AN INVARIANT OF TANGLE COBORDISMS

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ABSTRACT. We construct a new invariant of tangle cobordisms. The invariant of a tangle is a complex of bimodules over certain rings, well-defined up to chain homotopy equivalence. The invariant of a tangle cobordism is a homomorphism between complexes of bimodules assigned to boundaries of the cobordism.

1. INTRODUCTION

In [9] to a plane diagram D of an oriented tangle T with $2n$ bottom and $2m$ top endpoints we associated a complex $\mathcal{F}(D)$ of (H^m, H^n)-bimodules, for certain rings H^n. We proved that the isomorphism class of this complex in the homotopy category is an invariant of T. In this paper we give a short argument that our construction yields an invariant of tangle cobordisms. To a diagram of an oriented cobordism between diagrams D_1 and D_2 of tangles T_1 and T_2 we assign a homomorphism of complexes $\mathcal{F}(D_1) \to \mathcal{F}(D_2)$ and then check that (in the homotopy category) this homomorphism depends on the choice of a diagram of the cobordism only up to the overall minus sign. The result follows from the basic properties of rings H^n and H^n-bimodules assigned to tangle diagrams.

For link cobordisms this result was recently obtained by Magnus Jacobsson [6] in a different way (see also [10] for motivations).

2. 2-TANGLES

The analogue of Reidemeister moves for surfaces embedded in $\mathbb{R}^4$ was found by Roseman [11] and investigated in depth by Carter and Saito [3], [4].

The framework for studying 2-tangles was developed by Fischer [5], Kharlamov and Turaev [7], Carter, Rieger, and Saito [2], and Baez and Langford [1]. We will use a combinatorial realization of the 2-tangle 2-category described in [2], [4, Section 2.5], and [1, Section 3]. We assume familiarity with [1]. Baez and Langford [1] work with unoriented 2-tangles, but combinatorial description can be easily modified to the oriented case. We briefly review this description, referring the reader to [1] for details.

We consider oriented unframed tangles with an even number of bottom endpoints and oriented cobordisms between these tangles.

The objects of the 2-category $\mathcal{C}$ are even length sequences s of pluses and minuses (indicating orientations of tangles near endpoints). Let $|s|$ denote half the length of s.

1-morphisms of $\mathcal{C}$ represent planar diagrams of generic tangles. The generating 1-morphisms are positive and negative crossings, U-turns, and the identity 1-morphisms. They are depicted in Figure 1. Different orientations of arcs in a diagram lead to different 1-morphisms. We denote U-turns by $\cap_{i,n}$ and $\cup_{i,n-1}$ and identity morphisms by Vert_n (in [9] we used Vert_{2n} instead).
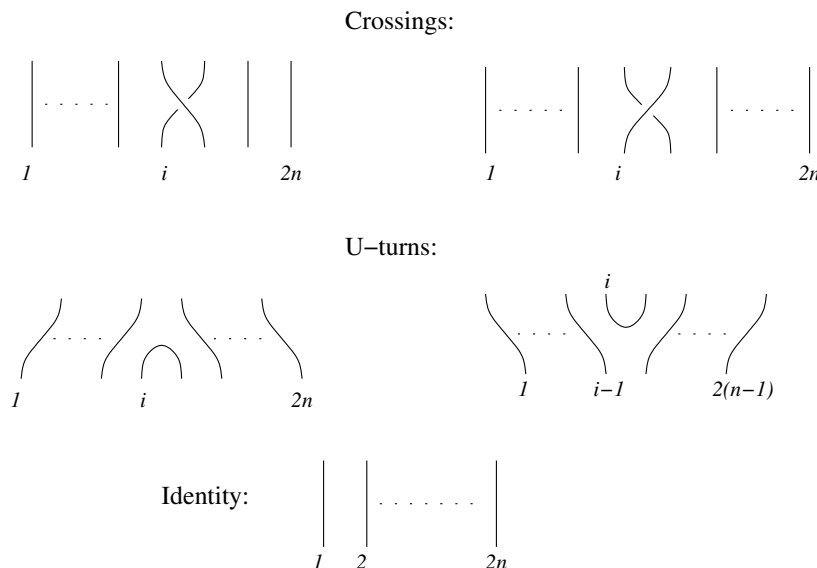
Crossings:

U−turns:

Identity:

FIGURE 1. Generating 1-morphisms of $\mathcal{C}$

1-morphisms are products of generating 1-morphisms. Orientations of arcs should be compatible when the diagrams are concatenated.

2-morphisms are combinatorial diagrams of tangle cobordisms, and depicted by "movies" of Roseman and Carter-Saito. The generating 2-morphisms are birth and death of a circle, saddle point (with compatible orientations), Reidemeister moves, a double point arc crossing a fold line, a cusp on a fold line, shifting relative heights of distant crossings and local extrema, and identity 2-morphisms. Generating 2-morphisms (except for identity morphisms) are depicted in Figures 2 and 3.

Each generating 2-morphism has several versions, obtained by

(a) reading the film from bottom to top rather than from top to bottom,

(b) changing between positive and negative crossings (the third Reidemeister move has many such versions),

(c) reflecting each frame about the x-axis,

(d) reflecting each frame about the y-axis,

(e) orienting strings in various ways.

Of course, for some moves some of these operations produce identical moves (and, for instance, operation (a) on a birth move produces a death move).

The two-morphisms in Figure 3 will be called T-move, H-move, and N-move, since these moves were labelled by T, H, and N in [1] (with subscripts which we omit).
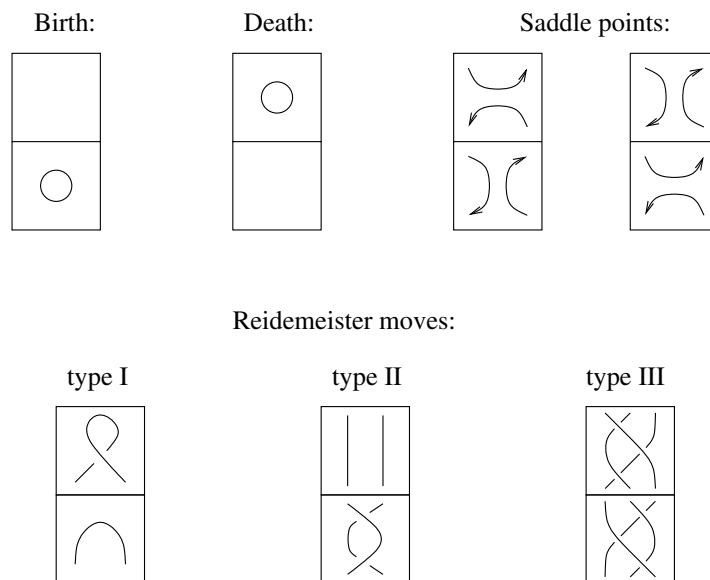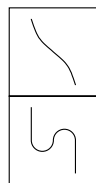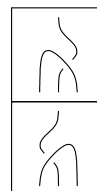
Birth: Death: Saddle points:

Reidemeister moves:

type I   type II   type III

FIGURE 2. Generating 2-morphisms

T−move: A cusp on a
        fold line.

H−move: A double point arc
        crossing a fold line.

N−move: Shifting relative heights
        of distant crossings
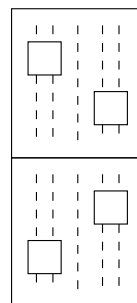   and local extrema:

FIGURE 3. Generating 2-morphisms

The height shifting morphism (N-move) has many versions, as we are free to put a U-turn or a crossing inside each small square of the top frame, add any number of strings separating the two small squares, and possibly invert the order of the film. An example is given in Figure 4.
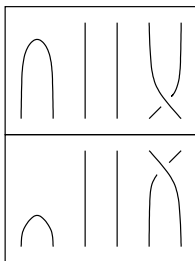

FIGURE 4. An example of height shifting

A complete set of defining relations on 2-morphisms is given by the 31 movie moves (see [2], [4, Section 2.5], or [1]). The first 30 of these moves are shown in Figures 5-9 at the end of the paper. Similar to modifications (a)-(e) of generating morphisms, there are modifications (a)-(e) of movie moves and they should be included in the list. See [1] for a detailed discussion.

Move 31 is not shown. It says that given horizontally composable 2-morphisms $\alpha : f \Rightarrow f'$ and $\beta : g \Rightarrow g'$, there is an equality $(\alpha \cdot \mathrm{Id})(\mathrm{Id} \cdot \beta) = (\mathrm{Id} \cdot \beta)(\alpha \cdot \mathrm{Id})$ of 2-morphisms from fg to $f'g'$.

Figures 5-7 show local moves, while Figures 8, 9 show semi-local moves. Little squares in semi-local moves could be U-turns or crossings.

3. Bimodule homomorphisms

For a ring A denote by $\mathcal{K}(A)$ the category of bounded complexes of A-bimodules up to homotopies of complexes. The objects of $\mathcal{K}(A)$ are bounded complexes of A-bimodules, and the morphisms are morphisms of complexes of bimodules quotiented by homotopic to 0 morphisms.

We say that a complex of bimodules $M \in \mathcal{K}(A)$ is *invertible* if there exists $N \in \mathcal{K}(A)$ such that $N \otimes_A M \cong A$ and $M \otimes_A N \cong A$ in $\mathcal{K}(A)$. Here A denotes the complex $0 \longrightarrow A \longrightarrow 0$ with A in cohomological degree 0 and the usual left and right multiplication action of A on itself.

Let $Z(A)$ be the center of A.

Proposition 1. *If M is invertible, then*

$$\mathrm{Hom}_{\mathcal{K}(A)}(M, M) \cong \mathrm{Hom}_{A \otimes A^\circ}(A, A) \cong Z(A).$$

Proof. The second isomorphism is obvious, since endomorphisms of A as an A-bimodule are multiplications by central elements.

Consider the following sequence of ring homomorphisms:

$$\mathrm{End}_{\mathcal{K}(A)}(M) \xrightarrow{f} \mathrm{End}_{\mathcal{K}(A)}(M \otimes_A N) \xrightarrow{g} \mathrm{End}_{\mathcal{K}(A)}((M \otimes_A N) \otimes_A M)$$
$$\cong \mathrm{End}_{\mathcal{K}(A)}(M \otimes_A (N \otimes_A M)) \cong \mathrm{End}_{\mathcal{K}(A)}(M \otimes_A A) \cong \mathrm{End}_{\mathcal{K}(A)}(M),$$

where f, respectively g, is tensoring with the identity endomorphism of N, respectively M. The composition gf is the identity, thus f is injective. Multiplication on the left by central elements makes each of the above rings a $Z(A)$-module, and f and g are $Z(A)$-module homomorphisms. f and g take identity endomorphisms to identity endomorphisms, and

$$\mathrm{End}_{\mathcal{K}(A)}(M \otimes_A N) \cong \mathrm{End}_{\mathcal{K}(A)}(A) = Z(A).$$

Therefore, f is surjective, since $f(\mathrm{id}) = \mathrm{id}$ generates $\mathrm{End}_{\mathcal{K}(A)}(M \otimes_A N)$ as a $Z(A)$-module. Thus, f and g are isomorphisms. $\qquad\square$

If A is graded, denote by $\mathcal{K}(A)$ the category of bounded complexes of graded A-bimodules (with grading-preserving differential) up to homotopies. The morphisms are grading-preserving homomorphisms of complexes (modulo homotopies). If M is an invertible complex in $\mathcal{K}(A)$, then $\mathrm{Hom}_{\mathcal{K}(A)}(M, M) \cong Z_0(A)$, the degree 0 component of the center of A. Furthermore, the group of automorphisms of M in $\mathcal{K}(A)$ is isomorphic to $Z_0^*(A)$, the group of invertible elements in $Z_0(A)$.

We now specialize to the rings H^n.

Proposition 2. *The only invertible central elements of degree 0 in H^n are ± 1:*

$$Z_0^*(H^n) \cong \{\pm 1\}.$$

Proof. A degree 0 element of H^n has the form $v = \sum v_a e_a$, where e_a is the minimal idempotent corresponding to the crossingless matching a and $v_a \in \mathbb{Z}$. For any a, b choose $x \in {}_a(H^n)_b, x \neq 0$. Then $vx = v_a x$ and $xv = v_b x$. Therefore, if v is central, $v_a = v_b$ for all a, b, and $v = m \sum e_a = m$, for some integer m, so that $Z_0(H^n) \cong \mathbb{Z}$. The proposition follows. $\qquad\square$

Remark. We investigated the center of H^n (and not just its degree 0 component) in [8]. It turned out to be isomorphic to the cohomology ring of the (n, n) Springer fiber.

Corollary 1. *If M is an invertible complex of graded H^n-bimodules, then* Id *and* $-$Id *are the only degree 0 automorphisms of M.*

We use notation $\mathcal{K}_n^m$ from [9] for the category of bounded complexes of geometric (H^m, H^n)-bimodules up to chain homotopies. A bimodule is geometric if it is isomorphic to a finite direct sum of bimodules $\mathcal{F}(a)\{i\}$, for flat tangles a and $i \in \mathbb{Z}$ (recall that $\{i\}$ denotes shift in the grading by i). Morphisms in $\mathcal{K}_n^m$ are grading-preserving homomorphisms of complexes up to chain homotopies.

From Corollary 1 we derive

Corollary 2. *If $f : M \to N$ is an isomorphism of invertible objects in $\mathcal{K}_n^n$, then the only other isomorphism of M and N is $-f$.*

For now on we assume that the reader is familiar with the construction of [9, Section 2], which to a surface S embedded in $\mathbb{R}^3$ and viewed as a cobordism between flat tangles a and b assigns a bimodule homomorphism $\mathcal{F}(S) : \mathcal{F}(a) \longrightarrow \mathcal{F}(b)$.

Standard cobordisms in $\mathbb{R}^3$ (the first is a birth move, the second and third are saddle points, the fourth is a death move)

$$\begin{aligned}
\mathrm{Vert}_{n-1} &\implies \cap_{i,n}\cup_{i,n-1}, \\
\cup_{i,n-1}\cap_{i,n} &\implies \mathrm{Vert}_n, \\
\mathrm{Vert}_n &\implies \cup_{i,n-1}\cap_{i,n}, \\
\cap_{i,n}\cup_{i,n-1} &\implies \mathrm{Vert}_{n-1}
\end{aligned}$$

induce grading-preserving bimodule homomorphisms

$$(1) \qquad\qquad H^{n-1} \longrightarrow \mathcal{F}(\cap_{i,n}) \otimes_{H^n} \mathcal{F}(\cup_{i,n-1})\{1\},$$

$$(2) \qquad\qquad \mathcal{F}(\cup_{i,n-1}) \otimes_{H^{n-1}} \mathcal{F}(\cap_{i,n})\{1\} \longrightarrow H^n,$$

$$(3) \qquad\qquad H^n \longrightarrow \mathcal{F}(\cup_{i,n-1}) \otimes_{H^{n-1}} \mathcal{F}(\cap_{i,n})\{-1\},$$

$$(4) \qquad\qquad \mathcal{F}(\cap_{i,n}) \otimes_{H^n} \mathcal{F}(\cup_{i,n-1})\{-1\} \longrightarrow H^{n-1}$$

(we used that $\mathcal{F}(\mathrm{Vert}_n) \cong H^n, \mathcal{F}(\cup_{i,n-1}\cap_{i,n}) \cong \mathcal{F}(\cup_{i,n-1}) \otimes_{H^{n-1}} \mathcal{F}(\cap_{i,n})$, etc.). Isotopies between compositions of these cobordisms translate into relations between homomorphisms. These relations imply that the functors of tensoring with $\mathcal{F}(\cap_{i,n})$ and $\mathcal{F}(\cup_{i,n-1})$ are biadjoint, up to grading shifts. Precisely, let $F_\cup$ be the functor of tensoring with $\mathcal{F}(\cup_{i,n-1})$ and $F_\cap$ the functor of tensoring with $\mathcal{F}(\cap_{i,n})$ (viewed as functors between categories of H^n and H^{n-1}-modules).

Proposition 3. *$F_\cup\{1\}$ is left adjoint to $F_\cap$, and $F_\cap\{-1\}$ is left adjoint to $F_\cup$.*

Corollary 3. *The only grading-preserving endomorphisms of bimodules $\mathcal{F}(\cap_{i,n})$ and $\mathcal{F}(\cup_{i,n-1})$ are multiplications by integers. The only grading-preserving automorphisms of bimodules $\mathcal{F}(\cap_{i,n})$ and $\mathcal{F}(\cup_{i,n-1})$ are Id and $-\mathrm{Id}$. Moreover, these bimodules have no graded endomorphisms of negative degree.*

Proof of Corollary 3. From adjointness,

$$\begin{aligned}
\mathrm{Hom}_{(n,n-1)}(\mathcal{F}(\cup_{i,n-1}),\mathcal{F}(\cup_{i,n-1})) &\cong \mathrm{Hom}_{(n-1,n-1)}(H^{n-1},\mathcal{F}(\cap_{i,n}\cup_{i,n-1})\{1\}) \\
&\cong \mathrm{Hom}_{(n-1,n-1)}(H^{n-1},H^{n-1} \oplus H^{n-1}\{2\}) \\
&\cong \mathrm{Hom}_{(n-1,n-1)}(H^{n-1},H^{n-1}) \\
&\cong \mathbb{Z}.
\end{aligned}$$

Subscripts of the form (m,n) in the above formula mean that the homomorphisms considered are those of graded (H^m,H^n)-bimodules. We used that

$$\mathrm{Hom}_{(n-1,n-1)}(H^{n-1},H^{n-1}\{k\}) = 0$$

for any positive k, since the ring H^{n-1} is $\mathbb{Z}_+$-graded. Similar computations establish the result for $\mathcal{F}(\cap_{i,n})$ and the last claim of the corollary. $\qquad\square$

Corollary 4. *If M is a tensor product of $\mathcal{F}(\cap_{i,n})$ and invertible complexes of bimodules, and if $f : M \to N$ is an isomorphism in $\mathcal{K}_n^{n-1}$, then the only other isomorphism from M to N is $-f$. The same with $\cup_{i,n-1}$ instead of $\cap_{i,n}$.*

More generally, let b be a flat tangle without closed components, (circles), with k arcs connecting bottom endpoints, l arcs connecting top endpoints, and some number of arcs connecting a top endpoint with a bottom endpoint. Let $W(b)$ be the reflection of b about the x-axis. Representing b as a product of U-turns and using Proposition 3 repeatedly we obtain the following.

Proposition 4. *The functor of tensoring with the bimodule $\mathcal{F}(W(b))\{k-l\}$ is left adjoint to tensoring with $\mathcal{F}(b)$ and the functor of tensoring with $\mathcal{F}(W(b))\{l-k\}$ is right adjoint to tensoring with $\mathcal{F}(b)$.*

Corollary 5. *If b is a flat tangle without closed components, then the only grading-preserving endomorphisms of the bimodule $\mathcal{F}(b)$ are multiplications by integers, the only grading-preserving automorphisms are Id and $-\mathrm{Id}$, and $\mathcal{F}(b)$ has no endomorphisms of negative degree. If $f : M \to \mathcal{F}(b)$ is an isomorphism in $\mathcal{K}_n^m$, then the only other isomorphism between M and $\mathcal{F}(b)$ is $-f$.*

4. The 2-functor

We introduce two 2-categories $\mathbb{K}$ and $\widehat{\mathbb{K}}$.

Objects of $\mathbb{K}$ are non-negative integers, 1-morphisms from n to m are objects of $\mathcal{K}_n^m$, and 2-morphisms between $M, N \in \mathcal{K}_n^m$ are $\mathrm{Hom}_{\mathcal{K}_n^m}(M, N)$, grading-preserving morphisms of complexes of bimodules up to chain homotopies. Composition of 1-morphisms is given by tensor product of complexes.

The 2-category $\widehat{\mathbb{K}}$ has the same objects and 1-morphisms as $\mathbb{K}$ but the 2-morphisms are

$$\mathrm{Hom}_{\widehat{\mathbb{K}}}(M, N) \overset{\mathrm{def}}{=} \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{K}_n^m}(M, N\{i\})/\{\pm 1\},$$

that is, the morphisms are all homomorphisms (not just grading-preserving), and each homomorphism is identified with its negative. The set of 2-morphisms between two 1-morphisms is no longer an abelian group.

We next construct a 2-functor $\mathcal{F} : \mathcal{C} \to \widehat{\mathbb{K}}$. This 2-functor takes object s of $\mathcal{C}$ to the object $|s|$ of $\widehat{\mathbb{K}}$. It takes generating 1-morphisms of $\mathcal{C}$ to complexes of bimodules in the same way as in [9, Sections 2.7, 3.4]. Recall that a U-turn b (and, more generally, any flat tangle) is taken to the complex $0 \longrightarrow \mathcal{F}(b) \longrightarrow 0$, where $\mathcal{F}(b)$ is the bimodule associated to b. A crossing r gives rise to its two resolutions $r(0)$ and $r(1)$ and a grading-preserving bimodule map $\psi : \mathcal{F}(r(0)) \longrightarrow \mathcal{F}(r(1))\{-1\}$. Then $\mathcal{F}(r)$ is defined as the complex

$$0 \longrightarrow \mathcal{F}(r(0)) \overset{\psi}{\longrightarrow} \mathcal{F}(r(1))\{-1\} \longrightarrow 0$$

with a suitable grading shift computed from the orientation of r near its crossing.

The 2-functor $\mathcal{F}$ takes composition of 1-morphisms to the tensor product of complexes:

$$\mathcal{F}(ab) \overset{\mathrm{def}}{=} \mathcal{F}(a) \otimes_{H^n} \mathcal{F}(b),$$

where a, respectively b, has $2n$ bottom, respectively, $2n$ top endpoints.

To a Reidemeister move between diagrams a and b (see Figure 1) we assign an isomorphism of bimodule complexes $\mathcal{F}(a) \overset{\cong}{\longrightarrow} \mathcal{F}(b)$ constructed in [9, Section 4]. The Reidemeister III move has several versions, depending on the directions of overcrossings. In [9] we described an isomorphism in $\mathcal{K}_n^n$ between complexes $\mathcal{F}(a)$ and $\mathcal{F}(b)$ for only one version of this move. Other versions can be expressed via compositions of this version with isotopies and type II moves. The compositions induce isomorphisms between $\mathcal{F}(a)$ and $\mathcal{F}(b)$ which we assign to these other version of the Reidemeister III move. Note that, since Reidemeister move diagrams in Figure 1 are either braids or composition of a U-turn and a braid, a grading-preserving isomorphism between $\mathcal{F}(a)$ and $\mathcal{F}(b)$ is unique up to minus sign, by Corollaries 2, 4.

The diagrams of birth, death, saddle point, and T-move do not involve crossings. These movies can be viewed as presentations of surfaces embedded in $\mathbb{R}^3$, and to them we assign bimodule homomorphisms using the construction of Proposition 5 of [9]. To the birth 2-morphism we assign the unit map $\iota : \mathbb{Z} \to \mathcal{A}$, and to the death 2-morphism the counit map $\epsilon : \mathcal{A} \to \mathbb{Z}$. More precisely, birth and death moves happen inside Vert_n diagrams, and the maps are

$$\iota \otimes \mathrm{Id}_{H^n} : H^n \longrightarrow \mathcal{A} \otimes_{\mathbb{Z}} H^n, \quad \epsilon \otimes \mathrm{Id}_{H^n} : \mathcal{A} \otimes_{\mathbb{Z}} H^n \longrightarrow H^n.$$

To the diagrams of saddle point cobordisms between Vert_n and $\cup_{i,n-1}\cap_{i,n}$ we assign bimodule maps (2), (3). These maps are, up to sign, the only maps of degree 1 between $\mathcal{F}(\mathrm{Vert}_n) \cong H^n$ and $\mathcal{F}(\cup_{i,n-1}\cap_{i,n})$ that generate the abelian group (isomorphic to $\mathbb{Z}$) of all degree 1 homomorphisms between these bimodules.

The natural isotopy between the two diagrams a, b in the H-move (Figure 3) induces an isomorphism of complexes $\mathcal{F}(a) \cong \mathcal{F}(b)$, which we assign to this 2-morphism.

A frame of an N-move between diagrams a and b contains two little squares, and each square is either a U-turn or a crossing. The N-move is an isotopy from a to b. This isotopy induces a canonical isomorphism of complexes $\mathcal{F}(a) \xrightarrow{\cong} \mathcal{F}(b)$ (see [9, Sections 4.1, 4.2]), which we assign to the N-move.

Theorem 1. *The above correspondence extends uniquely to a 2-functor*

$$\mathcal{F} : \mathcal{C} \to \widehat{\mathbb{K}}.$$

This theorem is proved in Section 5.

$\mathcal{C}$ is a combinatorial realization of the 2-category $\mathcal{T}$ of tangle cobordisms, that is, the natural 2-functor $\mathcal{C} \to \mathcal{T}$ is an equivalence of 2-categories; see [1]. This result is also valid for oriented tangles.

As a corollary, we obtain a 2-functor, also denoted $\mathcal{F}$, from the 2-category $\mathcal{T}$ of even unframed oriented tangle cobordisms to $\widehat{\mathbb{K}}$. The homomorphism of complexes of graded (H^m, H^n)-bimodules assigned to the cobordism S between (m,n)-tangles has degree $n + m - \chi(S)$, where $\chi(S)$ is the Euler characteristic of S.

5. Proof

When looking at a particular movie move we denote the top frame by b_1, the bottom frame by b_2, the left movie by S_l and the right movie by S_r.

Movies S_l and S_r induce homomorphisms $\mathcal{F}(S_l)$ and $\mathcal{F}(S_r)$ from $\mathcal{F}(b_1)$ to $\mathcal{F}(b_2)$. We need to show that $\mathcal{F}(S_l) = \pm\mathcal{F}(S_r)$ in $\mathcal{K}_n^m$.

Moves 1, 2, 3, 4, 5 say that composing a Reidemeister move with its inverse is equivalent to doing nothing. The isomorphism in $\mathcal{K}_n^m$ assigned to the inverse of a Reidemeister move equals the inverse of the isomorphism assigned to the move. Therefore, $\mathcal{F}(S_l) = \mathcal{F}(S_r)$ for each of these moves.

Movies S_l and S_r in move 6 consist of Reidemeister moves and relative height shifts of distant crossings. The complexes $\mathcal{F}(b_1)$ and $\mathcal{F}(b_2)$ are invertible (since b_1 and b_2 are braids), and

$$\mathcal{F}(S_l), \mathcal{F}(S_r) : \mathcal{F}(b_1) \longrightarrow \mathcal{F}(b_2)$$

are two isomorphisms of these complexes in $\mathcal{K}_n^n$. By Corollary 2, either $\mathcal{F}(S_l) = \mathcal{F}(S_r)$ or $\mathcal{F}(S_l) = -\mathcal{F}(S_r)$.

This proof works simultaneously for all versions of move 6. Identical argument takes care of moves 12, 13, 23a and 25 (and of moves 3, 4, 5 as well, although the latter have already been dealt with).

Each movie in move 7 is a composition of Reidemeister moves, thus, $\mathcal{F}(S_l)$ and $\mathcal{F}(S_r)$ are grading-preserving isomorphisms (in $\mathcal{K}_n^{n-1}$) of complexes $\mathcal{F}(b_1)$ and $\mathcal{F}(b_2)$. Since b_1 and b_2 are given by composing $\cap_{i,n}$ with braids, $\mathcal{F}(b_1)$ and $\mathcal{F}(b_2)$ are tensor products of $\mathcal{F}(\cap_{i,n})$ with invertible complexes (the index i is different for b_1 and b_2). By Corollary 4, $\mathcal{F}(S_l)$ differs from $\mathcal{F}(S_r)$ by at most a minus sign. Other versions of this move follow suit.

Identical arguments takes care of moves 11, 14, and 26.

Moves 8, 9, 10, 23b, 24 do not involve any crossings and the invariance of $\mathcal{F}$ follows from Proposition 6 of [9], since these moves are saying that certain surfaces in $\mathbb{R}^3$ are isotopic.

Both movies in move 21 consist of isotopies and a Reidemeister move. Therefore, $\mathcal{F}(S_l)$ and $\mathcal{F}(S_r)$ are isomorphisms in $\mathcal{K}_n^n$. The bottom diagram b_2 is a flat tangle without closed components (circles). Corollary 5 implies that any two isomorphisms from $\mathcal{F}(b_1)$ to $\mathcal{F}(b_2)$ differ by sign at most. Similar arguments take care of moves 15-20 (use Corollary 5 and its generalization from b to $b_1 b b_2$, where b_1 and b_2 are braids). Alternatively, the invariance of $\mathcal{F}$ under semi-local moves 15-20, 22 follows by observing that height shifts of U-turns and crossings do not do anything to our complexes of bimodules and maps between them.

The first frame change in both movies in move 28 is birth, which is then followed by a Reidemeister move and an isotopy (H-move). Decompose $S_l = R_l Q_l$ and $S_r = R_r Q_r$, where Q_l, Q_r are births. Denote by b_l', b_r' second frames from the top in the left and right movies. Note that $\mathcal{F}(b_l') \cong \mathcal{F}(b_r') \cong \mathcal{A} \otimes H^n$, and $\mathcal{F}(R_r)^{-1}\mathcal{F}(R_l) : \mathcal{F}(b_l') \to \mathcal{F}(b_r')$ is a grading-preserving isomorphism, while $\mathcal{F}(Q_l), \mathcal{F}(Q_r) : H^n \longrightarrow \mathcal{A} \otimes_{\mathbb{Z}} H^n$ have degree -1. Both $\mathcal{F}(Q_r)$ and $\mathcal{F}(R_r)^{-1}\mathcal{F}(S_l)$ generate the abelian group $\mathbb{Z}$ of degree -1 homomorphisms from $H^n \cong \mathcal{F}(b_1)$ to $\mathcal{A} \otimes_{\mathbb{Z}} H^n \cong \mathcal{F}(b_r')$. Therefore, $\mathcal{F}(Q_r)$ and $\mathcal{F}(R_r)^{-1}\mathcal{F}(S_l)$ differ by at most a minus sign, and $\mathcal{F}(S_l), \mathcal{F}(S_r)$ differ by at most a minus sign.

Invariance of $\pm\mathcal{F}$ under moves 22 and 27 follows from similar arguments.

Both movies in move 29 consist of a Reidemeister move followed by a saddle point 2-morphism. Both movies induce degree 1 homomorphisms from $\mathcal{F}(b_1)$ to $\mathcal{F}(b_2)$. Since the homomorphism assigned to the saddle point generates the abelian group (isomorphic to $\mathbb{Z}$) of degree 1 homomorphisms from $\mathcal{F}(\cup_{i,n-1}\cap_{i,n})$ to $H^n \cong \mathcal{F}(\text{Vert}_n)$, we see that both $\mathcal{F}(S_l)$ and $\mathcal{F}(S_r)$ are generators of

$$\text{Hom}_{\mathcal{K}_n^n}(\mathcal{F}(b_1)\{1\}, \mathcal{F}(b_2)) \cong \mathbb{Z},$$

and differ by at a most minus sign. Move 30 follows similarly.

Given a ring A and homomorphisms $f_1 : M_1 \to N_1$, resp. $f_2 : M_2 \to N_2$ of complexes of right, resp. left, A-modules, the map

$$f_1 \otimes f_2 : M_1 \otimes_A M_2 \longrightarrow N_1 \otimes_A N_2$$

can be written in two ways: as $(f_1 \otimes \text{Id})(\text{Id} \otimes f_2)$ and as $(\text{Id} \otimes f_2)(f_1 \otimes \text{Id})$. This observation takes care of move 31. $\qquad\square$
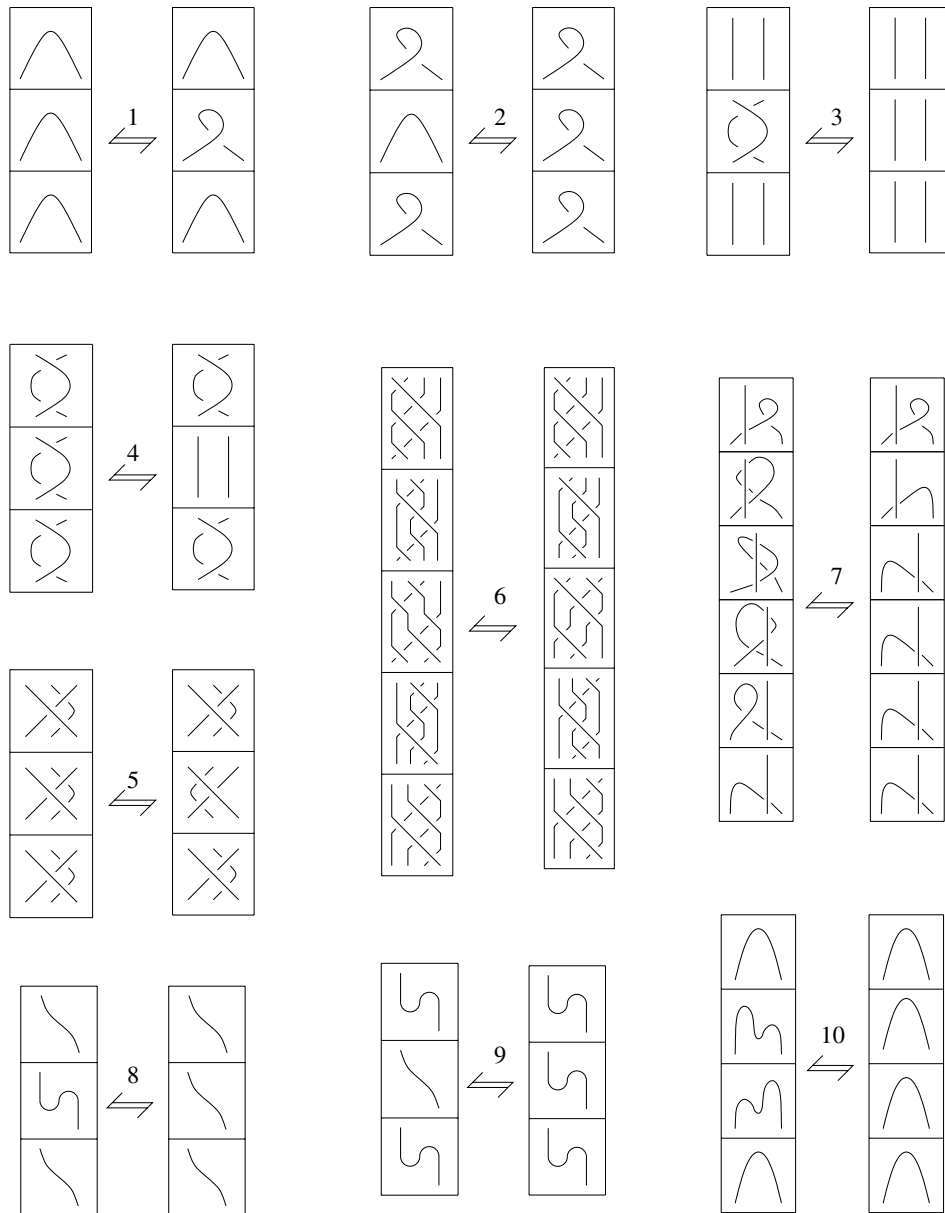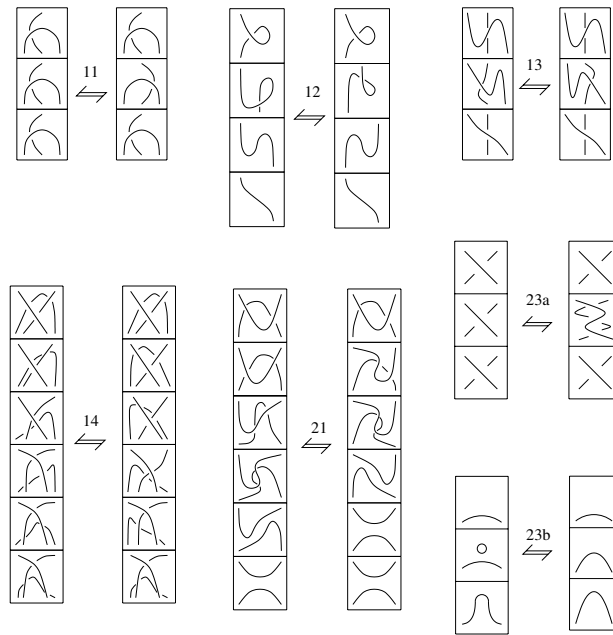
FIGURE 5. Movie moves 1-10

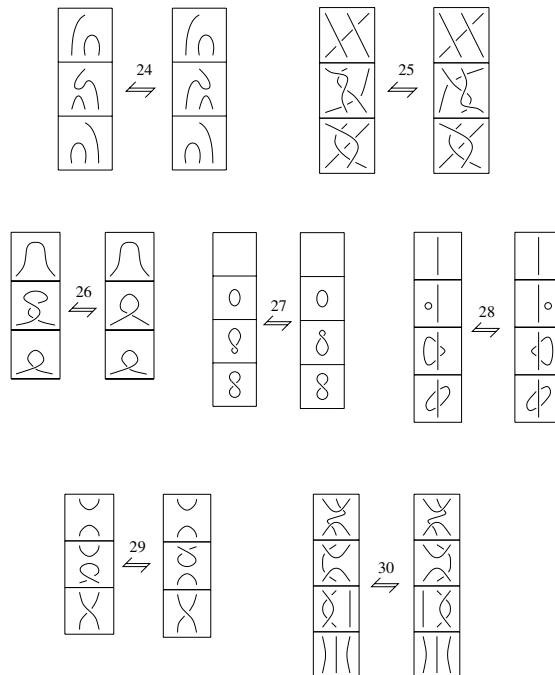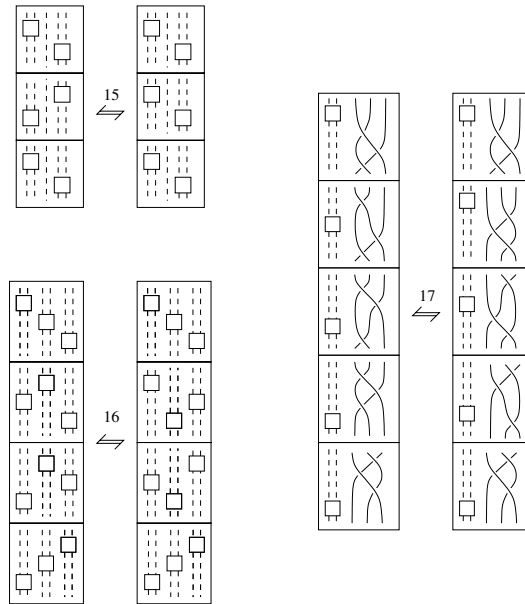FIGURE 6. Movie moves 11-14, 21, 23

FIGURE 7. Movie moves 24-30
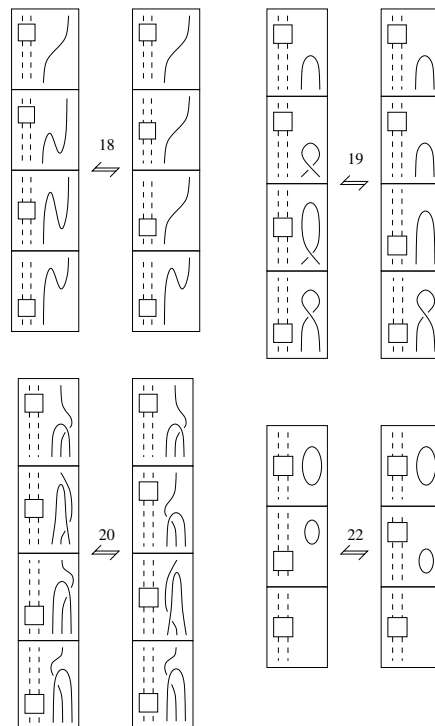
FIGURE 8. Movie moves 15-17

FIGURE 9. Movie moves 18-20, 22

References

[1] J. C. Baez and L. Langford, Higher-dimensional algebra IV: 2-Tangles, *Adv. Math.*, 180(2):705–764, 2003. MR2020556

[2] J. S. Carter, J. H. Rieger, and M. Saito, A combinatorial description of knotted surfaces and their isotopies, *Advances in mathematics*, 127:1–51, 1997. MR1445361 (98c:57023)

[3] J. S. Carter and M. Saito, Reidemeister moves for surface isotopies and their interpretation as moves to movies, *Journal of Knot Theory and its Ramifications*, 2(3):251–284, 1993. MR1238875 (94i:57007)

[4] J. S. Carter and M. Saito, *Knotted surfaces and their diagrams*, Mathematical surveys and monographs, 55. AMS, 1998. MR1487374 (98m:57027)

[5] J. Fischer, 2-categories and 2-knots, *Duke Mathematical Journal*, 75(2):493–526, 1994. MR1290200 (95k:18002)

[6] M. Jacobsson, An invariant of link cobordisms from Khovanov's homology theory, arXiv:math.GT/0206303.

[7] V. M. Kharlamov and V. G. Turaev, On the definition of the 2-category of 2-knots, In *Mathematics in St. Petersburg*, Amer. Math. Soc. Transl. Ser. 2, 174. Amer. Math. Soc., Providence, RI, 1996. MR1386661 (98g:18006)

[8] M. Khovanov, Crossingless matchings and the cohomology of (n, n) Springer varieties, preprint arXiv:math.QA/0202113.

[9] M. Khovanov, A functor-valued invariant of tangles, Algebraic and Geometric Topology **2** (2002), 665–741. MR1928174 (2004d:57016)

[10] M. Khovanov, A categorification of the Jones polynomial, *Duke Math J.*, 101(3):359–426, 1999. MR1740682 (2002j:57025)

[11] D. Roseman, Reidemeister-type moves for surfaces in four dimensional space, *Knot theory* (Warsaw, 1995), Banach Center Publ., 42. Polish Acad. Sci., Warsaw, 1998, pp. 347–380.

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