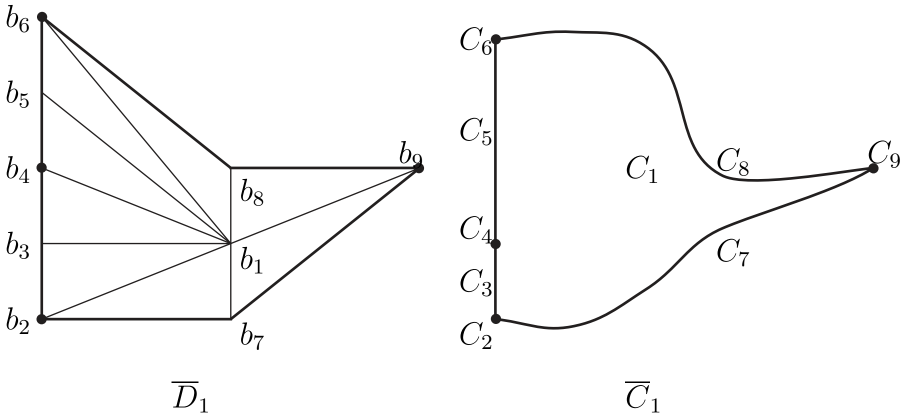


Figure 4.4: The construction of K

We finish the proof by constructing a definable homeomorphism $\Phi : |K| \to A$ such that $p \circ \Phi = p_{|K|}$ and $\Phi(\overline{D}_\alpha) = \overline{C}_\alpha$ for $\alpha = 1, \ldots, \ell$. We cannot just take $\Phi|_{\overline{D}_\alpha} = \theta_\alpha$, because $\theta_\alpha|_{\overline{D}_\beta}$ may be different from θ_β for $\overline{D}_\beta \subset \partial D_\alpha$ (see what happens for $\overline{D}_3 \subset \partial D_1$ in Figure 4.4). So we modify θ_α to obtain a definable homeomorphism $\Phi_\alpha : \overline{D}_\alpha \to \overline{C}_\alpha$ verifying $\Phi_\alpha|_{\overline{D}_\beta} = \Phi_\beta$ and $p \circ \Phi_\alpha = p|_{\overline{D}_\alpha}$. We proceed by induction on $\dim(\overline{D}_\alpha)$. We take $\Phi_\alpha = \theta_\alpha$ if $\overline{D}_\alpha$ is a point. If $\dim(\overline{D}_\alpha) > 0$, denote by $\rho_\alpha : \partial D_\alpha \to \partial D_\alpha$ the homeomorphism defined by $\rho_\alpha|_{\overline{D}_\beta} = \theta_\alpha^{-1} \circ \Phi_\beta$ for all $\overline{D}_\beta \subset \partial D_\alpha$. We extend the homeomorphism ρ_α to a homeomorphism $\eta_\alpha : \overline{D}_\alpha \to \overline{D}_\alpha$ using the conic structure of $\overline{D}_\alpha = b_\alpha * \partial D_\alpha$: for every $x \in \partial D_\alpha$ and $t \in [0,1]$, we set $\eta_\alpha(tx + (1-t)b_\alpha) = t\rho_\alpha(x) + (1-t)b_\alpha$. Then we set $\Phi_\alpha = \theta_\alpha \circ \eta_\alpha$. Observe that $p \circ \eta = p|_{\overline{D}_\alpha}$ and $p \circ \Phi_\alpha = p|_{\overline{D}_\alpha}$. Now we can take Φ defined by $\Phi_{\overline{D}_\alpha} = \Phi_\alpha$ for $\alpha = 1, \ldots, \ell$. $\square$

4.4 Triangulation of Definable Functions

Theorem 4.5 *Let X be a closed and bounded definable subset of R^n and $f : X \to R$ a continuous definable function. Then there exist a finite simplicial complex K in R^{n+1} and a definable homeomorphism $\rho : |K|_R \to X$ such that $f \circ \rho$ is an affine function on each simplex of K.*

Moreover, given finitely many definable subsets $B_1, \dots, B_k$ of X, we may choose the triangulation $\rho : |K|_R \to X$ so that each B_i is a union of images of open simplices of K.

We replace X with the set $A = \{(f(x), x) \in R \times R^n \ ; \ x \in X\}$ which is definably homeomorphic to X. This set is a closed and bounded definable subset of R^{n+1}. Let $\pi = p_{n+1,1} : R^{n+1} \to R$ be the projection on the first coordinate. In order to prove Theorem 4.5, it is sufficient to construct a finite simplicial complex K in R^{n+1} and a definable homeomorphism $\Phi : |K| \to A$ such that $\pi \circ \Phi = \pi|_{|K|}$. Indeed, the composition of π with the homeomorphism $X \to A$ is f. So Theorem 4.5 follows from the next proposition.

Proposition 4.6 *Let A be a closed and bounded definable subset of $R \times R^n$ and let B_i, $i = 1, \dots, k$, be definable subsets of A. Let $\pi : R \times R^n \to R$ be the projection on the first coordinate. Then there exist a finite simplicial complex K with vertices in $R \times \mathbb{Q}^n$ and a definable homeomorphism $\Phi : |K| \to A$ such that $\pi \circ \Phi = \pi|_{|K|}$ and each B_i is a union of images by Φ of open simplices of K.*

Proof. We proceed by induction on n. The case of $n = 0$ is obvious. We can subdivide R with finitely many points $x_1 < \ldots < x_p$ such that A and the B_i are unions of points x_i and intervals (x_j, x_{j+1}). We choose for K the collection of points x_i and closed and bounded intervals $[x_j, x_{j+1}]$ contained in A.

Assume that $n > 0$ and the proposition is proved for $n-1$. As in the proof of Theorem 4.4, we can assume that all B_i are closed.

Let G_0 be the boundary of A and G_i the boundary of B_i, for $i = 1, \dots, k$. Set $G = \bigcup_{i=0}^k G_i$. Then G is a closed and bounded definable set of dimension at most n. Denote by C the finite set of points $c \in R$ such that $\{x \in R^n \ ; \ (c, x) \in G\}$ is of dimension n. Let F_i, $i = 0, \dots, k$ be the union of the closure of $G_i \setminus (C \times R^n)$ with the boundary of $G_i \cap (C \times R^n)$ in $C \times R^n$. Set $F = \bigcup_{i=0}^k F_i$. Each F_i is a closed and bounded definable set. We claim that, for every t in R, the dimension of $F_t = \{x \in R^n \ ; \ (t, x) \in F\}$ is $< n$. Since $G_i \cap (C \times R^n)$ has dimension n, the dimension of its boundary is at most $n - 1$. Hence, it suffices to check that the dimension of $(\mathrm{clos}(G_i \setminus (C \times R^n)))_t$ is not greater than $n - 1$. This follows from the next lemma applied to $X = G_i \setminus (C \times R^n)$.

Lemma 4.7 *Let X be a definable subset of $R \times R^n$ such that, for every $t \in R$, the dimension of X_t is $< n$. Then, for every $t \in R$, we have* $\dim((\mathrm{clos}(X))_t) < n$.

Proof. It follows from the assumption that $\dim X \leq n$. Hence, the dimension of the boundary of X is $< n$. Since $(\mathrm{clos}(X))_t \setminus X_t$ is contained in the boundary of X, we have $\dim((\mathrm{clos}(X))_t) < n$. □

We return to the proof of Proposition 4.6. Let $p = p_{n+1,n} : R \times R^n \to R \times R^{n-1}$ be the projection on the first n coordinates. By Proposition 4.2, since $\dim(F_t) < n$ for every $t \in R$, we can assume that for all $(t, x') \in R \times R^{n-1}$, $(p^{-1}(t, x') \cap F)$ is finite. We choose a cdcd of $R \times R^n$ adapted to $F_0, \ldots, F_k$ and to $\{c\} \times R^n$ for every $c \in C$. We get a finite partition of $p(A)$ into definably connected definable subsets X_λ of $R \times R^{n-1}$, and definable continuous functions

$$\zeta_{\lambda,1} < \ldots < \zeta_{\lambda,m_\lambda} : X_\lambda \longrightarrow R \; ,$$

such that every graph of $\zeta_{\lambda,\mu}$ is contained in some F_i and every F_i is a union of graphs of $\zeta_{\lambda,\mu}$. Note that every graph of $\zeta_{\lambda,\mu}$ and every band $(\zeta_{\lambda,\mu}, \zeta_{\lambda,\mu+1})$ is either contained in G_i or disjoint from G_i. It follows that A and the B_i are unions of such cells.

Applying the inductive assumption, we may assume that there is a simplicial complex L with vertices in $R \times \mathbb{Q}^{n-1}$ and a definable homeomorphism $\Psi : |L| \to p(A)$ such that $p_{n,1} \circ \Psi = p_{n,1}|_{|L|}$ and all X_λ are unions of images of open simplices of L by Ψ. From now on, we can follow the proof of Theorem 4.4 and construct a triangulation $\Phi : |K| \to A$ such that all vertices of K are in $R \times \mathbb{Q}^n$, each B_i is the image by Φ of a union of open simplices of K, and $p \circ \Phi = \Psi \circ p|_{|K|}$. It follows that

$$\pi \circ \Phi = p_{n,1} \circ p \circ \Phi = p_{n,1} \circ \Psi \circ p|_{|K|} = p_{n,1} \circ p|_{|K|} = \pi|_{|K|} \; .$$

□

The vertices of the simplicial complex constructed in Proposition 4.6 have all their coordinates rational but the first. We shall need to have also the first coordinate rational. But this has to be paid with some extra complication.

Proposition 4.8 *Let A be a closed and bounded definable subset of $R \times R^n$ and let B_i, $i = 1, \ldots, k$, be definable subsets of A. Let $\pi : R \times R^n \to R$ be the projection on the first factor. Then there exist a finite simplicial complex K with vertices in $\mathbb{Q} \times \mathbb{Q}^n$ and definable homeomorphisms $\Phi : |K| \to A$ and $\tau : R \to R$ such that $\tau \circ \pi \circ \Phi = \pi|_{|K|}$ and each B_i is a union of images by Φ of open simplices of K.*

Proof. We just modify the step $n = 0$ of the proof of the preceding proposition. We can subdivide R with finitely many points $x_1 < \ldots < x_p$ such that A

and the B_i are unions of points x_i and intervals (x_j, x_{j+1}). We choose for K the collection of points $1, \ldots, p$ and segments $[j, j+1]$ such that $[x_j, x_{j+1}]$ is contained in A. We choose for τ a piecewise affine homeomorphism sending x_i to i. The rest of the proof is the same, using the appropriate inductive assumption. $\square$

The result of triangulation of continuous definable functions cannot be generalized to all continuous definable maps. Consider for instance, the map $f : [0,1]^2 \to \mathbb{R}^2$ defined by $f(x,y) = (x, xy)$. There is no way to choose triangulations $\Phi : |K| \to [0,1]^2$ and $\Psi : |L| \to f([0,1]^2)$ such that $\Psi^{-1} \circ f \circ \Phi$ is affine on every simplex of K.

Exercise 4.9 Prove the preceding assertion. Hint: Setting $a = (0,0) \in R^2$ and $A = f((0,1)^2)$, remark that $a \in \operatorname{clos}(A)$, $\dim(\operatorname{clos}(f^{-1}(A)) \cap f^{-1}(a)) = 1$ and $\dim(f^{-1}(x)) < 1$ for all $x \in A$. Show that this cannot happen for a map which is defined on a polyhedron and affine on each simplex of a simplicial decomposition.

Note that the crucial point in the proof of Theorem 4.5 which is only valid for functions with values in R is the claim that $\dim(F_t)$ is $< n$ for every t in R. Indeed, Lemma 4.7 is no longer true if the parameter t varies in R^m with $m > 1$. The counterexample above also gives a counterexample to the lemma for $m > 1$: take

$$X = \{(x, z, y) \in R^2 \times R \; ; \; x \neq 0 \;\text{ and }\; z = xy\} \; .$$

Then, for all $(x,z) \in R^2$, $\dim(X_{(x,z)}) < 1$, but $\dim(\operatorname{clos}(X)_{(0,0)}) = 1$.

We can deduce from the triangulation of definable functions a result which is usually proved in other ways (as a consequence of Hardt's theorem in [vD]). If $a \in R^n$ and $r > 0$, we denote by $S(a,r)$ the sphere with center a and radius r.

Theorem 4.10 (Local Conic Structure) *Let $A \subset R^n$ be a closed definable set, a a point in A. There is $r > 0$ such that there exists a definable homeomorphism h from the cone with vertex a and base $S(a,r) \cap A$ onto $\overline{B}(a,r) \cap A$, satisfying $h|_{S(a,r)\cap A} = \mathrm{Id}$ and $\|h(x) - a\| = \|x - a\|$ for all x in the cone.*

Proof. We triangulate the function $f : x \mapsto \|x - a\|$ restricted to $A \cap \overline{B}(a,1)$. We obtain a definable homeomorphism $\Phi : |K| \to A \cap \overline{B}(a,1)$ such that $f \circ \Phi$ is affine on each simplex of K. Since f takes its minimum in a, this point a

is the image of vertex w of K. Let $\mu > 0$ be the minimum of the values of $f \circ \Phi$ at all other vertices of K, and take r such that $0 < r < \mu$. Take a point $y \in |K|$ such that $f \circ \Phi(y) = r$, i.e. $\Phi(y) \in S(a, r) \cap A$. The point y belongs to a simplex $[v_0, \ldots, v_d]$ of K. We have $y = \sum_{i=0}^d \lambda_i v_i$ with $\sum_{i=0}^d \lambda_i = 1$. Since $f \circ \Phi(y) = \sum_{i=0}^d \lambda_i f \circ \Phi(v_i)$, one of the v_i, say v_0, must be w. Define h on the cone with vertex a and base $S(a,r) \cap A$ by $h(ta + (1-t)\Phi(y)) = \Phi(tw + (1-t)y)$. It is easily checked that h has the properties stated in the theorem. $\square$

The following exercise is a generalization of the local conic structure theorem.

Exercise 4.11 Prove the following result:

Let $Z \subset S$ be two closed and bounded definable sets. Let f be a nonnegative continuous definable function on S such that $f^{-1}(0) = Z$. Then there are $\delta > 0$ and a continuous definable map $h : f^{-1}(\delta) \times [0, \delta] \to f^{-1}([0, \delta])$, such that $f(h(x,t)) = t$ for every $(x,t) \in f^{-1}(\delta) \times [0, \delta]$, $h(x, \delta) = x$ for every $x \in f^{-1}(\delta)$, and $h|_{f^{-1}(\delta) \times\,]0,\delta]}$ is a homeomorphism onto $f^{-1}(]0, \delta])$.

Hints: 1) Triangulate f, so that one can assume $S = |K|$ for a finite simplicial complex, Z is a union of simplices and f is affine on each simplex.

2) Choose $\delta > 0$ so small that, for every vertex a of K such that $a \notin Z$, $\delta < f(a)$.

3) Let $x \in f^{-1}(\delta)$. The point x belongs to a simplex $[a_0, \ldots, a_d]$ of K. We can assume that $a_i \in Z$, for $i = 0, \ldots, k$, and $a_i \notin Z$, for $i = k+1, \ldots, d$. Let $x = \sum_{i=0}^d \lambda_i a_i$ with $\sum_{i=0}^d \lambda_i = 1$. Show that $\alpha = \sum_{i=0}^k \lambda_i$ satisfies $0 < \alpha < 1$.

4) For $t \in [0, \delta]$, define $h(x,t)$ as the point y in the segment joining $\sum_{i=0}^k (\lambda_i/\alpha) a_i$ to x such that $f(y) = t$. Check that h is well defined and satisfies the required properties.

Chapter 5

Generic Fibers for Definable Families

5.1 The Program

We motivate what we are going to do with a very simple example from algebraic geometry. Consider the family of conics $x^2 - t(1+y^2) = 0$ in $\mathbb{C}^2$, parametrized by $t \in \mathbb{C}$. We can also regard $x^2 - t(1+y^2) = 0$ as the equation of a conic defined over the field $\mathbb{C}(t)$. This conic is the "generic fiber of the family". The fact that the generic fiber of the family is a nondegenerate conic (the discriminant is t^2, which is invertible in $\mathbb{C}(t)$) corresponds to the fact that the conics in the family are non degenerate for almost all values of t (the only exception is $t = 0$). This is formalized by using the prime spectrum $\mathrm{Spec}(\mathbb{C}[t])$ of the ring $\mathbb{C}[t]$. We have $\mathbb{C}$ embedded in $\mathrm{Spec}(\mathbb{C}[t])$, each $z \in \mathbb{C}$ corresponding to the maximal ideal $(t - z)$ of $\mathbb{C}[t]$. The residue field at these maximal ideals is $\mathbb{C}$. There is another point in $\mathrm{Spec}(\mathbb{C}[t])$: the ideal (0), whose residue field is $\mathbb{C}(t)$. This point is the "generic point" of $\mathbb{C}$.

We are going to embed R^m into a bigger space $\widetilde{R^m}$. We shall associate to each point $\alpha \in \widetilde{R^m}$ a real closed field $\kappa(\alpha)$ with an o-minimal structure. Then, given a definable family $X \subset R^m \times R^n$ and $\alpha \in \widetilde{R^m}$, we shall define the fiber X_α which will be a definable subset of $\kappa(\alpha)^n$. If α is the image by the imbedding of a point $t \in R^m$, then $\kappa(\alpha)$ will be R and X_α will be the usual fiber X_t. The interesting things will happen for "generic fibers" X_α corresponding to points $\alpha \in \widetilde{R^m} \setminus R^m$. We shall relate the properties of the generic fiber X_α with properties of the family X over "large" subsets of the parameter space R^m. This will be used to deduce results of triviality for definable families from the triangulation theorems of the preceding chapter.

The tools that we are going to introduce are actually a reformulation of classical model-theoretic notions and results (m-types, definable ultrapower,...). This reformulation is modelled upon the theory of the real spectrum (cf. [BCR]).

5.2 The Space of Ultrafilters of Definable Sets

Let S_m be the Boolean algebra of definable subsets of R^m. The content of this section is nothing but the construction of the Stone space of the Boolean algebra S_m (cf. [BS]). Nevertheless, we give a more or less self-contained account of this construction for the reader who is not familiar with ultrafilters and Stone spaces.

We denote by $\widetilde{R^m}$ the set of ultrafilters of $\mathcal{S}_m$. Recall that an *ultrafilter of* S_m is a subset α of S_m such that

1. $R^m \in \alpha$
2. $A \cap B \in \alpha$ if and only if $A \in \alpha$ and $B \in \alpha$
3. $\emptyset \notin \alpha$
4. $A \cup B \in \alpha$ if and only if $A \in \alpha$ or $B \in \alpha$

We say that a family $\mathcal{F}$ of definable subsets of R^m *generates the ultrafilter* α if α is the set of definable sets $A \subset R^m$ such that there exists $B \in \mathcal{F}$ with $B \subset A$. If $\mathcal{F}$ is a nonempty family of definable subsets of R^m, closed under finite intersections, then it generates an ultrafilter if and only if $\emptyset \notin \mathcal{F}$ and, for every $A \in S_m$, either A or $R^m \setminus A$ contains a $B \in \mathcal{F}$.

If t is a point of R^m, the definable subsets of R^m containing t form an ultrafilter α_t. This is called the *principal ultrafilter* generated by t. The map $t \mapsto \alpha_t$ embeds R^m as a subset of $\widetilde{R^m}$. The following exercises give examples of points of $\widetilde{R^m} \setminus R^m$

Exercise 5.1 Points of $\widetilde{R}$. Show that that the following families generate ultrafilters of S_1:
1) The family of all intervals $(x, +\infty)$ for $x \in R$.
2) The family of all intervals $(-\infty, x)$ for $x \in R$.
3) For a fixed $a \in R$, the family of all intervals $(a, a+\varepsilon)$ for $\varepsilon > 0$.
4) For a fixed $a \in R$, the family of all intervals $(a-\varepsilon, a)$ for $\varepsilon > 0$.
If $R = \mathbb{R}$, show that all points of $\widetilde{\mathbb{R}} \setminus \mathbb{R}$ correspond to one of the four cases above.

Exercise 5.2 Construction of points of $\widetilde{R^m}$ by induction. Let α be a point of $\widetilde{R^m}$. Show that the family of definable subsets of R^{m+1} of the form

$$\{(x,y) \in R^m \times R \; ; \; x \in A \;\text{ and }\; 0 < y < f(x)\} ,$$

where $A \in \alpha$ and $f : A \to R$ is a positive definable function, generates an ultrafilter $\alpha_{\uparrow}$ of S_{m+1}. Hint: given a definable subset B of R^{m+1}, use a cdcd of R^{m+1} to show that either B or its complement belongs to $\alpha_{\uparrow}$.

Exercise 5.3 Dimension of a point of $\widetilde{R^m}$. For $\alpha \in \widetilde{R^m}$, define $\dim \alpha$ as the minimum of $\dim A$ for $A \in \alpha$. Show that $\dim \alpha = d$ if and only if α is generated by a family of definable sets of dimension d. Show that $\dim(\alpha_{\uparrow}) = \dim \alpha + 1$ (with the notation of Exercise 5.2). Show that there is an ultrafilter of dimension m in $\widetilde{R^m}$. Show that $\dim A$ is the maximum of $\dim \alpha$ for $\alpha \ni A$.

Now we put a topology on $\widetilde{R^m}$. We consider $\widetilde{R^m}$ as a subset of the powerset 2^{S_m}, identifying an ultrafilter α with its characteristic function $\mathbf{1}_\alpha : S_m \to 2 = \{0,1\}$. We equip 2 with the discrete topology and 2^{S_m} with the product topology. By Tychonoff theorem, 2^{S_m} is compact Hausdorf. We take as topology on $\widetilde{R^m}$ the topology induced by the topology of 2^{S_m}.

Proposition 5.4 *The topology of $\widetilde{R^m}$ has a basis of open and closed sets consisting of all $\widetilde{A} = \{\alpha \in \widetilde{R^m} \; ; \; \alpha \ni A\}$ for $A \in S_m$. The space $\widetilde{R^m}$ is compact Hausdorf.*

Proof. First note that the operation $A \mapsto \widetilde{A}$ preserves finite unions, finite intersections and taking complement.

The product topology of 2^{S_m} induces on $\widetilde{R^m}$ the topology which has as basis of open sets the sets of the form

$$U = \{\alpha \in \widetilde{R^m} \; ; \; \mathbf{1}_\alpha(A_1) = \ldots = \mathbf{1}_\alpha(A_k) = 1 \;\text{ and }\; \mathbf{1}_\alpha(B_1) = \ldots = \mathbf{1}_\alpha(B_\ell) = 0\} ,$$

where $(A_1, \ldots, A_k)$ and $(B_1, \ldots, B_\ell)$ are finite families of elements of S_m. Note that $U = \widetilde{A}$, where $A = \bigcap_{i=1}^k A_i \cap \bigcap_{j=1}^\ell (R^m \setminus B_j)$. This proves the first assertion of the proposition.

Finally, note that each one of the four conditions of the definiton of an ultrafilter defines a closed subset of 2^{S_m}. Hence, $\widetilde{R^m}$ is closed in 2^{S_m}. This proves the second assertion. $\square$

Exercise 5.5 Prove that the application $A \mapsto \widetilde{A}$ is a bijection from S_m to the set of open and closed subsets of $\widetilde{R^m}$.

5.3 O-minimal Structure Associated with an Ultrafilter

If A is a definable subset of R^m, we denote by $\mathrm{Def}(A, R)$ the ring of definable functions $A \to R$. Given $\alpha \in \widetilde{R^m}$, we define $\kappa(\alpha)$ as the inductive limit of $\mathrm{Def}(A, R)$ for $A \in \alpha$. This means the following. We form the disjoint union $\bigsqcup_{A\in\alpha} \mathrm{Def}(A, R)$. We say that two elements $f : A \to R$ and $g : B \to R$ of this disjoint union are equivalent if there exists $C \in \alpha$, $C \subset A \cap B$, such that $f|_C = g|_C$. Then $\kappa(\alpha)$ is the set of equivalence classes for this equivalence relation. We denote by $f(\alpha) \in \kappa(\alpha)$ the equivalence class of $f : A \to R$. By definition, $f(\alpha) = g(\alpha)$ if and only if f and g coincide on a definable set belonging to α. The inductive limit $\kappa(\alpha)$ has a canonical structure of commutative R-algebra: The sum $f(\alpha) + g(\alpha)$ is $(f + g)(\alpha)$, where $f + g$ is defined on the intersection of the domains of f and g, which belongs to α. We have a similar definition for the product $f(\alpha)g(\alpha)$. The images of elements of R are the classes of the constant functions. The verifications are easy.

We define $f(\alpha) \in \kappa(\alpha)$ to be positive if f is positive on a definable set belonging to α. This does not depend on the choice of the representant f.

Proposition 5.6 *The commutative R-algebra $\kappa(\alpha)$, with positive elements defined as above, is an ordered field which is an ordered extension of R.*

Proof. We check that, for an element $f(\alpha)$ of $\kappa(\alpha)$, we have exactly one of the three possibilities $f(\alpha) > 0$, $f(\alpha) = 0$ and $-f(\alpha) > 0$. This is because the domain of f is partitioned into the three definable sets where f is positive, zero or negative, respectively. Exactly one of these three sets belongs to α. Moreover, if $f(\alpha) \neq 0$, then $1/f$ is defined on a definable set belonging to α. Hence, $(1/f)(\alpha)$ is the inverse of $f(\alpha)$ in $\kappa(\alpha)$. The other verifications are also easy. □

Now we shall construct an o-minimal structure on $\kappa(\alpha)$. For this we need the notion of the fiber of a definable family $X \subset R^m \times R^n$ at $\alpha \in \widetilde{R^m}$. If $f = (f_1, \ldots, f_n) : A \to R^n$ is a definable map and $A \in \alpha$, we denote by $f(\alpha)$ the point $(f_1(\alpha), \ldots, f_n(\alpha)) \in \kappa(\alpha)^n$.

Definition 5.7 *Let X be a definable subset of $R^m \times R^n$. The* fiber *of X at $\alpha \in \widetilde{R^m}$ is the set X_α of those $f(\alpha)$ in $\kappa(\alpha)^n$ such that there exist $A \in \alpha$ on which f is defined and $(t, f(t)) \in X$ for all $t \in A$.*

In other words, $f(\alpha)$ belongs to X_α if and only if there is $A \in \alpha$ such that $f(t) \in X_t$ for all $t \in A$. This definition makes sense because, if we take another representant g of $f(\alpha)$ (i.e. $g(\alpha) = f(\alpha)$), g and f coincide on some $B \in \alpha$ and, therefore, $g(t) \in X_t$ for all $t \in A \cap B$, with $A \cap B \in \alpha$.

It is worth looking at what the preceding constructions and definitions give in the case of a principal ultrafilter α_t corresponding to a point $t \in R^m$. Since the ultrafilter α_t is generated by the singleton $\{t\}$, the inductive limit $\kappa(\alpha_t)$ is (canonically isomorphic to) $\mathrm{Def}(\{t\}, R) = R$. Moreover, if $f : A \to R$ is a definable function and $A \ni t$, then $f(\alpha_t) = f(t) \in R$. Finally, given a definable set $X \subset R^m \times R^n$, its fiber X_{α_t} is simply the set of $x \in R^n$ such that $(t, x) \in X$, i.e. X_t.

Now we can define the o-minimal structure on $\kappa(\alpha)$. Let $S_n(\alpha)$ be the family of all fibers X_α, for X a definable subset of $R^m \times R^n$.

Theorem 5.8 *The collection $(S_n(\alpha))_{n\in\mathbb{N}}$ is an o-minimal structure expanding the ordered field $\kappa(\alpha)$. In particular, $\kappa(\alpha)$ is real closed.*

Proof. We have to check the five properties in the definition of an o-minimal structure.

1) *For every $n \in \mathbb{N}$, $S_n(\alpha)$ is a Boolean subalgebra of the powerset of $\kappa(\alpha)^n$.*

It is sufficient to check the following fact. Let X and Y be definable subsets of $R^m \times R^n$. Then we have:

$$(X \cap Y)_\alpha = X_\alpha \cap Y_\alpha\ , \qquad (X \cup Y)_\alpha = X_\alpha \cup Y_\alpha\ ,$$
$$((R^m \times R^n) \setminus X)_\alpha = \kappa(\alpha)^n \setminus X_\alpha\ .$$

Let us check the last equality. Let $f(\alpha)$ be an element of $\kappa(\alpha)^n$, with $f_i \in \mathrm{Def}(A, R)$ and $A \in \alpha$. The two definable sets $B = \{t \in A\ ;\ f(t) \in X_t\}$ and $C = \{t \in A\ ;\ f(t) \in ((R^m \times R^n) \setminus X)_t\}$ partition A. Therefore, exactly one of B and C belongs to α. This means that $f(\alpha)$ belongs to $((R^m \times R^n) \setminus X)_\alpha$ if and only if it does not belong to X_α.

The verification of the other equalities is left to the reader.

2) *All algebraic subsets of $\kappa(\alpha)^n$ are in $S_n(\alpha)$.* It is sufficient to check this for an algebraic set $V \subset \kappa(\alpha)^n$ given by one equation

$$\sum_{i\in I} a_i(\alpha) x^i = 0\ ,$$

where I is a finite subset of $\mathbb{N}^n$, $i = (i_1, \dots, i_n)$ and $x^i = x_1^{i_1} \cdots x_n^{i_n}$. Let $A \in \alpha$ be such that all definable functions a_i are defined on A. Set

$$X = \{(t,x) \in R^m \times R^n \ ; \ t \in A \ \text{ and } \ \sum_{i \in I} a_i(t) x^i = 0\} \ .$$

Then X is a definable set. We have $f(\alpha) \in V$ if and only if there exists $B \in \alpha$ such that $\sum_{i \in I} a_i(t) f^i(t) = 0$ for all $t \in B$, i.e. $f(t) \in X_t$ for all $t \in B$. Hence, $V = X_\alpha$.

3) *If $X_\alpha \in S_n(\alpha)$ and $Y_\alpha \in S_p(\alpha)$, then $X_\alpha \times Y_\alpha \in S_{n+p}(\alpha)$.* Set

$$X \times_{R^m} Y = \{(t,x,y) \in R^m \times R^n \times R^p \ ; \ (t,x) \in X \ \text{ and } \ (t,y) \in Y\} \ .$$

Note that $(X \times_{R^m} Y)_t = X_t \times Y_t$ for all $t \in R^m$. It follows that $(X \times_{R^m} Y)_\alpha = X_\alpha \times Y_\alpha$.

4) *If $p : \kappa(\alpha)^{n+1} \to \kappa(\alpha)^n$ is the projection on the first n coordinates and $X_\alpha \in S_{n+1}(\alpha)$, then $p(X_\alpha) \in S_n(\alpha)$.* Let $\pi : R^m \times R^{n+1} \to R^m \times R^n$ be the projection on the first $m+n$ coordinates. It is sufficient to check that

$$p(X_\alpha) = \pi(X)_\alpha \ .$$

Let $f(\alpha)$ be an element of $\kappa(\alpha)^n$. If $f(\alpha)$ belongs to $p(X_\alpha)$, there is $g(\alpha) \in \kappa(\alpha)$ and $A \in \alpha$ such that $(f(t), g(t)) \in X_t$ for all $t \in A$. Hence $f(t) \in \pi(X)_t$ for all $t \in A$, which shows $f(\alpha) \in \pi(X)_\alpha$. Conversely, assume $f(\alpha) \in \pi(X)_\alpha$. Then there is $B \in \alpha$ such that, for all $t \in B$, there exists $y \in R$ with $(f(t), y) \in X_t$. By Definable Choice 3.1, we can choose such a y as a definable function $g : B \to R$. It follows that $(f(\alpha), g(\alpha)) \in X_\alpha$ and $f(\alpha) \in p(X_\alpha)$.

5) *The elements of $S_1(\alpha)$ are precisely the finite unions of points and intervals in $\kappa(\alpha)$.* Take $X_\alpha \in S_1(\alpha)$ and choose a cdcd of $R^m \times R$ adapted to X. We have a partition of R^m into cells $C_1, \dots, C_k$ and, for each cell C_i, continuous definable functions $\zeta_{i,1} < \dots < \zeta_{i,\ell(i)} : C_i \to R$ such that X is a union of graphs of $\zeta_{i,j}$ and bands $(\zeta_{i,j}, \zeta_{i,j+1})$. Exactly one of the C_i belongs to the ultrafilter α, say $C_1 \in \alpha$. We claim that X_α is the union of the points $\zeta_{1,j}(\alpha)$ such that the graph of $\zeta_{1,j}$ is contained in X and the intervals $(\zeta_{1,j}(\alpha), \zeta_{1,j+1}(\alpha))$ such that the band $(\zeta_{1,j}, \zeta_{1,j+1})$ is contained in X. Since $C_1 \in \alpha$, we can replace X with $X \cap (C_1 \times R)$. Since taking the fiber at α commutes with union, we can assume that X is a graph or a band over C_1. If X is the band $(\zeta_{1,j}, \zeta_{1,j+1})$, then $f(\alpha) \in X_\alpha$ if and only if there is $B \in \alpha$ such that $\zeta_{1,j}(t) < f(t) < \zeta_{1,j+1}(t)$ for all $t \in B$, i.e. $f(\alpha) \in (\zeta_{1,j}(\alpha), \zeta_{1,j+1}(\alpha))$. $\square$

Note that there are two crucial points in the proof: the use of Definable Choice in the proof of the stability by projection (point 4) and the use of the cdcd to prove the o-minimality (point 5). All the rest is rather formal.

The proof of Theorem 5.8 shows that taking fibers at α commutes with the boolean operations and the projections. We shall formalize this remark in order to have a useful tool for translating properties of the fiber X_α to properties of the fibers X_t for all t in a definable set belonging to α. First we introduce some notation. We consider the *definable families of formulas* $\Phi_t(x)$, where t ranges over R^m, which are constructed according to the following rules:

1. If $X \subset R^m \times R^n$ is definable, $x \in X_t$ is a definable family of formulas.
2. A polynomial equation $P(x) = 0$ or inequality $P(x) > 0$ with coefficients in R is a definable (constant) family of formulas.
3. If $\Phi_t(x)$ and $\Psi_t(x)$ are definable families of formulas, then $\Phi_t(x) * \Psi_t(x)$ (where $*$ is one of "and", "or", $\Rightarrow$, $\Leftrightarrow$) and the negation "not $\Phi_t(x)$" are definable families of formulas.
4. If $\Phi_t(x, y)$ is a definable family of formulas (with y ranging over R^p) and $Y \subset R^m \times R^p$ is definable, the existential quantification $\exists y \in Y_t \ \Phi_t(x, y)$ and the universal quantification $\forall y \in Y_t \ \Phi_t(x, y)$ are definable families of formulas.

Given a definable family of formulas $\Phi_t(x)$ and $\alpha \in \widetilde{R^m}$, we define the *fiber formula* $\Phi_\alpha(x)$ in the following way:

1. If $\Phi_t(x)$ is $x \in X_t$, $\Phi_\alpha(x)$ is $x \in X_\alpha$
2. If $\Phi_t(x)$ is the constant family $P(x) = 0$ or $P(x) > 0$, $\Phi_\alpha(x)$ is $P(x) = 0$ or $P(x) > 0$.
3. If $\Phi_t(x)$ is $\Theta_t(x) * \Psi_t(x)$, or "not $\Theta_t(x)$", $\Phi_\alpha(x)$ is $\Theta_\alpha(x) * \Psi_\alpha(x)$ or "not $\Theta_\alpha(x)$", respectively.
4. If $\Phi_t(x)$ is $\exists y \in Y_t \ \Psi_t(x, y)$ or $\forall y \in Y_t \ \Psi_t(x, y)$, $\Phi_\alpha(x)$ is $\exists y \in Y_\alpha \ \Psi_\alpha(x, y)$ or $\forall y \in Y_\alpha \ \Psi_\alpha(x, y)$, respectively.

Note that, if $\Phi_\alpha(x)$ is a fiber formula with $x = (x_1, \ldots, x_n)$, the set of $x \in \kappa(\alpha)^n$ such that $\Phi_\alpha(x)$ holds is definable in the o-minimal structure on $\kappa(\alpha)$.

Let us take an example. To the definable family of formulas

$$\forall x \in X_t \quad \exists \varepsilon > 0 \quad \forall y \in R^n \quad (\|x - y\| < \varepsilon \Rightarrow y \in X_t)$$

corresponds the fiber formula

$$\forall x \in X_\alpha \quad \exists \varepsilon > 0 \quad \forall y \in \kappa(\alpha)^n \quad (\|x - y\| < \varepsilon \Rightarrow y \in X_\alpha)\ .$$

Note that $\|x - y\| < \varepsilon$ can be expressed as a polynomial inequality.

Proposition 5.9 *Let X be a definable subset of $R^m \times R^n$ and $\Phi_\alpha(x)$ the fiber formula of a definable family of formulas $\Phi_t(x)$, with $x = (x_1, \ldots, x_n)$. The equality*

$$X_\alpha = \{x \in \kappa(\alpha)^n\ ;\ \Phi_\alpha(x)\}$$

holds if and only if there exists $A \in \alpha$ such that the equality

$$X_t = \{x \in R^n\ ;\ \Phi_t(x)\}$$

holds for all $t \in A$. In particular, if $n = 0$ (i.e. there is no free variable), the fiber formula Φ_α holds if and only if there exists $A \in \alpha$ such that Φ_t holds for all $t \in A$.

Proof. We proceed by induction on the construction of the definable family of formulas according to rules 1-4.

Rule 1: $\Phi_t(x)$ is $x \in Y_t$. We have to show that $X_\alpha = Y_\alpha$ if and only if there is $A \in \alpha$ such that $X_t = Y_t$ for all $t \in A$. The "if" part is easy. To show the "only if" part, it suffices to consider the case $Y = \emptyset$, since taking the fiber at α preserves the boolean operations. Suppose that the set of t such that $X_t = \emptyset$ does not belong to α. Then the projection of X on the space R^m of the first m coordinates belongs to α. By Definable Choice, there is a definable map f from this projection to R^n whose graph is contained in X. It follows that $f(\alpha) \in X_\alpha$, which contradicts $X_\alpha = \emptyset$.

Rule 2. Note that we can actually omit this rule. Indeed, if $X_t = \{x \in R^n\ ;\ P(x) = 0\}$ for all $t \in R^m$, then $X_\alpha = \{x \in \kappa(\alpha)^n\ ;\ P(x) = 0\}$ and we are reduced to Rule 1 (the same with $>$ instead of $=$).

Rule 3. Here we use the observation that taking fibers at α commutes with the boolean operations.

Rule 4. Here we use the observation that taking fibers at α commutes with the projections (for existential quantification). The universal quantification reduces to negations and existential quantification. □

Exercise 5.10 Let X and Y be definable subsets of $R^m \times R^n$. Show that X_α is closed in Y_α if and only if there exists $A \in \alpha$ such that X_t is closed in Y_t for every $t \in A$.

5.4 Extension of Definable Sets

Let α be an element of $\widetilde{R^m}$. We begin with the extension of polyhedra from R to $\kappa(\alpha)$. This will be needed for the proof of Hardt's Theorem 5.22.

Let $a_0, \ldots, a_d$ be points of R^n which are affine independent. We denote by $[a_0, \ldots, a_d]_R$ the simplex they generate in R^n, and by $[a_0, \ldots, a_d]_{\kappa(\alpha)}$ the simplex they generate in $\kappa(\alpha)^n$. If K is a finite simplicial complex with vertices in R^n, we denote by $|K|_R$ its realization in R^n, and by $|K|_{\kappa(\alpha)}$ its realization in $\kappa(\alpha)^n$, i.e. the union of the simplices $[a_0, \ldots, a_d]_{\kappa(\alpha)}$ for all $[a_0, \ldots, a_d]_R$ in K. If V_R is a union of open simplices of K, we define $V_{\kappa(\alpha)}$ as the union of the corresponding open simplices in $\kappa(\alpha)^n$.

Lemma 5.11 $|K|_{\kappa(\alpha)} = (R^m \times |K|_R)_\alpha$. $V_{\kappa(\alpha)} = (R^m \times V_R)_\alpha$.

Proof. Since taking fibers at α preserves finite unions, it is sufficient to prove the lemma for an open simplex σ with vertices $a_0, \ldots, a_d$. The open simplex σ is the set of $x \in R^n$ satisfying the formula

$$\exists \lambda_1 \in R \ldots \exists \lambda_d \in R \ \Big(\lambda_1 > 0 \ \text{ and } \ \ldots \ \text{ and } \ \lambda_d > 0 \ \text{ and } \ \sum_{i=1}^{d} \lambda_i = 1 \ \text{ and } \ \sum_{i=1}^{d} \lambda_i a_i = x\Big) \, .$$

Hence, the fiber $(R^m \times \sigma)_\alpha$ is described by the fiber formula of the constant family of formulas:

$$\exists \lambda_1 \in \kappa(\alpha) \ldots \exists \lambda_d \in \kappa(\alpha) \ \Big(\lambda_1 > 0 \ \text{ and } \ \ldots \ \text{ and } \ \lambda_d > 0 \ \text{ and } \ \sum_{i=1}^{d} \lambda_i = 1 \ \text{ and } \ \sum_{i=1}^{d} \lambda_i a_i = x\Big) \, .$$

This fiber formula describes $\sigma_{\kappa(\alpha)}$. □

The notation for polyhedra will also be used for fibers of constant definable families. Let S be a definable subset of R^n. Then $R^m \times S$ can be regarded as a constant definable family. We denote by $S_{\kappa(\alpha)} \subset \kappa(\alpha)^n$ the fiber $(R^m \times S)_\alpha$, and we call $S_{\kappa(\alpha)}$ the *extension of* S to $\kappa(\alpha)$.

Exercise 5.12 Show that $S_{\kappa(\alpha)} \cap R^n = S$.

In model-theoretic terms, the o-minimal structure over $\kappa(\alpha)$ is an extension of the o-minimal stucture over R. Any first-order formula of the o-minimal structure over R can be interpreted in the o-minimal structure over $\kappa(\alpha)$, taking the extension to $\kappa(\alpha)$ of the definable sets appearing in this formula. Proposition 5.9 implies that $\kappa(\alpha)$ is an elementary extension of R: every formula without free variables holds over R if an only if it holds over $\kappa(\alpha)$.

5.5 Definable Families of Maps

Let X and Y be definable subsets of $R^m \times R^n$ and $R^m \times R^p$, respectively. We regard them as definable families parametrized by R^m. A *definable family of maps* from X to Y is a definable map $f : X \to Y$ such that the following diagram commutes,

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & R^m & \end{tikzcd}$$

where the maps to R^m are the projections on the first m coordinates. We obtain a family of maps $f_t : X_t \to Y_t$ for $t \in R^m$ defined by $f(t,x) = (t, f_t(x))$.

Given $\alpha \in \widetilde{R^m}$, we shall define $f_\alpha : X_\alpha \to Y_\alpha$, which will be a definable map for the o-minimal structure on $\kappa(\alpha)$. Set

$$\Gamma = \{(t,x,y) \in R^m \times R^n \times R^p \ : \ (t,x) \in X \ \text{ and } \ f(t,x) = (t,y)\} \, .$$

The set Γ is a definable family parametrized by R^m and, for every $t \in R^m$, Γ_t is the graph of $f_t : X_t \to Y_t$. Hence, the formulas in the definable families of formulas

$$(*)_t \qquad \begin{cases} \forall x \in R^n \ ((\exists y \in R^p \ (x,y) \in \Gamma_t) \Leftrightarrow x \in X_t) \\ \forall x \in R^n \ \forall y \in R^p \ ((x,y) \in \Gamma_t \Rightarrow y \in Y_t) \\ \forall x \in R^n \ \forall y \in R^p \ \forall z \in R^p \Big(((x,y) \in \Gamma_t \ \text{ and } \ (x,z) \in \Gamma_t) \Rightarrow y = z \Big) \end{cases}$$

hold true for every $t \in R^m$. By Proposition 5.9, the fiber formulas for α, which are

$$(*)_\alpha \qquad \begin{cases} \forall x \in \kappa(\alpha)^n \ ((\exists y \in \kappa(\alpha)^p \ (x,y) \in \Gamma_\alpha) \Leftrightarrow x \in X_\alpha) \\ \forall x \in \kappa(\alpha)^n \ \forall y \in \kappa(\alpha)^p \ ((x,y) \in \Gamma_\alpha \Rightarrow y \in Y_\alpha) \\ \forall x \in \kappa(\alpha)^n \ \forall y \in \kappa(\alpha)^p \ \forall z \in \kappa(\alpha)^p \Big(((x,y) \in \Gamma_\alpha \ \text{ and } \ (x,z) \in \Gamma_\alpha) \\ \hfill \Rightarrow y = z \Big) \end{cases}$$

also hold true. These fiber formulas express the fact that Γ_α is the graph of a map $f_\alpha : X_\alpha \to Y_\alpha$. Since Γ_α is definable, the map f_α is definable for the o-minimal structure on $\kappa(\alpha)$.

The next proposition says that all definable maps are obtained in this way. We introduce a notation. If X is a definable subset of $R^m \times R^n$ and A a definable subset of R^m, we denote by X_A the definable subset $X \cap (A \times R^n)$ of $R^m \times R^n$. In other words, X_A is the definable family X restricted to A.

Proposition 5.13 *Let $\varphi : X_\alpha \to Y_\alpha$ be a definable map for the o-minimal structure on $\kappa(\alpha)$. Then there exist $A \in \alpha$ and a definable family of maps $f : X_A \to Y_A$ such that $\varphi = f_\alpha$.*

Proof. Let Γ_α be the graph of φ. Then the fiber formulas $(*)_\alpha$ above hold true. By Proposition 5.9, there exists $A \in \alpha$ such that the formulas $(*)_t$ hold true for every $t \in A$. This means that, for every $t \in A$, Γ_t is the graph of a map $f_t : X_t \to Y_t$. We obtain in this way a definable family of maps $f : X_A \to Y_A$, which satisfies $f_\alpha = \varphi$. □

Exercise 5.14 Show that $f_\alpha = g_\alpha$ if and only if there is $A \in \alpha$ such that $f_t = g_t$ for every $t \in A$.

Exercise 5.15 Let $f : X \to Y$ and $g : Y \to Z$ be definable families of maps parametrized by R^m, X' a definable subset of X. For $\alpha \in \widetilde{R^m}$, show that $(g \circ f)_\alpha = g_\alpha \circ f_\alpha$. Show that $f(X')_\alpha = f_\alpha(X'_\alpha)$.

Exercise 5.16 Show that f_α is continuous if and only if there is $A \in \alpha$ such that f_t is continuous for every $t \in A$. (Use Proposition 5.9.)

5.6 Fiberwise and Global Properties

We have seen several examples which show that a property holds for the fiber at α if and only if there is $A \in \alpha$ such that the same property holds for the fibers at t for all $t \in A$. In this section we are interested in topological properties which hold globally for the family. An example of a global property is the property "X is closed", which is stronger than the fiberwise property "every fiber X_t is closed".

The main tool will be the following.

Lemma 5.17 *Let X be a definable subset of $R^m \times R^n$. There is a partition of R^m into definable sets $C_1, \dots, C_k$ such that, for $i = 1, \dots, k$, and for every $t \in C_i$, we have* $\operatorname{clos}(X_t) = (\operatorname{clos} X_{C_i})_t$ *(recall that X_{C_i} denotes $X \cap (C_i \times R^n)$).*

Proof. We proceed by induction on m. So we assume that the lemma is proved for all d, $0 \le d < m$. Let G be the definable subset of those $t \in R^m$ such that $\operatorname{clos}(X_t) = (\operatorname{clos} X)_t$. Lemma 3.21 implies that the complement of G in R^m has dimension $< m$. Choose a cdcd of R^m adapted to G. Every open cell C of this cdcd is contained in G, and for every $t \in C$ we have $\operatorname{clos}(X_t) = (\operatorname{clos} X)_t = (\operatorname{clos} X_C)_t$. Now let D be a cell of dimension $d < m$. There is a definable homeomorphism $\theta_D : D \to R^d$. We apply the inductive assumption to the definable family of $(u, x) \in R^d \times R^n$ such that $(\theta_D^{-1}(u), x) \in X$. It follows that we can partition D into finitely many definable subsets D_j such that $\operatorname{clos}(X_t) = (\operatorname{clos} X_{D_j})_t$ for every $t \in D_j$. The lemma is proved. □

Proposition 5.18 *Let $X \subset Y$ be definable subsets of $R^m \times R^n$. Let $\alpha \in \widetilde{R^m}$. The fiber X_α is closed (resp. open) in Y_α if and only if there exists $A \in \alpha$ such that X_A is closed (resp. open) in Y_A.*

Proof. It suffices to prove the closed version. The open version follows by taking the relative complement of X in Y.

If X_A is closed in Y_A, then X_t is closed in Y_t for every $t \in A$. It follows (cf. Exercise 5.10) that X_α is closed in Y_α.

Conversely, assume that X_α is closed in Y_α. We use Lemma 5.17 to obtain a partition of R^m into finitely many definable sets $C_1, \dots, C_k$ such that $\operatorname{clos}(X_t) = (\operatorname{clos} X_{C_i})_t$ for every $t \in C_i$. Since α is an ultrafilter of definable sets, it contains exactly one of the C_i, say C_1. From the assumption that X_α is closed in Y_α, it follows that there is $A \in \alpha$ such that X_t is closed in Y_t for every $t \in A$ (cf. Exercise 5.10). Replacing A with $A \cap C_1$, we can assume $A \subset C_1$. Then $\operatorname{clos}(X_t) = (\operatorname{clos} X_A)_t$ for every $t \in A$. We deduce $(\operatorname{clos} X_A)_t \cap Y_t = \operatorname{clos}(X_t) \cap Y_t = X_t$ for every $t \in A$ (for the second equality we used the fact that X_t is closed in Y_t). This implies $(\operatorname{clos} X_A) \cap Y_A = X_A$, i.e. X_A is closed in Y_A. □

Theorem 5.19 *Let $f : X \to Y$ be a definable family of maps parametrized by R^m, and $\alpha \in \widetilde{R^m}$. Then f_α is continuous if and only if there exists $A \in \alpha$ such that $f_A : X_A \to Y_A$ is continuous.*

Proof. Say Y is a definable subset of $R^m \times R^p$. We can assume that, for all t, Y_t is contained in the bounded box $(-1, 1)^p$ (then Y_α is contained in the bounded box $(-1, 1)^p_{\kappa(\alpha)}$ in $\kappa(\alpha)^p$). The reason for this is the following. Let $\mu : R^p \to (-1, 1)^p$ be the semialgebraic homeomorphism defined by

$$\mu(y_1, \dots, y_p) = \left(\frac{y_1}{\sqrt{1 + y_1^2}}, \dots, \frac{y_p}{\sqrt{1 + y_p^2}} \right) .$$

We can replace f with the composition

$$g = (\mathrm{Id}_{R^m} \times \mu) \circ f : X \to R^m \times (-1,1)^p \ .$$

Note that g_α is $\mu_{\kappa(\alpha)} \circ f_\alpha$, where $\mu_{\kappa(\alpha)} : \kappa(\alpha)^p \to (-1,1)^p_{\kappa(\alpha)}$ is the semialgebraic homeomorphism defined by the same formula as μ. The continuity of g_A is equivalent to the continuity of f_A, and the continuity of g_α is equivalent to the continuity of f_α.

Now let

$$\Gamma = \{(t,x,y) \in X \times (-1,1)^p \ ; \ f(t,x) = (t,y)\} \ .$$

The map $f_A : X_A \to A \times (-1,1)^p$ is continuous if and only if Γ_A is closed in $X_A \times R^p$. The map $f_\alpha : X_\alpha \to (-1,1)^p_{\kappa(\alpha)}$ is continuous if and only if Γ_α is closed in $X_\alpha \times \kappa(\alpha)^p$. Hence, the theorem follows from Proposition 5.18. $\square$

Exercise 5.20 Assume that $\varphi : X_\alpha \to Y_\alpha$ is a definable homeomorphism. Show that there is $A \in \alpha$ and a definable family of maps $f : X_A \to Y_A$, such that f is a homeomorphism and $f_\alpha = \varphi$.

Exercise 5.21 Let f be a definable family of maps parametrized by R^m. Assume that f_t is continuous for every $t \in R^m$. Show that there is a finite partition of R^m into definable subsets $C_1, \dots, C_k$ such that f_{C_i} is continuous for $i = 1, \dots, k$. Hint: one can use Lemma 5.17 and the proof of Theorem 5.19. Another possibility is as follows:
1) Show that for every $\alpha \in \widetilde{R^m}$, there is $C(\alpha) \in \alpha$ such that $f_{C(\alpha)}$ is continuous.
2) Use the compactness of $\widetilde{R^m}$ to show that R^m is a finite union $R^m = C(\alpha_1) \cup \ldots \cup C(\alpha_k)$. Modify $C(\alpha_1)$,…, $C(\alpha_k)$ in order to get a partition of R^m.

5.7 Triviality Theorems

Let $X \subset R^m \times R^n$ be a definable family. Let A be a definable subset of R^m. We say that the family X is *definably trivial over* A if there exist a definable set F and a definable homeomorphism $h : A \times F \to X_A$ such that the folllowing diagram commutes:

$$\begin{tikzcd} A \times F \arrow[rr, "h"] \arrow[dr, "\text{projection}"'] & & X_A \arrow[dl, "\text{projection}"] \\ & A \subset \mathbb{R}^m & \end{tikzcd}$$

We say that h is a *definable trivialization of* X *over* A. Now let Y be a definable subset of X. We say that the trivialization h is *compatible with* Y if there is a definable subset G of F such that $h(A \times G) = Y_A$. Note that if h is compatible with Y, its restriction to Y_A is a trivialization of Y over A.

Theorem 5.22 (Hardt's Theorem for Definable Families)
Let $X \subset R^m \times R^n$ be a definable family. Let $Y_1, \dots, Y_\ell$ be definable subsets of X. There exists a finite partition of R^m into definable sets $C_1, \dots, C_k$ such that X is definably trivial over each C_i and, moreover, the trivializations over each C_i are compatible with $Y_1, \dots, Y_\ell$.

This theorem was proved by R. Hardt in the semialgebraic case.
Proof. We can assume that X is closed and contained in $R^m \times [-1,1]^n$. The reason for this is the following. Let $\mu : R^n \to (-1,1)^n$ be the semialgebraic homeomorphism defined by

$$\mu(x_1, \dots, x_n) = \left(\frac{x_1}{\sqrt{1+x_1^2}}, \dots, \frac{x_n}{\sqrt{1+x_n^2}} \right) .$$

First, we can replace X (and each Y_j) with its image by $\mathrm{Id}_{R^m} \times \mu$, which is contained in $R^m \times (-1,1)^n$. Then, we can replace X with $\operatorname{clos} X$, which is closed and contained in $R^m \times [-1,1]^n$, and add X to the list of definable subsets $Y_1, \dots, Y_\ell$. A definable trivialization of $\operatorname{clos}(X)$ over C_i compatible with X will induce a definable trivialization of X over C_i.

Let α be an element of $\widetilde{R^m}$. The assumption above implies that X_α is bounded and closed in $\kappa(\alpha)^n$. Hence, by the triangulation theorem 4.4, there is a finite simplicial complex K with vertices in $\mathbb{Q}^n \subset R^n$ and a definable homeomorphism $\Phi : |K|_{\kappa(\alpha)} \to X_\alpha$, such that each $(Y_j)_\alpha$ is the image by Φ of a union $(V_j)_{\kappa(\alpha)}$ of open simplices of K. By Lemma 5.11, we have $|K|_{\kappa(\alpha)} = (R^m \times |K|_R)_\alpha$. Hence (cf. Exercise 5.20), there exists $C(\alpha) \in \alpha$ and a definable family of maps $h : C(\alpha) \times |K|_R \to X_A$ such that h is a homeomorphism and $h_\alpha = \Phi$. Moreover, since

$$h_\alpha((R^m \times (V_j)_R)_\alpha) = h_\alpha((V_j)_{\kappa(\alpha)}) = (Y_j)_\alpha ,$$

we can assume, replacing $C(\alpha)$ with a smaller definable set still in α, that $h(C(\alpha) \times (V_j)_R) = (Y_j)_{C(\alpha)}$. We have proved the following fact: for every $\alpha \in \widetilde{R^m}$, there exist $C(\alpha) \in \alpha$ and a definable trivialization of X over $C(\alpha)$ compatible with each Y_j.

The open and closed subset $\widetilde{C(\alpha)}$ cover $\widetilde{R^m}$. Since $\widetilde{R^m}$ is compact, we can extract a finite subcover from this cover. Hence, there are $\alpha_1, \ldots, \alpha_k$ such that $R^m = C(\alpha_1) \cup \ldots \cup C(\alpha_k)$. Replacing $C(\alpha_i)$ with $C_i = C(\alpha_i) \setminus \bigcup_{j<i} C(\alpha_j)$, we obtain a partition of R^m into definable sets $C_1, \ldots, C_k$, and, for each i, a definable trivialization of X over C_i compatible with each Y_j. $\square$

Exercise 5.23 Use Hardt's theorem to give another proof of the local conic structure of definable sets (cf. 4.10). Hint: trivialize the definable family of
$$A_\varepsilon = \{x \in A \; ; \; \|x - a\| = \varepsilon\} \; .$$

We have a similar trivialization theorem for definable familes of functions (with values in R). First, we have to define the notion of trivialization for a family of functions.

Let $f : X \to R^m \times R$ be definable family of functions parametrized by R^m and A a definable subset of R^m. We say that f is *definably trivial over* A if there exist a definable function $g : F \to R$ and definable homeomorphisms $\rho : A \times F \to X_A$ and $\lambda : A \times R \to A \times R$ such that the following diagram commutes,

$$\begin{tikzcd}
A \times F \arrow[rr, "\rho"] \arrow[d, "\mathrm{Id}_A \times g"'] & & X_A \arrow[d, "f"] \\
A \times R \arrow[dr] & & A \times R \arrow[ll, "\lambda"] \arrow[dl] \\
& A &
\end{tikzcd}$$

where the maps to A are the projections on the first factor. The couple (ρ, λ) is called a definable trivialization of f over A. If Y is a definable subset of X, we say that the trivialization (ρ, λ) is compatible with Y if there is a definable subset G of F such that $\rho(A \times G) = Y_A$. Note that, in this case, (ρ, λ) induces a definable trivialization of $f|_Y$ over A.

Theorem 5.24 (Definable Triviality of Families of Functions)
Let $f : X \to R^m \times R$ be a definable family of functions parametrized by R^m.

Let $Y_1, \dots, Y_\ell$ be definable subsets of X. There exists a finite partition of R^m into definable sets $C_1, \dots, C_k$ such that f is definably trivial over each C_i and, moreover, the trivializations over each C_i are compatible with $Y_1, \dots, Y_\ell$.

Proof. We can assume that X is closed in $R^m \times [-1,1] \times [-1,1]^n$ and that f is the restriction to X of the projection p on the first $m+1$ coordinates. The reason for this is the following. Let $\nu : R \to (-1,1)$ be the semialgebraic homeomorphism defined by $\nu(y) = y/\sqrt{1+y^2}$, and $\mu : R^n \to (-1,1)^n$ the semialgebraic homeomorphism defined by $\mu(x_1, \dots, x_n) = (\nu(x_1), \dots, \nu(x_n))$. First we replace X with its homeomorphic image $\Gamma \subset R^m \times (-1,1) \times (-1,1)^n$ by the definable map

$$(t, x) \longmapsto (t, \nu(f_t(x)), \mu(x)) \; .$$

We replace also the Y_j with their images $\Delta_j \subset \Gamma$. We obtain the following commutative diagram, where the horizontal maps are definable homeomorphisms and the maps to R^m are the projections on the m first coordinates.

$$\begin{tikzcd}
\Gamma \arrow[rr] \arrow[d, "p"'] & & X \arrow[d, "f"] \\
R^m \times (-1,1) \arrow[dr] & & R^m \times R \arrow[ll, "\mathrm{Id}_{R^m} \times \nu"] \arrow[dl] \\
& R^m &
\end{tikzcd}$$

Now we replace Γ with $\mathrm{clos}(\Gamma) \subset R^m \times [-1,1] \times [-1,1]^n$, and we add Γ to the list of definable subsets $\Delta_1, \dots, \Delta_\ell$. A definable trivialization of $p|_{\mathrm{clos}(\Gamma)}$ compatible with $\Gamma, \Delta_1, \dots, \Delta_\ell$ will induce a definable trivialization of $p|_\Gamma$ compatible with $\Delta_1, \dots, \Delta_\ell$. Composing with the definable homeomorphisms represented by the horizontal arrows in the diagram above, we obtain a definable trivialization of f compatible with $Y_1, \dots, Y_\ell$.

Let $\alpha \in \widetilde{R^m}$. The assumption above implies that X_α is closed and bounded in $\kappa(\alpha) \times \kappa(\alpha)^n$ and f_α is the restriction to X_α of the projection π on the first coordinate. By Proposition 4.8, there exist a finite simplicial complex K with vertices in $\mathbb{Q} \times \mathbb{Q}^n$ and definable homeomorphisms $\Phi : |K|_{\kappa(\alpha)} \to X_\alpha$ and $\tau : \kappa(\alpha) \to \kappa(\alpha)$ such that the following left-hand side diagram commutes and each Y_j is a union of images by Φ of open simplices.

$$\begin{tikzcd} {|K|_{\kappa(\alpha)}} \arrow[r, "\Phi"] \arrow[d, "\pi"'] & X_\alpha \arrow[d, "f_\alpha"] \\ \kappa(\alpha) & \kappa(\alpha) \arrow[l, "\tau"] \end{tikzcd} \qquad \begin{tikzcd} C(\alpha) \times |K|_R \arrow[r, "\rho"] \arrow[d, "p"'] & X_{C(\alpha)} \arrow[d, "f"] \\ C(\alpha) \times R \arrow[dr] & C(\alpha) \times R \arrow[l, "\lambda"] \arrow[d] \\ & C(\alpha) \end{tikzcd}$$

We obtain $C(\alpha) \in \alpha$ and definable families of maps $\rho : C(\alpha) \times |K|_R \to X_{C(\alpha)}$ and $\lambda : C(\alpha) \times R \to C(\alpha) \times R$ which are homeomorphisms and such that $\rho_\alpha = \Phi$ and $\lambda_\alpha = \tau$. After possibly shrinking $C(\alpha)$, (ρ, λ) is a definable trivialization of f over $C(\alpha)$, compatible with the Y_j. We conclude as in the proof of Hardt's theorem. $\square$

5.8 Topological Types of Sets and Functions

Two embeddings $Y \subset X$ and $Y' \subset X'$, where all sets are definable, are said to have the same *(definable) topological type* if there is a (definable) homeomorphism $h : X \to X'$ such that $h(Y) = Y'$. Two definable maps $f : X \to Y$ and $f' : X' \to Y'$ are said to have the same (definable) topological type if there are (definable) homeomorphisms $\rho : X \to X'$ and $\lambda : Y' \to Y$ such that $\lambda \circ f' \circ \rho = f$.

Assume that the definable family $X \subset R^m \times R^n$ is definably trivial over $A \subset R^m$, and that the trivialization is compatible with $Y \subset X$. This means that there is a definable homeomorphism $h : A \times F \to X_A$ of the form $h(t, x) = (t, h_t(x))$, and a definable subset G of F such that $h(A \times G) = Y_A$. Now let t and t' be two points of A. Then $h_{t'} \circ h_t^{-1}$ is a definable homeomorphism from X_t onto X'_t, and the image of Y_t by this homeomorphism is $Y_{t'}$. Hence, all embeddings $Y_t \subset X_t$ for $t \in A$ have the same topological type. From this discussion and Hardt's theorem we obtain the following.

Corollary 5.25 *Let $X \subset R^m \times R^n$ be a definable family, Y a definable subset of X. There are finitely many definable topological types of embeddings $Y_t \subset X_t$ for $t \in R^m$.*

We give an example of application.

Theorem 5.26 *Given n and d positive integers, there are finitely many topological types of embedding $V \subset R^n$, where V is an algebraic subset defined by equations of degrees at most d.*

Proof. If V is defined by equations $P_1 = \ldots = P_\ell = 0$ of degrees at most d, it is also defined by the single equation $P_1^2 + \cdots + P_\ell^2 = 0$ of degree at most $2d$. Hence, it suffices to show that there are finitely many topological types of embedding $V \subset R^n$, where V is defined by one equation of degree at most $2d$. If $i = (i_1, \ldots, i_n) \in \mathbb{N}^n$, we set $|i| = i_1 + \cdots + i_n$ and $x^i = x_1^{i_1} \cdots x_n^{i_n}$. Let I_{2d} be the finite set of $i \in \mathbb{N}^n$ such that $|i| \le 2d$. We denote by $a = (a_i)_{i \in I_{2d}}$ an element of the affine space $R^{I_{2d}}$. Set

$$Y = \{(a, x) \in R^{I_{2d}} \times R^n \; ; \; \sum_{i \in I_{2d}} a_i x^i = 0\} \subset R^{I_{2d}} \times R^n \; .$$

For $a \in R^{I_{2d}}$, Y_a is an algebraic subset of R^n defined by one equation of degree $\le 2d$, and we obtain in this way all such algebraic subsets. Hence, the theorem follows from Corollary 5.25 applied to the semialgebraic families $Y \subset R^{I_{2d}} \times R^n$ parametrized by $R^{I_{2d}}$. $\square$

We now turn to the case of families of functions. Assume that the definable family of functions $f_t : X_t \to R$ parametrized by $t \in R^m$ is trivial over A. This means that there are definable homeomorphism $\rho : A \times F \to X_A$ of the form $\rho(t, x) = (t, \rho_t(x))$ and $\lambda : A \times R \to A \times R$ of the form $\lambda(t, y) = (t, \lambda_t(y))$ and a definable function $g : F \to R$, such that $\lambda \circ f \circ \rho = \mathrm{Id}_A \times g$. Let t and t' be two points of A. Then $\rho_{t'} \circ \rho_t^{-1} : X_t \to X_{t'}$ and $\lambda_t^{-1} \circ \lambda_{t'} : R \to R$ are definable homeomorphisms which satisfy

$$\lambda_t^{-1} \circ \lambda_{t'} \circ f_{t'} \circ \rho_{t'} \circ \rho_t^{-1} = f_t \; .$$

This shows that all f_t for $t \in A$ have the same definable topological type. From this discussion and Theorem 5.24 we obtain the following.

Corollary 5.27 *In a definable family of functions $f_t : X_t \to R$ parametrized by $t \in R^m$, there are finitely many definable topological types.*

The similar statement for definable families of maps is not true. For instance, there is a semialgebraic family of maps $\mathbb{R}^2 \to \mathbb{R}^2$ in which there are infinitely many topological types (cf. [CR]).

We give an example of application for the o-minimal structure $\mathbb{R}_{\exp}$ expanding $\mathbb{R}$ and which is generated by the exponential function. This means that

$\mathbb{R}_{\exp}$ is a structure $(S_n)_{n\in\mathbb{N}}$, where $S_n \subset \mathcal{P}(\mathbb{R}^n)$ and S_2 contains the graph of the exponential function, and $S_n \subset T_n$ for every structure $(T_n)_{n\in\mathbb{N}}$ with these properties. The fact that $\mathbb{R}_{\exp}$ is o-minimal (i.e. S_1 consists of finite unions of points and intervals) is a fundamental result of A. Wilkie [Wi]. In $\mathbb{R}_{\exp}$ the function $(x,\lambda) \mapsto x^\lambda$ on $\{x>0\} \times \mathbb{R}$ is definable, since

$$y = x^\lambda \Leftrightarrow \exists z \in \mathbb{R}(x = \exp(z) \ \text{ and } \ y = \exp(\lambda z)) .$$

Theorem 5.28 *Given n and k two positive integers, there are finitely many topological types in the family of polynomials $\mathbb{R}^n \to \mathbb{R}$ with at most k monomials (and no bound on the degree).*

Proof. We extend the power function $(x,\lambda) \mapsto x^\lambda$ to two definable functions $M_\epsilon : \mathbb{R}^2 \to \mathbb{R}$ for $\epsilon = 0, 1$, defined by

$$M_\epsilon(x,\lambda) = \begin{cases} x^\lambda & \text{if } x > 0 \ , \\ 0 & \text{if } x = 0 \text{ and } \lambda \neq 0 \ , \\ 1 & \text{if } x = 0 \text{ and } \lambda = 0 \ , \\ (-1)^\epsilon \, |x|^\lambda & \text{if } x < 0 \ . \end{cases}$$

Now consider the family of all functions $\mathbb{R}^n \to \mathbb{R}$

$$(x_1, \ldots, x_n) \mapsto \sum_{i=1}^{k} \left(a_i \prod_{j=1}^{n} M_{\epsilon_{i,j}}(x_j, \lambda_{i,j}) \right) \ .$$

This is a definable family of functions parametrized by the $((a_i), (\lambda_{i,j}), (\epsilon_{i,j})) \in \mathbb{R}^k \times \mathbb{R}^{kn} \times \{0,1\}^{kn}$, which is a definable subset of $\mathbb{R}^{k+2kn}$. In this family we have all polynomials in n variables with at most k monomials. Hence, the theorem follows from Corollary 5.27. $\square$

Note that the finiteness of topological types for polynomials $\mathbb{R}^n \to \mathbb{R}$ with bounded degree (a result of Fukuda [Fu] obtained by completely different methods) can be obtained by working in the semialgebraic structure (it is not difficult to put all polynomials $\mathbb{R}^n \to \mathbb{R}$ with degree $\leq d$ in a semialgebraic family). Theorem 5.28 is stronger. Its statement does not refer to o-minimal structures, but its proof uses this theory in an essential way.

Chapter 6

Smoothness

In this chapter, we assume for simplicity that $R = \mathbb{R}$, i.e. we consider an o-minimal expansion of the field $\mathbb{R}$ of real numbers. The results of this chapter still hold for o-minimal structures expanding an arbitrary real closed field. In order to prove these results, one would first have to establish the basic facts of analysis (such as the local inversion theorem) for definable mappings. This is done in [vD].

6.1 Derivability of Definable Functions in One Variable

If A is a definable subset of $\mathbb{R}^n$, we say that a function $g : A \to \mathbb{R} \cup \{-\infty, +\infty\}$ is *definable* if the inverse images of $-\infty$ and of $+\infty$ are definable subsets of A and the restriction of g to $g^{-1}(\mathbb{R})$ is a definable function with values in $\mathbb{R}$.

Lemma 6.1 *Let $f : I \to \mathbb{R}$ be a definable continuous function on an open interval I of $\mathbb{R}$. Then f has left and right derivatives in $\mathbb{R} \cup \{-\infty, +\infty\}$ at every point x in I. These derivatives are denoted respectively by $f'_\ell(x)$ and $f'_r(x)$. The functions f'_ℓ and f'_r from I to $\mathbb{R} \cup \{-\infty, +\infty\}$ are definable.*

Proof. Apply the monotonicity theorem to the function $y \mapsto (f(y) - f(x))/(y - x)$ to show that it has a limit in $\mathbb{R} \cup \{-\infty, +\infty\}$ as y tends to x from the left or from the right (cf. Exercise 2.3). The definability of the left and right derivatives is left as an exercise. □

Lemma 6.2 *Let $f : I \to \mathbb{R}$ be a definable continuous function on an open interval I of $\mathbb{R}$. If $f'_\ell > 0$ (resp. $f'_r > 0$) on I, f is strictly increasing on I.*

Proof. Apply the monotonicity theorem to the function f, plus the fact that there is no subinterval of I on which f is constant or f is strictly decreasing (otherwise, one would have $f'_\ell \leq 0$ and $f'_r \leq 0$ on such a subinterval). $\square$

Remark that the preceding lemma can be proved without the assumption that f is definable.

Exercise 6.3 Assume that f is any function having a positive (or $+\infty$) left derivative at every point of I. Take $a \in I$. Show that

$$\sup\{x \in I \ ; \ f \geq f(a) \text{ on } [a, x]\}$$

is the right endpoint of I (or $+\infty$ if I is not bounded from above). Deduce that f is non decreasing on I and, hence, that f is strictly increasing.

Theorem 6.4 *Let $f : I \to \mathbb{R}$ be a definable continuous function on an open interval I of $\mathbb{R}$. Then, for all but finitely many points in I, we have $f'_\ell(x) = f'_r(x) \in \mathbb{R}$. Hence, f is derivable outside a finite subset of I.*

Proof. First we prove that there is no subinterval of I on which $f'_\ell = +\infty$, or $f'_\ell = -\infty$, or $f'_r = +\infty$, or $f'_r = -\infty$. Suppose for instance that $f'_\ell = +\infty$ on a subinterval J of I. Take $a < b$, both in J and set

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}\, x \qquad \text{for } x \in J\ .$$

We have $g'_\ell = +\infty$ on J. By Lemma 6.2, g is strictly increasing on J. We deduce $g(b) > g(a)$, which is clearly impossible. We can show in a similar way that the other cases lead to a contradiction.

Suppose now that there is a subinterval J of I on which f'_ℓ and f'_r have values in $\mathbb{R}$ and $f'_\ell < f'_r$. Replacing J with a smaller subinterval, we can assume that f'_ℓ and f'_r are continuous. Taking a still smaller subinterval, we can assume that there is $c \in \mathbb{R}$ such that $f'_\ell < c < f'_r$. Lemma 6.2 implies that $x \mapsto f(x) - cx$ is at the same time strictly increasing and strictly decreasing on this subinterval, which is impossible. We can show in a similar way that there is no subinterval on which f'_ℓ and f'_r have values in $\mathbb{R}$ and $f'_\ell > f'_r$.

The definability of f'_ℓ and f'_r, together with the facts just proved, imply the conclusion of the theorem. $\square$

Corollary 6.5 *Let $f : I \to \mathbb{R}$ be a definable function. For all $k \in \mathbb{N}$, there exists a finite subset $M(k)$ of I such that f is of class $\mathcal{C}^k$ on $I \setminus M(k)$.*

Proof. By induction on k, using Theorem 6.4, the definability of the derivative, and piecewise continuity. $\square$

6.2 $\mathcal{C}^k$ Cell Decomposition

A $\mathcal{C}^k$ cylindrical definable cell decomposition of $\mathbb{R}^n$ is a cdcd satisfying extra smoothness conditions which imply, in particular, that each cell is a $\mathcal{C}^k$ submanifold of $\mathbb{R}^n$.

- A $\mathcal{C}^k$ cdcd of $\mathbb{R}$ is any cdcd of $\mathbb{R}$ (i.e. a finite subdivision of $\mathbb{R}$).
- If $n > 1$, a $\mathcal{C}^k$ cdcd of $\mathbb{R}^n$ is given by a $\mathcal{C}^k$ cdcd of $\mathbb{R}^{n-1}$ and, for each cell D of $\mathbb{R}^{n-1}$, definable functions of class $\mathcal{C}^k$

$$\zeta_{D,1} < \ldots < \zeta_{D,\ell(D)} : D \to \mathbb{R} .$$

  The cells of $\mathbb{R}^n$ are, of course, the graphs of the $\zeta_{D,i}$ and the bands delimited by these graphs.

It is clear from the description by induction that the cells of a $\mathcal{C}^k$ cdcd are $\mathcal{C}^k$ submanifolds of $\mathbb{R}^n$. Moreover, for each cell C, there is a definable $\mathcal{C}^k$ diffeomorphism $\theta_C : C \to \mathbb{R}^{\dim C}$. The diffeomorphism θ_C can be defined by induction on n, by the same formulas as in Proposition 2.5.

We now state the two theorems of this section.

Theorem 6.6 ($\mathcal{C}^k$ Cell Decomposition: $\mathcal{C}^k\mathrm{CDCD}_n$) *Given finitely many definable subsets $X_1, \ldots, X_\ell$ of $\mathbb{R}^n$, there is a $\mathcal{C}^k$ cdcd of $\mathbb{R}^n$ adapted to $X_1, \ldots, X_\ell$ (i.e. each X_i is a union of cells).*

Theorem 6.7 (Piecewise $\mathcal{C}^k$: $\mathrm{P}\mathcal{C}^k_n$) *Given a definable function $f : A \to \mathbb{R}$, where A is a definable subset of $\mathbb{R}^n$, there is a finite partition of A into definable $\mathcal{C}^k$ submanifolds $C_1, \ldots, C_\ell$, such that each restriction $f|_{C_i}$ is $\mathcal{C}^k$.*

Proof. We prove the two theorems simultaneously, by induction on n. The case $n = 1$ is already done: $\mathcal{C}^k\mathrm{CDCD}_1$ is obvious, and $\mathrm{P}\mathcal{C}^k_1$ follows from Theorem 6.4. So we assume $n > 1$ and the theorems proved for smaller dimensions.

1) $\mathcal{C}^k\mathrm{CDCD}_n$. Start with a cdcd of $\mathbb{R}^n$ adapted to $X_1, \ldots, X_\ell$. Using $\mathrm{P}\mathcal{C}^k_{n-1}$, partition the cells of the induced cdcd of $\mathbb{R}^{n-1}$ in order to have all $\zeta_{C,j}$ of class $\mathcal{C}^k$. Then, using $\mathcal{C}^k\mathrm{CDCD}_{n-1}$, refine to a $\mathcal{C}^k$ cdcd of $\mathbb{R}^{n-1}$.

2) $\mathrm{P}\mathcal{C}^k_n$. We use the following lemma.

Lemma 6.8 *Let $g : U \to \mathbb{R}$ be a definable function, with U definable open subset of $\mathbb{R}^n$. Then there is a definable open subset V of U such that $f|_V$ is $\mathcal{C}^k$ and $\dim(U \setminus V) < n$.*

Proof. By induction on k, it is sufficient to show that the set G_i where the partial derivative $\partial g/\partial x_i$ exists is definable, that its complement in U has empty interior, and that $\partial g/\partial x_i$ is definable on G_i. We leave the checking of definability to the reader and prove that $U \setminus G_i$ has empty interior. Otherwise, there would exist a nonempty open box where $\partial g/\partial x_i$ does not exists. Considering the restriction of g to an interval of a line parallel to the x_i axis contained in this box, we obtain a contradiction with Theorem 6.4. □

We return to the proof of $\mathrm{P}\mathcal{C}^k_n$. We choose a cdcd of $\mathbb{R}^n$ adapted to A. By Lemma 6.8, for each open cell C_i of $\mathbb{R}^n$ contained in A, there is a definable open subset V_i such that $f|_{V_i}$ is $\mathcal{C}^k$ and $\dim(C_i \setminus V_i) < n$. By $\mathcal{C}^k\mathrm{CDCD}_n$, we can refine the cdcd to a $\mathcal{C}^k$ cdcd of $\mathbb{R}^n$ adapted to A and the V_i. On each open cell of this new cdcd contained in A, f is $\mathcal{C}^k$. Let D be a cell of dimension $< n$ contained in A. Using a $\mathcal{C}^k$ definable diffeomorphism from D to $\mathbb{R}^{\dim D}$ and the inductive assumption, we can partition D into finitely many definable $\mathcal{C}^k$ submanifolds on which f is $\mathcal{C}^k$. This completes the proof of the two theorems. □

6.3 Definable Manifolds and Tubular Neighborhoods

Let $M \subset \mathbb{R}^n$ be a definable $\mathcal{C}^k$ submanifold (we always assume $1 \leq k < \infty$). We introduce the tangent and normal bundles of M.

The *tangent bundle* $\mathrm{T}M$ is the set of $(x, v) \in M \times \mathbb{R}^n$ such that v is a tangent vector to M at x. We denote by $p : TM \to M$ the projection defined by $p(x, v) = x$ and by T_xM the tangent space to M at x, i.e. $\mathrm{T}_xM = p^{-1}(x)$ for $x \in M$. We can argue as follows in order to prove that TM is a definable subset of $\mathbb{R}^n \times \mathbb{R}^n$. We consider the set S of triples (x, y, v), where v is a vector parallel to the line joining two distinct points x and y of M, that is

$$S = \{(x, y, \lambda(y - x)) \in M \times M \times \mathbb{R}^n \ ; \ x \neq y \text{ and } \lambda \in \mathbb{R}\} .$$

This set S is obviously definable. Its closure $\mathrm{clos}(S)$ in $M \times M \times \mathbb{R}^n$ is also definable. We claim that

$$\mathrm{T}M = \{(x, v) \in M \times \mathbb{R}^n \ ; \ (x, x, v) \in \mathrm{clos}(S)\} ,$$

which shows that it is definable.

Exercise 6.9 Prove the claim and show in the same time that $\mathrm{T}M$ is a $\mathcal{C}^{k-1}$ submanifold (this is a classical fact of differential geometry). Hint: One can assume that the origin 0 belongs to M and work in a neighborhood of 0. One can also assume that the first $d = \dim(M)$ cordinates $x_1, \ldots, x_d$ are the coordinates of a chart of M in a neighborhood of 0. Then the other coordinates are $\mathcal{C}^k$ functions $x_j = \xi_j(x_1, \ldots, x_d)$ on a neighborhood of 0 in M (for $j = d+1, \ldots, n$). Then show that $(0, 0, v)$ belongs to the closure of S if and only if v is a linear combination of the vectors

$$e_i + \sum_{j=d+1}^{n} \frac{\partial \xi_j}{x_i}(0) e_j \qquad \text{for } i = 1, \ldots, d\ ,$$

where e_i denotes the i-th vector of the canonical basis of $\mathbb{R}^n$. These vectors form a basis of $\mathrm{T}_0 M$.

Exercise 6.10 Let $f : M \to \mathbb{R}$ be a definable $\mathcal{C}^k$ function. Prove that $\mathrm{d}f : \mathrm{T}M \to \mathrm{T}\mathbb{R} = \mathbb{R} \times \mathbb{R}$ is a definable map (of class $\mathcal{C}^{k-1}$). Hint: construct the graph of df replacing the set S above with the set of

$$(x, y, \lambda(y - x), f(x), \lambda(f(y) - f(x)))\ ,$$

where x and y are distinct point of M and $\lambda \in \mathbb{R}$.

The normal bundle $\mathrm{N}M$ is the set of (x, v) in $M \times \mathbb{R}^n$ such that v is orthogonal to $\mathrm{T}_x M$. This is a $\mathcal{C}^{k-1}$ submanifold of $\mathbb{R}^n \times \mathbb{R}^n$, and it is definable since $\mathrm{T}M$ is definable.

Now we introduce a definable $\mathcal{C}^{k-1}$ map

$$\begin{array}{rcl} \varphi : \mathrm{N}M & \longrightarrow & \mathbb{R}^n \\ (x, v) & \longmapsto & (x + v) \end{array}$$

The map φ induces the canonical diffeomorphism from the zero section $M \times \{0\}$ of the normal bundle onto M, and it is a local diffeomomorphism at each point $(x, 0)$. Indeed, the derivative $\mathrm{d}_{(x,0)}\varphi : \mathrm{T}_x M \times \mathrm{N}_x M \to \mathbb{R}^n$ is the isomorphism which maps (ξ, v) to $\xi + v$.

Our aim in this section is to prove the following result, which is the definable version of the tubular neighborhood theorem.

Theorem 6.11 (Definable Tubular Neighborhood) *Let M be a definable $\mathcal{C}^k$ submanifold of $\mathbb{R}^n$. There exists a definable open neighborhood U of the zero-section $M \times \{0\}$ in the normal bundle $\mathrm{N}M$ such that the restriction $\varphi|_U$ is a*

$\mathcal{C}^{k-1}$ diffeomorphism onto an open neighborhood Ω of M in $\mathbb{R}^n$. Moreover, we can take U of the form

$$U = \{(x,v) \in \mathrm{N}M \ ; \ \|v\| < \varepsilon(x)\} \ ,$$

where ε is a positive definable $\mathcal{C}^k$ function on M.

A neighborhood Ω as in the theorem above is called a *definable $\mathcal{C}^{k-1}$ tubular neighborhood of M*. We have on Ω a definable $\mathcal{C}^{k-1}$ retraction $\pi : \Omega \to M$ and a definable $\mathcal{C}^{k-1}$ "square of distance function" $\rho : \Omega \to \mathbb{R}$, which are defined by

$$\begin{aligned} \pi(\varphi(x,v)) &= x \\ \rho(\varphi(x,v)) &= \|v\|^2 \ . \end{aligned}$$

For the proof of Theorem 6.11, we need to consider first the case of closed definable submanifolds.

Lemma 6.12 *Let M be a definable $\mathcal{C}^k$ submanifold of $\mathbb{R}^n$, closed in $\mathbb{R}^n$. Let $\psi : M \to \mathbb{R}$ be a positive definable function, which is locally bounded from below by positive constants (for every x in M, there exist $c > 0$ and a neighborhood V of x in M such that $\psi > c$ on V). Then there exists a positive definable $\mathcal{C}^k$ function $\varepsilon : M \to \mathbb{R}$ such that $\varepsilon < \psi$ on M.*

Proof. For $r \in \mathbb{R}$, set

$$M_r = \{ \ x \in M \ ; \ \|x\|^2 \leq r\} \ .$$

We can assume $M_{r_0} \neq \emptyset$ for some r_0. Observe that M_r is compact for $r \geq r_0$. For such r, we define $\mu(r)$ by $\mu(r) = \inf\{\psi(x) \ ; \ x \in M_r\}$. The function $\mu : [r_0, +\infty) \to \mathbb{R}$ is definable and nonincreasing. It is positive, since M_r is covered by finitely many subsets where ψ is bounded from below by positive constants. By Theorem 6.4, there is $a > r_0$ such that μ is $\mathcal{C}^k$ on an open interval containing $[a, +\infty)$. Choose a definable $\mathcal{C}^k$ function $\theta : \mathbb{R} \to \mathbb{R}$ such that $\theta = 0$ on $(-\infty, a]$, θ increases from 0 to 1 on $[a, a+1]$ and $\theta = 1$ on $[a+1, +\infty)$. Define the function $\mu_1 : \mathbb{R} \to \mathbb{R}$ by

$$\mu_1(r) = \theta(r)\mu(r) + (1 - \theta(r))\mu(a+1) \ .$$

Observe that μ_1 is $\mathcal{C}^k$, definable, and satisfies $\mu_1 \leq \mu$ on $[r_0, +\infty)$. We set $\varepsilon(x) = \frac{1}{2}\,\mu_1(\|x\|^2))$ for $x \in M$. By construction, the positive, definable, $\mathcal{C}^k$ function ε satisfies $\varepsilon < \psi$ on M. $\square$

Note that Lemma 6.12 also holds if we assume only that there is a definable $\mathcal{C}^k$ diffeomorphism from M onto a closed submanifold of some $\mathbb{R}^m$. Later we shall show that it holds for every definable submanifold.

Exercise 6.13 Give an explicit formula for θ in the proof above (you can take for θ on $[a, a+1]$ a primitive of a well-chosen polynomial).

Exercise 6.14 Having in view the generalization of the lemma to an o-minimal structure expanding an arbitrary real closed field, replace the argument for the positivity of μ in the proof of the preceding lemma by an argument valid in this general situation. (Hint: if $\psi(r) = 0$, show that there is $b > 0$ and a definable path $\gamma : (0, b) \to M_r$ such that $\psi(\gamma(t)) = t$.)

Lemma 6.15 *The definable tubular neighborhood 6.11 holds if M is closed in $\mathbb{R}^n$, or definably $\mathcal{C}^k$ diffeomorphic to a closed submanifold in some $\mathbb{R}^m$ (e.g. if M is a cell of a $\mathcal{C}^k$ cdcd).*

Proof. We want to avoid the following two "bad" situations:

1. φ is not a local diffeomorphism at (x, v).
2. there are (x, v) and (y, w) such that $\varphi(x, v) = \varphi(y, w)$.

Let Z be the subset of (x, v) in $\mathrm{N}M$ such that

$$\mathrm{d}_{(x,v)}\varphi : \mathrm{T}_{(x,v)}(\mathrm{N}M) \longrightarrow \mathbb{R}^n$$

is not an isomorphism. The set Z is definable, closed in $\mathrm{N}M$ and disjoint from the zero section $M \times \{0\}$. For x in M, let $\psi(x)$ be the minimum of 1, $\mathrm{dist}((x, 0), Z)$ and the infimum of $r \in \mathbb{R}$ such that

$$\exists (y, w) \in \mathrm{N}M \quad \exists v \in \mathrm{N}_x M \quad \|w\| \le \|v\| = r \;\text{ and }\; y + w = x + v\ .$$

The function $\psi : M \to \mathbb{R}$ is definable.

We claim that it is locally bounded from below by positive constants. Take $x \in M$. The inverse function theorem implies that there is $c_x > 0$ such that the restriction of φ to the intersection of $\mathrm{N}M$ with the open ball $B((x, 0), c_x) \subset \mathbb{R}^{2n}$ is a diffeomorphism onto a neighborhood of x in $\mathbb{R}^n$. We show that $\psi \ge c_x/4$ on the intersection of M with the open ball $B(x, c_x/4) \subset R^n$. We can, of course, take $c_x/4 \le 1$. Since $B((x, 0), c_x)$ is disjoint from Z we have $c_x/4 \le \mathrm{dist}((y, 0), Z)$ for every $y \in M \cap B(x, c_x/4)$. Suppose now that there are (y, v) and (z, w) in $\mathrm{N}M$ such that $y \in B(x, c_x/4)$, $\|w\| \le \|v\| < c_x/4$ and

$y + v = z + w$. Then (y, v) and (z, w) both belong to $B((x, 0), c_x)$, and we arrive to a contradiction with the fact that φ is injective in restriction to the intersection of this ball with $\mathrm{N}M$. Hence, the claim is proved. We can apply Lemma 6.12 to the function ψ. We obtain a definable, positive $\mathcal{C}^k$ function $\varepsilon : M \to \mathbb{R}$ such that, in restriction to the definable open subset

$$U = \{(x, v) \in \mathrm{N}M \ ; \ \|v\| < \varepsilon(x)\} \ ,$$

the map φ is an injective local diffeomorphism. We conclude that $\varphi|_U$ is a diffeomorphism onto an open definable neighborhood of M in $\mathbb{R}^n$. □

Proposition 6.16 *Let M be a definable $\mathcal{C}^k$ submanifold of $\mathbb{R}^n$, closed in $\mathbb{R}^n$ or definably $\mathcal{C}^k$ diffeomorphic to a closed submanifold of some $\mathbb{R}^m$. Then M has a nonnegative definable $\mathcal{C}^{k-1}$ equation in the complement of $\partial M = \mathrm{clos}(M) \setminus M$. This means that there exists a nonnegative definable $\mathcal{C}^{k-1}$ function $f : \mathbb{R}^n \setminus \partial M \to \mathbb{R}$ such that $M = f^{-1}(0)$. Moreover, one can choose $f \leq 1$.*

Proof. By Lemma 6.15, we can choose a definable tubular neighborhood Ω of M of radius $\varepsilon : M \to (0, +\infty)$. By Lemma 6.12, we can assume that $\varepsilon(x) < \min(1, \mathrm{dist}(x, \partial M))$ for all x in M. Observe that the square of distance function $\rho : \Omega \to [0, +\infty)$ is a definable nonnegative $\mathcal{C}^{k-1}$ equation of M in Ω. We are going to extend it to $\mathbb{R}^n \setminus \partial M$ by using a partition of unity. We choose a $\mathcal{C}^{k-1}$ definable function $\theta : \mathbb{R} \to [0, 1]$ which is 0 on $(-\infty, 0]$, 1 on $[1, +\infty)$ and increases from 0 to 1 on $[0, 1]$. Then we set

$$f(x) = \begin{cases} \theta(2\rho(x)/\varepsilon^2(x)) & \text{on } \Omega \\ 1 & \text{outside } \partial M \cup \{x \in \Omega \ ; \ \rho(x) \leq \frac{1}{2}\varepsilon^2(x)\} \end{cases} .$$

The function $f : \mathbb{R}^n \setminus \partial M \to \mathbb{R}$ is nonnegative, definable, and $f^{-1}(0) = M$. The assumption $\varepsilon(x) < \min(1, \mathrm{dist}(x, \partial M))$ implies that $\partial M \cup \{x \in \Omega \ ; \ \rho(x) \leq \frac{1}{2}\varepsilon^2(x)\}$ is closed (we leave the proof as an exercise). This shows that f is indeed $\mathcal{C}^{k-1}$, and completes the proof of the proposition. □

Theorem 6.17 *Let F be a closed definable subset of $\mathbb{R}^n$. For every positive integer k, there exists a definable, continuous, nonnegative function $f : \mathbb{R}^n \to \mathbb{R}$ such that $f^{-1}(0) = F$ and the restriction of f to $\mathbb{R}^n \setminus F$ is of class $\mathcal{C}^k$.*

Proof. We proceed by induction on the dimension of F. If F is empty, we can take for f any positive constant function. Now let $d = \dim F \geq 0$ and assume that the theorem holds for all closed definable subsets of $\mathbb{R}^n$ of dimension $< d$. Take a $\mathcal{C}^{k+1}$ cdcd of $\mathbb{R}^n$ adapted to F. Let $\mathcal{D}$ be the finite set of cells of dimension d contained in F. By Proposition 6.16, every d-cell $C \in \mathcal{D}$ has a nonnegative, definable $\mathcal{C}^k$ equation g_C in $\mathbb{R}^n \setminus \partial C$. Let Z be the union of all cells of dimension $< d$ contained in F and all ∂C, for $C \in \mathcal{D}$. The set Z is definable and closed in $\mathbb{R}^n$. The product of all functions g_C for $C \in \mathcal{D}$ defines a function $g : U = \mathbb{R}^n \setminus Z \to \mathbb{R}$. The function g is of class $\mathcal{C}^k$, and it is a definable, nonnegative equation of $F \cap U$ in U. Moreover, we can take all g_C bounded by 1 and, hence, we have $g \leq 1$. By the inductive assumption, there is a definable, continuous, nonnegative function $h : \mathbb{R}^n \to \mathbb{R}$ such that $h^{-1}(0) = Z$ and the restriction of h to U is of class $\mathcal{C}^k$. The product $f = gh$ is defined and $\mathcal{C}^k$ on U. It can be continuously extended to $\mathbb{R}^n$ by setting $f = 0$ on Z. The zeroset of F is then $Z \cup (F \cap U) = F$. The function f satisfies the properties of the theorem. $\square$

Corollary 6.18 *Let M be a definable $\mathcal{C}^k$ submanifold of $\mathbb{R}^n$. Then there is a definable $\mathcal{C}^k$ submanifold N of $\mathbb{R}^{n+1}$, closed in $\mathbb{R}^{n+1}$, such that the projection $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ on the first n coordinates induces a diffeomorphism from N onto M. Hence, every definable $\mathcal{C}^k$ submanifold of $\mathbb{R}^n$ is definably $\mathcal{C}^k$ diffeomorphic to a closed submanifold of $\mathbb{R}^{n+1}$, and Lemma 6.12 and Proposition 6.16 hold for every definable $\mathcal{C}^k$ submanifold.*

Proof. Let M be a definable $\mathcal{C}^k$ submanifold of $\mathbb{R}^n$. Since M is locally closed, $\partial M = \operatorname{clos}(M) \setminus M$ is closed in $\mathbb{R}^n$. By Theorem 6.17, there is a continuous, nonnegative definable function $f : \mathbb{R}^n \to \mathbb{R}$ such that $f^{-1}(0) = \partial M$ and f is $\mathcal{C}^k$ on $\mathbb{R}^n \setminus \partial M$. Let

$$N = \{(x,t) \in \mathbb{R}^{n+1} \; ; \; x \in M \;\text{ and }\; tf(x) = 1\} \, .$$

Then N is a definable $\mathcal{C}^k$ submanifold, closed in $\mathbb{R}^{n+1}$, and the mapping $x \mapsto (x, 1/f(x))$ is a definable $\mathcal{C}^k$ diffeomorphism from M onto N, which is the inverse of $\pi|_N$. $\square$

The preceding corollary, together with Lemma 6.15, completes the proof of the definable tubular neighborhood theorem 6.11. We conclude with a property which plays an important role in the course of Macintyre on constructing o-minimal structures.

Proposition 6.19 (D$\mathcal{C}^k$ **- all** k) *Let A be a definable subset of $\mathbb{R}^n$. For every positive integer k, there exists a definable $\mathcal{C}^k$ function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ such that $A = \pi(f^{-1}(0))$, where $\pi = \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection on the first n coordinates.*

Proof. Fix k. Take a $\mathcal{C}^{k+1}$ cdcd of $\mathbb{R}^n$ adapted to A. By corollary 6.18, for every cell C there is a definable $\mathcal{C}^{k+1}$ submanifold D of $\mathbb{R}^{n+1}$, closed in $\mathbb{R}^{n+1}$, such that $\pi|_D$ is a diffeomorphism onto C. By Proposition 6.16, there is a definable $\mathcal{C}^k$ function $f_C : \mathbb{R}^{n+1} \to \mathbb{R}$ such that $f_C^{-1}(0) = D$. Let f be the product of the functions f_C for all cells C contained in A. Then f is a definable $\mathcal{C}^k$ function on $\mathbb{R}^{n+1}$ and $A = \pi(f^{-1}(0))$. $\square$

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