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An Invariance Principle and Some Convergence Rate Results for Branching Processes

C. C. Heyde and B. M. Brown

1. Introduction

Let $Z_0=1$, $Z_1, Z_2, \dots$ denote a super-critical Galton-Watson branching process with $1<EZ_1=m<\infty$ and $0<\operatorname{var} Z_1=\sigma^2<\infty$. It is well-known that there exists a non-degenerate random variable W such that $\lim_{n\to\infty} W_n=W$ almost surely, where $W_n=m^{-n}Z_n$ (e.g. Harris [3], p. 13). Also, a central limit analogue holds, namely that, conditional on $Z_n>0$, $(m^2-m)^{\frac{1}{2}}\sigma^{-1}Z_n^{-\frac{1}{2}}m^n(W-W_n)$ is asymptotical normal $N(0,1)$ as $n\to\infty$ (Heyde [4]). This result is called a central limit analogue by virtue of the analogy with the convergence of $\sigma^{-1}n^{\frac{1}{2}}(\mu-n^{-1}S_n)$ to $N(0,1)$ for a sum S_n of independent and identically distributed random variables with mean μ and variance σ^2.

In this paper, we explore the setting of the central limit analogue. An invariance principle in R^∞ is obtained from which it follows as a simple corollary. Also, results are obtained on the rate of convergence to normality. It is shown that, under mild restrictions, there is a geometrically decreasing bound on the L_∞ metric.

2. Results

Let the probability generating function (p.g.f.) of the offspring distribution Z_1 be $F(s)=\sum_{j=0}^{\infty} s^j P(Z_1=j)$. The p.g.f. $F_n(s)$ of Z_n is the n-th functional iterate of $F(s)$. For the super-critical process in question, the probability of eventual extinction is q, the unique real number in $[0,1)$ satisfying $q=F(q)$. Furthermore, in addition to the fact that $Z_n=0$ for n sufficiently large on the extinction set $[W=0]$, it is known that

$$Z_n\to\infty \quad \text{as} \quad n\to\infty \quad \text{on} \quad [W>0], \tag{1}$$

so that the process either dies out or becomes arbitrarily large.

The symbols P and E will, for convenience, be used to denote various different probability measures and their associated expectation operators. In addition, we use the symbol Z_n^* to denote the random variable (r.v.) Z_n under the probability measure conditional on $Z_n>0$. These conveniences should cause no ambiguities.

Let R^∞ denote the space of real sequences $x=(x_1, x_2, \dots)$ with the metric

$$\rho(x,y)=\sup_{n\geq 1}\left\{\frac{\left|\sum_{j=1}^{n}(x_j-y_j)\right|}{1+\left|\sum_{j=1}^{n}(x_j-y_j)\right|}\right\}.$$

Let $\mathscr{B}$ be the σ-field generated by the open sets of the topology given by ρ and let V be the probability measure on $(R^\infty, \mathscr{B})$ under which the co-ordinates $x_1, x_2, \ldots$ of a random element x of R^∞ become independent r.v.'s with x_j distributed as $N(0, \sigma^2 m^{-1-j})$, $j=1, 2, \ldots$. Write U for the probability measure on $(R^\infty, \mathscr{B})$ which assigns probability one to the zero sequence $(0, 0, \ldots)$ and define, for $j=0, 1, 2, \ldots$ and $n=1, 2, 3, \ldots$,

$$X_n=(X_{n1}, X_{n2}, \ldots)$$
$$X_{nj}=m^n(W_{n+j}-W_n)(Z_n+1)^{-\frac{1}{2}},$$

and for $j=1, 2, 3, \ldots$

$$Y_{nj}=X_{nj}-X_{n,j-1}.$$

Theorem 1. *Let P_n denote the probability measure on $(R^\infty, \mathscr{B})$ defined by the distributions of $Y_n=(Y_{n1}, Y_{n2}, \ldots)$. Then,*

$$P_n \Rightarrow VP(W>0)+UP(W=0)$$

as $n\to\infty$, where $\Rightarrow$ denotes weak convergence of probability measures.

We remark that, as simple corollaries to Theorem 1, $(m^2-m)^{\frac{1}{2}}\sigma^{-1} Z_n^{-\frac{1}{2}} \cdot m^n(W-W_n)$ conditional on $Z_n>0$ and $(m^2-m)^{\frac{1}{2}}\sigma^{-1} m^{-j/2}(m^j-1)^{-\frac{1}{2}} Z_n^{-\frac{1}{2}}(Z_{j+n}-m^j Z_n)$ conditional on $Z_n>0$ (fixed j) are both asymptotically $N(0,1)$. These results have previously been obtained by direct methods by Heyde [4] and by Bühler [2] respectively. They can here be deduced from Theorem 1 using the measurable mapping theorem (Billingsley [1], p. 30) with the mappings

$$h_1(x)=\limsup_{n\to\infty}\sum_{j=1}^{n} x_j$$

if this is finite, 0 otherwise and $h_2(x)=x_1+x_2+\cdots+x_j$ respectively.

The next theorem gives rates of convergence to normality in the limit results cited above.

Theorem 2. *Let $EZ_1^3<\infty$. Then,*

$$\sup_x \left|P\left((m^2-m)^{\frac{1}{2}}\sigma^{-1} Z_n^{-\frac{1}{2}} m^n(W-W_n)\le x \mid Z_n>0\right)-\Phi(x)\right| \le K\sigma^{-3}(m^2-m)^{\frac{3}{2}} E(1/Z_n^*)^{\frac{1}{2}} E|W-1|^3 \tag{2}$$

and

$$\sup_x \left|P\left(\sigma_r^{-1} Z_n^{-\frac{1}{2}}(Z_{r+n}-m^r Z_n)\le x \mid Z_n>0\right)-\Phi(x)\right| \le K\sigma_r^{-3} E|Z_r-m^r|^3 E(1/Z_n^*)^{\frac{1}{2}}, \tag{3}$$

where $\Phi(x)$ is the distribution function of $N(0,1)$, $\sigma_r^2=\operatorname{var} Z_r=\sigma^2 m^r(m^r-1)(m^2-m)^{-1}$ and K is the universal constant in the Berry-Esseen bound ($K<0.82$; see Zolotarev *[6]). Furthermore,*

$$E(1/Z_n^*)^{\frac{1}{2}} \le \left(E(1/Z_n^*)\right)^{\frac{1}{2}} \le \left\{(1-F(0))^{-1}\gamma^n\right\}^{\frac{1}{2}} \tag{4}$$

where $\gamma<1$ is given by

$$\gamma=\int_0^1 \left(F(s(1-q)+q)-q\right)s^{-1}(1-q)^{-1}\,ds,$$

with $\gamma=EZ_1^{-1}$ if $F(0)=0$.

3. Proof of Theorem 1

Firstly we show that the finite-dimensional distributions of $\{P_n\}$ converge weakly to those of $VP(W>0)+U\,P(W=0)$.

Let $\mathscr{F}_k$ be the σ-field generated by $Z_0, Z_1, \ldots, Z_k$ and for fixed r write

$$\begin{aligned}\phi_n(\theta_1, \theta_2, \ldots, \theta_r) &= E\left\{\exp\left(i\sum_{j=1}^{r}\theta_j Y_{nj}\right)\right\}\\ &= E\left(\prod_{j=1}^{r} A_j\right),\end{aligned} \tag{5}$$

where

$$A_j = \exp\left(i\,\theta_j\, m^{-j}(Z_{n+j} - m\,Z_{n+j-1})(Z_n+1)^{-\frac{1}{2}}\right). \tag{6}$$

Consider the r.v.

$$\begin{aligned}R_r &= \prod_{j=1}^{r} A_j - \exp\left(-\tfrac{1}{2}\sigma^2 I(Z_n>0)\sum_{j=1}^{r}\theta_j^2 m^{-1-j}\right)\\ &= \sum_{j=1}^{r} B_j,\end{aligned} \tag{7}$$

where I is the indicator function and for $j=1,2,\ldots,r$, using the convention that a product of the form $\prod_{k=s+1}^{s}$ is replaced by unity,

$$\begin{aligned}B_j = &\left(\prod_{k=1}^{j-1} A_k\right)\exp\left(-\tfrac{1}{2}\sigma^2 I(Z_n>0)\sum_{k=j+1}^{r}\theta_k^2 m^{-1-k}\right)\\ &\cdot\left(A_j - \exp(-\tfrac{1}{2}\sigma^2 I(Z_n>0)\,\theta_j^2\, m^{-1-j})\right),\end{aligned}$$

with

$$\begin{aligned}E(B_j|\mathscr{F}_{n+j-1}) = &\left(\prod_{k=1}^{j-1} A_k\right)\exp\left(-\tfrac{1}{2}\sigma^2 I(Z_n>0)\sum_{k=j+1}^{r}\theta_k^2 m^{-1-k}\right)\\ &\cdot\left(E(A_j|\mathscr{F}_{n+j-1}) - \exp(-\tfrac{1}{2}\sigma^2 I(Z_n>0)\,\theta_j^2\, m^{-1-j})\right).\end{aligned} \tag{8}$$

But, Z_{n+j} is the sum of Z_{n+j-1} independent and identically distributed r.v.'s, each with the distribution of Z_1, so that from (6),

$$E(A_j|\mathscr{F}_{n+j-1}) = \{\phi(\theta_j\, m^{-j}(Z_n+1)^{-\frac{1}{2}})\}^{Z_{n+j-1}}, \tag{9}$$

where $\phi(t) = E\{\exp(i\,t(Z_1-m))\}$. Therefore, $E(A_j|\mathscr{F}_{n+j-1}) \to 1$ almost surely (a.s.) as $n\to\infty$ on $[W=0]$, so that from (8),

$$\lim_{n\to\infty} E(B_j|\mathscr{F}_{n+j-1}) = 0 \qquad \text{a.s. on } [W=0], \tag{10}$$

since

$$I(Z_n>0) \to I(W>0) \qquad \text{a.s. as } n\to\infty. \tag{11}$$

Also, we know from the central limit theorem that for each fixed t,

$$\lim_{n\to\infty}\{\phi(t(n+1)^{-\frac{1}{2}})\}^n = \exp(-\tfrac{1}{2}\sigma^2 t^2).$$

Therefore, from (9), using (1) and since

$$\lim_{n\to\infty} Z_n^{-1} Z_{n+j-1} = m^{j-1} \qquad \text{a.s. on } [W>0],$$

we have

$$\lim_{n\to\infty} E(A_j|\mathscr{F}_{n+j-1}) = \exp(-\tfrac{1}{2}\sigma^2\theta_j^2 m^{-1-j}) \qquad \text{a.s. on } [W>0],$$

so that

$$\lim_{n\to\infty} E(B_j|\mathscr{F}_{n+j-1})=0 \qquad \text{a.s. on } [W>0], \tag{12}$$

using (8) and (11). Clearly $|B_j|\leqq 2$ a.s. for each $j=1,2,\ldots,r$, so that the combination of (10) and (12), with dominated convergence, gives

$$E R_r = E\sum_{j=1}^{r} B_j = E\sum_{j=1}^{r} E(B_j|\mathscr{F}_{n+j-1})\to 0$$

as $n\to\infty$. Referring now to (5) and (7), we have

$$\lim_{n\to\infty}\left\{\phi_n(\theta_1,\ldots,\theta_r)-E\exp\left(-\tfrac{1}{2}\sigma^2 I(Z_n>0)\sum_{j=1}^{r}\theta_j^2 m^{-1-j}\right)\right\}=0. \tag{13}$$

But,

$$\begin{aligned}\lim_{n\to\infty} E\exp&\left(-\tfrac{1}{2}\sigma^2 I(Z_n>0)\sum_{j=1}^{r}\theta_j^2 m^{-1-j}\right)\\ &=P(W=0)+P(W>0)\exp\left(-\tfrac{1}{2}\sigma^2\sum_{j=1}^{r}\theta_j^2 m^{-1-j}\right)\end{aligned} \tag{14}$$

from (11) and dominated convergence. (13) and (14) complete the proof of the convergence of finite-dimensional distributions.

To complete the proof of the theorem, we must show that the sequence $\{P_n\}$ is *tight*. That is, for each $\varepsilon>0$ there exists a compact set A such that $P_n(A)>1-\varepsilon$ for all $n=1,2,\ldots$.

Consider the sets

$$C=C_r=\left\{x:\ \sup_{s\geqq k}\left|\sum_{j=k}^{s} x_j\right|\leqq m^{-k/3} \text{ for all } k\geqq r\right\}$$

and

$$D=D_M=\left\{x:\ \sup_{s\geqq 1}\left|\sum_{j=1}^{s} x_j\right|\leqq M\right\}.$$

The set $C\cap D$ has compact closure, for any sequence belonging to it can readily be shown to possess a limit point ([1], p. 217). Tightness will be established if we can show that, given $\varepsilon>0$, $P_n(C\cap D)>1-\varepsilon$ for all n, when M and r are suitably large.

We first note that $C=\bigcap_{k\geqq r} A_k$, where

$$A_k=\left\{x:\ \sup_{s\geqq k}\left|\sum_{j=k}^{s} x_j\right|\leqq m^{-k/3}\right\}.$$

Kolmogorov's inequality for martingales, when applied to the martingale $\left\{\sum_{j=k}^{r} Y_{nj},\ r=k,k+1,\ldots\right\}$ yields the inequality

$$P\left(\max_{k\leqq r\leqq s}\left|\sum_{j=k}^{r} Y_{nj}\right|>C\right)\leqq C^{-2}\sum_{j=k}^{s} E\,Y_{nj}^2. \tag{15}$$

But,

$$\begin{aligned}E(Y_{nj}^2|\mathscr{F}_{n+j-1})&=E\left(m^{-2j}(Z_n+1)^{-1}(Z_{n+j}-m Z_{n+j-1})^2|\mathscr{F}_{n+j-1}\right)\\ &=m^{-2j}(Z_n+1)^{-1}\sigma^2 Z_{n+j-1},\end{aligned}$$

and

$$\begin{aligned} E Y_{nj}^2 &= E(E(Y_{nj}^2 \mid \mathscr{F}_{n+j-1})) \\ &= m^{-2j} \sigma^2 \sum_{k \geq 0} E((Z_n+1)^{-1} Z_{n+j-1} \mid Z_n = k) P(Z_n = k) \\ &= m^{-2j} \sigma^2 \sum_{k \geq 0} k(k+1)^{-1} E Z_{j-1} P(Z_n = k) \\ &\leq \sigma^2 m^{-1-j}. \end{aligned}$$

Therefore, from (15),

$$P\left(\max_{k \leq r \leq s} \left|\sum_{j=k}^{r} Y_{nj}\right| > C\right) \leq \sigma^2 m^{-k} C^{-2} (m-1)^{-1}. \tag{16}$$

By applying the inequality (16) to A_k' and D' (the prime denoting complementation), we obtain

$$P_n(A_k') \leq \sigma^2 (m-1)^{-1} m^{-k/3}$$

and

$$P_n(D') \leq \sigma^2 m^{-1} (m-1)^{-1} M^{-2},$$

for all $n = 1, 2, \ldots$. It now follows that

$$\begin{aligned} P_n(C) &= 1 - P_n\Big(\bigcup_{k \geq r} A_k'\Big) \\ &\geq 1 - \sum_{k \geq r} P_n(A_k') \\ &\geq 1 - \sigma^2 (m-1)^{-1} (1 - m^{-\frac{1}{3}})^{-1} m^{-r/3} \end{aligned} \tag{17}$$

and

$$P_n(D) \geq 1 - \sigma^2 (m-1)^{-1} m^{-1} M^{-2}, \tag{18}$$

for all $n = 1, 2, \ldots$. Thus, from (17) and (18), $P_n(C \cap D) > 1 - \varepsilon$ for all n if r and M are chosen suitably large, so that tightness is established. The proof of the theorem is thus complete.

It is worth remarking that the result of Theorem 1 can be generalized to cover the case where Z_1 belongs to the domain of attraction of a stable law of index α, $1 < \alpha \leq 2$. This provides a complete extension of the result of Theorem 3 of Heyde [4]. Theorem 1 has been given in the less general form in the interests of a unified exposition in this paper. In order to obtain the general form we would use

$$X_{nj} = m^n b_{Z_n+1}^{-1} (W_{n+j} - W_n),$$

in the notation of [4].

4. Proof of Theorem 2

We first need the following lemma.

Lemma. *Let Y_i, $i = 1, 2, 3, \ldots$ be independent and identically distributed r.v.'s with $E Y_1 = 0$, $\operatorname{var} Y_1 = \alpha^2$ and $E|Y_1|^3 < \infty$. Let N be a positive integer-valued r.v. which is independent of the $\{Y_i\}$. Then,*

$$\sup_x \left|P(\alpha^{-1} N^{-\frac{1}{2}} (Y_1 + \cdots + Y_N) \leq x) - \Phi(x)\right| \leq C E(N^{-\frac{1}{2}}),$$

where $C = K \alpha^{-3} E|Y_1|^3$, K being the universal constant of the Berry-Esseen bound.

Proof. Using the Berry-Esseen bound,

$$-CE(N^{-\frac{1}{2}}) = -C\sum_{j=1}^{\infty} j^{-\frac{1}{2}} P(N=j)$$

$$\leq \sum_{j=1}^{\infty} \{P(\alpha^{-1} j^{-\frac{1}{2}}(Y_1+\cdots+Y_j)\leq x) - \Phi(x)\}\, P(N=j) \tag{19}$$

$$\leq CE(N^{-\frac{1}{2}}).$$

But, since N is independent of the $\{Y_i\}$,

$$\sum_{j=1}^{\infty} \{P(\alpha^{-1} j^{-\frac{1}{2}}(Y_1+\cdots+Y_j)\leq x) - \Phi(x)\}\, P(N=j)$$

$$= \sum_{j=1}^{\infty} \{P(\alpha^{-1} j^{-\frac{1}{2}}(Y_1+\cdots+Y_j)\leq x \mid N=j) - \Phi(x)\}\, P(N=j) \tag{20}$$

$$= P(\alpha^{-1} N^{-\frac{1}{2}}(Y_1+\cdots+Y_N)\leq x) - \Phi(x),$$

and the required result follows from (19) and (20).

We now proceed with the proof of the theorem. Firstly we establish (2) and (3).

Heyde [4] has shown that, conditional on $Z_n>0$, $m^n Z_n^{-\frac{1}{2}}(W-W_n)$ has the same distribution as $(Z_n^*)^{-\frac{1}{2}}(U_1+\cdots+U_{Z_n^*})$, where the U_i are independent of Z_n^* and are independent and identically distributed, each with the distribution of $W-1$. Furthermore, $\operatorname{var} W=\sigma^2(m^2-m)^{-1}$, and hence

$$P((m^2-m)^{\frac{1}{2}}\sigma^{-1} Z_n^{-\frac{1}{2}} m^n (W-W_n)\leq x \mid Z_n>0)$$

$$= P((\operatorname{var} U_1)^{-\frac{1}{2}} (Z_n^*)^{-\frac{1}{2}} (U_1+\cdots+U_{Z_n^*})\leq x),$$

so that (2) follows immediately from the lemma. (3) is obtained similarly by noting that, conditional on $Z_n>0$, $Z_n^{-\frac{1}{2}}(Z_{r+n}-m^r Z_n)$ has the same distribution as $(Z_n^*)^{-\frac{1}{2}}(V_1+\cdots+V_{Z_n^*})$, where the V_i are independent of Z_n^* and are independent and identically distributed, each with the distribution of Z_r-m^r.

Next, in order to obtain (4) we note that

$$E(Z_n^*)^{-\frac{1}{2}} \leq (E(Z_n^*)^{-1})^{\frac{1}{2}}, \tag{21}$$

by a standard moment inequality, so that it suffices to study $E(Z_n^*)^{-1}$, which is given by

$$E(Z_n^*)^{-1} = \int_0^1 (F_n(s)-F_n(0))\, s^{-1}(1-F_n(0))^{-1}\, ds. \tag{22}$$

Now define a p.g.f.

$$H(s)=(1-q)^{-1}(F(s(1-q)+q)-q), \qquad 0\leq s\leq 1.$$

Then, if $H_n(s)$ is the n-th functional iterate of $H(s)$, it is easily seen that

$$H_n(s)=(1-q)^{-1}(F_n(s(1-q)+q)-q),$$

and $H_n(s)$ is the p.g.f. of a super-critical Galton-Watson process $\{Y_n\}$ (say) with $P(Y_n=0)=0$ for each n. The growth properties of $\{Y_n\}$ are closely related to those of $\{Z_n\}$; in particular $EY_n=EZ_n=m^n$. Also, we note that

$$U(s)=F_n(s(1-q)+q)-(1-q)\,F_n(s)$$

satisfies

$$U(0)=F_n(q)-(1-q)\,F_n(0)=q-(1-q)\,F_n(0)$$

and

$$U'(s)=(1-q)\big(F_n'(s(1-q)+q)-F_n'(s)\big)\geqq 0$$

since $s(1-q)+q\geqq s$ $(0\leqq s\leqq 1)$ and $F_n'(t)$ is monotone increasing in t. Thus for $0\leqq s\leqq 1$,

$$F_n\big(s(1-q)+q\big)-(1-q)\,F_n(s)\geqq q-(1-q)\,F_n(0),$$

that is,

$$H_n(s)\geqq F_n(s)-F_n(0).$$

Consequently, from (22),

$$\begin{aligned} E(Z_n^*)^{-1} &\leqq (1-F_n(0))^{-1}\int_0^1 s^{-1}\,H_n(s)\,ds \\ &=(1-F_n(0))^{-1}\,E(Y_n^{-1}). \end{aligned} \tag{23}$$

Now let $\mathscr{F}_n$ be the σ-field generated by $Y_1,\ldots,Y_n$, and note that $Y_n\geqq 1$ for each n. Then,

$$\begin{aligned} E(Y_n^{-1}) &= E\big(Y_{n-1}^{-1}\,E(Y_{n-1}\,Y_n^{-1}\,|\,\mathscr{F}_{n-1})\big) \\ &= E\big(Y_{n-1}^{-1}\,E(Y_{n-1}(Y_1^{(1)}+\cdots+Y_1^{(Y_{n-1})})^{-1}\,|\,\mathscr{F}_{n-1})\big), \end{aligned} \tag{24}$$

where the $Y_1^{(i)}$ are independent of Y_{n-1} and are independent and identically distributed, each with the distribution of Y_1. Furthermore, using the harmonic mean-arithmetic mean inequality,

$$Y_{n-1}(Y_1^{(1)}+\cdots+Y_1^{(Y_{n-1})})^{-1}\leqq Y_{n-1}^{-1}\big((Y_1^{(1)})^{-1}+\cdots+(Y_1^{(Y_{n-1})})^{-1}\big),$$

so that

$$E\big(Y_{n-1}(Y_1^{(1)}+\cdots+Y_1^{(Y_{n-1})})^{-1}\,|\,\mathscr{F}_{n-1}\big)\leqq E(Y_1^{-1}).$$

Hence, from (24),

$$E(Y_n^{-1})\leqq E(Y_1^{-1})\,E(Y_{n-1}^{-1}),$$

so that

$$E(Y_n^{-1})\leqq (E(Y_1^{-1}))^n. \tag{25}$$

(4) then follows from (21), (23) and (25) by setting $\gamma=E(Y_1^{-1})$, which is in turn given by

$$\gamma=E(Y_1^{-1})=\int_0^1 s^{-1}(1-q)^{-1}\big(F(s(1-q)+q)-q\big)\,ds,$$

with $\gamma=E(Z_1^{-1})$ when $P(Z_1=0)=0$, for then $q=0$. This completes the proof of Theorem 2.

5. Remarks on Theorem 2

The bound (4) of Theorem 2 appears to be rather crude and with reference to it we note that, by Jensen's inequality,

$$E(Y_1^{-1})\geqq (E\,Y_1)^{-1}=m^{-1}.$$

How, if $E(W^{-1}|W>0)<\infty$, it is plausible that under suitable conditions, $E(Z_n^*)^{-1}\sim m^{-n}E(W^{-1}|W>0)$, since $\lim_{n\to\infty} m^n Z_n^{-1}=W^{-1}$ a.s. on $[W>0]$. However, it emerges that $E(W^{-1}|W>0)$ is often not finite and there are wide ranging possibilities for the asymptotic behaviour of $E(Z_n^*)^{-1}$. For example, take

$$F(s)=s\big(m-(m-1)\,s^k\big)^{-1/k},\qquad m>1,$$

where k is a positive integer. Then,

$$F_n(s)=s\big(m^n-(m^n-1)\,s^k\big)^{-1/k},$$

so that

$$E(Z_n^*)^{-1} = E(Z_n^{-1}) = \int_0^1 s^{-1} F_n(s)\, ds$$

$$= \int_0^1 (m^n - (m^n - 1) s^k)^{-1/k}\, ds,$$

and, putting $y = 1 - s^k(1 - m^{-n})$, we obtain

$$E(Z_n^{-1}) = \frac{1}{k(m^n - 1)^{1/k}} \int_{m^{-n}}^{1} \frac{dy}{y^{1/k}(1-y)^{(k-1)/k}}.$$

Thus, if $k=1$,

$$E(Z_n^{-1}) = (m^n - 1)^{-1} n \log m,$$

and if $k>1$,

$$E(Z_n^{-1}) \sim k^{-1} m^{-n/k} B(1-k^{-1}, k^{-1})$$

$$\sim k^{-1} m^{-n/k} \pi (\sin(\pi/k))^{-1}$$

as $n \to \infty$ since

$$B(1-k^{-1}, k^{-1}) = \Gamma(k^{-1}) \Gamma(1-k^{-1}) = \pi (\sin(\pi/k))^{-1}.$$

In conclusion, we make two observations. Firstly, Theorem 2 may be extended to deal with the case $EZ_1^{2+\delta} < \infty$, $0 < \delta \leqq 1$, by using results of Ibragimov [5]. In this case we obtain bounds of the form $c(E(Z_n^*)^{-1})^{\delta/2}$ in (2) and (3), c denoting a positive constant. We have not framed Theorem 2 in this more general context because an explicit form for c is not available.

Secondly, an improved rate of convergence to normality can be obtained by conditioning on the value of Z_n as well as on $[Z_n > 0]$. In this case it is apparent that, after minor modifications in the foregoing proof, the bound in (2) and (3) becomes $O((Z_n^*)^{-\frac{1}{2}})$, which is $O(m^{-\frac{1}{2}n})$ as $n \to \infty$.

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Dr. C.C. Heyde
Department of Statistics
Australian National University
School of General Studies
Box 4, P.O.
Canberra, A.C.T. 2600
Australia

Dr. B.M. Brown
Department of Mathematics
La Trobe University
Bundoora
Victoria 3083
Australia

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