## AN INTRODUCTION TO THE FRACTIONAL CALCULUS AND FRACTIONAL DIFFERENTIAL EQUATIONS

KENNETH S. MILLER

Mathematical Consultant
Formerly Professor of Mathematics
New York University

## BERTRAM ROSS

University of New Haven


A Wiley-Interscience Publication JOHN WILEY \& SONS, INC.
New York - Chichester - Brisbane - Toronto - Singapore

## CONTENTS

Preface ..... xi
I. Historical Survey ..... 1

1. The Origin of the Fractional Calculus,
2. The Contributions of Abel and Liouville, ..... 3
3. A Longstanding Controversy, 6
4. Riemann's Contribution, Errors by Noted Mathematicians, ..... 7
5. The Mid-Nineteenth Century, 9 ..... 9
6. The Origin of the Riemann-Liouville Definition, ..... 9
7. The Last Decade of the Nineteenth Century, ..... 13
8. The Twentieth Century, ..... 15
9. Bibliography, ..... 16
II. The Modern Approach ..... 21
10. Introduction, ..... 21
11. The Iterated Integral Approach, ..... 23
12. The Differential Equation Approach, ..... 25
13. The Complex Variable Approach, ..... 28
14. The Weyl Transform, ..... 33
15. The Fractional Derivative, ..... 35
16. The Definitions of Grünwald and Marchaud, ..... 38
III. The Riemann-Liouville Fractional Integral ..... 44
17. Introduction, ..... 44
18. Definition of the Fractional Integral, ..... 45
19. Some Examples of Fractional Integrals, ..... 47
20. Dirichlet's Formula, ..... 56
21. Derivatives of the Fractional Integral and the Fractional Integral of Derivatives, ..... 59
22. Laplace Transform of the Fractional Integral, ..... 67
23. Leibniz's Formula for Fractional Integrals, ..... 73
IV. The Riemann-Liouville Fractional Calculus ..... 80
24. Introduction, ..... 80
25. The Fractional Derivative, ..... 82
26. A Class of Functions, ..... 87
27. Leibniz's Formula for Fractional Derivatives, ..... 95
28. Some Further Examples, ..... 97
29. The Law of Exponents, ..... 104
30. Integral Representations, ..... 111
31. Representations of Functions, ..... 116
32. Integral Relations, ..... 118
33. Laplace Transform of the Fractional Derivative, ..... 121
V. Fractional Differential Equations ..... 126
34. Introduction, ..... 126
35. Motivation: Direct Approach, ..... 128
36. Motivation: Laplace Transform, ..... 133
37. Motivation: Linearly Independent Solutions, ..... 136
38. Solution of the Homogeneous Equation, ..... 139
39. Explicit Representation of Solution, ..... 145
40. Relation to the Green's Function, ..... 153
41. Solution of the Nonhomogeneous Fractional Differential Equation, ..... 157
42. Convolution of Fractional Green's Functions, ..... 165
43. Reduction of Fractional Differential Equations to Ordinary Differential Equations, ..... 171
44. Semidifferential Equations, ..... 174
VI. Further Results Associated with Fractional Differential Equations ..... 185
45. Introduction, ..... 185
46. Fractional Integral Equations, ..... 186
47. Fractional Differential Equations with Nonconstant Coefficients, ..... 194
48. Sequential Fractional Differential Equations, ..... 209
49. Vector Fractional Differential Equations, ..... 217
50. Some Comparisons with Ordinary Differential Equations, ..... 229
VII. The Weyl Fractional Calculus ..... 236
51. Introduction, ..... 236
52. Good Functions, ..... 237
53. A Law of Exponents for Fractional Integrals, ..... 239
54. The Weyl Fractional Derivative, ..... 240
55. The Algebra of the Weyl Transform, ..... 244
56. A Leibniz Formula, ..... 245
57. Some Further Examples, ..... 247
58. An Application to Ordinary Differential Equations, ..... 251
VIII. Some Historical Arguments ..... 255
59. Introduction, ..... 255
60. Abel's Integral Equation and the Tautochrone Problem, ..... 255
61. Heaviside Operational Calculus and the FractionalCalculus, 261
62. Potential Theory and Liouville's Problem, ..... 264
63. Fluid Flow and the Design of a Weir Notch, ..... 269
Appendix A. Some Algebraic Results ..... 275
64. Introduction, ..... 275
65. Some Identities Associated with Partial FractionExpansions,275
66. Zeros of Multiplicity Greater than One, ..... 285
67. Complementary Polynomials, ..... 290
68. Some Reduction Formulas, ..... 292
69. Some Algebraic Identities, ..... 294
Appendix B. Higher Transcendental Functions ..... 297
70. Introduction, ..... 297
71. The Gamma Function and Related Functions, ..... 297
72. Bessel Functions, ..... 301
73. Hypergeometric Functions, ..... 303
74. Legendre and Laguerre Functions, ..... 307
Appendix C. The Incomplete Gamma Function and Related Functions ..... 308
75. Introduction, ..... 308
76. The Incomplete Gamma Function, ..... 309
77. Some Functions Related to the Incomplete Gamma Function, ..... 314
78. Laplace Transforms, ..... 321
79. Numerical Results, ..... 330
Appendix D. A Brief Table of Fractional Integrals and Derivatives ..... 352
References ..... 357
Index of Symbols ..... 361
Index ..... 363
