

AN INVARIANT FOR SINGULAR KNOTS

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ABSTRACT. In this paper we introduce a Jones-type invariant for singular knots, using a Markov trace on the Yokonuma–Hecke algebras $Y_{d,n}(u)$ and the theory of singular braids. The Yokonuma–Hecke algebras have a natural topological interpretation in the context of framed knots. Yet, we show that there is a homomorphism of the singular braid monoid SB_n into the algebra $Y_{d,n}(u)$. Surprisingly, the trace does not normalize directly to yield a singular link invariant, so a condition must be imposed on the trace variables. Assuming this condition, the invariant satisfies a skein relation involving singular crossings, which arises from a quadratic relation in the algebra $Y_{d,n}(u)$.

1. INTRODUCTION

A *singular link on n components* is the image of a smooth immersion of n copies of the circle in S^3 , that has finitely many singularities, called *singular crossings*, which are all ordinary double points. So, a singular link is like a classical link, but with a finite number of transversal self-intersections permitted. A singular link on one component is a *singular knot*. Some examples of singular knots and links are given in Figure 8. We shall say ‘knots’ throughout meaning ‘knots and links’.

Two singular links K_1, K_2 are *isotopic*, that is, topologically equivalent, if there is an orientation preserving self-homeomorphism of S^3 carrying one to the other, such that it preserves a small rigid disc around each singular crossing (rigid-vertex isotopy). In terms of diagrams, K_1, K_2 are isotopic if and only if any two diagrams of theirs differ by planar isotopy and a finite sequence of the classical and the singular Reidemeister moves. In Figure 1 we illustrate the main two singular Reidemeister moves. The others are the obvious variants of these, with different crossings.

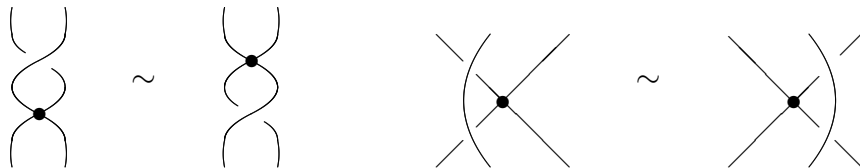


FIGURE 1. The singular Reidemeister moves

By their definition, singular links may admit a well-defined orientation on each component. Then the isotopy moves are considered with all possible orientations. It is a

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well-known fact that every isotopy invariant \mathcal{L} of classical oriented links extends to an invariant of singular oriented links, by means of the rule:

$$\mathcal{L}(L_\times) = \mathcal{L}(L_+) - \mathcal{L}(L_-)$$

where L_+, L_- and L_\times are identical diagrams, except for the place of one crossing, where it is positive, negative or singular respectively (see Figure 7).

A *singular braid on n strands* is the image of a smooth immersion of n arcs in S^3 , that has finitely many singularities, the singular crossings, which are all ordinary double points, such that the ends are arranged into n collinear top endpoints and into n collinear bottom endpoints and such that there are no local maxima or minima. So, a singular braid is like a classical braid, but with a finite number of singular crossings allowed.

Two singular braids are isotopic if there is a rigid-vertex isotopy taking one to the other, which fixes the endpoints of the strands and preserves the braid structure. Algebraically, the set of singular braids on n strands, denoted SB_n , forms a monoid with the usual concatenation of braids, the so-called *singular braid monoid*. It was introduced in different contexts by Baez[1], Birman[2] and Smolin[12]. SB_n is generated by the unit, by the classical elementary braids σ_i with their inverses, and by the corresponding elementary singular braids τ_i (view Figure 2):

$$1, \sigma_1, \dots, \sigma_{n-1}, \sigma_1^{-1}, \dots, \sigma_{n-1}^{-1}, \tau_1, \dots, \tau_{n-1}$$

which satisfy the relations below. These reflect precisely the singular braid isotopy.

$$(1) \quad \begin{array}{llll} \sigma_i \sigma_i^{-1} & = & \sigma_i^{-1} \sigma_i & = 1 & \text{for all } i \\ [\sigma_i, \sigma_j] & = & [\sigma_i, \tau_j] & = [\tau_i, \tau_j] = 0 & \text{for } |i - j| > 1 \\ [\sigma_i, \tau_i] & = & 0 & & \text{for all } i \\ \sigma_i \sigma_j \sigma_i & = & \sigma_j \sigma_i \sigma_j & & \text{for } |i - j| = 1 \\ \sigma_i \sigma_j \tau_i & = & \tau_j \sigma_i \sigma_j & & \text{for } |i - j| = 1. \end{array}$$



FIGURE 2. The elementary braids σ_i and τ_i

The singular braid monoid SB_n embeds in a group, the singular braid group, see [3].

The *closure* of a singular braid is defined like the ordinary closure of a classical braid, whereby we join the endpoints of corresponding strands by simple arcs. The closure of a singular braid ω shall be denoted $\widehat{\omega}$. In analogy to the classical setting, oriented singular links may be isotoped to closed singular braids. For a proof of Alexander's theorem for singular links see [2].

Let now $\cup_n SB_n$ denote the inductive limit associated to the natural monomorphisms of monoids $SB_n \hookrightarrow SB_{n+1}$. In analogy to the Markov theorem for classical braids, Gemein proved in [4] the following result (compare also with [10] for an L -move version).

Theorem 1 (Gemein, 1997). *Two singular braids in $\cup_n SB_n$ have isotopic closures if and only if they differ by singular braid relations and a finite sequence of the following moves:*

- (i) *Real conjugation:* $\sigma_i \omega \sim \omega \sigma_i$, $\omega, \sigma_i \in SB_n$
- (ii) *Singular commuting:* $\tau_i \omega \sim \omega \tau_i$, $\omega, \tau_i \in SB_n$
- (iii) *Real stabilization:* $\omega \sim \omega \sigma_n^{\pm 1}$, $\omega \in SB_n$

Moves (i) and (iii) of Theorem 1 are the two well-known moves of the classical Markov theorem for classical braids. Move (ii) is illustrated in Figure 3.

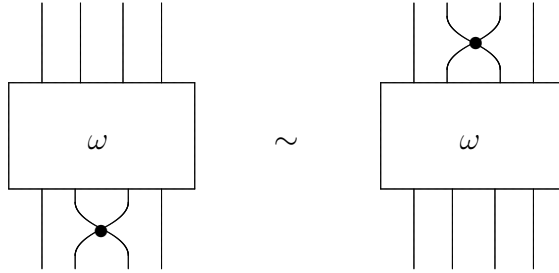


FIGURE 3. Singular commuting

Using the Alexander and Markov theorems for singular links and braids, it is possible to construct singular link invariants via Markov traces on quotient algebras. Baez[1] introduced the Vassiliev algebra as the quotient of $\mathbb{C} SB_n \otimes \mathbb{C}(\epsilon)$ by the ideal generated by the expressions $\sigma_i - \sigma_i^{-1} - \epsilon \tau_i$, which give rise to the following relations in the algebra:

$$g_i - g_i^{-1} = \epsilon \tau_i$$

He then showed that \mathbb{C} -valued Vassiliev–Gussarov invariants are in one-to-one correspondence with homogeneous (in ϵ) Markov traces on the algebra.

More recently, Paris and Rabenda[11] defined the singular Hecke algebra as the quotient of the algebra $\mathbb{C}(q)[SB_n]$ by the ideal generated by the well-known quadratic relations $\sigma_i^2 - (q - 1)\sigma_i - q 1$ of the classical Iwahori–Hecke algebra of type A , which give rise to the following relations in the algebra:

$$g_i - q g_i^{-1} = (q - 1) 1$$

They also constructed on these algebras singular Markov traces, from which –upon normalization, according to Theorem 1– they derived the universal HOMFLYPT (2-variable Jones polynomial) analogue for oriented singular links. Kauffman and Vogel[9] constructed analogues of the HOMFLYPT and the Kauffman polynomial for singular links by using diagrammatic methods. For their HOMFLYPT analogue it is shown in [11] that it is a specialization of the universal HOMFLYPT for singular links (but, as observed in [11], this specialization does not make much difference).

In the present paper we construct an invariant for singular links using the Yokonuma–Hecke algebras and Markov traces defined on them. The Yokonuma–Hecke algebra $Y_{d,n}(u)$ can be defined as a quotient of the modular framed braid group algebra $\mathcal{CF}_{d,n}$ (classical

framed braids with framings modulo d) by the quadratic relations $g_i^2 = 1 + (u-1)e_i(1-g_i)$ (here also σ_i corresponds to g_i). The elements e_i are certain idempotents in $\mathbb{C}\mathcal{F}_{d,n}$ and they are expressions of the framing generators t_i, t_{i+1} , see (4). Further, in [6] Juyumaya constructed a linear Markov trace on the Yokonuma–Hecke algebra $Y_{d,n}(u)$, that is, a linear trace which supports the Markov property: $\text{tr}(ag_n) = z \text{tr}(a)$ for $z \in \mathbb{C}$ and for any $a \in Y_{d,n}(u)$, see Theorem 3. For details and topological interpretations of the above we refer the reader to [7] and [8], where the algebras $Y_{d,n}(u)$ and the traces in [6] are used in the context of classical and p -adic framed braids and framed links.

Here, we first map homomorphically the singular braid monoid SB_n into the algebra $Y_{d,n}(u)$ via the map:

$$\begin{aligned} \delta : SB_n &\longrightarrow Y_{d,n}(u) \\ \sigma_i &\mapsto g_i \\ \tau_i &\mapsto p_i = e_i(1 - g_i) \end{aligned}$$

In the image $\delta(SB_n)$ the following relations hold:

$$(2) \quad g_i - g_i^{-1} = (u^{-1} - 1)p_i$$

For an illustration of the corresponding quadratic relations (Eq. 7) see Figure 4. We then consider the Markov trace on $Y_{d,n}(u)$ constructed in [6]. So, we obtain a map from the singular braid monoid SB_n to the complex numbers. A normalization of this trace according to the singular braid equivalence of Theorem 1 should yield a singular link invariant. But this turns out not to be the case (not even for the restriction to classical braids and links). In fact, we need to impose a condition on the variables of the trace, the ‘ E -condition’, see Definition 1. Surprisingly, there are non-trivial sets of complex numbers satisfying the E -condition. Given now the E -condition we normalize the trace to obtain an invariant Δ of singular links (Theorem 5).

To the best of our knowledge, the algebra $Y_{d,n}(u)$ is the first example of an algebra, which can admit so different topological interpretations: in the context of framed braids as well as in the context of singular braids. Also, the Markov trace in [6] is the only Markov trace we know of that does not normalize directly to yield a link invariant. Finally, it is worth adding that, for modulus d equal to 1, we have $e_i = 1$ and the algebra $Y_{1,n}(u)$ coincides with the classical Iwahori–Hecke algebra of type A . Also, the trace in [6] coincides with the Ocneanu trace [5].

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2. THE YOKONUMA–HECKE ALGEBRA AND A MARKOV TRACE

2.1. *Relations in $Y_{d,n}(u)$.* We fix a $u \in \mathbb{C} \setminus \{0, 1\}$. The Yokonuma–Hecke algebra, denoted by $Y_{d,n}(u)$, is a \mathbb{C} -associative algebra generated by the elements

$$1, g_1, \dots, g_{n-1}, t_1, \dots, t_n$$

subject to the following relations:

$$\begin{aligned} g_i g_j &= g_j g_i && \text{for } |i - j| > 1 \\ g_i g_j g_i &= g_j g_i g_j && \text{for } |i - j| = 1 \\ t_i t_j &= t_j t_i && \text{for all } i, j \\ t_j g_i &= g_i t_{s_i(j)} && \text{for all } i, j \\ t_j^d &= 1 && \text{for all } j \end{aligned}$$

where $s_i(j)$ is the result of applying the transposition $s_i = (i, i + 1)$ to j , together with the extra quadratic relations:

$$(3) \quad g_i^2 = 1 + (u - 1) e_i - (u - 1) e_i g_i \quad \text{for all } i$$

where

$$(4) \quad e_i := \frac{1}{d} \sum_{m=0}^{d-1} t_i^m t_{i+1}^{-m}$$

The first four relations are defining relations for the classical framed braid group, with the t_j 's being interpreted as the 'elementary framings' (framing 1 on the j th strand). The relations $t_j^d = 1$ mean that the framing of each strand is regarded modulo d . So, the algebra $Y_{d,n}(u)$ arises naturally as a quotient of the modular framed braid group algebra over the quadratic relations (3). But in the present paper we shall give a different topological interpretation to $Y_{d,n}(u)$, in relation to singular knots and links.

It is easily verified that the elements e_i are idempotents. Also, that the elements g_i are invertible in $Y_{d,n}(u)$. Indeed:

$$(5) \quad g_i^{-1} = g_i - (u^{-1} - 1) e_i + (u^{-1} - 1) e_i g_i$$

As noted in the Introduction, $d = 1$ implies $e_i = 1$ and $p_i = 1 - g_i$. So $g_i^2 = u + (1 - u)g_i$, and the algebra $Y_{1,n}(u)$ coincides with the Iwahori–Hecke algebra of type A . For more details on the algebra $Y_{d,n}(u)$ and for further topological interpretations see [7, 8] and references therein. In $Y_{d,n}(u)$ we have the following relations.

Lemma 1. *For the elements e_i and for $1 \leq i, j \leq n - 1$ the following relations hold:*

$$\begin{aligned} e_i e_j &= e_j e_i \\ e_i g_i &= g_i e_i \\ e_i g_j &= g_j e_i && \text{for } |i - j| > 1 \\ e_j g_i g_j &= g_i g_j e_i && \text{for } |i - j| = 1 \end{aligned}$$

Proof. The first three claims are easy to check (see Lemma 4 and Proposition 5 in [7]). We will check the last one. Let $j = i + 1$. From the defining relations in the algebra $Y_{d,n}(u)$ we have:

$t_{i+1} g_i g_{i+1} = g_i t_i g_{i+1} = g_i g_{i+1} t_i$. Similarly, $t_{i+2} g_i g_{i+1} = g_i g_{i+1} t_{i+1}$. Then

$$e_{i+1} g_i g_{i+1} = \frac{1}{d} \sum_{m=0}^{d-1} t_{i+1}^m t_{i+2}^{-m} g_i g_{i+1} = \frac{1}{d} \sum_{m=0}^{d-1} g_i g_{i+1} t_i^m t_{i+1}^{-m} = g_i g_{i+1} e_i.$$

The proof for $j = i - 1$ is completely analogous. □

2.2. *The elements p_i .* For $1 \leq i \leq n - 1$ we define the elements $p_i \in Y_{d,n}(u)$ by the formula:

$$(6) \quad p_i = e_i(1 - g_i)$$

Then, the quadratic relations (3) in the algebra $Y_{d,n}(u)$ may be rewritten as:

$$(7) \quad g_i^2 = 1 + (u - 1)p_i$$

Proposition 1. *For the elements p_i and for $1 \leq i, j \leq n - 1$, we have the relations:*

$$\begin{aligned} e_i p_i &= p_i e_i = p_i \\ p_i^k &= (u + 1)^{k-1} p_i \quad \text{for } k \in \mathbb{N} \\ g_i p_i &= p_i g_i = -u p_i \\ g_i p_j &= p_j g_i \quad \text{for } |i - j| > 1 \\ p_i p_j &= p_j p_i \quad \text{for } |i - j| > 1 \\ p_j g_i g_j &= g_i g_j p_i \quad \text{for } |i - j| = 1 \end{aligned}$$

Proof. The proofs follow from Eq. 6, from Lemma 1 and by direct computations. For example, we shall check the second relation. For $k = 2$ we have $p_i^2 = e_i(1 - g_i)e_i(1 - g_i) = e_i^2(1 - g_i)^2 = e_i(1 - 2g_i + g_i^2) = e_i(1 - 2g_i + 1 + (u - 1)p_i) = 2e_i(1 - g_i) + (u - 1)e_i p_i$. Then, by the first relation we have $p_i^2 = 2p_i + (u - 1)p_i = (u + 1)p_i$. For any $k > 2$ we apply induction. \square

Note that the elements p_i are not invertible in $Y_{d,n}(u)$. We now define for fixed $a \in \mathbb{C}$ the following map.

$$(8) \quad \begin{aligned} \delta_a : SB_n &\longrightarrow Y_{d,n}(u) \\ \sigma_i &\mapsto a g_i \\ \tau_i &\mapsto p_i \end{aligned}$$

In particular, we shall denote:

$$\delta := \delta_1$$

Theorem 2. *The map δ_a defines a monoid homomorphism.*

Proof. The proof follows immediately by comparing relations (1) in SB_n with the relations in Proposition 1. \square

2.3. *Topological interpretations.* We shall now give topological interpretations for the elements of the subalgebra $\delta(SB_n)$ of the algebra $Y_{d,n}(u)$. By Theorem 2, monomials in g_i, g_i^{-1}, p_i may be viewed as singular braids, such that g_i, g_i^{-1} correspond respectively to σ_i, σ_i^{-1} (for $a = 1$) and p_i corresponds to the singular crossing τ_i . These elementary braids are subject to the quadratic relations in Eq. 7. These relations are illustrated in Figure 4, where, for simplicity, we omit the identity strands. Multiplying Eq. 7 by g_i^{-1} and using Proposition 1 we obtain the equivalent relation (2).

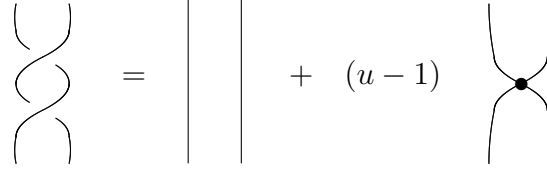


FIGURE 4. The quadratic relations involve singular crossings

Beyond relations (1), the images of the generators of SB_n under the map δ are also subject to the extra relations $p_i^k = (u + 1)^{k-1}p_i$ and $p_i g_i = -u p_i$ of Proposition 1. These are illustrated in Figures 5 and 6, where the identity strands are also omitted.

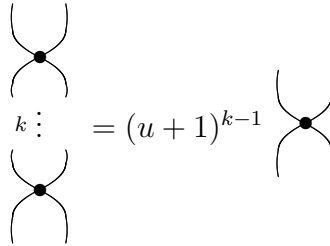


FIGURE 5. The relation $p_i^k = (u + 1)^{k-1}p_i$

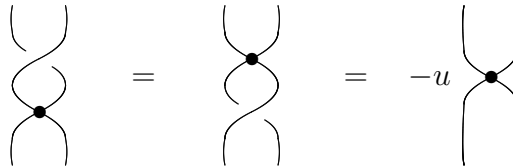


FIGURE 6. The relation $g_i p_i = p_i g_i = -u p_i$

Note that in this topological set-up there are no obvious interpretations for the generators t_i and the elements e_i .

2.4. *A Markov trace on $Y_{d,n}(u)$.* Let now $\cup_n Y_{d,n}(u)$ denote the inductive limit associated to the natural inclusions $Y_{d,n}(u) \subset Y_{d,n+1}(u)$. In [6] the following theorem is proved.

Theorem 3 (Juyumaya, 2004). *Let z, x_1, \dots, x_{d-1} be in \mathbb{C} . There exists a unique linear map tr on $\cup_n Y_{d,n}(u)$ with values in \mathbb{C} satisfying the rules:*

$$\begin{aligned} \text{tr}(ab) &= \text{tr}(ba) \\ \text{tr}(1) &= 1 \\ \text{tr}(ag_n) &= z \text{tr}(a) && (a \in Y_{d,n}(u)) \\ \text{tr}(at_{n+1}^m) &= x_m \text{tr}(a) && (a \in Y_{d,n}(u), 1 \leq m \leq d-1). \end{aligned}$$

As noted in the Introduction, for $d = 1$ the trace restricts to the first three rules and it coincides with Ocneanu’s trace on the Iwahori–Hecke algebra, which was used to construct the 2–variable Jones polynomial for classical knots and links, see [5].

3. THE E -CONDITION AND AN INVARIANT FOR SINGULAR KNOTS

In view of Theorems 1, 2 and 3 we would like to construct an isotopy invariant for singular knots and links. According to Theorem 1, such an invariant has to agree on the singular links $\widehat{\omega}$, $\widehat{\omega\sigma_n}$ and $\widehat{\omega\sigma_n^{-1}}$, for any $\omega \in SB_n$. Now, having present the recipe of Jones[5] for constructing the (2-variable) Jones polynomial for classical knots, we will try to define an invariant by re-scaling and normalizing the trace tr . By Eq. 5 we have:

$$\text{tr}(\omega g_n^{-1}) = \text{tr}(\omega g_n) - (u^{-1} - 1)\text{tr}(\omega e_n) + (u^{-1} - 1)\text{tr}(\omega e_n g_n)$$

In order that the invariant agrees on the closures of the braids $\omega\sigma_n^{-1}$ and $\omega\sigma_n$ we need that $\text{tr}(\omega g_n^{-1})$ factorizes through $\text{tr}(\omega)$. For the first term we have: $\text{tr}(\omega g_n) = z \text{tr}(\omega)$. Further:

$$(9) \quad \text{tr}(\omega e_n g_n) = \frac{1}{d} \sum_{m=0}^{d-1} \text{tr}(\omega t_n^m t_{n+1}^{-m} g_n) = \frac{1}{d} \sum_{m=0}^{d-1} z \text{tr}(\omega) = z \text{tr}(\omega)$$

since $\text{tr}(\omega t_n^m t_{n+1}^{-m} g_n) = \text{tr}(\omega t_n^m g_n t_n^{-m}) = z \text{tr}(\omega t_n^m t_n^{-m}) = z \text{tr}(\omega)$.

3.1. *The E -condition.* From the above analysis it is clear that tr needs to satisfy also the following multiplicative property:

$$(10) \quad \text{tr}(\omega e_n) = \text{tr}(e_n) \text{tr}(\omega)$$

Unfortunately, we do not have such a nice formula for $\text{tr}(\omega e_n)$. The underlying reason on the framed braid level (that is, for the natural interpretation for elements in $Y_{d,n}(u)$) is that e_n involves the n th strand of ω . Yet, by imposing some conditions on the indeterminates x_i it is possible to have property (10). Before giving these conditions let us define the following elements in $Y_{d,n}(u)$:

$$e_i^{(m)} := \frac{1}{d} \sum_{s=0}^{d-1} t_i^{m+s} t_{i+1}^{-s} \quad \text{and} \quad e_i := e_i^{(0)}$$

Also, the corresponding elements in $\mathbb{C}[z, x_1, \dots, x_{d-1}]$:

$$\zeta^{(m)} := \frac{1}{d} \sum_{s=0}^{d-1} x_{s+m} x_{d-s} = \text{tr}(e_i^{(m)}) \quad \text{and} \quad \zeta := \zeta^{(0)} = \text{tr}(e_i)$$

where the sub-indices of the indeterminates are regarded modulo d .

Definition 1. We shall say that the set $\mathcal{X}_d := \{x_1, \dots, x_{d-1}\}$ of complex numbers satisfies the E -condition if it satisfies the following system of $d-1$ non-linear equations in \mathbb{C} :

$$\zeta^{(m)} = x_m \zeta \quad (1 \leq m \leq d-1)$$

Or, equivalently:

$$(11) \quad \sum_{s=0}^{d-1} x_{m+s} x_{d-s} = x_m \sum_{s=0}^{d-1} x_s x_{d-s} \quad (1 \leq m \leq d-1)$$

where the sub-indices on the x_j 's are regarded modulo d and $x_0 = x_d := 1$.

Surprisingly, there exist non-trivial sets \mathcal{X}_d satisfying the E -condition. For example, taking $x_i = \theta^i$, where θ is a primitive d th root of unity. We note that for this solution we have $\zeta = \text{tr}(e_j) = 1$ and $\text{tr}(p_j) = 1 - z$. For $d = 3, 4$ and 5 we run the Mathematica program and we found other solutions of the E -system, for which:

$$\text{tr}(e_j) \neq 1.$$

For example in the case $d = 3$, where we have the E -system:

$$\begin{aligned} x_1 + x_2^2 &= 2x_1^2x_2 \\ x_1^2 + x_2 &= 2x_1x_2^2 \end{aligned}$$

we have the non-trivial solutions:

$$x_1 = x_2 = -\frac{1}{2} \quad \text{or} \quad x_1 = \frac{1}{3} \left(\frac{1}{3} - \frac{3i\sqrt{3}}{4} \right), \quad x_2 = \frac{1}{4} (1 + i\sqrt{3})$$

Also, the solution where we take the conjugates in the previous one. Another more interesting example is the set formed by the elements

$$x_i := \frac{-(-1)^{i(d-1)}}{d-1} \quad (1 \leq i \leq d-1)$$

We then have $\zeta = \text{tr}(e_j) = 1/(d-1)$. For explanations about the somewhat ‘mysterious’ E -condition and for a thorough discussion on the solutions of the E -system we refer the reader to [8].

3.2. A singular link invariant. We are now close to our aim. Indeed, assuming the E -condition we have the following.

Theorem 4. *If \mathcal{X}_d satisfies the E -condition, then for all $\omega \in Y_{d,n}(u)$ we have*

$$\text{tr}(\omega e_n) = \text{tr}(e_n) \text{tr}(\omega) = \zeta \text{tr}(\omega).$$

Proof. See [8]. □

Corollary 1. *If \mathcal{X}_d satisfies the E -condition, then for all $\omega \in Y_{d,n}(u)$ we have*

$$\text{tr}(\omega p_n) = \text{tr}(p_n) \text{tr}(\omega) = (\zeta - z) \text{tr}(\omega).$$

Proof. By (10) we have: $\text{tr}(\omega p_n) = \text{tr}(\omega e_n(1 - g_n)) = \text{tr}(\omega e_n) - \text{tr}(\omega e_n g_n)$. So, by Theorem 4 and by (9): $\text{tr}(\omega p_n) = (\zeta - z) \text{tr}(\omega) = \text{tr}(p_n) \text{tr}(\omega)$. □

We now proceed with the construction of our invariant. From the definition of SB_n , any element ω in SB_n can be written as

$$\omega_1^{\epsilon_1} \omega_2^{\epsilon_2} \dots \omega_m^{\epsilon_m},$$

where $\omega_j \in \{\sigma_i, \tau_i; 1 \leq i \leq n-1\}$ and $\epsilon_i = +1$ or -1 . If $\omega_j = \tau_j$ we set $\epsilon_j := +1$.

Definition 2. The *exponent* $\epsilon(\omega)$ of ω is defined as the sum $\epsilon_1 + \dots + \epsilon_m$. Since SB_n embeds in a group [3], $\epsilon(\omega)$ is well-defined.

Let now $\mathcal{X}_d = \{x_1, \dots, x_{d-1}\}$ be a set satisfying the E -condition and let \mathcal{S} be the set of oriented singular links. We define the following map on the set \mathcal{S} .

Definition 3. Let $\omega \in SB_n$. We define the map Δ on the closure $\widehat{\omega}$ of ω as follows:

$$\Delta(\widehat{\omega}) := \left(\frac{1 - \lambda u}{\sqrt{\lambda}(1 - u)\zeta} \right)^{n-1} (\text{tr} \circ \delta_{\sqrt{\lambda}})(\omega)$$

where:

$$\lambda := \frac{z - (1 - u)\zeta}{uz}$$

Equivalently, setting

$$D := \frac{1 - \lambda u}{\sqrt{\lambda}(1 - u)\zeta}$$

we can write:

$$\Delta(\widehat{\omega}) = D^{n-1}(\sqrt{\lambda})^{\epsilon(\omega)} \text{tr}(\delta(\omega))$$

For the definitions of $\delta_{\sqrt{\lambda}}$ and δ recall (8).

Theorem 5. *Assuming the E-condition, Δ is an isotopy invariant for oriented singular links.*

Proof. We need to show that Δ is well-defined on isotopy classes of oriented singular links. According to Theorem 1, it suffices to prove that Δ is consistent with moves (i), (ii) and (iii). From the facts that $\epsilon(\omega\omega') = \epsilon(\omega'\omega)$ and $\text{tr}(ab) = \text{tr}(ba)$, it follows that Δ respects moves (i) and (ii). Let now $\omega \in SB_n$. Then $\omega\sigma_n \in SB_{n+1}$ and $\epsilon(\omega\sigma_n) = \epsilon(\omega) + 1$. Hence:

$$\Delta(\widehat{\omega\sigma_n}) = D^n(\sqrt{\lambda})^{\epsilon(\omega\sigma_n)} \text{tr}(\delta(\omega\sigma_n)) = D^n(\sqrt{\lambda})^{\epsilon(\omega)+1} \text{tr}(\delta(\omega)g_n) = D\sqrt{\lambda}z \Delta(\widehat{\omega})$$

where we used that $\text{tr}(\delta(\omega)g_n) = z \text{tr}(\delta(\omega))$. Now $z = \frac{(1-u)\zeta}{1-\lambda u}$, so $D\sqrt{\lambda}z = 1$. Therefore, $\Delta(\widehat{\omega\sigma_n}) = \Delta(\widehat{\omega})$. Finally, we will prove that $\Delta(\widehat{\omega\sigma_n^{-1}}) = \Delta(\widehat{\omega})$. Indeed:

$$\Delta(\widehat{\omega\sigma_n^{-1}}) = D^n(\sqrt{\lambda})^{\epsilon(\omega\sigma_n^{-1})} \text{tr}(\delta(\omega\sigma_n^{-1})) = D^n(\sqrt{\lambda})^{\epsilon(\omega)-1} \text{tr}(\delta(\omega)g_n^{-1})$$

Resolving g_n^{-1} from Eq. 5 we obtain:

$$\Delta(\widehat{\omega\sigma_n^{-1}}) = D^n(\sqrt{\lambda})^{\epsilon(\omega)-1} [z - (u^{-1} - 1)\zeta + (u^{-1} - 1)z] \text{tr}(\delta(\omega))$$

Also, from Theorem 4 and Eq. 9 we have:

$$\text{tr}(\delta(\omega)e_n) = \zeta \text{tr}(\delta(\omega)) \quad \text{and} \quad \text{tr}(\delta(\omega)e_n g_n) = z \text{tr}(\delta(\omega)).$$

Therefore:

$$\Delta(\widehat{\omega\sigma_n^{-1}}) = D^n(\sqrt{\lambda})^{\epsilon(\omega)-1} \frac{z + (u - 1)\zeta}{u} \text{tr}(\delta(\omega)) = \frac{D}{\sqrt{\lambda}} \frac{z + (u - 1)\zeta}{u} \Delta(\widehat{\omega}) = \Delta(\widehat{\omega})$$

Move (iii) of Theorem 1 is now checked and the proof is concluded. \square

Remark 1. The invariant Δ is not of finite type. Indeed, take for example the link $\widehat{\tau_1^k}$, which contains k singular crossings. By the second relation of Proposition 1, we have: $\Delta(\widehat{\tau_1^k}) = (\sqrt{\lambda})^k (u+1)^{k-1} (\zeta - z)$, which is not equal to zero for all k . Of course, it would be interesting to consider an exponential variable change, and see if the coefficients become invariants of finite type as, for example, in the case of the Jones polynomial.

3.3. *Skein relations.* Let L_+ , L_- and L_\times be diagrams of three oriented singular links, which are identical, except near one crossing, where they are as follows:

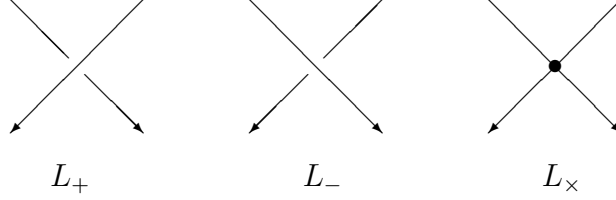


FIGURE 7. L_+ , L_- and L_\times

Then we have the following result.

Proposition 2. *The invariant Δ satisfies the following skein relation:*

$$(12) \quad \frac{1}{\sqrt{\lambda}} \Delta(L_+) - \sqrt{\lambda} \Delta(L_-) = \frac{u^{-1} - 1}{\sqrt{\lambda}} \Delta(L_\times)$$

Proof. The proof is standard. By the Alexander theorem for singular braids we may assume that L_+ is in braided form and that $L_+ = \widehat{\beta\sigma_i}$ for some $\beta \in SB_n$. Also that $L_- = \widehat{\beta\sigma_i^{-1}}$ and $L_\times = \widehat{\beta\tau_i}$. From the definition of Δ and by Theorem 5 we have:

$$\frac{1}{\sqrt{\lambda}} \Delta(L_+) - \sqrt{\lambda} \Delta(L_-) = D^{n-1} (\sqrt{\lambda})^{\epsilon(\beta)} (\text{tr}(\delta(\beta\sigma_i)) - \text{tr}(\delta(\beta\sigma_i^{-1})))$$

Now,

$$\begin{aligned} \text{tr}(\delta(\beta\sigma_i)) - \text{tr}(\delta(\beta\sigma_i^{-1})) &= \text{tr}(\delta(\beta)g_i) - \text{tr}(\delta_1(\beta)g_i^{-1}) \\ &= \text{tr}(\delta(\beta)(g_i - g_i^{-1})) \\ &= (u^{-1} - 1) \text{tr}(\delta(\beta)p_i) \quad (\text{from Eq. 2}) \\ &= (u^{-1} - 1) \text{tr}(\delta(\beta\tau_i)) \end{aligned}$$

Finally, substituting $(\sqrt{\lambda})^{\epsilon(\beta)} = (\sqrt{\lambda})^{-1} (\sqrt{\lambda})^{\epsilon(\beta\tau_i)}$ we deduce:

$$\frac{1}{\sqrt{\lambda}} \Delta(L_+) - \sqrt{\lambda} \Delta(L_-) = \frac{D^{n-1}}{\sqrt{\lambda}} (\sqrt{\lambda})^{\epsilon(\beta\tau_i)} (u^{-1} - 1) \text{tr}(\delta(\beta)p_i) = \frac{u^{-1} - 1}{\sqrt{\lambda}} \Delta(L_\times)$$

Thus the proof is concluded. \square

3.4. *Computations.* In this subsection we compute the values of the invariant Δ on some basic classical and singular knots, assuming always the E -condition. The singular ones are illustrated in Figure 8. We shall first give some formulas that are useful for computations. For powers of g_i we can easily deduce by induction the following formulae.

Lemma 2. *Let $m \in \mathbb{Z}, k \in \mathbb{N}$. (i) For m positive, define:*

$\alpha_m = (u - 1) \sum_{l=0}^{k-1} u^{2l}$ if $m = 2k$ and $\beta_m = u(u - 1) \sum_{l=0}^{k-1} u^{2l}$ if $m = 2k + 1$. Then:

$$g_i^m = \begin{cases} 1 + \alpha_m p_i & \text{if } m = 2k \\ g_i - \beta_m p_i & \text{if } m = 2k + 1 \end{cases}$$

(ii) For m negative, define:

$$\alpha'_m = u^{-1}(u^{-1} - 1) \sum_{l=0}^{k-1} u^{-2l} \text{ if } m = -2k \text{ and } \beta'_m = (u^{-1} - 1) \sum_{l=0}^{k-1} u^{-2l} \text{ if } m = -2k + 1.$$

Then:

$$g_i^m = \begin{cases} 1 + \alpha'_m p_i & \text{if } m = -2k \\ g_i - \beta'_m p_i & \text{if } m = -2k + 1 \end{cases}$$

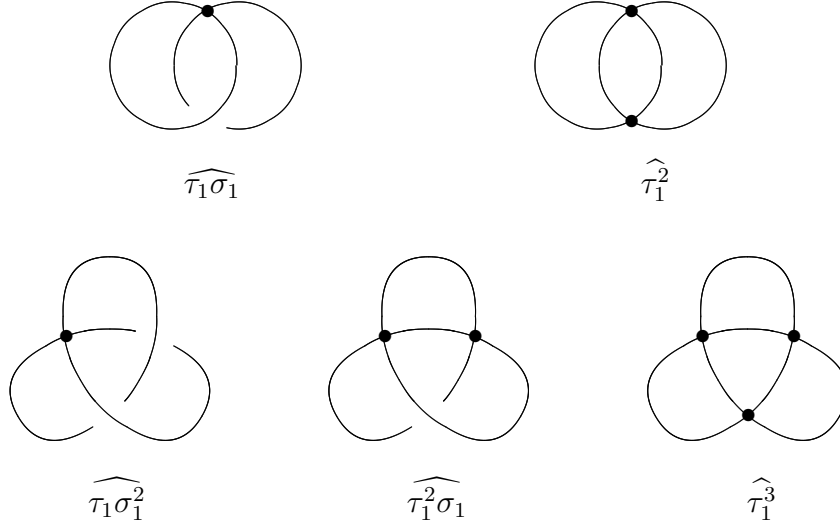


FIGURE 8. Examples of singular knots and links

We now proceed with our computations.

- Clearly, for the unknot O , $\Delta(O) = 1$.
- Let $K_1 = \widehat{\tau_1}$. Then $e(\tau_1) = 1$, so $\Delta(K_1) = D\sqrt{\lambda} \text{tr}(p_1) = D\sqrt{\lambda} [\text{tr}(e_1) - \text{tr}(e_1 g_1)] = D\sqrt{\lambda} [\zeta - z]$. Then:

$$\Delta(K_1) = \frac{\zeta - z}{z}$$

- Let $H = \widehat{\sigma_1^2}$, the Hopf link. We have $\text{tr}(g_1^2) = \text{tr}(1 + (u+1)p_1) = 1 + (u+1)(\zeta - z)$ and $e(\sigma_1^2) = 2$. Then:

$$\Delta(H) = \frac{1 - \lambda u}{(1 - u)\zeta} \sqrt{\lambda} (1 + (u+1)(\zeta - z)) = z^{-1} \sqrt{\lambda} (1 + (u+1)(\zeta - z)).$$

- Let $H_1 = \widehat{\sigma_1 \tau_1}$. We have $\text{tr}(g_1 p_1) = -u \text{tr}(p_1) = -u(\zeta - z)$ and $e(\sigma_1 \tau_1) = 2$. Then:

$$\Delta(H_1) = -\frac{1 - \lambda u}{(1 - u)\zeta} \sqrt{\lambda} u(\zeta - z) = z^{-1} \sqrt{\lambda} u(\zeta - z).$$

- Let $H_2 = \widehat{\tau_1^2}$. We have $e(\tau_1^2) = 2$. Then $\Delta(H_2) = D\lambda \text{tr}(p_1^2) = D\lambda (u+1) \text{tr}(p_1)$. So:

$$\Delta(H_2) = \frac{1 - \lambda u}{(1 - u)\zeta} \sqrt{\lambda} (u+1)(\zeta - z) = z^{-1} \sqrt{\lambda} (u+1)(\zeta - z).$$

- Let $T = \widehat{\sigma_1^3}$, the right-handed trefoil. We have $g_1^3 = g_1 - u(u-1)e_1 + u(u-1)e_1g_1$ from (Lemma 2). Hence: $\text{tr}(g_1^3) = z - u(u-1)\zeta + u(u-1)z$. Moreover $e(\sigma_1^3) = 3$. Then, using that $1 - \lambda u = z^{-1}\zeta(1 - u)$, we obtain:

$$\begin{aligned}\Delta(T) &= D(\sqrt{\lambda})^3 [(u(u-1) + 1)z - u(u-1)\zeta] \\ &= \frac{\lambda}{z} [(u(u-1) + 1)z - u(u-1)\zeta].\end{aligned}$$

- Let $T_1 = \widehat{\tau_1\sigma_1^2}$. We have $p_1g_1^2 = -up_1g_1 = u^2p_1$ so, $\text{tr}(p_1g_1^2) = u^2(\zeta - z)$. Moreover, $e(\tau_1\sigma_1^2) = 3$. Then

$$\Delta(T_1) = D(\sqrt{\lambda})^3 \text{tr}(p_1g_1^2) = \frac{u^2\lambda}{z}(\zeta - z).$$

- Let $T_2 = \widehat{\tau_1^2\sigma_1}$. We have $p_1^2g_1 = -u(u+1)p_1$ so, $\text{tr}(p_1^2g_1) = -u(u+1)(\zeta - z)$. Moreover, $e(\tau_1^2\sigma_1) = 3$. Then:

$$\Delta(T_2) = D(\sqrt{\lambda})^3 \text{tr}(p_1^2g_1) = \frac{-u(u+1)\lambda}{z}(\zeta - z).$$

- Let $T_3 = \widehat{\tau_1^3}$. We have $p_1^3 = (u+1)^2p_1$, so $\text{tr}(p_1^3) = (u+1)^2(\zeta - z)$. Moreover, $e(\tau_1^3) = 3$. Then:

$$\Delta(T_3) = D(\sqrt{\lambda})^3 \text{tr}(p_1^3) = \frac{(u+1)^2\lambda}{z}(\zeta - z).$$

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