An invariance principle for nonlinear hybrid and impulsive dynamical systems☆

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Abstract

In this paper we develop an invariance principle for dynamical systems possessing left-continuous flows. Specifically, we show that left-continuity of the system trajectories in time for each fixed state point and continuity of the system trajectory in the state for every time in some dense subset of the semi-infinite interval are sufficient for establishing an invariance principle for hybrid and impulsive dynamical systems. As a special case of this result we state and prove new invariant set stability theorems for a class of nonlinear impulsive dynamical systems; namely, state-dependent impulsive dynamical systems. These results provide less conservative stability conditions for impulsive systems as compared to classical results in the literature and allow us to address the stability of limit cycles and periodic orbits of impulsive systems.

Keywords: Left-continuous dynamical systems; Hybrid systems; Impulsive dynamical systems; Discontinuous flows; Positive limit sets; Invariant set theorems; Stability theorems; Lyapunov functions

1. Introduction

In light of the increasingly complex nature of engineering systems such as nonsmooth impact systems [5,7], biological systems [22], demographic systems [28], hybrid systems [6,36], sampled-data systems [16], discrete-event systems [30], systems with shock effects, and feedback systems with impulsive or resetting controls [8,14,15], dynamical systems exhibiting discontinuous flows on appropriate manifolds arise naturally. The mathematical descriptions of such systems can be characterized by impulsive differential equations [2,3,17,22,32]. Impulsive differential equations consist of three elements; namely, a continuous-time differential equation, which governs the motion of the dynamical system between impulsive or resetting events; a difference equation, which governs the way the system states are instantaneously changed when a resetting event occurs; and a criterion for determining when the states of the system are to be reset. Since impulsive systems can involve impulses at variable times, they are in general time-varying systems wherein the resetting events are both a function of time and the system's state. In the case where the resetting events are defined by a prescribed sequence of times which are independent of the system state, the equations are known as *time-dependent differential equations* [2,3,8,14,15,22]. Alternatively, in the case where the resetting events are defined by a manifold in the state space that is independent of time, the equations are autonomous and are known as *state-dependent differential equations* [2,3,8,14,15,22].

To analyze the stability of dynamical systems with impulsive effects, Lyapunov stability results have been presented in the literature [2,20,22–24,27,32–34,37]. In particular, local and global asymptotic stability conclusions of an equilibrium point of a given impulsive dynamical system are provided if a smooth (at least C^1) positive-definite function of the nonlinear system states (Lyapunov function) can be constructed for which its time rate of change over the continuous-time dynamics is strictly negative and its difference over the resetting times is negative. However, unlike dynamical systems possessing continuous flows, Barbashin–Krasovskii–LaSalle-type invariant set stability theorems [4,19,25,26] do not seem to have been addressed for impulsive dynamical systems. This is in spite of the fact that systems theory with impulsive effects has dominated the Russian and Eastern European literature [2,3,17,20,22–24,32–34]. This fact has been further substantiated by Lakshmikantham [21], Bainov [1], and Michel [29]. There appears to be (at least) two reasons for this state of affairs; namely, solutions of impulsive dynamical systems are *not* continuous in time and are *not* continuous functions of the system's initial conditions, which are two key properties needed to establish invariance of omega limit sets and hence an invariance principle.

In this paper we develop an invariance principle for a general class of dynamical systems possessing left-continuous flows; that is, left-continuous dynamical systems. A left-continuous dynamical system is defined on the semi-infinite interval $[0, \infty)$ as a mapping between vector spaces satisfying an appropriate set of axioms and includes hybrid and impulsive dynamical systems as special cases. In particular, invariant set theorems are derived wherein system trajectories converge to the largest invariant set of Lyapunov level surfaces of the left-continuous dynamical system. These systems are shown to specialize to hybrid systems and state-dependent nonlinear impulsive differential systems. For state-dependent impulsive dynamical systems with C^1 Lyapunov functions defined on a compact positively invariant set (with respect to the nonlinear impulsive system), the largest invariant set is contained in a hybrid level surface composed of a union involving vanishing Lyapunov derivatives and differences of the system dynamics over the continuous-time trajectories and the resetting instants, respectively. In addition, if the Lyapunov derivative along the continuous-time system trajectories is negative semidefinite and no system trajectories can stay indefinitely at points where the function's derivative or difference identically vanishes, then the system's equilibrium is asymptotically stable. These results provide less conservative conditions for examining the stability of state-dependent impulsive dynamical systems as compared to the classical results presented in [2,22,32,33,37]. Preliminary versions of the results developed in this paper can be found in the conference papers by the authors in [9,14]. Finally, the impulsive invariance principle developed in the paper can be used to establish the existence and investigate the stability of limit cycles and periodic orbits of impulsive systems.

The contents of the paper are as follows. In Section 2, we introduce the notion of a left-continuous dynamical system as a precise mathematical object satisfying a set of axioms. Furthermore, we show that hybrid dynamical systems are a specialization of left-continuous dynamical systems. Then in Section 3, we state and prove a fundamental result (Theorem 3.1) on positive limit sets for left-continuous dynamical systems. Specifically, it is shown that in order to establish an invariance principle for hybrid dynamical systems all that is needed is (i) left-continuity of the system trajectories in time for each fixed state point and (ii) continuity of the system trajectory in the state for every time in some dense subset of the semi-infinite interval. Using this result we generalize the Barbashin–Krasovskii–LaSalle invariance principle to left-continuous and hybrid dynamical systems. In Section 4, we develop several new results for state-dependent impulsive dynamical systems and provide necessary and sufficient conditions for guaranteeing that state-dependent impulsive systems satisfy the set of axioms of a left-continuous dynamical systems presented in Section 2. In Section 5, we use the results of Section 3 to state and prove new invariant set stability theorems for state-dependent impulsive dynamical systems. In Section 6, we present two illustrative examples to demonstrate the results of the paper. Finally, we draw conclusions in Section 7.

2. Left-continuous dynamical systems

In this section we establish definitions, notation, and introduce the notion of a left-continuous dynamical system.[1] Here, a left-continuous dynamical system is defined as a precise mathematical object satisfying a set of axioms whereas in Section 4 we specialize this notion to nonlinear dynamical systems characterized by impulsive differential equations. The following definition is concerned with left-continuous dynamical systems or, systems with left-continuous flows. For this definition $\mathscr{D} \subseteq \mathbb{R}^n$, $\|\cdot\|$ denotes the Euclidean norm, and $\mathscr{T}_{x_0} \subseteq [0,\infty)$, $x_0 \in \mathscr{D}$, is a dense subset

[1] Right-continuous dynamical systems; that is, systems possessing right-continuous flows, can also be analogously considered here.

of the semi-infinite interval $[0,\infty)$ such that $[0,\infty)\setminus\mathscr{T}_{x_0}$ is (finitely or infinitely) countable.

Definition 2.1. A *left-continuous dynamical system* on $\mathscr{D}$ is the triple $(\mathscr{D},[0,\infty),s)$, where $s:[0,\infty)\times\mathscr{D}\to\mathbb{R}^n$ is such that the following axioms hold:

(i) (Left-continuity): $s(\cdot,x_0)$ is left-continuous in t; that is, $\lim_{\tau\to t^-}s(\tau,x_0)=s(t,x_0)$ for all $x_0\in\mathscr{D}$ and $t\in(0,\infty)$.
(ii) (Consistency): $s(0,x_0)=x_0$ for all $x_0\in\mathscr{D}$.
(iii) (Semi-group property): $s(\tau,s(t,x_0))=s(t+\tau,x_0)$ for all $x_0\in\mathscr{D}$ and $t,\tau\in[0,\infty)$.
(iv) (Quasi-continuous dependence): For every $x_0\in\mathscr{D}$, there exists $\mathscr{T}_{x_0}\subseteq[0,\infty)$ such that $[0,\infty)\setminus\mathscr{T}_{x_0}$ is countable and for every $\varepsilon>0$ and $t\in\mathscr{T}_{x_0}$, there exists $\delta(\varepsilon,x_0,t)>0$ such that if $\|x_0-y\|<\delta(\varepsilon,x_0,t)$, $y\in\mathscr{D}$, then $\|s(t,x_0)-s(t,y)\|<\varepsilon$.

Henceforth, we denote the left-continuous dynamical system $(\mathscr{D},[0,\infty),s)$ by $\mathscr{G}$. Furthermore, we refer to $s(t,x_0)$, $t\geqslant 0$, as the *trajectory* of $\mathscr{G}$ corresponding to $x_0\in\mathscr{D}$, and for a given trajectory $s(t,x_0)$, $t\geqslant 0$, we refer to $x_0\in\mathscr{D}$ as an *initial condition* of $\mathscr{G}$. The trajectory $s(t,x_0)$, $t\geqslant 0$, of $\mathscr{G}$ is *bounded* if there exists $\gamma>0$ such that $\|s(t,x_0)\|<\gamma$, $t\geqslant 0$. Finally, for the remainder of the paper we refer to the left-continuous dynamical system $\mathscr{G}$ as the dynamical system $\mathscr{G}$.

The quasi-continuous dependence property (iv) is a generalization of the standard continuous dependence property for dynamical systems with continuous flows to dynamical systems with left-continuous flows. Specifically, by letting $\mathscr{T}_{x_0}=\bar{\mathscr{T}}_{x_0}=[0,\infty)$, where $\bar{\mathscr{T}}_{x_0}$ denotes the closure of the set $\mathscr{T}_{x_0}$, the quasi-continuous dependence property specializes to the classical continuous dependence of solutions of a given dynamical system with respect to the system's initial conditions $x_0\in\mathscr{D}$ [35]. If, in addition, $x_0=0$, $s(t,0)=0$, $t\geqslant 0$, and $\delta(\varepsilon,0,t)$ can be chosen independent of t then continuous dependence implies the classical Lyapunov stability of the zero trajectory $s(t,0)=0$, $t\geqslant 0$. Hence, Lyapunov stability of motion can be interpreted as continuous dependence of solutions uniformly in t for all $t\geqslant 0$. Conversely, continuous dependence of solutions can be interpreted as Lyapunov stability of motion for every fixed time t [35]. Analogously, Lyapunov stability of impulsive dynamical systems as defined in [22] can be interpreted as quasi-continuous dependence of solutions uniformly in t for all $t\in\mathscr{T}_{x_0}$. In applying Definition 2.1 it may be convenient to replace Axiom (iv) with a stronger condition which may be easier to verify in practice. The following proposition provides sufficient conditions for $\mathscr{G}$ to be a left-continuous dynamical system.

Proposition 2.1. *Let the triple* $(\mathscr{D},[0,\infty),s)$, *where* $s:[0,\infty)\times\mathscr{D}\to\mathscr{D}$, *be such that Axioms* (i)–(iii) *hold and*:

(iv)′ *For every* $x_0\in\mathscr{D}$, $\varepsilon,\eta>0$, *and* $T\in\mathscr{T}_{x_0}$, *there exists* $\delta(\varepsilon,x_0,T)>0$ *such that if* $\|x_0-y\|<\delta(\varepsilon,x_0,T)$, $y\in\mathscr{D}$, *then for every* $t\in\mathscr{T}_{x_0}\cap[0,T]$ *such that* $|t-\tau|>\eta$, *for all* $\tau\in[0,T]\setminus\mathscr{T}_{x_0}$, $\|s(t,x_0)-s(t,y)\|<\varepsilon$. *Furthermore, if* $t\in\mathscr{T}_{x_0}$ *is an accumulation point of* $[0,\infty)\setminus\mathscr{T}_{x_0}$, *then* $s(t,\cdot)$ *is continuous at* x_0.

Then $\mathscr{G}$ *is a left-continuous dynamical system.*

Proof. Let $x_0 \in \mathscr{D}$, $T \in \mathscr{T}_{x_0}$ be such that T is not an accumulation point of $[0,\infty)\backslash\mathscr{T}_{x_0}$. Furthermore, let $\eta(T) > 0$ be such that $|T - \tau| > \eta$, for every $\tau \in [0,T] \setminus \mathscr{T}_{x_0}$. Then it follows from (iv)$'$ that for every $\varepsilon > 0$ there exists $\delta(\varepsilon, x_0, T) > 0$, such that if $\|x_0 - y\| < \delta(\varepsilon, x_0, T)$, $y \in \mathscr{D}$, then $\|s(T, x_0) - s(T, y)\| < \varepsilon$, which implies (iv) with $t = T$. Now, the result is immediate since T is arbitrary in the set of all times such that T is not an accumulation point of $[0,\infty) \setminus \mathscr{T}_{x_0}$ and, by assumption, if $t \in \mathscr{T}_{x_0}$ is an accumulation point of $[0,\infty) \setminus \mathscr{T}_{x_0}$, then $s(t, x_0)$ is continuous in x_0. □

Definition 2.2. A *strong left-continuous dynamical system* on $\mathscr{D}$ is a left-continuous dynamical system on $\mathscr{D}$ and the triple $(\mathscr{D}, [0,\infty), s)$ is such that Axiom (iv)$'$ holds.

The next result shows that $\mathscr{G}$ is a strong left-continuous dynamical system if and only if the trajectory of $\mathscr{G}$ is *jointly continuous between resetting events*; that is, for every $\varepsilon > 0$ and $k \in \mathscr{N}$, where $\mathscr{N}$ denotes the set of nonnegative integers, there exists $\delta = \delta(\varepsilon, k) > 0$ such that if $|t - t'| + \|x_0 - y\| < \delta$, then $\|s(t, x_0) - s(t', y)\| < \varepsilon$, where $x_0, y \in \mathscr{D}$, $t \in (\tau_k(x_0), \tau_{k+1}(x_0)]$, and $t' \in (\tau_k(y), \tau_{k+1}(y)]$. For this result we assume that $\mathscr{T}_{x_0}$ in Definition 2.1 is given by $\mathscr{T}_{x_0} \triangleq \{t \in [0,\infty)\colon\ s(t, x_0) = s(t^+, x_0)\}$ so that $[0,\infty) \setminus \mathscr{T}_{x_0}$ corresponds to the (countable) set of *resetting times* where the trajectory $s(\cdot, x_0)$ is discontinuous. Furthermore, we let $\tau_i(x_0)$, $i = 1, 2, \ldots$, where $\tau_0(x_0) \triangleq 0$ and $\tau_1(x_0) < \tau_2(x_0) < \cdots$, denote the resetting times; that is, $\{\tau_1(x_0), \tau_2(x_0), \ldots\} = [0,\infty) \setminus \mathscr{T}_{x_0}$. Finally, we assume that for every $i = 1, 2, \ldots$, $\tau_i(\cdot)$ is continuous.

Proposition 2.2. *Consider the dynamical system $\mathscr{G}$ satisfying Axioms* (i)–(iii). *Then $\mathscr{G}$ is a strong left-continuous dynamical system if and only if the trajectory $s(t, x_0)$, $t \geqslant 0$, of $\mathscr{G}$ is jointly continuous between resetting events.*

Proof. Assume $\mathscr{G}$ is a strong left-continuous dynamical system, let $\varepsilon > 0$, and let $k \in \mathscr{N}$. Since, by assumption, $\tau_k(\cdot)$ is continuous, it follows that for sufficiently small $\delta_1 > 0$, $\tau_k(x)$ and $\tau_{k+1}(x)$, $x \in \mathscr{B}_{\delta_1}(x_0)$, where $\mathscr{B}_{\delta_1}(x_0)$ denotes the open ball centered at x_0 with radius δ_1, are well defined and finite. Hence, it follows from the strong quasi-continuity property (iv)$'$ that $s(t, \cdot), t \in (\tau_k(\cdot), \tau_{k+1}(\cdot)]$, is uniformly bounded on $\mathscr{B}_{\delta_1}(x_0)$. Now, since $\mathscr{G}$ is continuous between resetting events it follows that for $\varepsilon > 0$ and $k \in \mathscr{N}$ there exists $\hat{\delta} = \hat{\delta}(\varepsilon, k) > 0$ such that if $|t - t'| < \hat{\delta}$, then

$$\|s(t, x) - s(t', x)\| < \frac{\varepsilon}{3}, \quad x \in \mathscr{B}_{\delta_1}(x_0), \quad t, t' \in (\tau_k(x), \tau_{k+1}(x)]. \tag{1}$$

Next, it follows from the continuity of $\tau_k(\cdot)$ that for every sufficiently small $\lambda > 0$ and $k \in \mathscr{N}$, $\underline{\tau}_k(\lambda, x_0) \triangleq \inf_{x \in \mathscr{B}_\lambda(x_0)} \tau_k(x)$ and $\bar{\tau}_k(\lambda, x_0) \triangleq \sup_{x \in \mathscr{B}_\lambda(x_0)} \tau_k(x)$ are well defined and $\lim_{\lambda \to 0} \underline{\tau}_k(\lambda, x_0) = \lim_{\lambda \to 0} \bar{\tau}_k(\lambda, x_0) = \tau_k(x_0)$. (Note that $\underline{\tau}_k(\lambda, x_0) \leqslant \tau_k(x) \leqslant \bar{\tau}_k(\lambda, x_0)$, for all $x \in \mathscr{B}_\lambda(x_0)$.) Hence, there exists $\delta' = \delta'(\hat{\delta}) > 0$ such that $\bar{\tau}_k(\delta', x_0) - \underline{\tau}_k(\delta', x_0) < \hat{\delta}$ and $\bar{\tau}_{k+1}(\delta', x_0) - \underline{\tau}_{k+1}(\delta', x_0) < \hat{\delta}$. Next, let $\eta > 0$ be such that

$$\bar{\tau}_k(\delta', x_0) - \underline{\tau}_k(\delta', x_0) < \tau_k(x_0) - \underline{\tau}_k(\delta', x_0) + \eta < \hat{\delta}, \tag{2}$$

$$\bar{\tau}_{k+1}(\delta', x_0) - \underline{\tau}_{k+1}(\delta', x_0) < \eta + \bar{\tau}_{k+1}(\delta', x_0) - \tau_{k+1}(x_0) < \hat{\delta}. \tag{3}$$

Then, it follows from the strong quasi-continuity of $\mathscr{G}$ that there exists $\delta''=\delta''(\varepsilon,\eta,k)>0$ such that

$$\|s(t,x_0)-s(t,y)\|<\frac{\varepsilon}{3},\quad y\in\mathscr{B}_{\delta''}(x_0),\quad t\in(\tau_k(x_0)+\eta,\tau_{k+1}(x_0)-\eta). \tag{4}$$

Now, if $|t-t'|+\|x_0-y\|<\delta$, where $\delta=\min\{\delta_1,\delta',\delta'',\hat{\delta}\}$, $t\in(\tau_k(x_0)+\eta,\tau_{k+1}(x_0)-\eta)$, and $t'\in(\tau_k(y),\tau_{k+1}(y)]$, then it follows from (1), (4), and triangular inequality for vector norms that

$$\|s(t,x_0)-s(t',y)\|\leqslant\|s(t,x_0)-s(t,y)\|+\|s(t,y)-s(t',y)\|<\frac{2}{3}\varepsilon<\varepsilon. \tag{5}$$

Finally, if $|t-t'|+\|x_0-y\|<\delta$, where $t\in(\tau_k(x_0),\tau_{k+1}(x_0)]\setminus(\tau_k(x_0)+\eta,\tau_{k+1}(x_0)-\eta)$ and $t'\in(\tau_k(y),\tau_{k+1}(y)]$, then conditions (2), (3) imply that there exists $t''\in(\tau_k(x_0)+\eta,\tau_{k+1}(x_0)-\eta)$ such that $|t-t''|<\hat{\delta}$ and $|t'-t''|<\hat{\delta}$. Hence, by (1) and (4) it follows that

$$\|s(t,x_0)-s(t',y)\|\leqslant\|s(t,x_0)-s(t'',x_0)\|+\|s(t'',x_0)-s(t'',y)\|$$
$$+\|s(t'',y)-s(t',y)\|<\varepsilon, \tag{6}$$

which establishes that $\mathscr{G}$ is jointly continuous between resetting events.

To show that joint continuity of $s(t,x_0)$, $t\geqslant 0$, between resetting events implies strong left-continuity of $\mathscr{G}$, let $\varepsilon,\eta>0$, $T\in\mathscr{T}_{x_0}$, and suppose $\tau_k(x_0)<T<\tau_{k+1}(x_0)$. Then, it follows from the joint continuity of $\mathscr{G}$ that there exists $\delta'=\delta'(\varepsilon,k)>0$ such that if $|t-t'|+\|x_0-y\|<\delta'$, then $\|s(t,x_0)-s(t',y)\|<\varepsilon$, where $x_0,y\in\mathscr{D}$, $t\in(\tau_k(x_0),\tau_{k+1}(x_0)]$, and $t'\in(\tau_k(y),\tau_{k+1}(y)]$. Now, it follows that there exists $\delta''=\delta''(x_0,\eta,k)>0$ such that $\bar{\tau}_k(\delta'',x_0)-\tau_k(x_0)<\eta$ and $\tau_{k+1}(x_0)-\underline{\tau}_{k+1}(\delta'',x_0)<\eta$. Note that the above inequalities guarantee that if $t=t'\in(\tau_k(x_0)+\eta,\tau_{k+1}(x_0)-\eta)$, then $t\in(\tau_k(x_0),\tau_{k+1}(x_0)]$ and $t'\in(\tau_k(y),\tau_{k+1}(y)]$, $y\in\mathscr{B}_{\delta''}(x_0)$. Furthermore, letting $\delta_k=\delta_k(\varepsilon,\eta,x_0,k)=\min\{\delta',\delta''\}$, it follows from the joint continuity of $\mathscr{G}$ that for $t=t'\in(\tau_k(x_0)+\eta,\tau_{k+1}(x_0)-\eta)$, $\|s(t,x_0)-s(t,y)\|<\varepsilon$, $y\in\mathscr{B}_{\delta_k}(x_0)$. Similarly, we can obtain $\delta_{k-1}=\delta_{k-1}(\varepsilon,\eta,x_0,k)>0$ such that an analogous inequality can be constructed for all $y\in\mathscr{B}_{\delta_{k-1}}(x_0)$ and $t\in(\tau_{k-1}(x_0)+\eta,\tau_k(x_0)-\eta)$. Recursively repeating this procedure for $m=k-2,\ldots,1$, and choosing $\delta=\delta(\varepsilon,\eta,x_0,k)=\delta(\varepsilon,\eta,x_0,T)=\min\{\delta_1,\ldots,\delta_k\}$, it follows that $\|s(t,x_0)-s(t,y)\|<\varepsilon$, $y\in\mathscr{B}_\delta(x_0)$, $t\in[0,T]$, and $|t-\tau_l(x_0)|>\eta$, $l=1,\ldots,k$, which implies that $\mathscr{G}$ is a strong left-continuous dynamical system. □

Next, we show that hybrid dynamical systems [6,36] are a specialization of left-continuous dynamical systems. To see this, we recall the definition of an uncontrolled hybrid dynamical system given in [6]. For this definition let $\mathscr{Q}\subseteq\mathscr{N}$.

Definition 2.3. A *hybrid dynamical system* $\mathscr{G}_\mathrm{H}$ is the septuple $(\mathscr{D},\mathscr{Q},q,x,s_\mathrm{c},f_\mathrm{d},\mathscr{S})$, where $q:[0,\infty)\times\mathscr{D}\times\mathscr{Q}\to\mathscr{Q}$, $x:[0,\infty)\times\mathscr{D}\times\mathscr{Q}\to\mathbb{R}^n$, $s_\mathrm{c}=\{s_{\mathrm{c}q}\}_{q\in\mathscr{Q}}$, $s_{\mathrm{c}q}:[0,\infty)\times\mathscr{D}\to\mathbb{R}^n$, $\mathscr{S}=\{\mathscr{S}_q\}_{q\in\mathscr{Q}}$, $\mathscr{S}_q\subset\mathscr{D}$, $f_\mathrm{d}=\{f_{\mathrm{d}q}\}_{q\in\mathscr{Q}}$, and $f_{\mathrm{d}q}:\mathscr{S}_q\to\mathscr{D}\times\mathscr{Q}$ are such that the following axioms hold:

(i) For every $q\in\mathscr{Q}$, $s_{\mathrm{c}q}(\cdot,\cdot)$ is jointly continuous on $[0,\infty)\times\mathscr{D}$.
(ii) For every $q\in\mathscr{Q}$ and $x_0\in\mathscr{D}$, $s_{\mathrm{c}q}(0,x_0)=x_0$.

(iii) For every $q \in \mathscr{Q}$, $t_1, t_2 \in [0,\infty)$, $t_1 \leqslant t_2$, and $x_0 \in \mathscr{D}$, $s_{cq}(t_2,x_0) = s_{cq}(t_2 - t_1, s_{cq}(t_1,x_0))$.
(iv) For every $q_0 \in \mathscr{Q}$ and $x_0 \in \mathscr{D}$, $q(\cdot)$ and $x(\cdot)$ are such that $q(t,x_0,q_0) = q_0$ and $x(t,x_0,q_0) = s_{cq_0}(t,x_0)$, for all $0 \leqslant t \leqslant \tau_1$, where $\tau_1 \triangleq \min\{t \geqslant 0\colon s_{cq_0}(t,x_0) \notin \mathscr{S}_{q_0}\}$ exists. Furthermore, $[x^{\mathrm{T}}(\tau_1^+,x_0,q_0), q^{\mathrm{T}}(\tau_1^+,x_0,q_0)]^{\mathrm{T}} = f_{\mathrm{d}q_0}(x(\tau_1)) + [x^{\mathrm{T}}(\tau_1,x_0,q_0), q^{\mathrm{T}}(\tau_1,x_0,q_0)]^{\mathrm{T}}$ and for $(x_1,q_1) \triangleq (x(\tau_1^+,x_0,q_0), q(\tau_1^+,x_0,q_0))$, $q(\cdot)$ and $x(\cdot)$ are such that $q(t,x_0,q_0) = q_1$ and $x(t,x_0,q_0) = s_{cq_1}(\tau_1,x_1)$, for all $\tau_1 < t \leqslant \tau_2$, where $\tau_2 \triangleq \min\{t > \tau_1\colon s_{cq_1}(t,x_1) \notin \mathscr{S}_{q_1}\}$ exists, and so on.

In the above definition we assume that s_{c}, $\mathscr{S}$, and f_{d} are such that $\tau_1, \tau_2, \ldots,$ are well defined. Next, to show that $\mathscr{G}_{\mathrm{H}}$ is a left-continuous dynamical system, let $s\colon [0,\infty) \times (\mathscr{D} \times \mathscr{Q}) \to \mathbb{R}^n \times \mathscr{Q}$ be such that $s(0,(x_0,q_0)) = (x_0,q_0)$ and for every $k = 1,2,\ldots$,

$$s(t,(x_0,q_0)) = (s_{cq_{k-1}}(t,x_{k-1}), q_{k-1}), \qquad \tau_{k-1} < t \leqslant \tau_k, \tag{7}$$

$$s(\tau_k^+,(x_0,q_0)) = f_{\mathrm{d}q_{k-1}}(x_k) + [x_k^{\mathrm{T}},\ q_k^{\mathrm{T}}]^{\mathrm{T}}. \tag{8}$$

Note that s satisfies Axioms (i)–(iii) of Definition 4.1. Furthermore, if the resetting times τ_k are well posed (see Proposition 4.1 below), then the hybrid dynamical system $\mathscr{G}_{\mathrm{H}}$ generates a left-continuous dynamical system on $\mathscr{D} \times \mathscr{Q}$ given by the triple $(\mathscr{D} \times \mathscr{Q}, [0,\infty), s)$.

3. An invariance principle for left-continuous dynamical systems

In this section we develop an invariance principle for left-continuous dynamical systems. First, however, the following definitions are necessary for the main results of the paper. For the next definition let the map $s_t\colon \mathscr{D} \to \mathbb{R}^n$ be defined by $s_t(x) \triangleq s(t,x_0)$, $x_0 \in \mathscr{D}$, for a given $t \geqslant 0$.

Definition 3.1. A set $\mathscr{M} \subseteq \mathscr{D}$ is a *positively invariant set* for the dynamical system $\mathscr{G}$ if $s_t(\mathscr{M}) \subseteq \mathscr{M}$, for all $t \geqslant 0$, where $s_t(\mathscr{M}) \triangleq \{s_t(x)\colon x \in \mathscr{M}\}$. A set $\mathscr{M} \subseteq \mathscr{D}$ is an *invariant set* for the dynamical system $\mathscr{G}$ if $s_t(\mathscr{M}) = \mathscr{M}$ for all $t \geqslant 0$.

Definition 3.2. $p \in \mathscr{D}$ is a *positive limit point* of the trajectory $s(t,x_0)$, $t \geqslant 0$, if there exists a monotonic sequence $\{t_n\}_{n=0}^{\infty}$ of nonnegative real numbers, with $t_n \to \infty$ as $n \to \infty$, such that $s(t_n,x_0) \to p$ as $n \to \infty$. The set of all positive limit points of $s(t,x_0)$, $t \geqslant 0$, is the *positive limit set* $\omega(x_0)$ of $s(t,x_0)$, $t \geqslant 0$.

Note that $p \in \omega(x_0)$ if and only if there exists a monotonic sequence $\{t_n\}_{n=0}^{\infty} \subset \mathscr{T}_{x_0}$, with $t_n \to \infty$ as $n \to \infty$, such that $s(t_n,x_0) \to p$ as $n \to \infty$. To see this, let $p \in \omega(x_0)$ and let $\mathscr{T}_{x_0}$ be a dense subset of the semi-infinite interval $[0,\infty)$. In this case, it follows that there exists an unbounded sequence $\{t_n\}_{n=0}^{\infty}$, such that $\lim_{n\to\infty} s(t_n,x_0) = p$. Hence, for every $\varepsilon > 0$, there exists $n > 0$ such that $\|s(t_n,x_0) - p\| < \varepsilon/2$. Furthermore, since $s(\cdot,x_0)$ is left-continuous and $\mathscr{T}_{x_0}$ is a dense subset of $[0,\infty)$, there exists $\hat{t}_n \in \mathscr{T}_{x_0}$, $\hat{t}_n \leqslant t_n$, such that $\|s(\hat{t}_n,x_0) - s(t_n,x_0)\| < \varepsilon/2$ and hence

$\|s(\hat{t}_n,x_0)-p\| \leqslant \|s(t_n,x_0)-p\| + \|s(\hat{t}_n,x_0)-s(t_n,x_0)\| < \varepsilon$. Using this procedure, with $\varepsilon = 1, 1/2, 1/3, \ldots$, we can construct an unbounded sequence $\{\hat{t}_k\}_{k=1}^{\infty} \subset \mathscr{T}_{x_0}$, such that $\lim_{k\to\infty} s(\hat{t}_k,x_0) = p$. Hence, $p \in \omega(x_0)$ if and only if there exists a monotonic sequence $\{t_n\}_{n=0}^{\infty} \subset \mathscr{T}_{x_0}$, with $t_n \to \infty$ as $n \to \infty$, such that $s(t_n,x_0) \to p$ as $n \to \infty$.

Next, we state and prove a fundamental result on positive limit sets for left-continuous dynamical systems which forms the basis for all later developments. The result generalizes the classical results on positive limit sets to systems with left-continuous flows. For the remainder of the paper the notation $s(t,x_0) \to \mathscr{M} \subseteq \mathscr{D}$ as $t \to \infty$ denotes the fact that $\lim_{t\to\infty} s(t,x_0)$ evolves in $\mathscr{M}$; that is, for each $\varepsilon > 0$ there exists $T > 0$ such that $\mathrm{dist}(s(t,x_0),\mathscr{M}) < \varepsilon$ for all $t > T$, where $\mathrm{dist}(p,\mathscr{M}) \triangleq \inf_{x\in\mathscr{M}} \|p-x\|$.

Theorem 3.1. *Suppose the trajectory $s(t,x_0)$ of the left-continuous dynamical system $\mathscr{G}$ is bounded for all $t \geqslant 0$. Then the positive limit set $\omega(x_0)$ of $s(t,x_0)$, $t \geqslant 0$, is a nonempty, compact invariant set. Furthermore, $s(t,x_0) \to \omega(x_0)$ as $t \to \infty$.*

Proof. Since $s(t,x_0)$ is bounded for all $t \geqslant 0$, it follows from the Bolzano–Weierstrass theorem [31] that every sequence in the positive orbit $\gamma^+(x_0) \triangleq \{s(t,x_0)\colon t \in [0,\infty)\}$ has at least one accumulation point $y \in \mathscr{D}$ as $t \to \infty$ and hence $\omega(x_0)$ is nonempty. Furthermore, since $s(t,x_0)$, $t \geqslant 0$, is bounded it follows that $\omega(x_0)$ is bounded. To show that $\omega(x_0)$ is closed let $\{y_i\}_{i=0}^{\infty}$ be a sequence contained in $\omega(x_0)$ such that $\lim_{i\to\infty} y_i = y$. Now, since $y_i \to y$ as $i \to \infty$ it follows that for every $\varepsilon > 0$ there exists i such that $\|y - y_i\| < \varepsilon/2$. Next, since $y_i \in \omega(x_0)$ it follows that for every $T > 0$ there exists $t \geqslant T$ such that $\|s(t,x_0) - y_i\| < \varepsilon/2$. Hence, it follows that for every $\varepsilon > 0$ and $T > 0$ there exists $t \geqslant T$ such that $\|s(t,x_0) - y\| \leqslant \|s(t,x_0) - y_i\| + \|y_i - y\| < \varepsilon$ which implies that $y \in \omega(x_0)$ and hence $\omega(x_0)$ is closed. Thus, since $\omega(x_0)$ is closed and bounded, $\omega(x_0)$ is compact.

Next, to show positive invariance of $\omega(x_0)$ let $y \in \omega(x_0)$ so that there exists an increasing unbounded sequence $\{t_n\}_{n=0}^{\infty} \subset \mathscr{T}_{x_0}$ such that $s(t_n,x_0) \to y$ as $n \to \infty$. Now, it follows from the quasi-continuous dependence property (iv) that for every $\varepsilon > 0$ and $t \in \mathscr{T}_y$ there exists $\delta(\varepsilon,y,t) > 0$ such that $\|y - z\| < \delta(\varepsilon,y,t)$, $z \in \mathscr{D}$, implies $\|s(t,y) - s(t,z)\| < \varepsilon$ or, equivalently, for every sequence $\{y_i\}_{i=1}^{\infty}$ converging to y and $t \in \mathscr{T}_y$, $\lim_{i\to\infty} s(t,y_i) = s(t,y)$. Now, since $s(t_n,x_0) \to y$ as $n \to \infty$, it follows from the semi-group property (iii) that $s(t,y) = s(t, \lim_{n\to\infty} s(t_n,x_0)) = \lim_{n\to\infty} s(t+t_n,x_0) \in \omega(x_0)$ for all $t \in \mathscr{T}_y$. Hence, $s(t,y) \in \omega(x_0)$ for all $t \in \mathscr{T}_y$. Next, let $t \in [0,\infty)\backslash\mathscr{T}_y$ and note that, since $\mathscr{T}_y$ is dense in $[0,\infty)$, there exists a sequence $\{\tau_n\}_{n=0}^{\infty}$ such that $\tau_n \leqslant t$, $\tau_n \in \mathscr{T}_y$, and $\lim_{n\to\infty}\tau_n = t$. Now, since $s(\cdot,y)$ is left-continuous it follows that $\lim_{n\to\infty} s(\tau_n,y) = s(t,y)$. Finally, since $\omega(x_0)$ is closed and $s(\tau_n,y) \in \omega(x_0)$, $n = 1,2,\ldots$, it follows that $s(t,y) = \lim_{n\to\infty} s(\tau_n,y) \in \omega(x_0)$. Hence, $s_t(\omega(x_0)) \subseteq \omega(x_0)$, $t \geqslant 0$, establishing positive invariance of $\omega(x_0)$.

Now, to show invariance of $\omega(x_0)$ let $y \in \omega(x_0)$ so that there exists an increasing unbounded sequence $\{t_n\}_{n=0}^{\infty}$ such that $s(t_n,x_0) \to y$ as $n \to \infty$. Next, let $t \in \mathscr{T}_{x_0}$ and note that there exists N such that $t_n > t$, $n \geqslant N$. Hence, it follows from the semi-group property (iii) that $s(t, s(t_n - t, x_0)) = s(t_n,x_0) \to y$ as $n \to \infty$. Now, it follows from the Bolzano–Weierstass theorem [31] that there exists a subsequence z_{n_k} of the sequence $z_n = s(t_n - t, x_0)$, $n = N, N+1, \ldots$, such that $z_{n_k} \to z \in \mathscr{D}$ and, by definition, $z \in \omega(x_0)$. Next, it follows from the quasi-continuous dependence property (iv) that $\lim_{k\to\infty} s(t,z_{n_k}) = s(t,\lim_{k\to\infty} z_{n_k})$ and hence $y = s(t,z)$ which implies that $\omega(x_0) \subseteq s_t(\omega(x_0))$, $t \in \mathcal{T}_{x_0}$. Next, let $t \in [0,\infty)\setminus\mathcal{T}_{x_0}$, let $\hat{t} \in \mathcal{T}_{x_0}$ be such that $\hat{t} > t$, and consider $y \in \omega(x_0)$. Now, there exists $\hat{z} \in \omega(x_0)$ such that $y = s(\hat{t},\hat{z})$, and it follows from the positive invariance of $\omega(x_0)$ that $z = s(\hat{t}-t,\hat{z}) \in \omega(x_0)$. Furthermore, it follows from the semi-group property (iii) that $s(t,z) = s(t,s(\hat{t}-t,z)) = s(\hat{t},\hat{z}) = y$ which implies that for all $t \in [0,\infty)\setminus\mathcal{T}_{x_0}$ and for every $y \in \omega(x_0)$ there exists $z \in \omega(x_0)$ such that $y = s(t,z)$. Hence, $\omega(x_0) \subseteq s_t(\omega(x_0))$, $t \geqslant 0$. Now, using the positive invariance of $\omega(x_0)$ it follows that $s_t(\omega(x_0)) = \omega(x_0)$, $t \geqslant 0$, establishing invariance of the positive limit set $\omega(x_0)$.

Finally, to show $s(t,x_0) \to \omega(x_0)$ as $t \to \infty$, suppose, ad absurdum, $s(t,x_0) \nrightarrow \omega(x_0)$ as $t \to \infty$. In this case, there exists an $\varepsilon > 0$ and a sequence $\{t_n\}_{n=0}^{\infty}$, with $t_n \to \infty$ as $n \to \infty$, such that

$$\inf_{p\in\omega(x_0)} \|s(t_n,x_0) - p\| \geqslant \varepsilon, \quad n \geqslant 0.$$

However, since $s(t,x_0)$, $t \geqslant 0$, is bounded, the bounded sequence $\{s(t_n,x_0)\}_{n=0}^{\infty}$ contains a convergent subsequence $\{s(t_n^*,x_0)\}_{n=0}^{\infty}$ such that $s(t_n^*,x_0) \to p^* \in \omega(x_0)$ as $n \to \infty$ which contradicts the original supposition. Hence, $s(t,x_0) \to \omega(x_0)$ as $t \to \infty$. □

Remark 3.1. Note that the compactness of the positive limit set $\omega(x_0)$ depends only on the boundedness of the trajectory $s(t,x_0)$, $t \geqslant 0$, whereas the left-continuity and quasi-dependence properties are key in proving invariance of the positive limit set $\omega(x_0)$. In classical dynamical systems where the trajectory $s(\cdot,\cdot)$ is assumed to be continuous in both its arguments, both the left-continuity and the quasi-continuous dependence properties are trivially satisfied. Finally, we note that unlike dynamical systems with continuous flows, the omega limit set of a left-continuous dynamical system may not be connected.

Remark 3.2. To demonstrate the importance of the quasi-continuous dependence property for the invariance of positive limit sets, consider the left-continuous dynamical system characterized by the state-dependent impulsive differential equation (see Section 4 for details)

$$\dot{x}(t) = -x(t), \quad x(t) \neq 0, \tag{9}$$

$$\Delta x(t) = 1, \quad x(t) = 0, \tag{10}$$

where $t \geqslant 0$, $x(t) \in \mathbb{R}$, $x(0) = x_0$, and $\Delta x(t) \triangleq x(t^+) - x(t)$. In this case, the trajectory $s(\cdot,\cdot)$ is given by $s(0,x_0) = x_0$, $x_0 \in \mathbb{R}$, and for all $t > 0$,

$$s(t,x_0) = \begin{cases} e^{-t}x_0, & x_0 \neq 0, \\ e^{-t}, & x_0 = 0, \end{cases} \tag{11}$$

which shows that for every $x_0 \in \mathbb{R}$, the trajectory $s(t,x_0)$ is left-continuous in t and approaches the positive limit set containing only the origin. However, note that the dynamical system does not satisfy the quasi-continuous property and the origin is *not* an invariant set.

Next, we present a generalization of the Barbashin–Krasovskii–LaSalle invariance principle [4,18,19,25,26] to left-continuous dynamical systems. For the remainder of the paper define the notation

$$V^{-1}(\gamma) \triangleq \{x \in \mathscr{D}_{\mathrm{c}}\colon\ V(x) = \gamma\},$$

where $\gamma \in \mathbb{R}$, $\mathscr{D}_{\mathrm{c}} \subseteq \mathscr{D}$, and $V : \mathscr{D}_{\mathrm{c}} \to \mathbb{R}$ is a C^0 function, and let $\mathscr{M}_\gamma$ denote the largest invariant set (with respect to the dynamical system $\mathscr{G}$) contained in $V^{-1}(\gamma)$.

Theorem 3.2. *Let $s(t,x_0)$, $t \geqslant 0$, denote a trajectory of a left-continuous dynamical system $\mathscr{G}$ and let $\mathscr{D}_{\mathrm{c}} \subset \mathscr{D}$ be a compact positively invariant set with respect to $\mathscr{G}$. Assume there exists a C^0 function $V : \mathscr{D}_{\mathrm{c}} \to \mathbb{R}$ such that $V(s(t,x_0)) \leqslant V(s(\tau,x_0))$, $0 \leqslant \tau \leqslant t$, for all $x_0 \in \mathscr{D}_{\mathrm{c}}$. If $x_0 \in \mathscr{D}_{\mathrm{c}}$, then $s(t,x_0) \to \mathscr{M} \triangleq \bigcup_{\gamma\in\mathbb{R}} \mathscr{M}_\gamma$ as $t \to \infty$.*

Proof. Since $V(\cdot)$ is continuous on the compact set $\mathscr{D}_{\mathrm{c}}$, there exists $\beta \in \mathbb{R}$ such that $V(x) \geqslant \beta$, $x \in \mathscr{D}_{\mathrm{c}}$. Hence, since $V(s(t,x_0))$, $t \geqslant 0$, is nonincreasing, $\gamma_{x_0} \triangleq \lim_{t\to\infty} V(s(t,x_0))$, $x_0 \in \mathscr{D}_{\mathrm{c}}$, exists. Now, for every $y \in \omega(x_0)$ there exists an increasing unbounded sequence $\{t_n\}_{n=0}^{\infty}$ such that $s(t_n,x_0) \to y$ as $n \to \infty$, and, since $V(\cdot)$ is continuous, it follows that $V(y) = V(\lim_{n\to\infty} s(t_n,x_0)) = \lim_{n\to\infty} V(s(t_n,x_0)) = \gamma_{x_0}$. Hence, $y \in V^{-1}(\gamma_{x_0})$ for all $y \in \omega(x_0)$, or, equivalently, $\omega(x_0) \subseteq V^{-1}(\gamma_{x_0})$. Now, since $\mathscr{D}_{\mathrm{c}}$ is compact and positively invariant, it follows that $s(t,x_0)$, $t \geqslant 0$, is bounded for all $x_0 \in \mathscr{D}_{\mathrm{c}}$ and hence it follows from Theorem 3.1 that $\omega(x_0)$ is a nonempty, compact invariant set. Thus, $\omega(x_0)$ is a subset of the largest invariant set contained in $V^{-1}(\gamma_{x_0})$; that is, $\omega(x_0) \subseteq \mathscr{M}_{\gamma_{x_0}}$. Hence, for all $x_0 \in \mathscr{D}_{\mathrm{c}}$, $\omega(x_0) \subseteq \mathscr{M}$. Finally, since $s(t,x_0) \to \omega(x_0)$ as $t \to \infty$ it follows that $x(t) \to \mathscr{M}$ as $t \to \infty$. □

Remark 3.3. Since $V : \mathscr{D}_{\mathrm{c}} \to \mathbb{R}$ is a continuous function, it follows from the proof of Theorem 3.2 that for every $x_0 \in \mathscr{D}_{\mathrm{c}}$ there exists $\gamma_{x_0} \leqslant V(x_0)$ such that $\omega(x_0) \subseteq \mathscr{M}_{\gamma_{x_0}} \subseteq \mathscr{M}$.

Remark 3.4. If $V : \mathscr{D}_{\mathrm{c}} \to \mathbb{R}$ is a lower semicontinuous function such that all the conditions of Theorem 3.2 are satisfied then, for all $x_0 \in \mathscr{D}_{\mathrm{c}}$, $\omega(x_0) \subseteq \hat{\mathscr{M}} \triangleq \bigcup_{\gamma\in\mathbb{R}} \hat{\mathscr{M}}_\gamma$, where $\hat{\mathscr{M}}_\gamma$ denotes the largest invariant set contained in the set

$$\mathscr{R}_\gamma \triangleq \bigcap_{c>\gamma} \overline{\{x \in \mathscr{D}_{\mathrm{c}}\colon\ \gamma \leqslant V(x) \leqslant c\}}.$$

For further details on invariant set theorems for dynamical systems with lower semicontinuous Lyapunov functions see [10].

Finally, we close this section by noting that Theorems 3.1 and 3.2 can be easily extended to infinite-dimensional dynamical systems. Specifically, let $\mathscr{D} \subseteq \mathscr{X}$, where $\mathscr{X}$ denotes a metric space with metric $d : \mathscr{X} \times \mathscr{X} \to [0,\infty)$. Now, let the triple $(\mathscr{D},[0,\infty),s)$, where $s : [0,\infty) \times \mathscr{D} \to \chi$, be such that Axioms (i)–(iii) hold and (iv) is replaced by

(iv)″ (Quasi-continuous dependence): For every $x_0 \in \mathscr{D}$, there exists $\mathscr{T}_{x_0} \subseteq [0,\infty)$ such that $[0,\infty)\setminus\mathscr{T}_{x_0}$ is countable and for every $\varepsilon > 0$ and $t \in \mathscr{T}_{x_0}$, there exists $\delta(\varepsilon,x_0,t) > 0$ such that if $d(x_0,y) < \delta(\varepsilon,x_0,t)$, $y \in \mathscr{D}$, then $d(s(t,x_0),s(t,y)) < \varepsilon$.

In this case, it can be shown that if the closure of the positive orbit

$$\overline{\gamma^+(x_0)} = \overline{\{s(t,x_0)\colon\ t\in[0,\infty)\}}$$

is compact in $\mathscr{X}$, then the positive limit set $\omega(x_0)$ of $s(t,x_0)$, $t\geqslant 0$, is a nonempty, compact invariant set. Furthermore, $d(s(t,x_0),\omega(x_0))\to 0$ as $t\to\infty$. Now, if $\mathscr{D}_\mathrm{c}\subseteq\mathscr{D}$ is a positively invariant set with respect to $\mathscr{G}$, every positive orbit $\gamma^+(x_0)$ in $\mathscr{D}_\mathrm{c}$ is contained in a compact set, and there exists a C^0 function $V:\mathscr{D}_\mathrm{c}\to\mathbb{R}$ such that $V(s(t,x_0))\leqslant V(s(\tau,x_0))$, $0\leqslant\tau\leqslant t$, for all $x_0\in\mathscr{D}_\mathrm{c}$, then for every $x_0\in\mathscr{D}_\mathrm{c}$, $d(s(t,x_0),\mathscr{M})\to 0$ as $t\to\infty$. Of course, in this case the notions of openness, convergence, continuity, and compactness used in Theorems 3.1 and 3.2 refer to the topology generated on $\mathscr{X}$ by the metric $d(\cdot,\cdot)$.

4. State-dependent impulsive dynamical systems

A state-dependent impulsive dynamical system has the form

$$\dot{x}(t) = f_\mathrm{c}(x(t)),\quad x(0)=x_0,\quad x(t)\notin\mathscr{S}, \tag{12}$$

$$\Delta x(t) = f_\mathrm{d}(x(t)),\quad x(t)\in\mathscr{S}, \tag{13}$$

where $t\geqslant 0$, $x(t)\in\mathscr{D}\subseteq\mathbb{R}^n$, $\mathscr{D}$ is an open set with $0\in\mathscr{D}$, $\Delta x(t)\triangleq x(t^+)-x(t)$, $f_\mathrm{c}:\mathscr{D}\to\mathbb{R}^n$ is such that $f_\mathrm{c}(0)=0$, $f_\mathrm{d}:\mathscr{S}\to\mathbb{R}^n$ is continuous, and $\mathscr{S}\subset\mathscr{D}$ is the *resetting set*. We refer to the differential equation (12) as the *continuous-time dynamics*, and we refer to the difference equation (13) as the *resetting law*. We assume that the continuous-time dynamics $f_\mathrm{c}(\cdot)$ are such that the solution to (12) is jointly continuous in t and x_0 between resetting events. A sufficient condition ensuring this is Lipschitz continuity of $f_\mathrm{c}(\cdot)$. Alternatively, uniqueness of solutions in forward time along with the continuity of $f_\mathrm{c}(\cdot)$ ensure that solutions to (12) between resetting events are continuous functions of the initial conditions $x_0\in\mathscr{D}$ even when $f_\mathrm{c}(\cdot)$ is not Lipschitz continuous on $\mathscr{D}$ (see [11, Theorem 4.3, p. 59]). More generally, $f_\mathrm{c}(\cdot)$ need not be continuous. In particular, if $f_\mathrm{c}(\cdot)$ is discontinuous but bounded and $x(\cdot)$ is the unique solution to (12) between resetting events in the sense of Filippov [12], then continuous dependence of solutions with respect to the initial conditions hold [12]. Finally, note that since the resetting set $\mathscr{S}$ is a subset of the state space $\mathscr{D}$ and is independent of time, state-dependent impulsive dynamical systems are time-invariant.

For a particular trajectory $x(t)$, we let $\tau_k(x_0)$ denote the kth instant of time at which $x(t)$ intersects $\mathscr{S}$, and we call the times $\tau_k(x_0)$ the *resetting times*. Thus the trajectory of the system (12), (13) from the initial condition $x(0)=x_0$ is given by $\psi(t,x_0)$ for $0<t\leqslant\tau_1(x_0)$ where $\psi(t,x_0)$ denotes the solution to the continuous-time dynamics (12). If and when the trajectory reaches a state $x_1\triangleq x(\tau_1(x_0))$ satisfying $x_1\in\mathscr{S}$, then the state is instantaneously transferred to $x_1^+\triangleq x_1+f_\mathrm{d}(x_1)$ according to the resetting law (13). The solution $x(t)$, $\tau_1(x_0)<t\leqslant\tau_2(x_0)$, is then given by $\psi(t-\tau_1(x_0),x_1^+)$, and so on. Note that the solution $x(t)$ of (12), (13) is left-continuous; that is, it is

continuous everywhere except at the resetting times $\tau_k(x_0)$, and

$$x_k \triangleq x(\tau_k(x_0)) = \lim_{\varepsilon\to 0^+} x(\tau_k(x_0) - \varepsilon), \tag{14}$$

$$x_k^+ \triangleq x(\tau_k(x_0)) + f_{\mathrm{d}}(x(\tau_k(x_0))) \tag{15}$$

for $k = 1, 2, \ldots$.

We make the following additional assumptions:

A1. If $x(t) \in \bar{\mathscr{S}} \setminus \mathscr{S}$, then there exists $\varepsilon > 0$ such that, for all $0 < \delta < \varepsilon$, $s(\delta, x(t)) \notin \mathscr{S}$.
A2. If $x \in \mathscr{S}$, then $x + f_{\mathrm{d}}(x) \notin \mathscr{S}$.

Assumption A1 ensures that if a trajectory reaches the closure of $\mathscr{S}$ at a point that does not belong to $\mathscr{S}$, then the trajectory must be directed away from $\mathscr{S}$; that is, a trajectory cannot enter $\mathscr{S}$ through a point that belongs to the closure of $\mathscr{S}$ but not to $\mathscr{S}$. Furthermore, A2 ensures that when a trajectory intersects the resetting set $\mathscr{S}$, it instantaneously exists $\mathscr{S}$. Finally, we note that if $x_0 \in \mathscr{S}$, then the system initially resets to $x_0^+ = x_0 + f_{\mathrm{d}}(x_0) \notin \mathscr{S}$ which serves as the initial condition for the continuous-time dynamics (12).

Remark 4.1. It follows from A2 that resetting removes $x(\tau_k(x_0))=x_k$ from the resetting set $\mathscr{S}$. Thus, immediately after resetting occurs, the continuous-time dynamics (12), and not the resetting law (13), becomes the active element of the impulsive dynamical system.

Remark 4.2. It follows from A1 and A2 that no trajectory starting outside of $\mathscr{S}$ can intersect the interior of $\mathscr{S}$. Specifically, it follows from A1 that a trajectory can only reach $\mathscr{S}$ through a point belonging to both $\mathscr{S}$ and its boundary. And from A2, it follows that if a trajectory reaches a point in $\mathscr{S}$ that is on the boundary of $\mathscr{S}$, then the trajectory is instantaneously removed from $\mathscr{S}$. Since a continuous trajectory starting outside of $\mathscr{S}$ and intersecting the interior of $\mathscr{S}$ must first intersect the boundary of $\mathscr{S}$, it follows that no trajectory can reach the interior of $\mathscr{S}$.

Remark 4.3. Let $x^* \in \mathscr{D}$ satisfy $f_{\mathrm{d}}(x^*)=0$. Then $x^* \notin \mathscr{S}$. To see this, suppose $x^* \in \mathscr{S}$. Then $x^* + f_{\mathrm{d}}(x^*) = x^* \in \mathscr{S}$, contradicting A2. Specifically, we note that $0 \notin \mathscr{S}$.

Remark 4.4. Note that it follows from the definition of $\tau_k(\cdot)$ that $\tau_1(x) > 0$, $x \notin \mathscr{S}$, and $\tau_1(x) = 0$, $x \in \mathscr{S}$. Furthermore, since for every $x \in \mathscr{S}$, $x + f_{\mathrm{d}}(x) \notin \mathscr{S}$, it follows that $\tau_2(x) = \tau_1(x) + \tau_1(x + f_{\mathrm{d}}(x)) > 0$.

To show that the resetting times $\tau_k(x_0)$ are well defined and distinct, assume $T = \inf\{t\colon \psi(t,x_0) \in \mathscr{S}\} < \infty$. Now, ad absurdum, suppose $\tau_1(x_0)$ is not well defined; that is, $\min\{t\colon \psi(t,x_0) \in \mathscr{S}\}$ does not exist. Since $\psi(\cdot,x_0)$ is continuous, it follows that $\psi(T,x_0) \in \partial\mathscr{S}$ and since, by assumption, $\min\{t\colon \psi(t,x_0) \in \mathscr{S}\}$ does not exist it follows that $\psi(T,x_0) \in \bar{\mathscr{S}} \setminus \mathscr{S}$. Note that $\psi(t,x_0) = s(t,x_0)$, for every t such that $\psi(\tau,x) \notin \mathscr{S}$ for all $0 \leqslant \tau \leqslant t$. Now, it follows from A1 that there exists $\varepsilon > 0$ such that $s(T + \delta, x_0) = \psi(T + \delta, x_0)$, $\delta \in (0, \varepsilon)$, which implies that $\inf\{t\colon \psi(t,x_0) \in \mathscr{S}\} > T$ which is a contradiction. Hence, $\psi(T,x_0)\in\partial\mathcal{S}\cap\mathcal{S}$ and $\inf\{t:\psi(t,x_0)\in\mathcal{S}\}=\min\{t:\psi(t,x_0)\in\mathcal{D}\}$ which implies that the first resetting time $\tau_1(x_0)$ is well defined for all initial conditions $x_0\in\mathcal{D}$. Next, it follows from A2 that $\tau_2(x_0)$ is also well defined and $\tau_2(x_0)\neq\tau_1(x_0)$. Repeating the above arguments it follows that the resetting times $\tau_k(x_0)$ are well defined and distinct.

Since the resetting times are well defined and distinct, and since the solution to (12) exists and is unique, it follows that the solution of the impulsive dynamical system (12), (13) also exists and is unique over a forward time interval. However, it is important to note that the analysis of impulsive dynamical systems can be quite involved. In particular, such systems can exhibit Zenoness, beating, as well as confluence wherein solutions exhibit infinitely many resettings in a finite-time, encounter the same resetting surface a finite or infinite number of times in zero time, and coincide after a given point in time. In this paper we allow for the possibility of confluence and Zeno solutions, however, A2 precludes the possibility of beating. Furthermore, since *not* every bounded solution of an impulsive dynamical system over a forward time interval can be extended to infinity due to Zeno solutions, we assume that existence and uniqueness of solutions are satisfied in forward time. For details see [2,3,22].

Next, we show that the dynamical system $\mathcal{G}$ given by (12), (13) under an additional set of assumptions is a left-continuous dynamical system. To show that state-dependent impulsive dynamical systems[2] satisfy Axioms (i)–(iii) of a left-continuous dynamical system, note that $s(0,x_0)=x_0$ for all $x_0\in\mathcal{D}$ and

$$s(t,x_0)=\begin{cases}\psi(t,x_0), & 0\leqslant t\leqslant\tau_1(x_0),\\ \psi(t-\tau_k(x_0),s(\tau_k(x_0),x_0) & \\ \quad +f_{\mathrm{d}}(s(\tau_k(x_0),x_0))), & \tau_k(x_0)<t\leqslant\tau_{k+1}(x_0),\\ \psi(t-\tau(x_0),s(\tau(x_0),x_0)), & t\geqslant\tau(x_0),\end{cases}\tag{16}$$

where $\tau(x_0)\triangleq\sup_{k\geqslant 0}\tau_k(x_0)$ which implies that $s(\cdot,x_0)$ is left-continuous. Furthermore, uniqueness of the solutions implies that $s(t,x_0)$ satisfies the semi-group property $s(\tau,s(t,x_0))=s(t+\tau,x_0)$ for all $x_0\in\mathcal{D}$ and $t,\tau\in[0,\infty)$. The following result provides sufficient conditions that guarantee that the dynamical system $\mathcal{G}$ given by (12), (13) satisfies the quasi-continuous dependence property.

Proposition 4.1. *Consider the nonlinear impulsive dynamical system $\mathcal{G}$ given by* (12), (13). *Assume* A1 *and* A2 *hold, and assume that either of the following statements holds*:

(i) *For all $x_0\notin\mathcal{S}$, $0<\tau_1(x_0)<\infty$, $\tau_1(\cdot)$ is continuous, and $\lim_{k\to\infty}\tau_k(x_0)\to\infty$.*
(ii) *For all $x_0\notin\bar{\mathcal{S}}$, $0<\tau_1(x_0)<\infty$, $\tau_1(\cdot)$ is continuous, $\lim_{k\to\infty}\tau_k(x_0)\to\infty$, and $\bar{\mathcal{S}}\setminus\mathcal{S}$ is a positively invariant set with respect to the dynamical system $\mathcal{G}$ given by* (12), (13).

Then $\mathcal{G}$ satisfies the quasi-continuous dependence property given by Axiom (iv).

[2] Here we assume that if the solution is Zeno, then it is convergent.

Proof. To show that (i) implies the quasi-continuous dependence property, assume that for all $x_0 \notin \mathscr{S}$, $0 < \tau_1(x_0) < \infty$, $\tau_1(\cdot)$ is continuous, and $\lim_{k\to\infty} \tau_k(x_0) \to \infty$. In this case, it follows from the definition of $\tau_k(x_0)$ that for every $x_0 \in \mathscr{D}$ and $k \in \{1,2,\ldots,\}$,

$$\tau_k(x_0) = \tau_{k-j}(x_0) + \tau_j[s(\tau_{k-j}(x_0),x_0) + f_{\rm d}(s(\tau_{k-j}(x_0),x_0))], \quad j = 1,\ldots,k, \quad (17)$$

where $\tau_0(x_0) \triangleq 0$. Since $f_{\rm c}(\cdot)$ is such that the solutions to (12) are continuous with respect to the initial conditions between resetting events, it follows that for every $k=0,1,\ldots$, and $t \in (\tau_k(x_0),\tau_{k+1}(x_0)]$, $\psi(\cdot,\cdot)$ is continuous in both its arguments. Specifically, note that since $\tau_1(x_0)$ is continuous it follows that $\eta_1(x_0) \triangleq s(\tau_1(x_0),x_0) = \psi(\tau_1(x_0),x_0)$ is continuous on $\mathscr{D}$. Hence, it follows from (17) and the continuity of $f_{\rm d}(\cdot)$ that $\tau_2(x_0)=\tau_1(x_0)+\tau_1[s(\tau_1(x_0),x_0)+f_{\rm d}(s(\tau_1(x_0),x_0))]$ is also continuous which implies that $\eta_2(x_0) \triangleq s(\tau_2(x_0),x_0)=\psi(\tau_2(x_0)-\tau_1(x_0),\eta_1(x_0)+f_{\rm d}(\eta_1(x_0))$ is continuous on $\mathscr{D}$. By recursively repeating this procedure for $k=3,4,\ldots$, it follows that $\tau_k(x_0)$ and $\eta_k(x_0) \triangleq s(\tau_k(x_0),x_0)$ are continuous on $\mathscr{D}$. Next, let $\mathscr{T}_{x_0} = \{t \in [0,\infty),\ t \neq \tau_k(x_0)\}$ and let $t \in \mathscr{T}_{x_0}$ be such that $\tau_k(x_0) < t < \tau_{k+1}(x_0)$. Now, noting that $s(t,x_0) = \psi(t - \tau_k(x_0), s(\tau_k(x_0),x_0)) + f_{\rm d}(s(\tau_k(x_0),x_0))$, it follows from the continuity of $f_{\rm d}(\cdot)$ and $\tau_k(\cdot)$ that $s(t,x_0)$ is a continuous function of x_0 for all $t \in \mathscr{T}_{x_0}$ such that $\tau_k(x_0) < t < \tau_{k+1}(x_0)$ for some k. Hence, since $\lim_{k\to\infty} \tau_k(x_0) \to \infty$, $\mathscr{G}$ satisfies the quasi-continuous dependence property given by Axiom (iv). Next, consider the case in which $x_0 \in \mathscr{S}$. Note that in this case $\tau_1(x_0) = 0$ and $\tau_2(x_0) = \tau_1(x_0 + f_{\rm d}(x_0))$. Since $x_0 \in \mathscr{S}$, it follows that $x_0 + f_{\rm d}(x_0) \notin \mathscr{S}$ and since $\tau_1(\cdot)$ is continuous on $\mathscr{D}\backslash\mathscr{S}$ and $f_{\rm d}(\cdot)$ is continuous on $\mathscr{S}$ it follows that $\tau_2(\cdot)$ is continuous on $\mathscr{S}$. Now the quasi-continuous dependence property given by Axiom (iv) for all $x_0 \in \mathscr{S}$ can be shown as above.

Alternatively, if (ii) is satisfied then as in the proof of (i) it can be shown that for all $x_0 \in \mathscr{D}$, $x_0 \notin \bar{\mathscr{S}}\backslash\mathscr{S}$, $s(t,x_0)$ is a continuous function of x_0 for all $t \in \mathscr{T}_{x_0}$. Next, if $\bar{\mathscr{S}}\backslash\mathscr{S}$ is a positively invariant set with respect to $\mathscr{G}$, then for all $x_0 \in \bar{\mathscr{S}}\backslash\mathscr{S}$, $\mathscr{T}_{x_0} = [0,\infty)$. Now, the continuity of $s(t,x_0)$ for all $t \in [0,\infty)$ follows from the fact that $f_{\rm c}(\cdot)$ is such that the solutions to (12) are continuous with respect to the initial conditions between resetting events, and hence $\mathscr{G}$ satisfies the quasi-continuous dependence property given by Axiom (iv). □

Remark 4.5. If, for every $x_0 \in \mathscr{D}$, the solution $s(t,x_0)$ to (12), (13) is a Zeno solution; that is, $\lim_{k\to\infty} \tau_k(x_0) \to \tau(x_0) < \infty$, and the resetting sequence $\{\tau_k(x_0)\}_{k=0}^{\infty}$ is uniformly convergent in x_0, then Condition (ii) of Proposition 4.1 implies that $\mathscr{G}$ satisfies the quasi-continuous dependence property (iv). To see this, note that since $\{\tau_k(\cdot)\}_{k=1}^{\infty}$ is a uniformly convergent sequence of continuous functions, it follows that $\tau(\cdot)$ is a continuous function. Now, noting that for all $t > \tau(x_0)$, $t \in \mathscr{T}_{x_0}$, $s(t,x_0) = \psi(t-\tau(x_0), s(\tau^+(x_0),x_0))$, it follows that $s(t,x_0)$ is a continuous function of x_0 for all $t \in \mathscr{T}_{x_0}$ which proves that $\mathscr{G}$ satisfies the quasi-continuous dependence property given by Axiom (iv).

Proposition 4.1 requires that the first resetting time $\tau_1(\cdot)$ be continuous at $x_0 \in \mathscr{D}$. The following result proven in [13] provides sufficient conditions for establishing the continuity of $\tau_1(\cdot)$ at $x_0 \in \mathscr{D}$.

Proposition 4.2 (Grizzle et al. [13]). *Consider the nonlinear impulsive dynamical system* $\mathscr{G}$ *given by* (12), (13). *Assume there exists a continuously differentiable function* $\mathscr{X}:\mathscr{D}\to\mathbb{R}$ *such that the resetting set is given by* $\mathscr{S}=\{x\colon \mathscr{X}(x)=0\}$ *and* $\mathscr{X}'(x)f_{\rm c}(x)\neq 0$, $x\in\mathscr{S}$. *Then* $\tau_1(\cdot)$ *is continuous at* $x_0\in\mathscr{D}$, *where* $0<\tau_1(x_0)<\infty$.

Remark 4.6. The first assumption in Proposition 4.2 implies that the resetting set $\mathscr{S}$ is an embedded submanifold while the second assumption assures that the solution of $\mathscr{G}$ is not tangent to the resetting set $\mathscr{S}$.

The next result provides a partial converse to Proposition 4.1. For this result, we introduce the following assumption in place of A1 and A2.

A3. $f_{\rm c}(\cdot)$ is locally Lipschitz, $\mathscr{S}$ is closed, and $f_{\rm d}(x)\neq 0$, $x\in\mathscr{S}\setminus\partial\mathscr{S}$. If $x\in\partial\mathscr{S}$ such that $f_{\rm d}(x)=0$, then $f_{\rm c}(x)=0$. If $x\in\mathscr{S}$ such that $f_{\rm d}(x)\neq 0$, then $x+f_{\rm d}(x)\notin\mathscr{S}$.

Proposition 4.3. *Consider the nonlinear impulsive dynamical system* $\mathscr{G}$ *given by* (12), (13) *and assume* A3 *holds. If* $\mathscr{G}$ *satisfies the quasi-continuous dependence property given by Axiom* (iv), *then* $\tau_1(\cdot)$ *is lower-semicontinuous at every* $x\notin\mathscr{S}$. *Furthermore, for every* $x\notin\mathscr{S}$ *such that* $\tau_1(x)<\infty$, $\tau_1(\cdot)$ *is continuous at* x. *Finally, for every* $x\notin\mathscr{S}$ *such that* $\tau_1(x)=\infty$, $\tau_1(x_n)\to\infty$ *for every sequence* $\{x_n\}_{n=1}^\infty$ *such that* $x_n\to x$.

Proof. Assume $\mathscr{G}$ satisfies the quasi-continuous dependence property *iv*). Let $x\notin\mathscr{S}$ and $\{x_n\}_{n=1}^\infty\notin\mathscr{S}$ be such that $x_n\to x$ and $\tau_1(x_1)\geqslant\tau_1(x_2)\geqslant\cdots\geqslant\tau_-\triangleq\lim_{n\to\infty}\tau_1(x_n)$. First, assume $\tau_1(x_1)<\infty$ so that $\tau_-,\tau_1(x_2),\ldots<\infty$. Since $f_{\rm c}(\cdot)$ is locally Lipschitz it follows that $\psi(\cdot,\cdot)$ is jointly continuous and hence it follows that $\psi(\tau_1(x_n),x_n)\to\psi(\tau_-,x)$. Next, since $\mathscr{S}$ is closed and $\psi(\tau_1(x_n),x_n)\in\mathscr{S}$ for every $n=1,2,\ldots$, it follows that $\psi(\tau_-,x)\in\mathscr{S}$ which implies that $\tau_-\geqslant\tau_1(x)=\inf\{t\colon \psi(t,x)\in\mathscr{S}\}$, establishing the lower semicontinuity of $\tau_1(\cdot)$ at x. Alternatively, if $\tau_-=\infty$ so that $\tau_1(x_1)=\tau_1(x_2)=\cdots=\infty$, lower semicontinuity follows trivially since $\tau_1(x)\leqslant\tau_-=\infty$.

Next, note that since $f_{\rm c}(\cdot)$ is locally Lipschitz it follows that $\psi(t,x)$, $t\geqslant 0$, cannot converge to any equilibrium in a finite time and hence $f_{\rm c}(\psi(\tau_1(x),x))\neq 0$, which implies that $f_{\rm d}(\psi(\tau_1(x),x))\neq 0$. Let $\{x_n\}_{n=1}^\infty\notin\mathscr{S}$ be such that $x_n\to x$ and $\tau_1(x_1)\leqslant\tau_1(x_2)\leqslant\cdots\leqslant\tau_+\triangleq\lim_{n\to\infty}\tau_1(x_n)$. Suppose, ad absurdum, $\tau_+>\tau_1(x)$, let $\varepsilon>0$ be such that $\tau_1(x)<\tau_+-\varepsilon<\tau_2(x)$, and let $M>0$ be such that $\tau_+-\varepsilon<\tau_1(x_n)$, $n>M$. Now, since $\mathscr{G}$ satisfies the quasi-continuous dependence property (iv) it follows that $s(\tau_+-\varepsilon,x_n)\to s(\tau_+-\varepsilon,x)$ as $n\to\infty$, and for every $n\geqslant M$, $s(\tau_+-\varepsilon,x_n)=\psi(\tau_+-\varepsilon,x_n)$. Furthermore, $\lim_{n\to\infty}s(\tau_+-\varepsilon,x_n)=\lim_{n\to\infty}\psi(\tau_+-\varepsilon,x_n)=\psi(\tau_+-\varepsilon,x)$. Hence,

$$\begin{aligned}\psi(\tau_+-\varepsilon-\tau_1(x),\psi(\tau_1(x),x)+f_{\rm d}(\psi(\tau_1(x),x))&=s(\tau_+-\varepsilon,x)\\&=\lim_{n\to\infty}s(\tau_+-\varepsilon,x_n)\\&=\psi(\tau_+-\varepsilon,x)\\&=\psi(\tau_+-\varepsilon-\tau_1(x),\psi(\tau_1(x),x)).\end{aligned}$$

Now, since $f_{\rm c}(\cdot)$ is locally Lipschitz it follows that the solution $\psi(t,x)$, $t\in\mathbb{R}$, is unique both forward and backward in time and hence it follows that $\psi(\tau_1(x),x)=\psi(\tau_1(x),x)+f_{\rm d}(\psi(\tau_1(x),x))$, or, equivalently, $f_{\rm d}(\psi(\tau_1(x),x))=0$, which is a contradiction. Hence,

$\tau^+ \leqslant \tau_1(x)$ and thus $\tau_1(\cdot)$ is upper semicontinuous at x. Hence, $\tau_1(\cdot)$ is continuous at x.

Finally, let $x \notin \mathscr{S}$ be such that $\tau_1(x) = \infty$ and let $\{x_n\}_{n=1}^{\infty} \in \mathscr{S}$ be such that $x_n \to x$. Suppose, ad absurdum, that $\{\tau_1(x_n)\}_{n=1}^{\infty}$ has a bounded subsequence $\{\tau_1(x_{n_j})\}_{j=1}^{\infty}$. Let $\tau \triangleq \lim_{j\to\infty} \tau_1(x_{n_j}) < \infty$. Now, since $\psi(\cdot,\cdot)$ is jointly continuous it follows that $\lim_{j\to\infty} \psi(\tau_1(x_{n_j}),\ x_{n_j}) = \psi(\tau, x)$. Next, since $\mathscr{S}$ is closed and $\psi(\tau_1(x_{n_j}), x_{n_j}) \in \mathscr{S}$, $j = 1, 2, \ldots$, it follows that $\psi(\tau, x) \in \mathscr{S}$ which implies that $\tau_1(x) = \inf\{t\colon \psi(t,x) \notin \mathscr{S}\} \leqslant \tau < \infty$ which is a contradiction. Hence, $\lim_{n\to\infty} \tau_1(x_n) = \infty$. $\square$

The following result shows that all convergent Zeno solutions to (12), (13) converge to $\bar{\mathscr{S}} \setminus \mathscr{S}$ if A1 and A2 hold while all convergent Zeno solutions converge to an equilibrium point if A3 holds.

Proposition 4.4. *Consider the nonlinear impulsive dynamical system $\mathscr{G}$ given by (12), (13). If the trajectory $s(t,x_0)$, $t \geqslant 0$, to (12), (13) is convergent, bounded and Zeno; that is, there exists $\tau(x_0) < \infty$ such that $\tau_k(x_0) \to \tau(x_0)$ as $k \to \infty$ and $\lim_{k\to\infty} s(\tau_k(x_0)) = s(\tau(x_0), x_0)$, then the following statements hold:*

(i) *If* A1 *and* A2 *hold and $\tau_2(\cdot)$ is continuous on $\mathscr{S}$, then $s(\tau(x_0), x_0) \in \bar{\mathscr{S}} \setminus \mathscr{S}$.*
(ii) *If* A3 *holds, then $s(\tau(x_0), x_0)$ is an equilibrium point.*

Proof. If the trajectory $s(t,x_0)$, $t \geqslant 0$, is Zeno then there exists $\tau(x_0) < \infty$ such that $\tau_k(x_0) \to \tau(x_0)$ as $k \to \infty$ and, since $\tau_1(x_0) < \tau_2(x_0) < \cdots < \tau(x_0)$, it follows that $\tau_1(x_0) < \infty$. Next, note that there exists $y_1 \in \mathscr{S}$ such that $s(\tau_1(x_0), x_0) = y_1$ and hence it follows that $\tau_2(x_0) = \tau_1(x_0) + \tau_1(y_1 + f_{\mathrm{d}}(y_1)) = \tau_1(x_0) + \tau_2(y_1)$. By recursively repeating this procedure for $k = 3, 4, \ldots$, it follows that

$$\tau_k(x_0) = \tau_1(x_0) + \sum_{i=1}^{k-1} \tau_2(y_i),$$

where $y_i \triangleq s(\tau_i(x_0), x_0)$, $i = 1, 2, \ldots$. Now, since $\tau(x_0) = \lim_{k\to\infty} \tau_k(x_0)$ it follows that $\tau(x_0) = \tau_1(x_0) + \sum_{k=1}^{\infty} \tau_2(y_k)$. Hence, it follows that $\tau_2(y_k) \to 0$ as $k \to \infty$. Now, if the trajectory $s(t,x_0)$, $t \geqslant 0$, is bounded then the sequence $\{y_k\}_{k=0}^{\infty}$ is also bounded and it follows from the Bolzano–Weierstrass theorem [31] that there exists a convergent subsequence $\{y_{k_i}\}_{i=1}^{\infty}$ such that $\lim_{i\to\infty} y_{k_i} = y \in \bar{\mathscr{S}}$. Hence, since $s(\cdot, x_0)$ is left-continuous, it follows that $y = \lim_{i\to\infty} y_{k_i} = \lim_{i\to\infty} s(\tau_{k_i}(x_0), x_0) = s(\lim_{i\to\infty} \tau_{k_i}(x_0), x_0) = s(\tau(x_0), x_0)$.

(i) Now, assume A1, A2 hold and $\tau_2(\cdot)$ is continuous on $\mathscr{S}$. Next, ad absurdum, suppose $y \in \mathscr{S}$. Since $\tau_2(\cdot)$ is continuous on $\mathscr{S}$ it follows that $\tau_2(y) = \tau_2(\lim_{i\to\infty} y_{k_i}) = \lim_{i\to\infty} \tau_2(y_{k_i}) = 0$, which contradicts the fact that $\tau_2(x) > 0$, $x \in \mathscr{S}$ (see Remark 4.4). Thus $y \in \bar{\mathscr{S}} \setminus \mathscr{S}$ or, equivalently, $s(\tau(x_0), x_0) \in \bar{\mathscr{S}} \setminus \mathscr{S}$.

(ii) Finally, assume A3 holds. Furthermore, note that $y_k = \psi(\tau_2(y_{k-1}), y_{k-1} + f_{\mathrm{d}}(y_{k-1}))$, $k = 2, 3, \ldots$, and since $\psi(\cdot,\cdot)$ is jointly continuous and $\tau_2(y_k) \to 0$ as $k \to \infty$

it follows that

$$y = \lim_{k\to\infty} y_k = \psi\left(\lim_{k\to\infty} \tau_2(y_k), \lim_{k\to\infty} y_k + f_{\mathrm{d}}(y_k)\right)$$
$$= \psi(0, y + f_{\mathrm{d}}(y)) = y + f_{\mathrm{d}}(y),$$

which implies that $f_{\mathrm{d}}(y) = 0$. Now, since $\mathscr{S}$ is closed it follows that $y \in \mathscr{S}$ and since $f_{\mathrm{d}}(y) = 0$ it follows from A3 that $f_{\mathrm{c}}(y) = 0$ which proves the result. □

5. Invariant set theorems for state-dependent impulsive dynamical systems

In this section we use the results on left-continuous dynamical systems to state and prove *new* invariant set stability theorems for a class of nonlinear impulsive dynamical systems; namely, state-dependent impulsive dynamical systems [2,3,8,14,15,22]. Our main result characterizes impulsive dynamical system limit sets in terms of C^1 functions. In particular, we show that the system trajectories converge to an invariant set contained in a union of level surfaces characterized by the continuous-time dynamics and the resetting system dynamics. In this section, we assume that $f_{\mathrm{c}}(\cdot)$, $f_{\mathrm{d}}(\cdot)$, and $\mathscr{S}$ are such that the dynamical system $\mathscr{G}$ given by (12), (13) satisfies the quasi-continuous dependence property (iv) so that $\mathscr{G}$ is a left-continuous dynamical system.

Theorem 5.1. *Consider the nonlinear impulsive dynamical system* (12), (13), *assume* $\mathscr{D}_{\mathrm{c}} \subset \mathscr{D}$ *is a compact positively invariant set with respect to* (12), (13), *and assume that there exists a* C^1 *function* $V : \mathscr{D}_{\mathrm{c}} \to \mathbb{R}$ *such that*

$$V'(x) f_{\mathrm{c}}(x) \leqslant 0, \quad x \in \mathscr{D}_{\mathrm{c}}, \quad x \notin \mathscr{S}, \tag{18}$$

$$V(x + f_{\mathrm{d}}(x)) - V(x) \leqslant 0, \quad x \in \mathscr{D}_{\mathrm{c}}, \quad x \in \mathscr{S}. \tag{19}$$

Let $\mathscr{R} \triangleq \{x \in \mathscr{D}_{\mathrm{c}}\colon x \notin \mathscr{S},\ V'(x) f_{\mathrm{c}}(x) = 0\} \cup \{x \in \mathscr{D}_{\mathrm{c}}\colon x \in \mathscr{S},\ V(x + f_{\mathrm{d}}(x)) = V(x)\}$ *and let* $\mathscr{M}$ *denote the largest invariant set contained in* $\mathscr{R}$. *If* $x_0 \in \mathscr{D}_{\mathrm{c}}$, *then* $x(t) \to \mathscr{M}$ *as* $t \to \infty$.

Proof. The result follows from Theorem 3.2. Specifically, prior to the first resetting time, we can determine the value of $V(x(t))$ as

$$V(x(t)) = V(x(0)) + \int_0^t V'(x(\tau)) f_{\mathrm{c}}(x(\tau))\,\mathrm{d}\tau, \quad t \in [0, \tau_1(x_0)]. \tag{20}$$

Between consecutive resetting times $\tau_k(x_0)$ and $\tau_{k+1}(x_0)$, we can determine the value of $V(x(t))$ as its initial value plus the integral of its rate of change along the trajectory $x(t)$, that is,

$$V(x(t)) = V(x(\tau_k(x_0)) + f_{\mathrm{d}}(x(\tau_k(x_0)))) + \int_{\tau_k(x_0)}^t V'(x(\tau)) f_{\mathrm{c}}(x(\tau))\,\mathrm{d}\tau,$$
$$t \in (\tau_k(x_0), \tau_{k+1}(x_0)] \tag{21}$$

for $k = 1, 2, \ldots$. Adding and subtracting $V(x(\tau_k(x_0)))$ to and from the right-hand side of (21) yields

$$V(x(t)) = V(x(\tau_k(x_0))) + [V(x(\tau_k(x_0)) + f_{\mathrm{d}}(x(\tau_k(x_0))) - V(x(\tau_k(x_0)))] + \int_{\tau_k(x_0)}^{t} V'(x(\tau)) f_{\mathrm{c}}(x(\tau))\,\mathrm{d}\tau,\quad t \in (\tau_k(x_0), \tau_{k+1}(x_0)] \tag{22}$$

and in particular at time $\tau_{k+1}(x_0)$,

$$V(x(\tau_{k+1}(x_0))) = V(x(\tau_k(x_0)) + f_{\mathrm{d}}(x(\tau_k(x_0)))) + \int_{\tau_k(x_0)}^{\tau_{k+1}(x_0)} V'(x(\tau)) f_{\mathrm{c}}(x(\tau))\,\mathrm{d}\tau. \tag{23}$$

By recursively substituting (23) into (22) and ultimately into (20), we obtain for all $t \in (\tau_k(x_0), \tau_{k+1}(x_0)]$,

$$V(x(t)) - V(x(0)) = \int_0^t V'(x(\tau)) f_{\mathrm{c}}(x(\tau))\,\mathrm{d}\tau + \sum_{i=1}^{k} [V(x(\tau_i(x_0)) + f_{\mathrm{d}}(x(\tau_i(x_0))) - V(x(\tau_i(x_0))]. \tag{24}$$

Now, it follows from (18) and (19) that $V(x(t)) \leqslant V(x(0))$, $t \geqslant 0$. Using a similar argument it follows that $V(x(t)) \leqslant V(x(\tau))$, $t \geqslant \tau$. Hence, it follows from Theorem 3.2 that for every $x_0 \in \mathscr{D}_{\mathrm{c}}$ there exists $\gamma_{x_0} \in \mathbb{R}$ such that $\omega(x_0) \subseteq \mathscr{M}_{\gamma_{x_0}}$, where $\mathscr{M}_{\gamma_{x_0}}$ is the largest invariant set contained in $V^{-1}(\gamma_{x_0})$. Hence, $V(x) = \gamma_{x_0}$, $x \in \omega(x_0)$. Now, since $\mathscr{M}_{\gamma_{x_0}}$ is an invariant set it follows that for all $x(0) \in \mathscr{M}_{\gamma_{x_0}}$, $x(t) \in \mathscr{M}_{\gamma_{x_0}}$, $t \geqslant 0$, and thus $\dot{V}(x(t)) \triangleq \mathrm{d}V(x(t))/\mathrm{d}t = V'(x(t)) f_{\mathrm{c}}(x(t)) = 0$, for all $x(t) \notin \mathscr{S}$, and $V(x(t) + f_{\mathrm{d}}(x(t))) = V(x(t))$, for all $x(t) \in \mathscr{S}$. Thus, $\mathscr{M}_{\gamma_{x_0}}$ is contained in $\mathscr{M}$ which is the largest invariant set contained in $\mathscr{R}$. Hence, $x(t) \to \mathscr{M}$ as $t \to \infty$. $\square$

The following corollaries to Theorem 5.1 present sufficient conditions that guarantee local asymptotic stability of the nonlinear impulsive dynamical system (12), (13). Note that for addressing the stability of the zero solution of an impulsive dynamical system the usual stability definitions are valid.

Corollary 5.1. *Consider the nonlinear impulsive dynamical system* (12), (13), *assume* $\mathscr{D}_{\mathrm{c}} \subset \mathscr{D}$ *is a compact positively invariant set with respect to* (12), (13) *such that* $0 \in \mathring{\mathscr{D}}_{\mathrm{c}}$, *and assume there exists a* C^1 *function* $V : \mathscr{D}_{\mathrm{c}} \to \mathbb{R}$ *such that* $V(0) = 0$, $V(x) > 0$, $x \neq 0$, *and* (18), (19) *are satisfied. Furthermore, assume that the set* $\mathscr{R} \triangleq \{x \in \mathscr{D}_{\mathrm{c}}\colon x \notin \mathscr{S},\ V'(x) f_{\mathrm{c}}(x) = 0\} \cup \{x \in \mathscr{D}_{\mathrm{c}}\colon x \in \mathscr{S},\ V(x + f_{\mathrm{d}}(x)) = V(x)\}$ *contains no invariant set other than the set* $\{0\}$. *Then the zero solution* $x(t) \equiv 0$ *to* (12), (13) *is asymptotically stable and* $\mathscr{D}_{\mathrm{c}}$ *is a subset of the domain of attraction of* (12), (13).

Proof. Lyapunov stability of the zero solution $x(t) \equiv 0$ to (12), (13) follows from the fact that $V(x(t)) \leqslant V(x(\tau))$, $t \geqslant \tau$, using standard arguments. For details see [8,14].

Next, it follows from Theorem 5.1 that if $x_0 \in \mathcal{D}_c$, then $\omega(x_0) \subseteq \mathcal{M}$, where $\mathcal{M}$ denotes the largest invariant set contained in $\mathcal{R}$, which implies that $\mathcal{M} = \{0\}$. Hence, $x(t) \to \mathcal{M} = \{0\}$ as $t \to \infty$ establishing asymptotic stability of the zero solution $x(t) \equiv 0$ to (12), (13). □

Remark 5.1. Setting $\mathcal{D} = \mathbb{R}^n$ and requiring $V(x) \to \infty$ as $\|x\| \to \infty$ in Corollary 5.1, it follows that the zero solution $x(t) \equiv 0$ to (12), (13) is globally asymptotically stable. Similar remarks hold for Corollaries 5.2 and 5.3 below.

Corollary 5.2. *Consider the nonlinear impulsive dynamical system* (12), (13), *assume* $\mathcal{D}_c \subset \mathcal{D}$ *is a compact positively invariant set with respect to* (12), (13) *such that* $0 \in \mathring{\mathcal{D}}_c$, and assume there exists a C^1 function $V : \mathcal{D}_c \to \mathbb{R}$ such that $V(0) = 0$, $V(x) > 0$, $x \neq 0$,

$$V'(x) f_c(x) < 0, \quad x \in \mathcal{D}_c, \quad x \notin \mathcal{S}, \quad x \neq 0, \tag{25}$$

and (19) *is satisfied. Then the zero solution* $x(t) \equiv 0$ *to* (12), (13) *is asymptotically stable and* $\mathcal{D}_c$ *is a subset of the domain of attraction of* (12), (13).

Proof. It follows from (25) that $V'(x) f_c(x) = 0$ for all $x \in \mathcal{D}_c \setminus \mathcal{S}$ if and only if $x = 0$. Hence, $\mathcal{R} = \{0\} \cup \{x \in \mathcal{D}_c\colon x \in \mathcal{S},\ V(x + f_d(x)) = V(x)\}$ which contains no invariant set other than $\{0\}$. Now, the result follows as a direct consequence of Corollary 5.1. □

Corollary 5.3. *Consider the nonlinear impulsive dynamical system* (12), (13), *assume* $\mathcal{D}_c \subset \mathcal{D}$ *is a compact positively invariant set with respect to* (12), (13) *such that* $0 \in \mathring{\mathcal{D}}_c$, *and assume that for all* $x_0 \in \mathcal{D}_c$, $x_0 \neq 0$, *there exists* $\tau \geqslant 0$ *such that* $x(\tau) \in \mathcal{S}$, *where* $x(t)$, $t \geqslant 0$, *denotes the solution to* (12), (13) *with the initial condition* x_0. *Furthermore, assume there exists a* C^1 *function* $V : \mathcal{D}_c \to \mathbb{R}$ *such that* $V(0) = 0$, $V(x) > 0$, $x \neq 0$,

$$V(x + f_d(x)) - V(x) < 0, \quad x \in \mathcal{D}_c, \quad x \in \mathcal{S}, \tag{26}$$

and (18) *is satisfied. Then the zero solution* $x(t) \equiv 0$ *to* (12), (13) *is asymptotically stable and* $\mathcal{D}_c$ *is a subset of the domain of attraction of* (12), (13).

Proof. It follows from (26) that $\mathcal{R} = \{x \in \mathcal{D}_c\colon x \notin \mathcal{S},\ V'(x) f_c(x) = 0\}$. Since for all $x_0 \in \mathcal{D}_c$, $x_0 \neq 0$, there exists $\tau \geqslant 0$ such that $x(\tau) \in \mathcal{S}$ it follows that the largest invariant set contained in $\mathcal{R}$ is $\{0\}$. Now, the result follows as a direct consequence of Corollary 5.1. □

6. Illustrative examples

In this section we present two illustrative examples to demonstrate the results of this paper.

Example 6.1. Consider a bouncing ball, with coefficient of restitution $e \in (0,1)$, on a horizontal surface under a normalized gravitational field. Modeling the surface collisions as instantaneous, it follows from Newton's equations of motion that the bouncing ball dynamics are characterized by the state-dependent impulsive differential equations

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -\mathrm{sgn}(x_1) \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}, \quad (x_1(t), x_2(t)) \notin \mathscr{S}, \tag{27}$$

$$\begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ -(1+e)x_2(t) \end{bmatrix}, \quad (x_1(t), x_2(t)) \in \mathscr{S}, \tag{28}$$

where $t \geqslant 0$, $x_1(t)$, $x_2(t) \in \mathbb{R}$, $x_1(t) \geqslant 0$, $\mathrm{sgn}(x_1) \triangleq x_1/|x_1|$, $x_1 \neq 0$, $\mathrm{sgn}(0) \triangleq 0$, $\mathscr{S} = \{(x_1, x_2) \in \mathscr{D}\colon x_1 = 0,\ x_2 < 0\}$, and $\mathscr{D} = \{(x_1, x_2) \in \mathbb{R}^2\colon x_1 \geqslant 0\}$.

First, we use Proposition 4.1 to show that the impulsive dynamical system (27), (28) is a left-continuous dynamical system. Note that $\bar{\mathscr{S}} = \{(x_1, x_2) \in \mathbb{R}^2\colon x_1 = 0,\ x_2 \leqslant 0\}$ and hence $\bar{\mathscr{S}} \setminus \mathscr{S} = \{(0,0)\}$ is an invariant set with respect to the dynamical system $\mathscr{G}$ given by (27), (28). Next, it can be shown that

$$\tau_1(x_1, x_2) = \begin{cases} x_2 + \sqrt{x_2^2 + 2x_1}, & x_1 > 0, \\ 2x_2, & x_1 = 0, \quad x_2 > 0, \end{cases} \tag{29}$$

which shows that $\tau_1(x_1, x_2)$ is continuous for all $(x_1, x_2) \notin \bar{\mathscr{S}}$. Furthermore, it can be easily shown that for all $(x_1, x_2) \in \mathscr{D}$, the sequence $\{\tau_k(x_1, x_2)\}_{k=1}^{\infty}$ is a uniformly convergent sequence. Now, it follows from (ii) of Proposition 4.1 and Remark 4.5 that the dynamical system $\mathscr{G}$ given by (27), (28) is a left-continuous dynamical system.

Next, (27), (28) can be written in the form of (12), (13) with $x \triangleq [x_1\ x_2]^{\mathrm{T}}$, $f_{\mathrm{c}}(x) = [x_2, -\mathrm{sgn}(x_1)]^{\mathrm{T}}$, and $f_{\mathrm{d}}(x) = [0, -(1+e)x_2]^{\mathrm{T}}$. Now, consider the function $V\colon \mathbb{R}^2 \to \mathbb{R}$ given by $V(x) = x_1 + \frac{1}{2}x_2^2$ and note that $V'(x) f_{\mathrm{c}}(x) = 0$ for all $x \notin \mathscr{S}$. Furthermore, since $e \in (0,1)$ note that $V(x + f_{\mathrm{d}}(x)) = V(x)$ if and only if $x_2 = 0$. Hence, the set $\{(x_1, x_2) \in \mathscr{S}\colon V(x + f_{\mathrm{d}}(x)) = V(x)\} = \varnothing$ and the set $\mathscr{R} = \{(x_1, x_2) \in \mathbb{R}^2\colon x_1 \geqslant 0\} \setminus \mathscr{S}$. Now, note that the largest invariant set $\mathscr{M}$ contained in $\mathscr{R} = \{(x_1, x_2) \in \mathbb{R}^2\colon x_1 \geqslant 0\} \setminus \mathscr{S}$ is $\{(0,0)\}$, and hence since $V(x)$ is radially unbounded it follows from Theorem 5.1 that $(x_1(t), x_2(t)) \to (0,0)$ as $t \to \infty$. Alternatively, this can also be shown using Proposition 4.4. Specifically, it follows from Proposition 4.4 that $(x_1(t), x_2(t)) \to \bar{\mathscr{S}} \setminus \mathscr{S} = \{(0,0)\}$ as $t \to \tau(x_0)$ and since $\bar{\mathscr{S}} \setminus \mathscr{S} = \{(0,0)\}$ is an invariant set it follows that $(x_1(t), x_2(t)) \to \{(0,0)\}$ as $t \to \infty$.

Example 6.2. Consider the state-dependent impulsive differential equations

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} x_3(t) \\ x_4(t) \\ x_1(t) - 2x_4(t) - \dfrac{x_1(t)}{\sqrt{x_1^2(t) + x_2^2(t)}} \\ x_2(t) + 2x_3(t) - \dfrac{x_2(t)}{\sqrt{x_1^2(t) + x_2^2(t)}} \end{bmatrix}, \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \\ x_{30} \\ x_{40} \end{bmatrix}, x(t) \notin \bar{\mathscr{S}}, \tag{30}$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} -x_2(t) \\ x_1(t) \\ -x_4(t) \\ x_3(t) \end{bmatrix}, \quad x(t) \in \bar{\mathscr{S}} \setminus \mathscr{S}, \tag{31}$$

$$\begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \\ \Delta x_3(t) \\ \Delta x_4(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -(1+e)(x_2(t) + x_3(t)) \\ (1+e)(x_1)(t) - x_4(t)) \end{bmatrix}, \quad x(t) \in \mathscr{S}, \tag{32}$$

where $t \geqslant 0$, $x_1(t), x_2(t), x_3(t), x_4(t) \in \mathbb{R}$, $e \in (0,1)$, $\mathscr{S} = \{x \in \mathscr{D}_c\colon x_1^2 + x_2^2 = 1,\ x_1x_3 + x_2x_4 < 0\}$, $x \triangleq [x_1\ x_2\ x_3\ x_4]^{\mathrm{T}}$, and $\mathscr{D} = \mathscr{D}_c = \{x \in \mathbb{R}^4\colon x_1^2 + x_2^2 \geqslant 1,\ x_1x_4 - x_2x_3 = x_1^2 + x_2^2\}$. First, note that $\bar{\mathscr{S}} = \{x \in \mathscr{D}_c\colon x_1^2 + x_2^2 = 1,\ x_1x_3 + x_2x_4 \leqslant 0\}$ and hence $\bar{\mathscr{S}} \setminus \mathscr{S} = \{x \in \mathscr{D}_c\colon x_1^2 + x_2^2 = 1,\ x_1x_3 + x_2x_4 = 0\}$ which can be shown to be an invariant set with respect to the dynamical system $\mathscr{G}$ given by (30)–(32). Furthermore, note that $\mathscr{D}_c$ is an invariant set with respect to the impulsive dynamics (30)–(32). To see this, consider the function $\phi(x) \triangleq x_1x_4 - x_2x_3 - x_1^2 + x_2^2$ and note that $\dot{\phi}(x)$ along the solutions to (30) is zero and $\phi(x + \Delta x) - \phi(x) = 0$, for all $x \in \mathscr{D}_c$.

Next, we use Proposition 4.1 to show that the dynamical system (30)–(32) is a left-continuous dynamical system. To see this, note that it can be shown that

$$\tau_1(x) = \begin{cases} \dfrac{x_1x_3 + x_2x_4 + \sqrt{2(x_1^2 + x_2^2)(\sqrt{x_1^2 + x_2^2} - 1) + (x_1x_3 + x_2x_4)^2}}{\sqrt{x_1^2 + x_2^2}}, & x_1^2 + x_2^2 > 1, \\ 2(x_1x_3 + x_2x_4), & x_1^2 + x_2^2 = 1,\ x_1x_3 + x_2x_4 > 0, \end{cases} \tag{33}$$

which shows that $\tau_1(x)$ is continuous for all $x \notin \bar{\mathscr{S}}$. Furthermore, it can be easily shown that for all $x \in \mathscr{D}$, the sequence $\{\tau_k(x)\}_{k=1}^{\infty}$ is a uniformly convergent sequence. Now, it follows from (ii) of Proposition 4.1 and Remark 4.5 that the dynamical system $\mathscr{G}$ given by (30)–(32) is a left-continuous dynamical system.

Next, (30)–(32) can be written in the form of (12),(13) with

$$f_c(x) = \begin{bmatrix} x_3 \\ x_4 \\ x_1 - 2x_4 - \dfrac{x_1}{\sqrt{x_1^2 + x_2^2}} \\ x_2 + 2x_3 - \dfrac{x_2}{\sqrt{x_1^2 + x_2^2}} \end{bmatrix}, \quad x \notin \bar{\mathscr{S}}, \quad f_c(x) = \begin{bmatrix} -x_2 \\ x_1 \\ -x_4 \\ x_3 \end{bmatrix}, \quad x \in \bar{\mathscr{S}} \setminus \mathscr{S},$$

and

$$f_d(x) = \begin{bmatrix} 0 \\ 0 \\ (1+e)(x_2 - x_3) \\ (1+e)(x_1 - x_4) \end{bmatrix}.$$

Now, consider the function $V : \mathscr{D}_c \to \mathbb{R}$ given by

$$V(x) = (x_1^2 + x_2^2)^{1/2} + \frac{1}{2}\frac{(x_1x_3 + x_2x_4)^2}{x_1^2 + x_2^2}$$

and note that $V'(x)f_c(x) = 0$ for all $x \notin \mathscr{S}$. Furthermore, since $e \in (0,1)$ note that $V(x + f_d(x)) = V(x)$ if and only if $x_1x_3 + x_2x_4 = 0$. Hence, the set $\{x \in \mathscr{S}\colon V(x + f_d(x)) = V(x)\} = \varnothing$ and the set $\mathscr{R} = \mathscr{D}_c \setminus \mathscr{S}$. Now, note that the largest invariant set $\mathscr{M}$ contained in $\mathscr{R} = \mathscr{D}_c \setminus \mathscr{S}$ is $\{x \in \mathscr{D}_c\colon x_1^2 + x_2^2 = 1,\ x_1x_3 + x_2x_4 = 0\}$, and hence it follows from Theorem 5.1 that the solution $x(t)$, $t \geqslant 0$, to (30)–(32) approaches the invariant set $\{x \in \mathscr{D}_c\colon x_1^2 + x_2^2 = 1,\ x_1x_3 + x_2x_4 = 0\}$ as $t \to \infty$ for all initial conditions contained in $\mathscr{D}_c$. Finally, Fig. 1 shows the phase portrait of the states x_1 versus x_2 for the initial condition $[x_1(0)\ x_2(0)\ x_3(0)\ x_4(0)]^T = [2\ 0\ 0\ 2]^T \in \mathscr{D}_c$. Alternatively, this can also be shown using Proposition 4.4. Specifically, it follows from Proposition 4.4 that $(x_1(t), x_2(t)) \to \bar{\mathscr{S}} \setminus \mathscr{S}$ as $t \to \tau(x_0)$ and since $\bar{\mathscr{S}} \setminus \mathscr{S}\}$ is an invariant set it follows that $(x_1(t), x_2(t)) \to \{x \in \mathscr{D}_c\colon x_1^2 + x_2^2 = 1,\ x_1x_3 + x_2x_4 = 0\}$ as $t \to \infty$.

7. Conclusion

An invariance principle was developed for left-continuous dynamical systems. As a special case of this result new invariant set stability theorems were established for nonlinear impulsive dynamical systems. These results provide generalizations to previous stability conditions developed in the literature as well as allow the investigation of limit cycles and periodic orbits of impulsive dynamical systems.

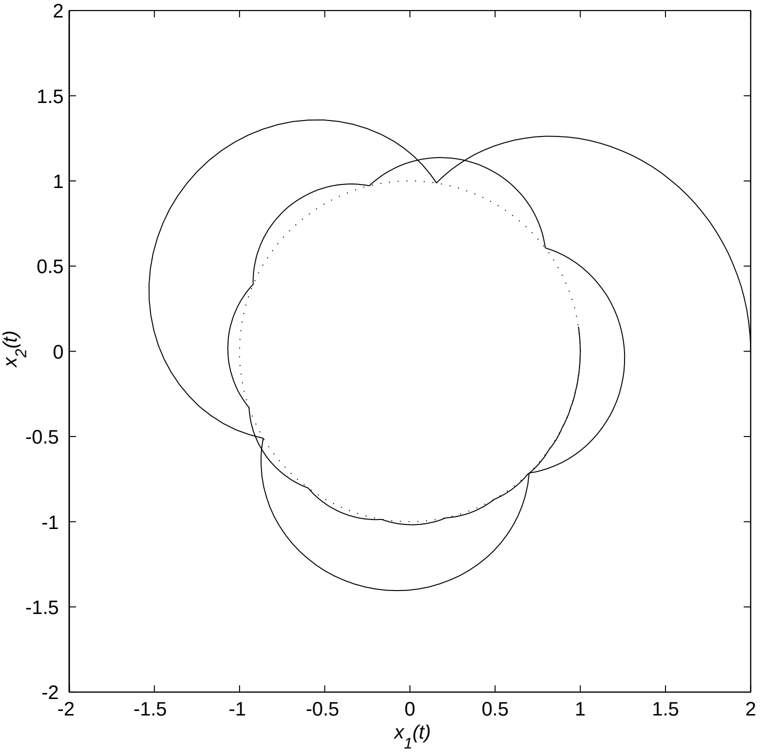

Fig. 1. Phase portrait of x_1 versus x_2.

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