J. Group Theory 5 (2002), 177–197

Journal of Group Theory

An invariant of elements of finite order in semisimple simply connected algebraic groups

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(Communicated by J. S. Wilson)

This paper is a continuation of the paper [14], henceforth cited as I. Let k be a field of characteristic $p \geqslant 0$ and k_s a separable closure of k. Let G/k be a semisimple simply connected algebraic group which is assumed absolutely almost k-simple, i.e. $G \times_k k_s$ is almost simple. Let $g \in G(k)$ be an element of order n. The goal of the paper is to define and study an invariant $M_G(g)$ with value in $K_2(k)/nK_2(k)$, where $K_2(k)$ denotes the second Milnor K-group of k. This invariant is characteristic-free and therefore it permits us to work in an arithmetical set-up.

The case when $n = p = \mathrm{char}(k) > 0$ and G is split is especially interesting, and an element of order p is then unipotent. In [28], Tits constructed explicit examples of anisotropic unipotent elements, i.e. unipotent elements which do not belong to any proper k-parabolic subgroup of G, and a large part of this paper is devoted to the computation of the invariant for such elements. For the split groups of type G_2 (resp. F_4, E_8) and $n = p = 2$ (resp. 3, 5), we show that this invariant classifies conjugacy classes of anisotropic elements of order p, and it turns out that any such element is conjugate to one of the anisotropic elements constructed by Tits. In particular, we prove the remaining case of Theorem I.3, i.e. that any anisotropic unipotent element of order 5 in the split group E_8/k normalizes a maximal k-split torus.

0 Notation

We use the following notation throughout.

${}_nA$ and A/n denote the kernel and the cokernel respectively of the multiplication map $A \xrightarrow{\times n} A$ by an integer n in an abelian group A.

$H^i(k, M)$ is the ith Galois cohomology group for some Galois sheaf M on $\mathrm{Spec}(k)$ (see [23]), and $\mathrm{Br}(k) = H^2(k, \mathbb{G}_m)$ is the Brauer group of k.

μ_n is the Galois sheaf of nth roots of unity (for $(n, \mathrm{char}(k)) = 1$).

$H^3(k)$ is the Kato cohomology group defined on the p-primary part with logarithmic differentials (see [17]) and on the tame part by

$$H^3(k)\{l\} = \varinjlim H^3(k, \mu_{l^n}^{\otimes 2})$$

for any prime l invertible in k.
$K_2(R)$ is the 2-group of Milnor K-theory for a commutative ring R with unit (see [3]), and the symbol $\{a_1, a_2\} \in K_2(R)$ for $a_1, a_2 \in R^\times$. Furthermore, for any positive integer n, there is a cup product

$$\cup : H^1(k, \mathbb{Z}/n\mathbb{Z}) \times K_2(k)/n \to H^3(k).$$

For a scheme X, $\mathcal{K}_2$ is the sheaf on X_{Zar} associated to the presheaf $U \to K_2(O_X(U))$.

If G is semisimple and absolutely almost simple, we denote by $S(G)$ the finite set of torsion primes of G as defined by Serre in [24], and by d_G the Dynkin index of G (cf. [19, §2], [10, Appendix 6]). Note that prime factors of d_G belongs to $S(G)$. We recall that an element g of $G(k)$ of finite order is called *k-good* if it lies in the radical of some k-parabolic subgroup of G and k-bad otherwise; and that g is *k-anisotropic* if g does not belong to any proper parabolic subgroup of G.

1 Definition of the invariant $M_G(g)$

1.1. Let $g \in G(k)$ be an element of order n. We view g also as a morphism $g : \mathbb{Z}/n\mathbb{Z} \to G(k)$ mapping 1 to g. We consider the Brylinski–Deligne central extension [8]

$$0 \to K_2(k) \to E_k \to G(k) \to 1. \qquad (\mathscr{E})$$

If G is split and not of type C_n (resp. of type C_n), this extension is (resp. is 2 times) the Steinberg–Matsumoto extension of $G(k)$ defined in [20]. We denote by $[\mathscr{E}] \in H^2(G(k), K_2(k))$ the class of this extension. We define $M_G(g) \in K_2(k)/nK_2(k)$ by

$$M_G(g) := g^*([\mathscr{E}]) \in H^2(\mathbb{Z}/n\mathbb{Z}, K_2(k)) \xrightarrow{\sim} K_2(k)/nK_2(k).$$

1.2 Functoriality and specialization. Because of the way our invariant is constructed, functoriality properties of Brylinski–Deligne extension have counterparts for the invariant itself. First, as Brylinski–Deligne extension is functorial in k, for any field extension k'/k, one has $M_{G_{k'}}(g_{k'}) = M_G(g)_{k'}$.

Next, we discuss change of group, which is especially interesting. Let $f : G \to G'$ be a morphism with G' also a semisimple simply connected absolutely almost k-simple group. Recall that $H^1_{\mathrm{Zar}}(G, \mathcal{K}_2) = \mathbb{Z}$, so that the map

$$f^* : \mathbb{Z} = H^1_{\mathrm{Zar}}(G', \mathcal{K}_2) \to H^1_{\mathrm{Zar}}(G, \mathcal{K}_2) = \mathbb{Z}$$

is the multiplication by an integer d_f which is the Dynkin index of f (see [10, Appendix B6]). Denoting by $\mathscr{E}'$ the Brylinski–Deligne extension of $G'(k)$ by $K_2(k)$

and by $f_k : G(k) \to G'(k)$ the induced map on k-points, one has $[f_k^* \mathscr{E}'] = d_f \times [\mathscr{E}]$ in $H^2(G(k), K_2(k))$. The behaviour of M_G is thus given by

$$M_{G'}(g) = d_f \times M_G(g).$$

The specialization property of a Brylinski–Deligne extension will be helpful. Let A be a complete discrete valuation ring with residue field k and field of fractions F_A. Let $\mathfrak{G}/\mathrm{Spec}(A)$ be a semisimple simply connected group scheme with special fiber G/k and generic fiber $\mathfrak{G}_{F_A}$. Then there exist an extension of $\mathfrak{G}(A)$ by $K_2(A)$ and a commutative diagram [8, Proposition 5.2.8]

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_2(F_A) & \longrightarrow & E_{F_A} & \longrightarrow & \mathfrak{G}(F_A) & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & K_2(A) & \longrightarrow & \mathfrak{E} & \longrightarrow & \mathfrak{G}(A) & \longrightarrow & 1 \\
 & & {\scriptstyle \mathrm{sp}}\downarrow & & {\scriptstyle \mathrm{sp}}\downarrow & & {\scriptstyle \mathrm{sp}}\downarrow & & \\
0 & \longrightarrow & K_2(k) & \longrightarrow & E_k & \longrightarrow & G(k) & \longrightarrow & 1.
\end{array}$$

If $g \in \mathfrak{G}(A)$ is an element of order n whose specialization $\bar{g}$ has order n, then by viewing g as an element $\mathfrak{G}(F_A)$, we see that $M_{\mathfrak{G}_{F_A}}(g)$ belongs to $K_2(A)/nK_2(A) \subset K_2(F_A)/nK_2(F_A)$, so that $\mathrm{sp}(M_{\mathfrak{G}_{F_A}}(g)) = M_G(\bar{g})$.

2 Link with Galois cohomology

We set $K = k((t))$ and denote by K_{mod} a maximal tamely ramified extension of K. We assume that there exists a character $\chi \in H^1(K, \mathbb{Z}/n\mathbb{Z})$ such that the corresponding extension L/K is totally ramified (this hypothesis is satisfied if either $n = p = \mathrm{char}(k)$ or k contains a primitive nth root of unity). As in [13, 14], one defines the 1-cocycle

$$f_\chi(g) := g_*(\chi) \in Z^1(\mathbb{Z}/n\mathbb{Z}, G(L)) \subset Z^1(K, G),$$

and the corresponding cohomology class

$$\gamma_\chi(g) = [f_\chi(g)] \in H^1(K, G).$$

2.1 A formula. We denote by $r_K : H^1(K, G) \to H^3(K)$ the Rost invariant of G/K as defined in [10, Appendix B]. The Rost invariant of $\gamma_\chi(g)$ is given by the following formula.

Lemma 1. *$r_K(\gamma_\chi(g)) = \chi \cup M_G(g)$ in $H^3(K)$.*

Proof. We denote by $\eta_K^L : H^1(\mathbb{Z}/n\mathbb{Z}, G(L)) \to H^2(\mathbb{Z}/n\mathbb{Z}, K_2(L))$ the boundary map associated to the Brylinski–Deligne extension $1 \to K_2(L) \to E_L \to G(L) \to 1$. Lemma 5 of [12] yields the following (-1)-commutative diagram

$$\begin{array}{ccc} H^1(\mathbb{Z}/n\mathbb{Z}, G(L)) & \xrightarrow{r_K} & \mathrm{Ker}(H^3(K) \to H^3(L)) \\ & \searrow^{\eta_K^L} & \downarrow a_K^L \\ & & H^2(\mathbb{Z}/n\mathbb{Z}, K_2(L)), \end{array}$$

where

$$a_K^L : \mathrm{Ker}(H^3(K) \to H^3(L)) \to H^2(\mathbb{Z}/n\mathbb{Z}, K_2(L))$$

is the map introduced by Kahn in [16]. We denote by θ the generator of $H^2(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z})$ and we prove first that

$$a_K^L(r_K(\gamma_\chi(g))) = -\theta \cup M_G(g) \in H^2(\mathbb{Z}/n\mathbb{Z}, K_2(L)).$$

From the commutative diagram of $\mathbb{Z}/n\mathbb{Z}$-groups

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_2(k) & \longrightarrow & E_k & \longrightarrow & G(k) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K_2(L) & \longrightarrow & E_L & \longrightarrow & G(L) & \longrightarrow & 1, \end{array}$$

we get the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, G(k)) & \xrightarrow{\partial} & K_2(k)/nK_2(k) \\ \| & & \| \\ H^1(\mathbb{Z}/n\mathbb{Z}, G(k)) & \longrightarrow & H^2(\mathbb{Z}/n\mathbb{Z}, K_2(k)) \\ \downarrow & & \downarrow \\ H^1(\mathbb{Z}/n\mathbb{Z}, G(L)) & \xrightarrow{\eta_K^L} & H^2(\mathbb{Z}/n\mathbb{Z}, K_2(L)). \end{array}$$

The map

$$K_2(k) \to K_2(k)/nK_2(k) \to H^2(\mathbb{Z}/n\mathbb{Z}, K_2(k))$$

is precisely the multiplication by θ, and $\partial(g) = M_G(g)$ by definition, so that one gets

$$a_K^L(\gamma_\chi(g)) = -\theta \cup M_G(g).$$

To finish the proof, we discuss the p-primary and p'-primary parts of the invariant.

The p-primary part. Lemma 2 of [12] shows that

$$a_L^K(\chi \cup h_{p,F}(\{x,y\})) = \theta \cup \mathrm{Res}_K^L(\{x,y\}) \in H^2(\mathbb{Z}/n\mathbb{Z}, K_2(L)).$$

Because a_K^L is an isomorphism on the p-primary part (see [16]), the formula holds.

The p′-primary part. We consider the commutative exact diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^3(k,\mu_m^{\otimes 2}) & \longrightarrow & H^3(K,\mu_m^{\otimes 2}) & \xrightarrow{\partial_K} & {}_m\mathrm{Br}(k) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \mathrm{Res}_K^L} & & \downarrow{\scriptstyle \times m} & & \\
0 & \longrightarrow & H^3(k,\mu_m^{\otimes 2}) & \longrightarrow & H^3(L,\mu_m^{\otimes 2}) & \xrightarrow{\partial_L} & {}_m\mathrm{Br}(k) & \longrightarrow & 0.
\end{array}$$

The classes $r_{K,G}(\beta)$ and $\chi \cup M_G(g)$ have similar residues and are killed by L, and so a diagram chase shows that they are equal.

The formula established in Lemma 1 proves that the invariant $M_G(g)$ is an obstruction to the property of being good.

Corollary 1. *If g is k-good, then $M_G(g) = 0$ in $K_2(k)/nK_2(k)$.*

Proof. In the tame case, if g is k-good, then $\gamma_\chi(g) = 1$ by Proposition I.3.b, so that $\chi \cup M_G(g) = 0$ in $H^3(K)$ by Lemma 1, and $M_G(g) = 0$ by applying the residue map

$$H^3(K,\mu_n^{\otimes 2}) \to H^2(k,\mu_n) \approx K_2(k)/n,$$

where the last isomorphism is the Merkurjev–Suslin Theorem [21]. In the wild case, by a result of Izhboldin [15, Theorem D], the cup-product by χ induces an isomorphism

$$K_2(K)/N_{L/K}(K_2(L)) \xrightarrow{\sim} \mathrm{Ker}(H^3(K) \to H^3(L))$$

and a diagram chase shows that $K_2(O)/N_{L/K}(K_2(O_L)) \xrightarrow{\sim} K_2(K)/N_{L/K}(K_2(L))$, so that

$$K_2(k)/nK_2(k) \hookrightarrow \mathrm{Ker}(H^3(K) \to H^3(L)).$$

If g is k-good, then $\gamma_\chi(g) = 1$ by Proposition I.3.a, so that $\chi \cup M_G(g) = 0$ by Lemma 1 and $M_G(g) = 0$.

This formula and Proposition I.3.b give the following corollary.

Corollary 2. *Assume that the Rost invariant*

$$r_{K_{\mathrm{mod}}} : H^1(K_{\mathrm{mod}}, G) \to H^3(K_{\mathrm{mod}})\{p\}$$

has trivial kernel. Then for any unipotent element u of order p, the following assertions are equivalent:

(i) $M_G(u) = 0$ *in* $K_2(k)/pK_2(k)$;

(ii) *u is k-good.*

2.2 Case-by-case consequences for unipotent elements. In this subsection, we assume that $G/\mathbb{Z}$ is a simply connected almost simple Chevalley group of type X. To use Corollary 2, we need to know whether or not the Rost invariant has trivial kernel for fields whose Galois group is a pro-p group, where p is a torsion prime of G. Let us recall what is known about this kernel.

Theorem 1. (a) *The Rost invariant* $H^1(k, G) \to H^3(k)$ *has trivial kernel for* $X = D_4$, B_4, G_2, F_4, E_6 *and* E_7.

(b) *Assume that* $\mathrm{Gal}(k_s/k)$ *is a pro-5 group. Then the Rost invariant* $H^1(k, E_8) \to H^3(k)$ *has trivial kernel.*

By the main theorem of [12], it is sufficient to consider the characteristic 0 case. Cases D_4, B_4 are reformulations of Arason's theorem [1, Satz 5.6] and cases G_2, F_4 are known (cf. [24] and [18, §40]). Cases E_6 and E_7 are proved by Garibaldi [11] and case E_8 is due to Chernousov [9]. However, in the E_8 case, the kernel is not trivial in general as shown in the Appendix below.

Corollary 3. *In each of the cases*

(1) $p = 2$, $X = G_2$, B_3, D_4, B_4, F_4, E_6, E_7,

(2) $p = 3$, $X = F_4$, E_6, E_7,

(3) $p = 3$, $X = E_8$,

for any unipotent element u of order p in $G(k)$, *the following conditions are equivalent*:

(i) $M_G(u) = 0$ *in* $K_2(k)/pK_2(k)$;

(ii) *u is k-good.*

The case of bad unipotent elements in spinor groups in characteristic 2, which can be deduced easily from the case of orthogonal groups [26, §4.3.3] shows that Corollary 3 is not true in general; more precisely, there exist bad unipotent elements in the split group Spin_{16} with trivial invariant.

Lemma 2. *Assume that* $G = G_2$ (*resp.* F_4, E_8) *and that* $p = 2$ (*resp.* 3, 5). *Let u be a unipotent element in* $G(k)$ *of order p. The following are equivalent*:

(i) *u is k-bad*;

(ii) *u is k-anisotropic*;

(iii) $M_G(u) \neq 0$ *in* $K_2(k)/pK_2(k)$.

Proof. The equivalence of (i) and (iii) is already in Theorem 1 and (ii) $\Rightarrow$ (i) is obvious. So we have to prove that if u is k-isotropic, then $M_G(u) = 0$. Assume that u is k-isotropic. We keep the notation of the beginning of Section 2. By Proposition I.5, the class $\gamma_\chi(u)$ in $H^1(K, G)$ is isotropic, so that there exist a parabolic group $P/\mathbb{Z}$ of $G/\mathbb{Z}$ and a Levi subgroup $Z/\mathbb{Z}$ of $P/\mathbb{Z}$ such that $\gamma_\chi(g) \in \operatorname{Im}(H^1(k, P) \to H^1(K, G))$ and as the map $H^1(K, DZ) \to H^1(K, P)$ is surjective; this yields that $\gamma_\chi(g) \in \operatorname{Im}(H^1(K, DZ) \to H^1(K, G))$. The group DZ is a split semisimple simply connected group, so that one has a decomposition $DZ/K = \prod_i H_i/K$ where each H_i is an almost simple simply connected group. The composite

$$\Psi : H^1(K, DZ) = \prod_i H^1(K, H_i) \to H^1(K, G) \to H^3(K)$$

is the sum of the restrictions $r_i : H^1(K, H_i) \to H^1(K, G) \to H^3(K)$ which are the Rost invariants of the groups H_i. But by a case-by-case analysis of the Dynkin diagram, left to the reader, it turns out that the Dynkin indices of the groups H_i are prime to p, so that the composite Ψ is trivial on the p-primary part of $H^3(K)$, and $r_K(\gamma_\chi(u)) = 0$. By the argument of Corollary 1 in Section 2, we get $M_G(u) = 0$.

We will have more to say about groups of types G_2, F_4 and E_8 in Section 4 below.

3 Value of M_G on Tits' bad elements

We recall the following result.

Theorem 2 ([28, Proposition S1] and [29, Theorème 7]). *If $[k : k^p] \geqslant p^2$ and p divides d_G, then the group $G(k)$ contains a bad unipotent element of order p.*

Tits reduced the theorem to the cases G_2, $p = 2$, F_4, $p = 3$, E_8, $p = 5$, and base field $\mathbb{F}_p(x, y)$, and produced explicit anisotropic elements, the anisotropy being guaranteed by a nice building argument. Here we compute the invariant M_G of such elements and show their non-triviality (thus giving a tedious proof of their badness). Let T be a maximal k-split torus of G, $\hat{T}$ (resp. $\hat{T}^0$) its character group (resp. cocharacter group), $\Phi = \Phi(G, T)$ the root system and $(U_\alpha)_{\alpha \in \Phi}$ the set of root subgroups of G. Let Δ be a basis of Φ, and let α_0 be the opposite of the highest root of Φ with respect to Δ. Finally, let $\tilde{\Delta} = \Delta \cup \{\alpha_0\}$. So we have

$$\alpha_0 + \sum_{\alpha \in \Delta} c_\alpha \alpha = 0,$$

where the coefficients c_α are positive integers. As Δ is a $\mathbb{Z}$-basis of $\hat{T}$, we denote by $(\alpha^*)_{\alpha \in \Delta}$ the dual basis of $\hat{T}^0 \otimes_{\mathbb{Z}} \mathbb{R}$. We also consider the base $\Delta^\vee = (\alpha^\vee)$ of the coroot system in $\hat{T}^0 \otimes_{\mathbb{Z}} \mathbb{R}$ and its dual basis consisting of fundamental weights $(\varpi_\alpha)_{\alpha \in \Delta}$, which is a $\mathbb{Z}$-basis of $\hat{T}$.

Proposition 1. *Let α be a simple root of Δ and $H_\alpha/k \subset G/k$ the maximal reductive* *subgroup associated to α, i.e. the subgroup of G generated by $T, U_{\pm\beta}$ ($\beta \in \Delta\backslash\{\alpha\}$). Assume that there exists an element $\tilde{w} \in N_{H_\alpha}(T)(k)$ such that*

(i) *$\tilde{w}$ has order $n := \alpha_0^\vee(\varpi_\alpha)$,*

(ii) *its image, w say, in W has order n and acts anisotropically on T (i.e. T^w is finite),*

(iii) *$\tilde{w}$ is k-good in $H_\alpha(k)$.*

For $x, y \in k^\times$, set

$$g_{x,y} := \mathrm{Ad}(\alpha^*(\sqrt[n]{y})) \cdot (\alpha^\vee(x) \times \tilde{w}) \quad \text{in } G(k(\sqrt[n]{y})).$$

Then $g_{x,y}$ belongs to $G(k)$, has order n, normalizes T and

$$M_G(g_{x,y}) = \{x, y\} \quad \text{in } K_2(k)/nK_2(k).$$

Remarks. As α^* defines a cocharacter in the adjoint group G_{ad} of G, this formula makes sense. Tits' bad unipotent elements are the unipotent elements $g_{x,y}$ given in Proposition 1 in the case when $n = p = \mathrm{char}(k)$ and $k = \mathbb{F}_p(x, y)$, so that $\{x, y\} \neq 0$ in $K_2(\mathbb{F}_p(x, y))/pK_2(\mathbb{F}_p(x, y))$. To apply Proposition 1 for a given pair $H_\alpha \subset G$, we need to see if such a $\tilde{w}$ exists. This works in the three following 'irreducible' cases considered by Tits and we sketch the proof of the following botanical statement.

Lemma 3. *Assume that $G = G_2$ (resp. F_4, E_8), $n = 2$ (resp. 3, 5) and H_α is the k-maximal split subgroup of G of type $A_1 \times A_1$ (resp. $A_2 \times A_2$, $A_4 \times A_4$).*

(a) *The anisotropic elements of W of order n (those w for which T^w is finite) form a single conjugacy class W_{an} of elements of W.*

(b) *There exists $w \in W_{\mathrm{an}}$ and a lift $\tilde{w} \in N_G(T)(k) \cap H_\alpha(k)$ such that $\tilde{w}$ has order n and is k-good.*

Proof. We only do the case when $n = 5$, $G = E_8$ and H is of type $A_4 \times A_4$. We refer to the Atlas [2], p. 85. The 5-Sylow subgroup of W is $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ which is the 5-Sylow subgroup of the Weyl group $N_H(T)/T$ of H, and there are two conjugacy classes of elements order 5 in W. One obviously consists of isotropic elements, and the other consists of anisotropic elements since there exists one anisotropic element (see [25, p. 177, table 3]). Now as $H = \mathrm{SL}_5 \times \mathrm{SL}_5/\mu_5$, the image in H of (c, c) in $\mathrm{SL}_5(k) \times \mathrm{SL}_5(k)$ (where c denotes the standard cycle of length 5 in $\mathrm{SL}_5(k)$ defined by $c(e_i) = e_{i+1}$, i taken modulo 5) gives an element $\tilde{w} \in N_G(T)(k) \cap H(k)$ of order 5 and mapping to an anisotropic element of order 5 of W.

So using that the well-known inclusion of split groups

$$G_2 \subset F_4 \subset E_6 \subset E_7 \subset E_8$$

has total Dynkin index 1, we see from Subsection 1.2 that for any group G in the list, and for any p dividing d_G, there is a bad unipotent element of order p in $G(\mathbb{F}_p(x, y))$.

Proof of Proposition 1. For convenience, we denote H_α simply by H. The group H/k is the reductive quotient of the closed fiber of the parahoric group scheme $\mathfrak{P}_\alpha/\mathrm{Spec}(k[[t]])$ (see [6, II.3.2]). By [12, §IV], one has the following exact sequence of reductive groups

$$1 \to \mathbb{G}_m \to H^u \xrightarrow{\lambda} H \to 1, \tag{A}$$

which is a part of a larger exact diagram

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & \mathbb{G}_m & = & \mathbb{G}_m & & & & \\
 & & {\scriptstyle \times n}\uparrow & & \uparrow & & & & \\
1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & H^u & \xrightarrow{\lambda} & H & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \| & & \\
1 & \longrightarrow & \mu_n & \longrightarrow & H^{\mathrm{sc}} & \longrightarrow & H & \longrightarrow & 1, \\
 & & \uparrow & & \uparrow & & & & \\
 & & 1 & & 1 & & & &
\end{array} \tag{B}$$

such that H_α^{sc} is semisimple and simply connected. The construction of this diagram is based on the torus $\mathcal{T} = (\mathbb{G}_m)^{\tilde{\Delta}}$ introduced by Prasad and Raghunathan [22, §4.6], which is a maximal k-split torus of H^u/k. Writing $T^{\mathrm{sc}} = \mathcal{T} \cap H^{\mathrm{sc}}$, one has the commutative exact diagram of tori

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & \mathbb{G}_m & = & \mathbb{G}_m & & & & \\
 & & {\scriptstyle \times n}\uparrow & & {\scriptstyle p_\alpha}\uparrow & & & & \\
1 & \longrightarrow & \mathbb{G}_m & \xrightarrow{p_\alpha} & \mathcal{T} = (\mathbb{G}_m)^{\tilde{\Delta}} & \longrightarrow & T & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \| & & \\
1 & \longrightarrow & \mu_n & \longrightarrow & T^{\mathrm{sc}} & \longrightarrow & T & \longrightarrow & 1, \\
 & & \uparrow & & \uparrow & & & & \\
 & & 1 & & 1 & & & &
\end{array} \tag{C}$$

where the map $p_\alpha : \mathcal{T} = \mathbb{G}_m^{\tilde{\Delta}} \to \mathbb{G}_m$ is the projection along the factor α and the map $\mathcal{T} = (\mathbb{G}_m)^{\tilde{\Delta}} \to T$ sends $(x_\beta)_{\beta \in \tilde{\Delta}}$ to $\sum_{\beta \in \Delta} \beta^\vee(x_\beta)$. We consider the characteristic map of the isogeny $H^{\mathrm{sc}} \to H$

$$\delta : H(k) \to H^1_{fppf}(k, \mu_n) = k^\times / k^{\times n}.$$

By construction, one has $\delta(\alpha^\vee(x)) = (x)$ in $k^\times/k^{\times n}$. By Hilbert's Theorem 90, the exact sequence (A) gives an exact sequence

$$1 \to k^\times \to H^u(k) \to H(k) \to 1 \tag{D}$$

and a map $H^1(\mathbb{Z}/n\mathbb{Z}, H(k)) \to H^2(\mathbb{Z}/n\mathbb{Z}, k^\times)$, i.e. a map

$$\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, H(k))/\mathrm{Int}(H(k)) \to k^\times/k^{\times n}.$$

One checks easily that this map is the restriction of δ to the elements of $H(k)$ of order n. As $\tilde{w}$ is k-good, one has $\delta(\tilde{w}) = 1$ (cf. [26, Proposition 5.2]), and as δ is a group homomorphism, we get

$$\delta(\alpha^\vee(x).\tilde{w}) = \delta(\alpha^\vee(x)) \times \delta(\tilde{w}) = (x) \quad \text{in } k^\times/k^{\times n}.$$

We consider the case when y is an indeterminate over k and we denote by

$$\xi : H(k) \to G(k(\sqrt[n]{y}))$$

the restriction to $H(k)$ of the map $\mathrm{Ad}(\alpha^*(\sqrt[n]{y}))$. We have $\xi(H(k)) \subset G(k(y))$ by [27, p. 662]. We consider Milnor's exact sequence mod n

$$0 \longrightarrow K_2(k)/nK_2(k) \longrightarrow K_2(k(y))/nK_2(k(y))$$
$$\xrightarrow{\oplus \partial_{BT,M}} \bigoplus_{M \in \mathbb{A}^1_k} k(M)^\times/k(M)^{\times n} \longrightarrow 0.$$

As $\xi(h)$ is unramified on $\mathbb{P}^1_k \backslash \{0, \infty\}$ with trivial value in 1, and using the behaviour of M_G by specialization (Subsection 1.2), it is enough to show that for any $h \in H(k)$ of order n, one has

$$\partial_{BT,0}(M_G(\xi(h)) = (\delta(h)) \quad \text{in } k^\times/k^{\times n}.$$

This last identity is a straighforward consequence of the following result.

Lemma 4. *The extension $(\partial_{BT,0})_*[\xi^*(\mathrm{E}_{k((y))})]$ of $H(k)$ by $k^\times$ is isomorphic to the extension*

$$1 \to k^\times \to H^u(k) \to H(k) \to 1.$$

In other words, there exists a map $\psi : H^u(k) \to (\partial_{BT,0})_*(E_{k((y))})$ such that the following diagram commutes:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & K_2(k((y))) & \longrightarrow & \mathrm{E}_{k((y))} & \longrightarrow & G(k((y))) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \partial_{BT,0}} & & \downarrow & & \| & & \\
0 & \longrightarrow & k^{\times} & \longrightarrow & (\partial_{BT,0})_*(\mathrm{E}_{k((y))}) & \longrightarrow & G(k((y))) & \longrightarrow & 1 \\
 & & \| & & \uparrow{\scriptstyle \psi} & & \uparrow{\scriptstyle \xi} & & \\
0 & \longrightarrow & k^{\times} & \longrightarrow & H^u(k) & \xrightarrow{\lambda} & H(k) & \longrightarrow & 1.
\end{array}
$$

Proof of Lemma 4. We denote by $\partial'_{BT,0} : K_2(k((y'))) \to k^{\times}$ the Bass–Tate residue map for the field $k((y'))$; see [3]. By [12, proof of Theorem 4], one has the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & K_2(k((y'))) & \longrightarrow & \mathrm{E}_{k((y'))} & \longrightarrow & G(k((y'))) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \partial'_{BT,0}} & & \downarrow & & \| & & \\
0 & \longrightarrow & k^{\times} & \longrightarrow & \partial'_{BT,0}(\mathrm{E}_{k((y'))}) & \longrightarrow & G(k((y'))) & \longrightarrow & 1 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & k^{\times} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathfrak{P}_\alpha(k[[y']]) & \longrightarrow & 1. \\
 & & \| & & \downarrow{\scriptstyle \phi} & \nearrow{\scriptstyle \xi\circ\lambda} & \downarrow{\scriptstyle \mathrm{sp}} & & \\
0 & \longrightarrow & k^{\times} & \longrightarrow & H^u(k) & \xrightarrow{\lambda} & H(k) & \longrightarrow & 1,
\end{array}
$$

where $\mathfrak{P}_\alpha/k[[y']]$ denotes the parahoric subgroup corresponding to the root α. As we have $\mathrm{sp} \circ \xi = 1$, there exists a lift $\psi : H^u(k) \to \partial'_{BT,0}(\mathrm{E}_{k((y'))})$ of $\xi : H(k) \to G(k((y')))$. As

$$(\partial_{BT,0})_*(\mathrm{E}_{k((y'))}) = \mathrm{Res}^{G(k((y')))}_{G(k((y)))}[(\partial_{BT,0})_*(\mathrm{E}_{k((y))})],$$

the map ψ factorizes through $(\partial_{BT,0})_*(\mathrm{E}_{k((y))})$ and the lemma is proved.

By the specialization functorial properties of M_G (in Subsection 1.2) we can specialize the previous case where y was an indeterminate to the case where y is any scalar. The proof of Proposition 1 is now complete.

4 Conjugacy classes of anisotropic elements

4.1 The tame case. We assume that G has type G_2, F_4 or E_8, i.e. we require that G be simply connected, adjoint and almost simple. Our goal is to describe k-anisotropic morphisms $\mu_m \subset G$ (as defined in I.1) using Proposition I.5 together with the computation of $H^1(K, G)_{\mathrm{an}}$ (which is the Bruhat–Tits decomposition [7]). For $\alpha \in \Delta$, we consider the cocharacter $\alpha^* : \mathbb{G}_m \to T$ and the subgroup

$$\mu_{c_\alpha} = \mathrm{Ker}(\mathbb{G}_m \xrightarrow{\times c_\alpha} \mathbb{G}_m) \xrightarrow{\alpha^*} T \subset G.$$

It is known that

$$Z(H_\alpha) = \mu_{c_\alpha} \quad \text{and} \quad H_\alpha = Z_G(\mu_{c_\alpha}),$$

and we denote by $\phi_\alpha : \mu_{c_\alpha} \hookrightarrow G$ the resulting morphism.

Proposition 2. *Assume that $G = G_2$, F_4 or E_8 and that* $\mathrm{char}(k) = 0$.

(a) *Let m be a non-negative integer. Let $\phi : \mu_m \subset G$ be an anisotropic subgroup. Then there exists $\alpha \in \Delta$ such that $m = c_\alpha$ and such that ϕ is k_s-conjugate to ϕ_α.*

(b) *The set*

$$\coprod_{\alpha \in \Delta} \mathrm{Ker}[H^1(k, H_\alpha)_{\mathrm{an}} \to H^1(k, G)],$$

can be identified with the set of $G(k)$-conjugacy classes of k-anisotropic $\mu_m \hookrightarrow G$ for various m.

(c) *Let $\alpha \in \Delta$ and $[z] \in \mathrm{Ker}[H^1(k, H_\alpha)_{\mathrm{an}} \to H^1(k, G)]$. Denote by ${}_z\phi_\alpha : \mu_{c_\alpha} \subset G$ the twist by z of ϕ_α. Then $Z_G(\mathrm{Im}({}_z\phi_\alpha)) \xrightarrow{\sim} {}_zH_\alpha$.*

(d) *Consider the exact sequence of algebraic groups*

$$1 \to \mathbb{G}_m \to H_\alpha^u \to H_\alpha \to 1$$

of [12, p. 29, Theorem 3] *and the boundary of Galois cohomology*

$$\delta : H^1(k, H_\alpha) \to \mathrm{Br}(k) = H^2(k, \mathbb{G}_m).$$

Assume that k contains a primitive root of unity ζ and identify the group $K_2(k)/m \approx {}_m\mathrm{Br}(k)$ by the Merkurjev–Suslin isomorphism. Then for any anisotropic $g : \mathbb{Z}/m\mathbb{Z} \approx \mu_m \hookrightarrow G(k)$ and underlying class $\beta(g)$ in $\mathrm{Ker}[H^1(k, H_\alpha)_{\mathrm{an}} \to H^1(k, G)]$, *one has, up to the sign,*

$$\delta(\beta(g)) = M_G(g) \quad \text{in } {}_m\mathrm{Br}(k).$$

Proof. Let us recall first what the Bruhat–Tits decomposition means. As before, write $O = k[[t]]$ and $K = k((t))$. Let $\boldsymbol{P}_\alpha/\mathrm{Spec}(O)$ $(\alpha \in \Delta \cup \{\alpha_0\})$ be the maximal standard parahoric subgroups of G/K. There is an isomorphism

$$(\boldsymbol{P}_\alpha \times_O k)_{\mathrm{red}} \xrightarrow{\sim} H_\alpha/k.$$

Taking the inverse of the bijection

$$H^1(\mathcal{G}, \boldsymbol{P}_\alpha(O_{\mathrm{nr}})) \xrightarrow{\sim} H^1(\mathcal{G}, (\boldsymbol{P}_\alpha \times_O k)(k_s)) \xrightarrow{\sim} H^1(k, H_\alpha),$$

we have a map

$$\rho_\alpha : H^1(k, H_\alpha) \xrightarrow{\sim} H^1(\mathcal{G}, \boldsymbol{P}_\alpha(O_{\mathrm{nr}})) \to H^1(\mathcal{G}, G(K_{\mathrm{nr}})) = H^1(K_{\mathrm{nr}}/K, G).$$

We denote by $\rho_0 : H^1(k, G) \to H^1(K_{nr}/K, G)$ the restriction from k to K. The Bruhat –Tits decomposition (*loc. cit.*, Theorem 3.12) states that the map ρ_α is injective and that $\coprod_{\alpha \in \Delta \cup \{\alpha_0\}} \rho_\alpha$ induces a bijection

$$H^1(k, G)_{an} \sqcup \coprod_{\alpha \in \Delta} H^1(k, H_\alpha)_{an} \xrightarrow{\sim} H^1(K, G)_{an}.$$

We now identify in this the decomposition cohomology classes coming from k-anisotropic morphisms.

Lemma 5. *Let $\alpha \in \Delta$ and $[z] \in \mathrm{Ker}[H^1(k, H_\alpha) \to H^1(k, G)]$. Denote by ${}_z\phi_\alpha : \mu_{c_\alpha} \subset G$ the twist by z of ϕ_α. Taking $I = \mu_{c_\alpha}$, $L = k((\sqrt[c_\alpha]{t}))$, and $\theta = (t) \in H^1(K, \mu_{c_\alpha})$ as in Section* 2, *one has*

$$({}_z\phi_{\alpha,*})\theta = \rho_\alpha([z]) \quad \textit{in } H^1(K, G).$$

Let us prove this formula. Following [27, §2], we denote by $\tau_\alpha \in T(L)$ the element defined by

$$\tau_\alpha = \alpha^*(\sqrt[c_\alpha]{t}).$$

This element is related to our setting as follows. Denoting by $(f_\sigma)_{\sigma \in \mathrm{Gal}(K_{tr}/K)}$ the cocycle with value in μ_{c_α} associated to $t \in Z^1(K(\sqrt[c_\alpha]{t})/K, \mu_{c_\alpha})$, one has via α^*

$$f_\sigma = (\sqrt[c_\alpha]{t})^{-1} \cdot {}^\sigma(\sqrt[c_\alpha]{t}) = \tau_\alpha^{-1} \cdot {}^\sigma\tau_\alpha \quad (\sigma \in \mathrm{Gal}(K_{tr}/K)).$$

We know (*loc. cit.*) that $\xi := \mathrm{Ad}(\tau_\alpha) : G(L_{nr}) \to G(L_{nr})$ satisfies $\xi^{-1}(H_\alpha(k_s)) \subset G(K_{nr})$ and that the map $\rho_\alpha : H^1(k, H_\alpha) \to H^1(K, G)$ is given by

$$\rho_\alpha([a_s]) = [\xi^{-1}(a_s)] = [\tau_\alpha^{-1} a_s \tau_\alpha] \quad \text{in } H^1(K, G).$$

Let $[z] \in \mathrm{Ker}[H^1(k, H_\alpha) \to H^1(k, G)]$. Then $z_s = g^{-1s}g$ for some $g \in G(k_s)$. Using the morphism $\psi : \mathrm{Gal}(K_{tr}/K) \to \mathrm{Gal}(k_s/k)$, we have

$$\begin{aligned}
\rho_\alpha([z_s])_{s \in \mathrm{Gal}(k_s/k)} &= [\tau_\alpha^{-1} z_s \tau_\alpha] \\
&= [z_{\psi(\sigma)} (\tau_\alpha)^{1-\sigma}]_{\sigma \in \mathrm{Gal}(K_{tr}/K)} \\
&= [z_{\psi(\sigma)} f_\sigma] \\
&= [f_\sigma z_{\psi(\sigma)}] \quad (z \in H_\alpha = Z(\mu_{c_\alpha})) \\
&= [f_\sigma g^{-1} \cdot {}^{\psi(\sigma)}g] \\
&= [g f_\sigma g^{-1}] \\
&= ({}_z\phi_{\alpha,*})\theta \quad \text{in } H^1(K, G),
\end{aligned}$$

and Lemma 5 is proved.

We return to the proof of Proposition 2.

(a) Let $\gamma = [\phi_* \theta] \in H^1(L/K, G)$. By Proposition I.5.a we have $\gamma \in H^1(L/K, G)_{\mathrm{an}}$. Clearly γ does not come from the first set $H^1(k, G)_{\mathrm{an}}$ of constant classes, and so there exists $\alpha \in \Delta$ such that α comes from a class $[z] \in H^1(k, H_\alpha)_{\mathrm{an}}$. In view of $\tau_\alpha \in T(L)$, Lemma 5 shows that

$$[\rho_\alpha([z])]_L = [\tau_\alpha^{-1} z_s \tau_\alpha] = \rho_0([z]) \quad \text{in } H^1(L, G),$$

so that $[z] \in \mathrm{Ker}(H^1(k, H_\alpha)_{\mathrm{an}} \to H^1(k, G))$. Let d be the greatest common divisor of c_α and m. We consider the morphisms

$$\phi' : \mu_d \xrightarrow{\times m/d} \mu_m \xrightarrow{\phi} G,$$

$$_z\phi'_\alpha : \mu_d \xrightarrow{\times m/c_\alpha} \mu_{c_\alpha} \xrightarrow{_z\phi} G.$$

Now we take the extension $K(\sqrt[d]{t})/K$ and we apply the functoriality remark 1 of §I.2.1. As $\gamma(\phi) = \gamma(_z\phi_\alpha)$ in $H^1(K, G)$, one has $\gamma(\phi') = \gamma(_z\phi'_\alpha) \in H^1(K(\sqrt[d]{t})/K, G)$, and Proposition I.5 shows that ϕ' and $_z\phi'_\alpha$ are k-conjugate. The image of ϕ and ϕ' have the same cardinality, so that $m = c_\alpha = d$ and we conclude that ϕ and $_z\phi_\alpha$ are k-conjugate.

(b) From (a), the map from

$$\coprod_{\alpha \in \Delta} \mathrm{Ker}[H^1(k, H_\alpha)_{\mathrm{an}} \to H^1(k, G)],$$

to $G(k)$-conjugacy classes of k-anisotropic $\mu_m \hookrightarrow G$ is surjective; injectivity follows from Proposition I.5.b, the Bruhat–Tits decomposition and Lemma I.4. Assertion (c) is obvious.

(d) By definition,

$$\gamma_\chi(g) = \rho(\beta(g)) \in \mathrm{Im}(H^1(k, H)_{\mathrm{an}} \xrightarrow{\rho} H^1(K, G)).$$

We now apply the Rost invariant to both sides. By Lemma 1 one has

$$r_K(\gamma_\chi(g)) = \chi \cup M_G(g) \quad \text{in } H^3(K),$$

while by [12, Theorem 4], one has (up to the sign)

$$\partial_K[r_K(\rho(\beta(g)))] = M_G(g) \quad \text{in } \mathrm{Br}(k),$$

and the formula is proved.

We specialize to the case when $m = 2$ for G_2, $m = 3$ for F_4 and $m = 5$ for E_8. In these cases, there is exactly one $\alpha \in \Delta$ such that $c_\alpha = m$.

Corollary 4. *Assume that* $\mathrm{char}(k) = 0$ *and that* $G = G_2$ *(resp.* F_4, E_8*),* $m = 2$ *(resp.* 3, 5*) and* H_α *is the k-maximal split subgroup of G of type* $A_1 \times A_1$ *(resp.* $A_2 \times A_2$, $A_4 \times A_4$*). Then the map* ∂ *induces a one-to-one correspondence between the following two sets:*

(i) *the set of* $G(k)$*-conjugacy classes of k-anisotropic injective morphisms* $\mu_m \hookrightarrow G$;

(ii) $\mathrm{Ker}[H^1(k, H_\alpha)_{\mathrm{an}} \to H^1(k, G)]$.

Lemma 6. *With notation as in the previous corollary, the boundary map*

$$\delta : \mathrm{Ker}[H^1(k, H_\alpha) \to H^1(k, G)] \to \mathrm{Br}(k)$$

is injective.

Proof. We only treat the case when E_8, $p = 5$, the two other cases being similar. We proceed fiberwise to establish the injectivity of α. We have $H = H_\alpha = \mathrm{SL}_5 \times \mathrm{SL}_5/\mu_5$ where the embedding $\mu_5 \to \mu_5 \times \mu_5$ is defined by $x \mapsto (x, x^2)$. Let $[z] \in \mathrm{Ker}(H^1(k, H_\alpha) \to H^1(k, E_8))$ and denote by

$${}_zH = \mathrm{SL}(A) \times \mathrm{SL}(B)/\mu_5 \subset E_8$$

the twisted group by z, where A/k, B/k are central simple algebras of degree 5 such that $[B] = 3[A]$ in $\mathrm{Br}(k)$. The exact sequence

$$1 \to \mu_5 \to \mathrm{SL}(A) \times \mathrm{SL}(B) \to {}_zH \to 1$$

gives rise to the long exact sequence of pointed sets

$$\begin{array}{ccccccc}
H^1(k,\mu_5) & \longrightarrow & H^1(k,\mathrm{SL}(A)) \times H^1(k,\mathrm{SL}(B)) & \longrightarrow & H^1(k,{}_zH) & \xrightarrow{{}_z\partial} & H^2(k,\mu_5) \\
\uparrow\wr & & \uparrow\wr & & \| & & \uparrow\wr \\
k^\times/k^{\times 5} & \xrightarrow{(\times 1, \times 2)} & k^\times/\mathrm{Nrd}(A^\times) \times k^\times/\mathrm{Nrd}(B^\times) & \longrightarrow & H^1(k,{}_zH) & \xrightarrow{{}_z\partial} & {}_5\mathrm{Br}(k).
\end{array}$$

In the case when A is split, the map

$$\mathrm{Ker}(H^1(k, {}_zH) \longrightarrow H^1(k, E_8)) \xrightarrow{{}_z\partial} {}_5\mathrm{Br}(k)$$

has trivial kernel because $\mathrm{Nrd}(A^\times) = k^\times$. Assume now that A/k is a division algebra. By Dieudonné's theorem, one has $\mathrm{Nrd}(A^\times) = \mathrm{Nrd}(B^\times)$. By trivializing the second factor in the group $k^\times/\mathrm{Nrd}(A^\times) \times k^\times/\mathrm{Nrd}(A^\times)$, we get the exact sequence of pointed sets

$$1 \longrightarrow k^\times/\mathrm{Nrd}(A^\times) \longrightarrow H^1(k, {}_zH) \xrightarrow{{}_z\partial} {}_5\mathrm{Br}(k). \qquad (*)$$

According to [9], if $r : H^1(k, E_8) \to H^3(k)$ is the Rost invariant, then the composite map

$$k^\times/\mathrm{Nrd}(A^\times) \to H^1(k, {}_zH) \to H^1(k, E_8) \xrightarrow{r} H^3(k)$$

is in fact the Merkurjev–Suslin invariant $k^\times/\mathrm{Nrd}(A^\times) \xrightarrow{[A]\cup ?} H^3(k)$, which is injective: see [21]. The map

$$\mathrm{Ker}(H^1(k, {}_zH) \longrightarrow H^1(k, E_8)) \xrightarrow{{}_z\partial} {}_5\mathrm{Br}(k)$$

therefore has trivial kernel as desired.

4.2 The main statement for G_2, F_4 and E_8.

Theorem 3. *Assume that* $G = G_2$ *(resp.* F_4, E_8*) and let* $l = 2$ *(resp.* 3, 5*). Assume also that either* $\mathrm{char}(k) = 0$ *and k contains a primitive l-th root of unity, or* $p = \mathrm{char}(k) = l$.

(a) *Let g be an element of* $G(k)$ *of order l. The following are equivalent*:

  (i) *g is k-bad*;

  (ii) *g is k-anisotropic*;

  (iii) $M_G(g) \neq 0$ *in* $K_2(k)/lK_2(k)$.

(b) *For any k-anisotropic elements* g, g' *of order l in* $G(k)$, *the following conditions are equivalent*:

  (i) $M_G(g) = M_G(g')$ *in* $K_2(k)/lK_2(k)$;

  (ii) *g and g′ are conjugate under* $G(k)$.

(c) *Except possibly for the case of* E_8 *with* $l = 5$ *invertible in k, there is a one-to-one correspondence between the* $G(k)$*-conjugacy classes of k-anisotropic elements of order l in* $G(k)$ *and the symbols* $\{x, y\} \in K_2(k)/lK_2(k)$. *Each such conjugacy class is represented by the Tits anisotropic element* $g_{x,y}$ *given in Proposition* 1.

(d) *Assume that* $G = E_8$, *that* $l = 5$ *and that* 5 *is invertible in k. Let g be an anisotropic element of order* 5. *Then the following are equivalent*:

  (i) $M_G(g)$ *is a symbol* $\{x, y\} \in K_2(k)/5K_2(k)$;

  (ii) *g is conjugate to the Tits anisotropic element* $g_{x,y}$;

  (iii) *g normalizes a maximal k-split torus of G.*

Remark. Assertion (c) for E_8 with $l = p = 5$ finishes the proof of Theorem I.3.

Proof of Theorem 3. Everything is already known for G_2. The Rost invariant is then injective, and Theorem 3 follows from Lemma 1 and Proposition I.5. One can assume that l is odd.

(a) For $l = p$, this is Lemma 2 of Section 2. For $l \neq p$, it follows from Lemma 6 and Proposition 2.d.

(b) *Step* 1. *In case* E_8, $l = p = 5$, *any anisotropic element* g *normalizes a maximal* k-*split torus.*

By Lemma I.8, there exists an element $n \in N_G(T)(k)$ such that n is k_s-conjugate to g. By [5, Proposition 3.6], the element g is k_s-bad, so that n is k_s-bad and Lemma 2 shows that n is k_s-anisotropic and is *a fortiori* k-anisotropic. Moreover, as the map $K_2(k)/p \to K_2(k_s)/p$ is injective by the Bloch–Gabber–Kato Theorem [4, Theorem 1.2], one has $M_G(n) = M_G(g)$ in $K_2(k)/pK_2(k)$.

Let A be a complete discrete valuation ring with residue field k and field of fractions F_A of characteristic 0. By Lemma I.4.d, there exists a lift $\tilde{n} \in N_G(T)(A)$ of n of order p; by Lemmas I.7 and I.8, there exists a lift $\tilde{g} \in G(A)$ of g of order p. By Lemma I.6, the elements $\tilde{n}$, $\tilde{g}$ are anisotropic elements of $G(F_A)$. By Subsection 1.2, one has

$$M_G(\tilde{n}) - M_G(\tilde{g}') \in \mathrm{Ker}(K_2(A)/p \to K_2(k)/p).$$

Lemma 7. *Let* $[\beta] \in \mathrm{Ker}(K_2(A)/p \to K_2(k)/p)$. *Then there exists a totally ramified finite extension* A'/A *of local rings such that* $\beta_{A'} = 0 \in K_2(A')/p$.

Let us prove the lemma. Denote by π a uniformizing parameter of A. As the map $K_2(A) \to K_2(k)$ is onto, one can assume that $\beta \in \mathrm{Ker}(K_2(A) \to K_2(k))$. Embedding $K_2(A)$ in $K_2(F_A)$, we see from [3, Proposition 4.3] that there exist elements $a_1, \dots, a_n$ of A and elements $b_1, \dots, b_n$ of $F_A^\times$ such that

$$\beta = \sum_{i=1,\dots,n} \{1 + \pi a_i, b_i\}.$$

We set

$$A' = A[\sqrt[p]{1 + \pi a_1}, \dots, \sqrt[p]{1 + \pi a_n}].$$

Then A' is finite over A and A' is complete for the (unique) extended valuation; we denote by π' a uniformizing parameter of A'. By construction, one has $\beta_{A'} \in pK_2(F_{A'})$ and using the exact sequence (*loc. cit.*)

$$\begin{array}{ccccccccc}
1 & \longrightarrow & K_2(A') & \longrightarrow & K_2(F_{A'}) & \xrightarrow{\partial_{\pi'}} & k^\times & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \times p} & & \downarrow{\scriptstyle \times p} & & \downarrow{\scriptstyle \times p} & & \\
1 & \longrightarrow & K_2(A') & \longrightarrow & K_2(F_{A'}) & \xrightarrow{\partial_{\pi'}} & k^\times & \longrightarrow & 1,
\end{array}$$

and the fact that the Frobenius map is injective, we get that $\beta_{A'} \in pK_2(A')$, and so Lemma 7 is proved. Replacing A by A' if necessary, one can then assume that

$$M_G(\tilde{g}) = M_G(\tilde{g}') \in K_2(A)/p \subset K_2(F_A)/p.$$

By the characteristic zero case, there exists an element $h \in G(F_A)$ such that $\tilde{g} = h\tilde{n}h^{-1}$ and by Lemma I.6.b, g and n are conjugate under $G(k)$. This establishes the lemma and finishes the proof of Step 1.

Step 2. *In all cases, any anisotropic elements g, g' of order l in $G(k)$ such that $M_G(g) = M_G(g')$ are conjugate.*

Let g, g' be such elements.

Case 1. $\mathrm{char}(k) = 0$. It follows from Corollary 4 and Lemma 5 that g, g' are conjugate.

Case 2. $l = p$. Up to conjugation in $G(k)$, one can assume that $g, g' \in N_G(T)(k)$ and Lemma I.4.d shows that there exist $\tilde{g}, \tilde{g}' \in G(A)$ of order p lifting g, g'. The same argument as before shows that g and g' are conjugate under $G(k)$.

(c) and (d) By assertion (b), it is enough to prove that if g normalizes a maximal k-split torus, then the invariant $M_G(g)$ consists of one symbol.

Case 1. $\mathrm{char}(k) = 0$. Let g be an anisotropic element of order l. By Theorem I.3, one can assume that g belongs to $N_G(T)(k)$. But by Lemma 3, up to conjugation, there exist $\tau \in T(k)$ and $\tilde{w} \in N_G(T)(k) \cap H_\alpha(k)$ such that $g = \tau\tilde{w}$, so that $g \in H_\alpha(k)$. We then consider the simply connected covering

$$1 \to \mu_l \to \mathrm{SL}_l \times \mathrm{SL}_l \to H_\alpha \to 1$$

of H_α and we have the exact sequence of groups

$$1 \to \mu_l(k) \to \mathrm{SL}_l(k) \times \mathrm{SL}_l(k) \to H_\alpha(k) \to k^\times/k^{\times l}.$$

As g has order l in $H_\alpha(k)$, the obstruction to lifting g to an element of order l of $\mathrm{SL}_l(k) \times \mathrm{SL}_l(k)$ is given by some $(x) \in k^\times/k^{\times p}$. So $g_{k(\sqrt[l]{x})}$ lifts to an element of order p in $\mathrm{SL}_l(k(\sqrt[l]{x})) \times \mathrm{SL}_l(k(\sqrt[l]{x}))$ and is $k(\sqrt[l]{x})$-good. By Corollary 1 one has

$$M_G(g) \in \mathrm{Ker}(K_2(k)/l \to K_2(k(\sqrt[l]{x}))/l).$$

As $K_2(k)/p \approx {}_l\mathrm{Br}(k)$, there exists $y \in k^\times$ such that $M_G(g) = \{x, y\}$.

Case 2. $l = p$. As before, there exists $\tilde{g} \in G(A)$ of order p which lifts g and normalizes a maximal K-split torus. Case 1 then gives $x, y \in K^\times$ such that $M_G(\tilde{g}) = \{x, y\}$ in $K_2(K)/p$. Up to ramification of A, one can assume that x, y belong to $A^\times$. We remark that Tits' K-anisotropic element $\tilde{g}_{x,y}$ belongs to $G(A)$, and by (a), it is conjugate to $\tilde{g}$ in $G(K)$. Finally, by Lemma I.6.b, the elements g and $g_{x,y}$ are conjugate in $G(k)$ and $M_G(g) = \{x, y\}$.

Appendix. The kernel of the Rost invariant is not trivial in general for E_8

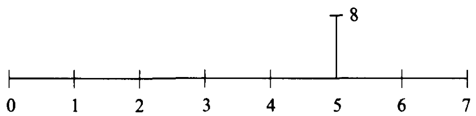
As an application of [12, Theorem 4], we get the following result.

Proposition 3. *Let k be a field of characteristic* 0 *such that there exists an anisotropic 4-fold Pfister form. Then the kernel Rost invariant*

$$r_{E_8} : H^1(k((t)), E_8) \to H^3(k((t)))$$

is non-trivial.

Proof. Write $K = k((t))$, $O = k[[t]]$ and let G/K be the split group of type E_8. We consider the extended Dynkin diagram



$\tilde{E}_8$

of E_8 and consider the standard maximal parahoric subgroup $\mathfrak{P}/\mathrm{Spec}(O)$ associated to the vertex 7. Then the reductive special fiber $M := (\mathfrak{P} \times_O k)_{\mathrm{red}}$ is a split semisimple group of type D_8 and we consider the exact sequence

$$1 \to \mathbb{G}_m \to M^u \to M \to 1$$

defined in [12, Theorem 4]. The diagram

$$\begin{array}{ccc} H^1(K, G) & \xrightarrow{r_K} & H^3(K) \\ \uparrow & & \Big| \\ H^1(O, \mathfrak{P}) & & \Big| \\ \downarrow\wr & & \Big|\, \partial_K \\ H^1(k, \mathfrak{P} \times_k O) & & \Big| \\ \wr\downarrow & & \downarrow \\ H^1(k, M) & \xrightarrow{\delta} & \mathrm{Br}(k) \end{array}$$

commutes, where δ denotes the boundary $\delta : H^1(k, M) \to \mathrm{Br}(k)$. Thus we have a natural map $H^1(k, M) \to H^1(K, G)$, which is injective according to [7, Lemme 3.9]. We recall that DM' is a semisimple simply connected group of type D_8, i.e. the group Spin_{16}. The previous diagram above shows that the composite

$$H^1(k, \mathrm{Spin}_{16}) \to H^1(k, M') \to H^1(k, M) \to H^1(K, G) \to H^3(K) \to \mathrm{Br}(k)$$

is trivial. But one has an exact sequence

$$0 \longrightarrow H^3(k) \longrightarrow H^3(K) \xrightarrow{\partial_K} \mathrm{Br}(k) \longrightarrow 0,$$

and therefore an invariant

$$f : H^1(k, \mathrm{Spin}_{16}) \to H^3(k)$$

which is functorial in k. By Rost's theorem (cf. [12, Theorem 1]) and a classic genericity argument (*ibid*, Proposition 3), f is a multiple of the Rost invariant $r_{D_8} : H^1(k, \mathrm{Spin}_{16}) \to H^3(k)$, so that $f = 1$ or $f = r_{D_8}$. Now we consider the exact sequence

$$1 \to \mu_2 \to \mathrm{Spin}_{16} \to \mathrm{SO}_{16} \to 1,$$

and the Galois cohomology exact sequence of pointed sets

$$1 \to H^1(k, \mathrm{Spin}_{16}) \to H^1(k, \mathrm{SO}_{16}) \to {}_2\mathrm{Br}(k)$$

(the first map has trivial kernel as the spinor norm is surjective). We pick an anisotropic 4-fold Pfister form φ whose class $[\varphi]$ belongs to the kernel $H^1(k, \mathrm{SO}_{16}) \to {}_2\mathrm{Br}(k)$, so that $[\varphi]$ comes from a class $c \in H^1(k, \mathrm{Spin}_{16})_{\mathrm{an}}$. As $r_{D_8}(c) = 0$ in $H^3(k)$, one has $f(c) = 0$ in $H^3(k)$ and the image of c under the composite map

$$H^1(k, \mathrm{Spin}_{16}) \to H^1(k, M') \to H^1(k, M) \to H^1(K, G) \to H^3(K)$$

is trivial. But c is anisotropic, so by Bruhat–Tits theory [7, I, Lemme 3.7.ii], the image of c in $H^1(K, G)$ is anisotropic. We have constructed an anisotropic class in $H^1(K, G)$ whose Rost invariant is trivial.

Remark. In fact, the invariant f is the Rost invariant; one way of proving this is to use the quadratic extension $K' = K(\sqrt{t})$ and the proof of Proposition 2.

Acknowledgement. The author is very grateful to Arturo Pianzola for helpful comments on a preliminary version of this paper.

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Received 9 July, 2001; revised 28 September, 2001

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