INTRODUCTION TO SYMPLECTIC TOPOLOGY

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1. LECTURE 1: SYMPLECTIC LINEAR ALGEBRA

Let V be a real vector space.

Definition 1.1. A symplectic form on V is a skew-symmetric bilinear nondegenerate form:
(1) $\omega(x, y) = -\omega(y, x)$ $(\Longrightarrow \omega(x, x) = 0)$;
(2) $\forall x, \exists y$ such that $\omega(x, y) \neq 0$.

For a general 2-form ω on a vector space, V, we denote $\ker(\omega)$ the subspace given by

$$\ker(\omega) = \{v \in V \mid \forall w \in V \omega(v, w) = 0\}$$

The second condition implies that $\ker(\omega)$ reduces to zero, so when ω is symplectic, there are no "preferred directions" in V.

There are special types of subspaces in symplectic manifolds. For a vector subspace F, we denote by

$$F^\omega = \{v \in V \mid \forall w \in F ,\ \omega(v, w) = 0\}$$

From Grassmann's formula it follows that $\dim(F^\omega) = codim(F) = \dim(V) - \dim(F)$
Also we have

Proposition 1.2.
$$(F^\omega)\omega = F$$

$$(F_1 + F_2)^\omega = F_1^\omega \cap F_2^\omega$$

Definition 1.3. A subspace F of V, ω is

- isotropic if $F \subset F^\omega$ $(\Longleftrightarrow \omega|_F = 0)$;
- coisotropic if $F^\omega \subset F$
- Lagrangian if it is maximal isotropic.

Proposition 1.4. (1) *Any symplectic vector space has even dimension*
(2) *Any isotropic subspace is contained in a Lagrangian subspace and Lagrangians have dimension equal to half the dimension of the total space.*
(3) *If (V_1, ω_1), (V_2, ω_2) are symplectic vector spaces with L_1, L_2 Lagrangian subspaces, and if $\dim(V_1) = \dim(V_2)$, then there is a linear isomorphism $\varphi : V_1 \to V_2$ such that $\varphi^* \omega_2 = \omega_1$ and $\varphi(L_1) = L_2$.*

Proof. We first prove that if I is an isotropic subspace it is contained in a Lagrangian subspace. Indeed, I is contained in a maximal isotropic subspace. We denote it again by I and we just have to prove $2\dim(I) = \dim(V)$.

Since $I \subset I^\omega$ we have $\dim(I) \leq \dim I^\omega = \dim(V) - \dim(I)$ so that $2\dim(I) \leq \dim(V)$Now assume the inequality is strict. Then there exist a non zero vector, e, in $I^\omega \setminus I$, and $I \oplus \mathbb{R}e$ is isotropic and contains I. Therefore I was not maximal, a contradiction.

We thus proved that $I = I^\omega$ and $2\dim(I) = \dim(V)$, and $\dim(V)$ is even.

Applying the above result to $\{0\}$, an obviously isotropic subspace, we conclude that we may always find a Lagrangian subspace, and V is always even-dimensional.

This proves (1) and (2).

Let us now prove (3).

We shall consider a standard symplectic vector space $(\mathbb{R}^2, \sigma)$ with canonical base e_x, e_y and the symplectic form given by

$$\sigma(x_1 e_x + y_1 e_y, x_2 e_x + y_2 e_y) = x_1 y_2 - y_1 x_2.$$

Similarly by orthogonal direct sum, we get the symplectic space $(\mathbb{R}^{2n}, \sigma_n)$

$$\sigma((x_1, ..., x_n, y_1, ..., y_n), (x'_1, ..., x'_n, y'_1, ..., y'_n)) = \sum_{j=1}^n x_j y'_j - x'_j y_j$$

It contains an obvious Lagrangian subspace,

$$Z_n = \mathbb{R}^n \oplus 0 = \{(x_1, ..., x_n, y_1, ..., y_n) \mid \forall 1 \leq j \leq n, y_j = 0\}$$

Let (V, ω) be a symplectic vector space and L a Lagrangian. Pick any $e_1 \in L$. Since ω is nondegenerate, there exists an $f_1 \in V$ such that $\omega(e_1, f_1) = 1$. Then $f_1 \notin L$. Define

$$V_2 = Vect(e_1, f_1)^\omega = \{x \in V | \omega(x, e_1) = \omega(x, f_1) = 0\}.$$

It is easy to see that $(V_2, \omega_{|V_2})$ is symplectic since only non-degeneracy is an issue, and it follows from the fact that

$$\ker(\omega_{|V_2}) = V_2 \cap V_2^\omega = \{0\}$$

We now claim that $L_2 = L \cap V_2$ is a Lagrangian in V_2. First, since $\omega_{|L_2}$ is the restriction of $\omega_{|L}$ L_2 is clearly isotropic. It is maximal isotropic, since otherwise, there would be an isotropic $V_2 \supset W \supsetneq L_2$, and then $W \oplus \mathbb{R}e_1$ would be a strictly larger isotropic subspace than L, which is impossible.

Now we claim by induction that there is a symplectic map, φ_{n-1} from $(\mathbb{R}^{2n-2}, \sigma)$ to (V_2, ω) sending Z_{n-1} to L_2. Now the map

$$
\begin{aligned}
\varphi_n : (\mathbb{R}^2, \sigma_2) \oplus (\mathbb{R}^{2n-2}, \sigma) &\longrightarrow (V, \omega) \\
(x_1, y_1; z) &\longrightarrow x_1 e_1 + y_1 f_1 + \varphi_{n-1}(z)
\end{aligned}
$$

is symplectic and sends Z_n to L.

Now given two symplectic manifolds, $(V_1, \omega_1), (V_2, \omega_2)$ of dimension $2n$, and two lagrangians L_1, L_2, we get two symplectic maps

$$\psi_j (\mathbb{R}^{2n}, \sigma_n) \longrightarrow (V_j, \omega_j)$$

sending Z_n to L_j. Then the map $\psi_2 \circ \psi_1^{-1}$ is a symplectic map from (V_1, ω_1) to (V_2, ω_2) sending L_1 to L_2.

$\square$

We now give a better description of the set of lagrangians

Proposition 1.5. (1) *The action of $Sp(n) = \{\varphi \in GL(V)|\varphi^*\omega = \omega\}$ acts bi-transitively in the set of Lagrangians;*
(2) *$\{\Lambda|Lagrangian \text{ and } \Lambda \cap L = \{0\}\} \longleftrightarrow \{quadratic \text{ forms on } L^*\}$.*

Proof. The first statement is a rephrasing of (3) of proposition 1.4 applied to $V_1 = V_2 = V$.

For (2), we notice that $W = L \oplus L^*$ with the symplectic form

$$\sigma((e, f), (e', f')) = \langle e', f \rangle - \langle e, f' \rangle$$

is a symplectic vector space and that $L \oplus 0$ is a Lagrangian subspace.

According to the previous proposition there is a symplectic map $\psi : V \longrightarrow W$ such that $\psi(L) = L \oplus 0$, so we can work in W.

Let Λ be a Lagrangian in W with $\Lambda \cap L = \{0\}$. Then Λ is the graph of $A : L^* \to L$, more precisely

$$\Lambda = \{(Ay^*, y^*)|y^* \in L^*\}.$$

The subspace Λ is Lagrangian if and only if

$$\sigma((Ay_1^*, y_1^*), (Ay_2^*, y_2)) = 0, \text{ for all } y_1, y_2$$

i.e. if and only if

$$\langle y_1^*, Ay_2^* \rangle = \langle y_2^*, Ay_1^* \rangle$$

that is if $\langle \cdot, A\cdot \rangle$ is a bilinear symmetric form on L^*. But such bilinear form are in 1-1 correspondence with quadratic forms. $\square$

Exercice 1: (Witt's Theorem) Let V_1 and V_2 be two symplectic vector spaces with the same dimension and $F_i \subset (V_i, \omega_i), i = 1, 2$. Assume that there exists $\varphi : F_1 \cong F_2$, i.e. $\varphi^*(\omega_2)_{|F_2} = (\omega_1)_{|F_1}$. Then φ extends to a symplectic map $\widetilde{\varphi} : (V_1, \omega_1) \to (V_2, \omega_2)$.

Exercice 2: Prove that the above results are valid over any field of any characteristic, but for the last statement , where in characteristic 2, quadratic forms and bilinear symmetric forms are not equivalent.

2. COMPLEX STRUCTURE

Let h be a hermitian form on a complex vector space V in the sense:
1) $h(z, z') = \overline{h(z', z)}$;
2) $h(\lambda z, z') = \lambda h(z, z')$ for $\lambda \in \mathbb{C}$;
3) $h(z, \lambda z') = \bar{\lambda} h(z, z')$ for $\lambda \in \mathbb{C}$;
4) $h(z, z) > 0$ for all $z \neq 0$.
Then

$$h(z, z') = g(z, z') + i\omega(z, z'),$$

where g is a scalar product and ω is symplectic.

Example: On $\mathbb{C}^n$, define

$$h((z_1, \cdots, z_n), (z_1', \cdots, z_n')) = \sum_{j=1}^{n} z_j \bar{z}_j' \in \mathbb{C}.$$

Then the symmetric part is the usual scalar product on $\mathbb{R}^{2n}$ while ω is the standard symplectic form.

Denote by J the multiplication by $i = \sqrt{-1}$.

Theorem 2.1.
$$\left\{ \begin{array}{l} g(Jz, z') = -\omega(z, z') \\ \omega(z, Jz') = -g(z, z') \end{array} \right.$$

Remark 2.2. ω is nondegenerate because $\omega(z, Jz) = -g(z, z) < 0$ for all $z \neq 0$.

Conclusion: Any hermitian space V has a canonical symplectic form.

We will now answer the following question: can a symplectic vector space be made into a hermitian space? In how many ways?

Proposition 2.3. *Let (V, ω) be a symplectic vector space. Then there is a complex structure on V such that $\omega(J\xi, \xi)$ is a scalar product. Moreover, the set of such J is contractible.*

Proof. Let $(\cdot, \cdot)$ be any fixed scalar product on V. Then there exists A such that
$$\omega(x, y) = (Ax, y).$$
Since ω is skew-symmetric, $A^* = -A$ where A^* is the adjoint of A with respect to $(\cdot, \cdot)$. Since any other scalar product can be given by a positive definite symmetric matrix M, we look for J such that $J^2 = -I$ and M such that $M^* = M$ and $(x, y)_M = (Mx, y)$ and $\omega(Jx, y) = (x, y)_M$. The last equality is
$$(AJx, y) = (Mx, y) \text{ for all } x, y.$$
It's easy to check that $M = (AA^*)^{1/2}$ and $J = A^{-1}M$ solves $AJ = M$, $J^2 = -I$ and $M^* = M$.

In summary, for any fixed scalar product $(\cdot, \cdot)$, we can find a pair (J_0, M_0) such that $\omega(J_0x, y)$ is the scalar product $(M_0 \cdot, \cdot)$. If we know (J_0, M_0) is such a pair and we start from the scalar product $(M_0 \cdot, \cdot)$, then we get the pair (J_0, id).

Define X to be the set of all J's such that $\omega(J\cdot, \cdot)$ is a scalar product. Define Y to be the set of all scalar product. By previous discussion, there is continuous map
$$\Psi : Y \to X.$$
Moreover, if J is in X, Ψ maps $\omega(J\cdot, \cdot)$ to J. On the other hand, we have a continuous embedding i from X to Y which maps J to $\omega(J\cdot, \cdot)$. Let $p \in Y$ be in the image. Since we know Y is contractible, there is a continuous family
$$F_t : Y \to Y$$
such that $F_0 = id$ and F_1 maps anything to p. Consider
$$\tilde{F}_t : X \to X$$
given by
$$\tilde{F}_t = \Psi \circ F_t \circ i.$$
By the definition of Ψ, we know $\tilde{F}_0 = id$ and $\tilde{F}_1 = J_p$. This shows that X is contractible. $\square$

Exercice: Let L be a Lagrangian subspace, show that JL is also a Lagrangian and $L \cap JL = \{0\}$.