AN INTRODUCTION TO THE THEORY OF PIEZOELECTRICITY

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AN INTRODUCTION TO THE THEORY OF PIEZOELECTRICITY

by

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Contents

Foreword

This book is based on lecture notes for a graduate course that has been offered at University of Nebraska-Lincoln on and off since 1998. The course is intended to provide graduate students with the basic aspects of the continuum modeling of electroelastic interactions in solids. A concise treatment of linear, nonlinear, static and dynamic theories and problems is presented. The emphasis is on formulation and understanding of problems useful in device applications rather than solution techniques of mathematical problems. The mathematics used in the book is minimal. The book is suitable for a one-semester graduate course on electroelasticity. It can also be used as a reference for researchers. I would like to take this opportunity to thank UNL for a Maude Hammond Fling Faculty Research Fellowship in 2003 for the preparation of the first draft of this book. I also wish to thank Ms. Deborah Derrick of the College of Engineering and Technology at UNL for editing assistance with the book, and Professor David Y. Gao of Virginia Polytechnic Institute and State University for recommending this book to Kluwer for publication in the series of Advances in Mechanics and Mathematics.

JSY
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Preface

Electroelastic materials exhibit electromechanical coupling. They experience mechanical deformations when placed in an electric field, and become electrically polarized under mechanical loads. Strictly speaking, piezoelectricity refers to linear electromechanical couplings only. Electrostriction may be the simplest nonlinear electromechanical coupling, where mechanical fields depend on electric fields quadratically in the simplest description. Electroelastic materials have been used for a long time to make many electromechanical devices. Examples include transducers for converting electrical energy to mechanical energy or vice versa, resonators and filters for frequency control and selection for telecommunication and precise timing and synchronization, and acoustic wave sensors.

Although most of the book is devoted to the linear theory of piezoelectricity, the book begins with a concise chapter on the nonlinear theory of electroelasticity. It is hoped that this will be helpful for a deeper understanding of the theory of piezoelectricity, because the linear theory is a linearization of the nonlinear theory about a natural state with zero fields. The presentation of the linear theory of piezoelectricity is rather independent so that readers who are not interested in nonlinear electroelasticity can begin directly with Section 2 of Chapter 2 on linear piezoelectricity.

Whereas the majority of books on elasticity treat static problems, the author believes that dynamic problems deserve more attention for piezoelectricity. Therefore, they occupy more space in this book. Chapter 3 is on linear statics and Chapters 4 and 5 are on linear dynamics. This is because in technological applications piezoelectric materials seem to be used in devices operating with vibration modes or propagating waves more than with static deformations. Chapters 2 to 5 form the core for a one-semester course on linear piezoelectricity.

Linear piezoelectricity assumes infinitesimal deviations from an ideal reference state in which there are no pre-existing mechanical and/or electric fields (initial or biasing fields). The presence of biasing fields makes a material apparently behave like a different material and renders the linear theory of piezoelectricity invalid. The behavior of electroelastic bodies under biasing fields can be described by the linear theory for infinitesimal incremental fields superposed on finite biasing fields, which is the subject of Chapter 6. The theory of the incremental fields is derived from the nonlinear theory of electroelasticity when the nonlinear theory is linearized about a bias.

Chapter 7 gives a brief presentation of nonlinear theory including the cubic effects of displacement gradient and electric potential gradient, linear nonlocal theory, linear theory of gradient effects of electrical variables, coupled thermal and dissipative effects, semiconduction, and dynamic theory with Maxwell equations.

The development of the theory of electroelasticity was strongly motivated and influenced by its applications in technology. A book on piezoelectricity does not seem to be complete without some discussion on the applications of the theory, which is given in Chapter 8. A piezoelectric gyroscope, a transformer, a pressure sensor, a temperature sensor, and a resonator are discussed in this chapter.

Throughout the book, effort has been made to present materials with mathematics that are necessary and minimal. Two-point Cartesian tensors with indices are assumed and are used from the very beginning, without which certain concepts of the nonlinear theory cannot be made fully clear. Some concepts from partial differential equations relevant to the well-postness of a boundary-value problem are helpful, but classical solution techniques of separation of variables and integral transforms, etc., are not necessary. Although most problems appear as boundary-value problems of partial differential equations, usually part of a solution is either known or can be guessed from physical reasoning. Therefore some solution techniques for ordinary differential equations are sufficient.

Many problems are analyzed in the book. Some exercise problems are also provided. The problems were chosen based on usefulness and simplicity. Most problems have applications in devices, and have closed-form solutions.

Due to the use of quite a few stress tensors and electric fields in nonlinear electroelasticity, a list of notation is provided in Appendix 1. Material constants used in the book are given in Appendix 2.

Chapter 1

NONLINEAR ELECTROELASTICITY FOR STRONG FIELDS

In this chapter we develop the nonlinear theory of electroelasticty for large deformations and strong electric fields. Readers who are only interested in linear theories may skip this chapter and begin with Chapter 2, Section 2. This chapter uses two-point Cartesian tensor notation, the summation convention for repeated tensor indices and the convention that a comma followed by an index denotes partial differentiation with respect to the coordinate associated with the index.

1. DEFORMATION AND MOTION OF A CONTINUUM

This section is on the kinematics of a deformable continuum. The section is not meant to be a complete treatment of the subject. Only results needed for the rest of the book are presented.

Consider a deformable continuum which, in the reference configuration at time t_0, occupies a region V with boundary surface S (see Figure 1.1-1). $\mathbf{N}$ is the unit exterior normal of S. In this state the body is free from deformation and fields. The position of a material point in this state may be denoted by a position vector $\mathbf{X} = X_K\mathbf{I}_K$ in a rectangular coordinate system X_K. X_K denotes the reference or material coordinates of the material point. They are a continuous labeling of material particles so that they are identifiable. At time t, the body occupies a region v with boundary surface s and exterior normal $\mathbf{n}$. The current position of the material point associated with $\mathbf{X}$ is given by $\mathbf{y} = y_k\mathbf{i}_k$, which denotes the present or spatial coordinates of the material point.

Since the coordinate systems are othogonal,

$$\mathbf{i}_k \cdot \mathbf{i}_l = \delta_{kl}, \quad \mathbf{I}_K \cdot \mathbf{I}_L = \delta_{KL}, \tag{1.1-1}$$

where δ_{kl} and δ_{KL} are the Kronecker delta. In matrix notation,

$$[\delta_{kl}] = [\delta_{KL}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.1-2}$$

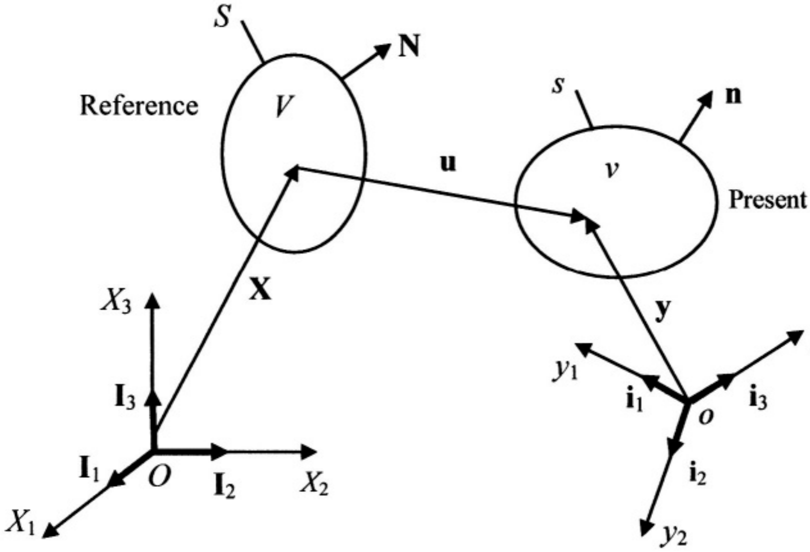

Figure 1.1-1. Motion of a continuum and coordinate systems.

The transformation coefficients (shifters) between the two coordinate systems are denoted by

$$\mathbf{i}_k \cdot \mathbf{I}_L = \delta_{kL}. \tag{1.1-3}$$

In the rest of this book the two coordinate systems are chosen to be coincident, i.e.,

$$o = O, \quad \mathbf{i}_1 = \mathbf{I}_1, \quad \mathbf{i}_2 = \mathbf{I}_2, \quad \mathbf{i}_3 = \mathbf{I}_3. \tag{1.1-4}$$

Then δ_{kL} becomes the Kronecker delta. A vector can be resolved into rectangular components in different coordinate systems. For example, we can also write

$$\mathbf{y} = y_K \mathbf{I}_K, \tag{1.1-5}$$

with

$$y_M = \delta_{Mi} y_i. \tag{1.1-6}$$

The motion of the body is described by

$$y_i = y_i(\mathbf{X}, t). \tag{1.1-7}$$

The displacement vector $\mathbf{u}$ of a material point is defined by

$$\mathbf{y} = \mathbf{X} + \mathbf{u}, \tag{1.1-8}$$

or

$$y_i = \delta_{iM}(X_M + u_M)\,. \tag{1.1-9}$$

A material line element $d\mathbf{X}$ at t_0 deforms into the following line element at t:

$$dy_i\big|_{t\text{ fixed}} = y_{i,K}dX_K\,, \tag{1.1-10}$$

where the deformation gradient

$$y_{k,K} = \delta_{kK} + \delta_{kL}u_{L,K} \tag{1.1-11}$$

is a two-point tensor. The following determinant is called the Jacobian of the deformation:

$$\begin{aligned} J = \det(y_{k,K}) &= \varepsilon_{ijk}y_{i,1}y_{j,2}y_{k,3} = \varepsilon_{KLM}y_{1,K}y_{2,L}y_{3,M} \\ &= \frac{1}{6}\varepsilon_{klm}\varepsilon_{KLM}y_{k,K}y_{l,L}y_{m,M}\,, \end{aligned} \tag{1.1-12}$$

where ε_{klm} and ε_{KLM} are the permutation tensor, and

$$\varepsilon_{ijk} = \mathbf{i}_i\cdot(\mathbf{i}_j\times\mathbf{i}_k) = \begin{cases} 1 & i,j,k = 1,2,3;\quad 2,3,1;\quad 3,1,2, \\ -1 & i,j,k = 3,2,1;\quad 2,1,3;\quad 1,3,2, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1-13}$$

The following relation exists (ε–δ identity):

$$\varepsilon_{ijk}\varepsilon_{pqr} = \begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{vmatrix}. \tag{1.1-14}$$

As a special case, when $i = p$, then

$$\varepsilon_{ijk}\varepsilon_{iqr} = \delta_{jq}\delta_{kr} - \delta_{jr}\delta_{kq}\,. \tag{1.1-15}$$

With Equation (1.1 -14) it can be shown from (1.1-12) that

$$J = \frac{1}{6}\left[2\frac{\partial y_K}{\partial X_L}\frac{\partial y_L}{\partial X_M}\frac{\partial y_M}{\partial X_K} - 3\frac{\partial y_K}{\partial X_K}\frac{\partial y_L}{\partial X_M}\frac{\partial y_M}{\partial X_L} + \left(\frac{\partial y_M}{\partial X_M}\right)^3\right]. \tag{1.1-16}$$

It can be verified that for all L, M, and N the following is true:

$$\varepsilon_{ijk}y_{i,L}y_{j,M}y_{k,N} = J\varepsilon_{LMN}\,. \tag{1.1-17}$$

From Equation (1.1-17) the following can be shown:

$$\varepsilon_{ijk}y_{j,M}y_{k,N} = J\varepsilon_{LMN}X_{L,i}\,. \tag{1.1-18}$$

Proof: Multiplying both sides of (1.1-17) by $X_{L,r}$, we have

$$\varepsilon_{ijk}y_{i,L}X_{L,r}y_{j,M}y_{k,N} = J\varepsilon_{LMN}X_{L,r}\,. \tag{1.1-19}$$

Then

$$\varepsilon_{ijk}\delta_{ir}y_{j,M}y_{k,N} = J\varepsilon_{LMN}X_{L,r}, \tag{1.1-20}$$

$$\varepsilon_{rjk}y_{j,M}y_{k,N} = J\varepsilon_{LMN}X_{L,r}. \tag{1.1-21}$$

Replacing the index r by i gives (1.1-18).

The following relation can then be derived:

$$\varepsilon_{ijk}\varepsilon_{LMN}y_{j,M}y_{k,N} = 2JX_{L,i}. \tag{1.1-22}$$

Proof. Multiply both sides of (1.1-18) by ε_{PMN}

$$\begin{aligned}\varepsilon_{ijk}\varepsilon_{PMN}y_{j,M}y_{k,N} &= J\varepsilon_{LMN}\varepsilon_{PMN}X_{L,i} = J\varepsilon_{MNL}\varepsilon_{MNP}X_{L,i}\\ &= J(\delta_{NN}\delta_{LP} - \delta_{NP}\delta_{LN})X_{L,i} = J(3\delta_{LP} - \delta_{LP})X_{L,i}\\ &= 2JX_{P,i}.\end{aligned} \tag{1.1-23}$$

where (1.1-15) has been used. Replacing the index P by L gives (1.1-22).

The derivative of the Jacobian with respect to one of its elements is

$$\frac{\partial J}{\partial y_{i,L}} = JX_{L,i}. \tag{1.1-24}$$

Proof: From Equation (1.1-12)

$$\begin{aligned}6\frac{\partial J}{\partial y_{p,Q}} &= \varepsilon_{ijk}\varepsilon_{LMN}\delta_{ip}\delta_{LQ}y_{j,M}y_{k,N}\\ &+ \varepsilon_{ijk}\varepsilon_{LMN}y_{i,L}\delta_{jp}\delta_{MQ}y_{k,N} + \varepsilon_{ijk}\varepsilon_{LMN}y_{i,L}y_{j,M}\delta_{kp}\delta_{NQ}\\ &= \varepsilon_{pjk}\varepsilon_{QMN}y_{j,M}y_{k,N} + \varepsilon_{ipk}\varepsilon_{LQN}y_{i,L}y_{k,N} + \varepsilon_{ijp}\varepsilon_{LMQ}y_{i,L}y_{j,M}\\ &= 3\varepsilon_{pjk}\varepsilon_{QMN}y_{j,M}y_{k,N} = 3(2JX_{Q,p}),\end{aligned} \tag{1.1-25}$$

where (1.1-22) has been used.

With (1.1-22) we can also show that

$$(JX_{K,k})_{,K} = 0. \tag{1.1-26}$$

Proof: Differentiate both sides of (1.1-22) with respect to X_L

$$2(JX_{L,i})_{,L} = \varepsilon_{ijk}\varepsilon_{LMN}(y_{j,ML}y_{k,N} + y_{j,M}y_{k,NL}) = 0, \tag{1.1-27}$$

because

$$\varepsilon_{LMN}y_{j,ML} = 0, \quad \varepsilon_{LMN}y_{k,NL} = 0. \tag{1.1-28}$$

Similarly, the following is true:

$$(J^{-1}y_{k,K})_{,k} = 0. \tag{1.1-29}$$

The length of a material line element before and after deformation is given by

$$(dS)^2 = dX_K dX_K = \delta_{KL}dX_K dX_L, \tag{1.1-30}$$

and

$$(ds)^2 = dy_i dy_i = y_{i,K} dX_K y_{i,L} dX_L = C_{KL} dX_K dX_L, \tag{1.1-31}$$

where C_{KL} is the deformation tensor

$$C_{KL} = y_{k,K} y_{k,L} = C_{LK}. \tag{1.1-32}$$

Its inverse is

$$C_{KL}^{-1} = X_{K,k} X_{L,k} = C_{LK}^{-1}, \quad C_{KL}^{-1} C_{LM} = \delta_{KM}. \tag{1.1-33}$$

From Equation (1.1-32)

$$\det(C_{KL}) = J^2, \tag{1.1-34}$$

which defines J as a function of **C**.

It can then be shown that

$$\frac{\partial J}{\partial C_{KL}} = \frac{1}{2} J C_{LK}^{-1}. \tag{1.1-35}$$

Proof:

$$\begin{aligned}
\frac{\partial J}{\partial y_{i,L}} &= \frac{\partial J}{\partial C_{KM}} \frac{\partial C_{KM}}{\partial y_{i,L}} = \frac{\partial J}{\partial C_{KM}} \frac{\partial}{\partial y_{i,L}} (y_{j,K} y_{j,M}) \\
&= \frac{\partial J}{\partial C_{KM}} (\delta_{ji} \delta_{KL} y_{j,M} + y_{j,K} \delta_{ji} \delta_{ML}) \\
&= \frac{\partial J}{\partial C_{KM}} (\delta_{KL} y_{i,M} + y_{i,K} \delta_{ML}) \\
&= \frac{\partial J}{\partial C_{LM}} y_{i,M} + \frac{\partial J}{\partial C_{KL}} y_{i,K} \\
&= \left(\frac{\partial J}{\partial C_{LK}} + \frac{\partial J}{\partial C_{KL}} \right) y_{i,K} = J X_{L,i},
\end{aligned} \tag{1.1-36}$$

where (1.1-24) has been used, and the components of **C** are treated as if they were independent in the partial differentiation. Equation (1.1-36) implies that

$$\frac{\partial J}{\partial C_{LK}} + \frac{\partial J}{\partial C_{KL}} = J X_{L,i} X_{K,i} = J C_{LK}^{-1}. \tag{1.1-37}$$

If J is written as a symmetric function of **C** in the sense that

$$\frac{\partial J}{\partial C_{LK}} = \frac{\partial J}{\partial C_{KL}}, \tag{1.1-38}$$

then Equation (1.1-35) is true.

The derivative of $\mathbf{C}^{-1}$ with respect to **C** is given by

$$\frac{\partial C_{KN}^{-1}}{\partial C_{LM}} = -\frac{1}{2}(C_{KL}^{-1}C_{MN}^{-1} + C_{KM}^{-1}C_{LN}^{-1}). \tag{1.1-39}$$

Proof: From Equation **(1.1-33)$_2$,** for a small variation of **C,**

$$\delta(C_{KL}^{-1}C_{LM}) = C_{LM}\frac{\partial C_{KL}^{-1}}{\partial C_{PQ}}\delta C_{PQ} + C_{KL}^{-1}\delta C_{LM} = 0, \tag{1.1-40}$$

where the components of **C** are treated as if they were independent in the partial differentiation. Multiply Equation (1.1-40) by C_{MN}^{-1}:

$$\frac{\partial C_{KN}^{-1}}{\partial C_{PQ}}\delta C_{PQ} + C_{KL}^{-1}C_{MN}^{-1}\delta C_{LM} = 0, \tag{1.1-41}$$

or

$$\begin{aligned}\left(\frac{\partial C_{KN}^{-1}}{\partial C_{LM}} + C_{KL}^{-1}C_{MN}^{-1}\right)\delta C_{LM} &= \left(\frac{\partial C_{KN}^{-1}}{\partial C_{11}} + C_{K1}^{-1}C_{1N}^{-1}\right)\delta C_{11}\\ &+\left(\frac{\partial C_{KN}^{-1}}{\partial C_{12}} + C_{K1}^{-1}C_{2N}^{-1} + \frac{\partial C_{KN}^{-1}}{\partial C_{21}} + C_{K2}^{-1}C_{1N}^{-1}\right)\delta C_{12} + \cdots = 0.\end{aligned} \tag{1.1-42}$$

Hence

$$\frac{\partial C_{KN}^{-1}}{\partial C_{LM}} + \frac{\partial C_{KN}^{-1}}{\partial C_{ML}} = -C_{KL}^{-1}C_{MN}^{-1} - C_{KM}^{-1}C_{LN}^{-1}. \tag{1.1-43}$$

Equation (1.1-39) follows when $\mathbf{C}^{-1}$ is written as a symmetric function of **C** similar to (1.1-38).

From Equations (1.1-30) and (1.1-31):

$$(ds)^2 - (dS)^2 = (C_{KL} - \delta_{KL})dX_K dX_L = 2S_{KL}dX_K dX_L, \tag{1.1-44}$$

where the finite strain tensor is defined by

$$\begin{aligned}S_{KL} &= (C_{KL} - \delta_{KL})/2\\ &= (u_{K,L} + u_{L,K} + u_{M,K}u_{M,L})/2 = S_{LK}.\end{aligned} \tag{1.1-45}$$

The unabbreviated form of (1.1-45) is given below:

$$\begin{aligned}S_{11} &= u_{1,1} + (u_{1,1}u_{1,1} + u_{2,1}u_{2,1} + u_{3,1}u_{3,1})/2,\\ S_{22} &= u_{2,2} + (u_{1,2}u_{1,2} + u_{2,2}u_{2,2} + u_{3,2}u_{3,2})/2,\\ S_{33} &= u_{3,3} + (u_{1,3}u_{1,3} + u_{2,3}u_{2,3} + u_{3,3}u_{3,3})/2,\\ S_{23} &= (u_{2,3} + u_{3,2} + u_{1,2}u_{1,3} + u_{2,2}u_{2,3} + u_{3,2}u_{3,3})/2,\\ S_{31} &= (u_{3,1} + u_{1,3} + u_{1,3}u_{1,1} + u_{2,3}u_{2,1} + u_{3,3}u_{3,1})/2,\\ S_{12} &= (u_{1,2} + u_{2,1} + u_{1,1}u_{1,2} + u_{2,1}u_{2,2} + u_{3,1}u_{3,2})/2.\end{aligned} \tag{1.1-46}$$

At the same material point consider two non-collinear material line elements $d\overset{(1)}{\mathbf{X}}$ and $d\overset{(2)}{\mathbf{X}}$ which deform into $d\overset{(1)}{\mathbf{y}}$ and $d\overset{(2)}{\mathbf{y}}$. The area of the parallelogram spanned by $d\overset{(1)}{\mathbf{X}}$ and $d\overset{(2)}{\mathbf{X}}$, and that by $d\overset{(1)}{\mathbf{y}}$ and $d\overset{(2)}{\mathbf{y}}$, can be represented by the following vectors, respectively:

$$N_L dA = dA_L = \varepsilon_{LMN} d\overset{(1)}{X}_M \, d\overset{(2)}{X}_N \,, \tag{1.1-47}$$

$$n_i da = da_i = \varepsilon_{ijk} d\overset{(1)}{y}_j \, d\overset{(2)}{y}_k \,. \tag{1.1-48}$$

They are related by

$$da_i = JX_{L,i} dA_L \,. \tag{1.1-49}$$

Proof:

$$\begin{aligned} da_i &= \varepsilon_{ijk} d\overset{(1)}{y}_j \, d\overset{(2)}{y}_k = \varepsilon_{ijk} y_{j,M} d\overset{(1)}{X}_M \, y_{k,N} d\overset{(2)}{X}_N \\ &= \varepsilon_{ijk} y_{j,M} y_{k,N} d\overset{(1)}{X}_M \, d\overset{(2)}{X}_N = JX_{L,i} \varepsilon_{LMN} d\overset{(1)}{X}_M \, d\overset{(2)}{X}_N \\ &= JX_{L,i} dA_L \,, \end{aligned} \tag{1.1-50}$$

where Equation (1.1-18) has been used.

At the same material point consider three non-coplanar material line elements $d\overset{(1)}{\mathbf{X}}$, $d\overset{(2)}{\mathbf{X}}$, and $d\overset{(3)}{\mathbf{X}}$ which deform into $d\overset{(1)}{\mathbf{y}}$, $d\overset{(2)}{\mathbf{y}}$, and $d\overset{(3)}{\mathbf{y}}$. The volume of the parallelepiped spanned by $d\overset{(1)}{\mathbf{X}}$, $d\overset{(2)}{\mathbf{X}}$ and $d\overset{(3)}{\mathbf{X}}$, and that by $d\overset{(1)}{\mathbf{y}}$, $d\overset{(2)}{\mathbf{y}}$, and $d\overset{(3)}{\mathbf{y}}$, are related by

$$dv = JdV \,. \tag{1.1-51}$$

Proof:

$$\begin{aligned} dv &= d\overset{(1)}{\mathbf{y}} \cdot (d\overset{(2)}{\mathbf{y}} \times d\overset{(3)}{\mathbf{y}}) = \varepsilon_{ijk} d\overset{(1)}{y}_i \, d\overset{(2)}{y}_j \, d\overset{(3)}{y}_k \\ &= \varepsilon_{ijk} y_{i,L} d\overset{(1)}{X}_L \, y_{j,M} d\overset{(2)}{X}_M \, y_{k,N} d\overset{(3)}{X}_N \\ &= \varepsilon_{ijk} y_{i,L} y_{j,M} y_{k,N} d\overset{(1)}{X}_L \, d\overset{(2)}{X}_M \, d\overset{(3)}{X}_N \\ &= J\varepsilon_{LMN} d\overset{(1)}{X}_L \, d\overset{(2)}{X}_M \, d\overset{(3)}{X}_N \\ &= Jd\overset{(1)}{\mathbf{X}} \cdot (d\overset{(2)}{\mathbf{X}} \times d\overset{(3)}{\mathbf{X}}) = JdV \,, \end{aligned} \tag{1.1-52}$$

where Equation (1.1-17) has been used.

The velocity and acceleration of a material point are given by the following material time derivatives:

$$v_i = \frac{Dy_i}{Dt} = \dot{y}_i = \left.\frac{\partial y_i(\mathbf{X},t)}{\partial t}\right|_{\mathbf{X}\text{ fixed}},$$
$$\dot{v}_i = \frac{Dv_i}{Dt} = \left.\frac{\partial^2 y_i(\mathbf{X},t)}{\partial t^2}\right|_{\mathbf{X}\text{ fixed}}. \tag{1.1-53}$$

The deformation rate tensor d_{ij} and the spin tensor ω_{ij} are introduced by decomposing the velocity gradient into symmetric and anti-symmetric parts

$$\partial_i v_j = v_{j,i} = d_{ij} + \omega_{ij},$$
$$d_{ij} = \frac{1}{2}(v_{j,i} + v_{i,j}), \quad \omega_{ij} = \frac{1}{2}(v_{j,i} - v_{i,j}). \tag{1.1-54}$$

We also have

$$\frac{D}{Dt}(dy_i) = \frac{D}{Dt}\left(\frac{\partial y_i}{\partial X_K}dX_K\right) = \frac{\partial}{\partial X_K}\left(\frac{Dy_i}{Dt}\right)dX_K$$
$$= \frac{\partial}{\partial X_K}(v_i)dX_K = v_{i,j}y_{j,K}dX_K. \tag{1.1-55}$$

The strain rate and the deformation rate are related by

$$\dot{S}_{KL} = d_{ij}y_{i,K}y_{j,L}\ . \tag{1.1-56}$$

Proof:

$$\dot{S}_{KL} = \frac{1}{2}(\dot{y}_{i,K}y_{i,L} + y_{i,K}\dot{y}_{i,L}) = \frac{1}{2}(v_{i,K}y_{i,L} + y_{i,K}v_{i,L})$$
$$= \frac{1}{2}(v_{i,j}y_{j,K}y_{i,L} + y_{i,K}v_{i,j}y_{j,L})$$
$$= \frac{1}{2}(v_{j,i}y_{i,K}y_{j,L} + y_{i,K}v_{i,j}y_{j,L})$$
$$= \frac{1}{2}(v_{j,i} + v_{i,j})y_{i,K}y_{j,L} = d_{ij}y_{i,K}y_{j,L}. \tag{1.1-57}$$

The material derivative of the Jacobian is

$$\dot{J} = Jv_{k,k}\ . \tag{1.1-58}$$

Proof: From Equation (1.1-12)

$$\dot{J} = \frac{1}{6}\varepsilon_{klm}\varepsilon_{KLM}(v_{k,K}y_{l,L}y_{m,M} + y_{k,K}v_{l,L}y_{m,M} + y_{k,K}y_{l,L}v_{m,M})$$
$$= \frac{1}{2}\varepsilon_{klm}\varepsilon_{KLM}v_{k,K}y_{l,L}y_{m,M} = \frac{1}{2}v_{k,K}\varepsilon_{klm}\varepsilon_{KLM}y_{l,L}y_{m,M} \tag{1.1-59}$$
$$= \frac{1}{2}v_{k,K}2JX_{K,k} = Jv_{k,k},$$

where Equation (1.1-22) has been used.

The following expression will be useful later in the book:

$$\frac{D}{Dt}(X_{L,j}) = -v_{i,K}X_{K,j}X_{L,i}. \tag{1.1-60}$$

Proof: Since

$$y_{i,K}X_{K,j} = \delta_{ij}, \tag{1.1-61}$$

we have, upon differentiating both sides,

$$\dot{y}_{i,K}X_{K,j} + y_{i,K}\frac{D}{Dt}(X_{K,j}) = 0. \tag{1.1-62}$$

Then

$$y_{i,K}\frac{D}{Dt}(X_{K,j}) = -v_{i,K}X_{K,j}. \tag{1.1-63}$$

Multiplication of both sides of (1.1-63) by $X_{L,i}$ gives

$$\frac{D}{Dt}(X_{L,j}) = -v_{i,K}X_{K,j}X_{L,i}. \tag{1.1-64}$$

Problems

1.1-1. Show (1.1-15) from (1.1-14).
1.1-2. Show (1.1-16).
1.1-3. Show (1.1-45).
1.1-4. Show that $\dfrac{\partial J}{\partial S_{KL}} = 2\dfrac{\partial J}{\partial C_{KL}} = JC_{KL}^{-1}$.

2. GLOBAL BALANCE LAWS

This section summarizes the fundamental physical laws that govern the motion of an elastic dielectric. They are experimental laws and are postulated as the foundation for a continuum theory.

2.1 Polarization

When a dielectric is placed in an electric field, the electric charges in its molecules redistribute themselves microscopically, resulting in a macroscopic polarization. The microscopic charge redistribution occurs in different ways (see Figure 1.2-1).

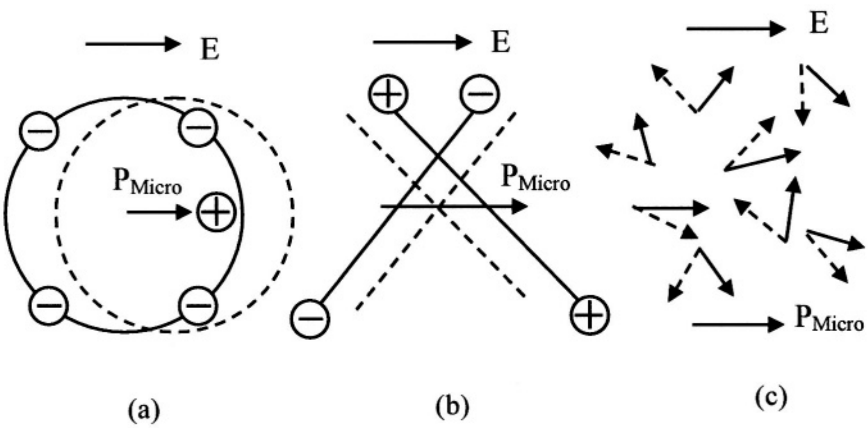

Figure 1.2-1. Microscopic polarization: (a) electronic, (b) ionic, (c) orientational.

At the macroscopic level the distinctions among different polarization mechanisms do not matter. A macroscopic polarization vector per unit present volume,

$$\mathbf{P} = \lim_{\Delta v \to 0} \frac{\sum_{\Delta v} \mathbf{P}_{\text{Micro}}}{\Delta v}, \tag{1.2-1}$$

is introduced which describes the macroscopic polarizing state of the material.

2.2 Piezoelectric Effects

Experiments show that in certain materials polarization can also be induced by mechanical loads. Figure 1.2-2(a) shows such a phenomenon called the direct piezoelectric effect. The induced polarization can be at an angle, e.g., perpendicular to the applied load, depending on the anisotropy of the material. When the load is reversed, so is the induced polarization. When a voltage is applied to a material possessing the direct piezoelectric effect, the material deforms. This is called the converse piezoelectric effect (see Figure 1.2-2(b)).

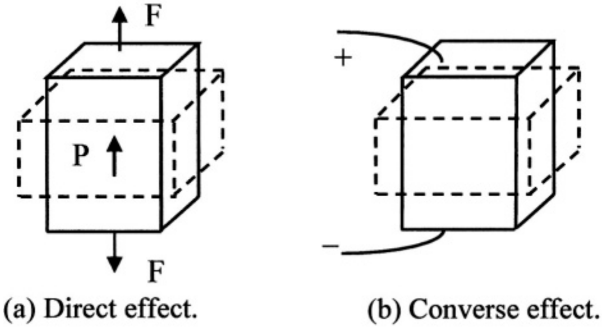

Figure 1.2-2. Macroscopic piezoelectric effects.

Whether a material is piezoelectric depends on its microscopic charge distribution. For example, the charge distribution in Figure 1.2-3(a), when deformed into Figure 1.2-3(b), results in a polarization.

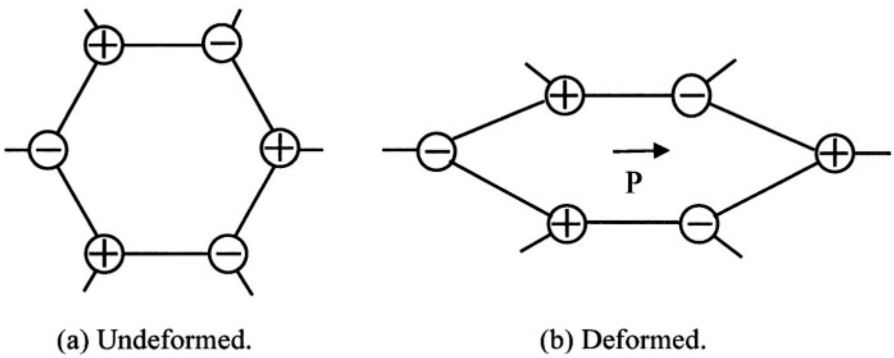

Figure 1.2-3. Origin of the direct piezoelectric effect.

2.3 Electric Body Force, Couple and Power

When a mechanically deformable and electrically polarizable material is subjected to an electric field, a differential element of the material experiences body force and couple due to the electric field. When such a material deforms and polarizes, the electric field also does work to the material. Fundamental to the development of the equations of electroelasticity is the derivation of the electric body force, couple, and power due to the electric field. This can be done by averaging fields associated with charged and interacting particles [1] or particles with internal degrees of freedom [2]. Tiersten [3] introduced a physical model of two mechanically and electrically interacting and interpenetrating continua to describe electric polarization macroscopically. One continuum is the lattice continuum which carries mass and positive charges. The other is the electronic continuum which is negatively charged and is without mass. Electric polarization is modeled by a small, relative displacement of the electronic continuum with respect to the lattice continuum. By systematic applications of the basic laws of physics to each continuum and combining the resulting equations, Tiersten [3] obtained the expressions for the electric body force $\mathbf{F}^E$, couple $\mathbf{C}^E$ and power w^E as

$$\begin{aligned} F_j^E &= \rho_e E_j + P_i E_{j,i}, \\ C_i^E &= \varepsilon_{ijk} P_j E_k, \\ w^E &= \rho E_i \dot{\pi}_i, \end{aligned} \tag{1.2-2}$$

where $\mathbf{E}$ is the electric field vector, ρ is the present mass density, ρ_e (a scalar) is the present free charge density, and $\pi_i = P_i / \rho$ is the polarization per unit mass. The presence of the mass density ρ in $(1.2\text{-}2)_3$ is not obvious. It is due to a relation between the density of the bound charge and mass density [3]. The problem at the end of this section is helpful for understanding (1.2-2).

2.4 Balance Laws

Let l be a closed curve. The continuum theory of electroelasticity postulates the following global balance laws in integral form:

$$\begin{aligned} &\int_s \mathbf{D}\cdot d\mathbf{a} = \int_v \rho_e dv, \\ &\int_l \mathbf{E}\cdot d\mathbf{l} = 0, \\ &\frac{D}{Dt}\int_v \rho dv = 0, \\ &\frac{D}{Dt}\int_v \rho \mathbf{v} dv = \int_v (\rho\mathbf{f} + \mathbf{F}^E) dv + \int_s \mathbf{t} da, \\ &\frac{D}{Dt}\int_v \mathbf{y}\times\rho\mathbf{v} dv = \int_v [\mathbf{y}\times(\rho\mathbf{f} + \mathbf{F}^E) + \mathbf{C}^E] dv + \int_s \mathbf{y}\times\mathbf{t} da, \\ &\frac{D}{Dt}\int_v \rho(\frac{1}{2}\mathbf{v}\cdot\mathbf{v} + e) dv = \int_v [(\rho\mathbf{f} + \mathbf{F}^E)\cdot\mathbf{v} + w^E] dv + \int_s \mathbf{t}\cdot\mathbf{v} da, \end{aligned} \tag{1.2-3}$$

where $\mathbf{D}$ is the electric displacement vector, $\mathbf{f}$ is the mechanical body force per unit mass, $\mathbf{t}$ is the surface traction on s, and e is the internal energy per unit mass. The equations in (1.2-3) are, respectively, Gauss's law (the charge equation), Faraday's law in quasistatic form, the conservation of mass, the conservation of linear momentum, the conservation of angular momentum, and the conservation of energy. In the above balance laws, the electric field appears to be static. This is the so-called quasistatic approximation [4]. The approximation is valid when we are considering phenomena at elastic wavelengths which are much shorter than electromagnetic wavelengths at the same frequency [4]. Quasistatic approximation can be considered as the lowest order approximation of the electrodynamic theory through a perturbation procedure [5], which will be shown in Chapter 7, Section 6 when discussing the dynamic theory. Within the quasistatic approximation, the electric field depends on time through coupling to the dynamic mechanical fields. The following relation exists among **D**, **E,** and **P:**

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}, \tag{1.2-4}$$

where ε_0 is the permittivity of free space.

Problem

1.2-1. Derive expressions for the force, couple, and power on a single, stretchable dipole in an electric field.

3. LOCAL BALANCE LAWS

From Equation $(1.2\text{-}3)_1$, and using the divergence theorem, we can write

$$\int_v n_i D_i da = \int_v D_{i,i} dv = \int_v \rho_e dv, \tag{1.3-1}$$

$$\int_v (D_{i,i} - \rho_e) dv = 0. \tag{1.3-2}$$

Equation (1.3-2) holds for any v. Assume a continuous integrand, then

$$D_{i,i} - \rho_e = 0. \tag{1.3-3}$$

From Equation $(1.2\text{-}3)_2$, with Stoke's theorem, the line integral along l can be converted to a surface integral over an area s whose boundary is l:

$$\int_l \mathbf{E} \cdot d\mathbf{l} = \int_s (\nabla \times \mathbf{E}) \cdot d\mathbf{a} = 0, \tag{1.3-4}$$

which implies that

$$\nabla \times \mathbf{E} = \varepsilon_{ijk} E_{k,j} \mathbf{i}_i = 0. \tag{1.3-5}$$

From Equation $(1.2\text{-}3)_3$, change the integral back to the reference configuration

$$\frac{D}{Dt}\int_v \rho dv = \frac{D}{Dt}\int_V \rho J dV = \int_V \frac{D}{Dt}(\rho J) dV$$
$$= \int_V (\dot{\rho} J + \rho \dot{J}) dV = \int_V (\dot{\rho} J + \rho J v_{i,i}) dV \qquad (1.3\text{-}6)$$
$$= \int_v (\dot{\rho} + \rho v_{i,i}) dv = 0,$$

where Equation (1.1-58) has been used. Hence

$$\dot{\rho} + \rho v_{i,i} = 0 . \qquad (1.3\text{-}7)$$

With Equations (1.1-58) and (1.3-7) it can be shown that

$$\frac{D}{Dt}\int_v \rho [\quad] dv = \int_v \rho \frac{D[\quad]}{Dt} dv . \qquad (1.3\text{-}8)$$

Proof: With the change of integration variables

$$\frac{D}{Dt}\int_v \rho [\quad] dv = \frac{D}{Dt}\int_V \rho [\quad] J dV = \int_V \frac{D}{Dt}\big(\rho [\quad] J\big) dV$$
$$= \int_V \left(\dot{\rho} [\quad] J + \rho \frac{D[\quad]}{Dt} J + \rho [\quad] \dot{J} \right) dV$$
$$= \int_V \left(-\rho v_{i,i} [\quad] J + \rho \frac{D[\quad]}{Dt} J + \rho [\quad] J v_{i,i} \right) dV \qquad (1.3\text{-}9)$$
$$= \int_V \rho \frac{D[\quad]}{Dt} J dV = \int_v \rho \frac{D[\quad]}{Dt} dv.$$

The Cauchy stress tensor σ_{kj} can be introduced by

$$t_i = \sigma_{ji} n_j , \qquad (1.3\text{-}10)$$

through the usual tetrahedron argument. Then from $(1.2\text{-}3)_4$, with (1.3-8) and the divergence theorem, the balance of linear momentum becomes

$$\int_v \rho \frac{Dv_i}{Dt} dv = \int_v (\rho f_i + F_i^E) dv + \int_s t_i da$$
$$= \int_v (\rho f_i + F_i^E) dv + \int_s \sigma_{ji} n_j da \qquad (1.3\text{-}11)$$
$$= \int_v (\rho f_i + F_i^E) dv + \int_v \sigma_{ji,j} dv.$$

Hence

$$\sigma_{ji,j} + \rho f_i + F_i^E = \rho \dot{v}_i . \qquad (1.3\text{-}12)$$

From $(1.2\text{-}3)_5$, the balance of angular momentum can be written as

$$\frac{D}{Dt}\int_v \varepsilon_{ijk} y_j \rho v_k dv \\ = \int_v [\varepsilon_{ijk} y_j (\rho f_k + F_k^E) + C_i^E] dv + \int_s \varepsilon_{ijk} y_j t_k da. \tag{1.3-13}$$

The term on the left-hand side can be written as

$$\frac{D}{Dt}\int_v \varepsilon_{ijk} y_j \rho v_k dv = \int_v \rho \varepsilon_{ijk} \frac{D}{Dt}(y_j v_k) dv \\ = \int_v \rho \varepsilon_{ijk} (\dot{y}_j v_k + y_j \dot{v}_k) dv \\ = \int_v \rho \varepsilon_{ijk} (v_j v_k + y_j \dot{v}_k) dv = \int_v \rho \varepsilon_{ijk} y_j \dot{v}_k dv. \tag{1.3-14}$$

The last term on the right-hand side can be written as

$$\int_s \varepsilon_{ijk} y_j t_k da = \int_s \varepsilon_{ijk} y_j \sigma_{lk} n_l da \\ = \int_v (\varepsilon_{ijk} y_j \sigma_{lk})_{,l} dv = \int_v \varepsilon_{ijk} (\delta_{jl} \sigma_{lk} + y_j \sigma_{lk,l}) dv \\ = \int_v \varepsilon_{ijk} (\sigma_{jk} + y_j \sigma_{lk,l}) dv. \tag{1.3-15}$$

Substituting Equations (1.3-14) and (1.3-15) back into (1.3-13), we obtain

$$\int_v \rho \varepsilon_{ijk} y_j \dot{v}_k dv = \int_v [\varepsilon_{ijk} y_j (\rho f_k + F_k^E) + C_i^E] dv \\ + \int_v \varepsilon_{ijk} (\sigma_{jk} + y_j \sigma_{lk,l}) dv, \tag{1.3-16}$$

or

$$\int_v \varepsilon_{ijk} y_j (\rho \dot{v}_k - \rho f_k - F_k^E - \sigma_{lk,l}) dv \\ = \int_v (\varepsilon_{ijk} \sigma_{jk} + C_i^E) dv. \tag{1.3-17}$$

Hence

$$\int_v (\varepsilon_{ijk} \sigma_{jk} + C_i^E) dv = 0, \tag{1.3-18}$$

which implies that

$$\varepsilon_{ijk} \sigma_{jk} + C_i^E = 0, \tag{1.3-19}$$

or

$$\varepsilon_{ijk} (\sigma_{jk} + P_j E_k) = 0. \tag{1.3-20}$$

It will be proven convenient to introduce an electrostatic stress tensor σ_{ij}^E whose divergence yields the electric body force

$$\sigma_{ij,i}^E = F_j^E. \tag{1.3-21}$$

For the existence of such σ_{ij}^E consider

$$\begin{aligned}\sigma_{ij}^E &= D_i E_j - \frac{1}{2}\varepsilon_0 E_k E_k \delta_{ij} \\ &= P_i E_j + \varepsilon_0 (E_i E_j - \frac{1}{2} E_k E_k \delta_{ij}).\end{aligned} \qquad (1.3\text{-}22)$$

We have

$$\sigma_{ij,i}^E = D_{i,i} E_j + P_i E_{j,i} = \rho_e E_j + P_i E_{j,i} = F_j^E, \qquad (1.3\text{-}23)$$

where Equation (1.2-4) has been used. We note that σ_{ij}^E is not unique in the sense that there are other tensors that also satisfy (1.3-21). For example, adding a second rank tensor with zero divergence to the σ_{ij}^E in (1.3-22) will not affect (1.3-21). In this book we will use (1.3-22).

With σ_{ij}^E, the balance of linear momentum, Equation (1.3-12), can be written as

$$(\sigma_{ij} + \sigma_{ij}^E)_{,i} + \rho f_j = \rho \dot{v}_j. \qquad (1.3\text{-}24)$$

The balance of angular momentum, Equation (1.3-20), can be written as

$$\varepsilon_{ijk}(\sigma_{jk} + P_j E_k) = \varepsilon_{ijk}(\sigma_{jk} + \sigma_{jk}^E) = 0, \qquad (1.3\text{-}25)$$

which shows that the sum of the Cauchy stress tensor σ_{ij} and the electrostatic stress tensor σ_{ij}^E is symmetric, which we call the total stress tensor and denote it by τ_{ij}

$$\begin{aligned}\tau_{ij} &= \sigma_{ij} + \sigma_{ij}^E \\ &= \sigma_{ij} + P_i E_j + \varepsilon_0 (E_i E_j - \frac{1}{2} E_k E_k \delta_{ij}) = \tau_{ji}.\end{aligned} \qquad (1.3\text{-}26)$$

τ_{ij} can also be decomposed into the sum of a symmetric tensor σ_{ij}^S and the symmetric Maxwell stress tensor σ_{ij}^M as follows:

$$\begin{aligned}\tau_{ij} &= \sigma_{ij}^S + \sigma_{ij}^M, \\ \sigma_{ij}^S &= \sigma_{ij} + P_i E_j = \sigma_{ji}^S, \\ \sigma_{ij}^M &= \varepsilon_0 (E_i E_j - \frac{1}{2} E_k E_k \delta_{ij}) = \sigma_{ji}^M.\end{aligned} \qquad (1.3\text{-}27)$$

From Equation $(1.2\text{-}3)_6$, the conservation of energy is

$$\frac{D}{Dt}\int_v \rho\left(\frac{1}{2}v_iv_i + e\right)dv$$
$$= \int_v (\rho f_k + F_k^E)v_k + w^E]dv + \int_s t_kv_kda. \quad (1.3\text{-}28)$$

The left-hand side can be written as

$$\frac{D}{Dt}\int_v \rho\left(\frac{1}{2}v_iv_i + e\right)dv = \int_v \rho\frac{D}{Dt}\left(\frac{1}{2}v_iv_i + e\right)dv$$
$$= \int_v \rho(v_i\dot{v}_i + \dot{e})dv. \quad (1.3\text{-}29)$$

The last term on the right-hand side can be written as

$$\int_s t_kv_kda = \int_s \sigma_{lk}n_lv_kda$$
$$= \int_v (\sigma_{lk}v_k)_{,l}dv = \int_v (\sigma_{lk,l}v_k + \sigma_{lk}v_{k,l})dv. \quad (1.3\text{-}30)$$

Substituting (1.3-29) and (1.3-30) back into (1.3-28) gives

$$\int_v \rho(v_k\dot{v}_k + \dot{e})dv = \int_v (\rho f_k + F_k^E)v_k + w^E]dv$$
$$+ \int_v (\sigma_{lk,l}v_k + \sigma_{lk}v_{k,l})dv, \quad (1.3\text{-}31)$$

or

$$\int_v v_k(\rho\dot{v}_k - \rho f_k - F_k^E - \sigma_{lk,l})dv = \int_v (\sigma_{lk}v_{k,l} + w^E - \rho\dot{e}]dv. \quad (1.3\text{-}32)$$

With the equation of motion (1.3-12), the left-hand side of (1.3-32) vanishes, and what is left is

$$\int_v (\sigma_{lk}v_{k,l} + w^E - \rho\dot{e})dv = 0, \quad (1.3\text{-}33)$$

which implies that

$$\rho\dot{e} = \sigma_{ij}v_{j,i} + \rho E_i\dot{\pi}_i. \quad (1.3\text{-}34)$$

A free energy ψ can be introduced through the following Legendre transform:

$$\psi = e - E_i\pi_i. \quad (1.3\text{-}35)$$

Then

$$\dot{\psi} = \dot{e} - \dot{E}_i\pi_i - E_i\dot{\pi}_i. \quad (1.3\text{-}36)$$

Substitute Equation (1.3-36) into (1.3-34)

$$\rho\dot{\psi} = \sigma_{ij}v_{j,i} - P_i\dot{E}_i. \quad (1.3\text{-}37)$$

In summary, the local balance laws are