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AN INVENTORY MODEL WITH PRICE AND CREDIT INSTALLMENTS-DEPENDENT DEMAND

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Abstract:
Financial capability is one of the primary drivers for buyers to make purchases. Therefore, sellers must set an optimum selling price and consider trade credit facilities to attract more demand. This paper proposes an inventory decision model in which customer demand depends on the price and number of credit installments to serve low-abled buyers. This study has developed a demand function with a positive impact on installment policies and the effect of the selling price. Two models have been formulated to optimize the selling price and positive stock time, m total profit, with and without installment policies. Then, numerical examples and sensitivity analysis illustrate the proposed model for different cases. The study has found that the selling price and positive stock time can be optimized. Profits can be higher in the case of an installment facility than in the case without an installment facility. It shows positive responses from the buyer to the installment policy.

Key words: *inventory, pricing, installment-dependent demand, trade credit*

INTRODUCTION

In business, it is essential to entice more customers to buy inventories. Sellers offer a discount on the selling price, ensure warranty, and promote products to attract customers. Another common practice is offering a delay in payment or a trade-credit facility with a certain amount of down payment. It will help the low-abled buyers pay only part of the initial purchase price and then complete the payment in several installments [1].

The benefit of delaying payment is that the buyer (e.g., a retailer) is not required to invest much capital to purchase inventories and may profit from selling them to other customers within the permissible delay period specified by suppliers. Moreover, the supplier can apply this agreement as a sales promotional tool to attract new customers and promote new and unproven products. Installment policy within or out of the permissible delay period is a tool for inventory management problems. Goyal [2] studied installments in the permitted deferral period, and Chand and Ward [3] analyzed Goyal's concern under the traditional financial request amount model's suppositions, acquiring various outcomes. In the same way, Mashud et al. [4] discussed the trade-credit period with preservation technology to reduce deterioration. In addition to the amount of the down payment and the duration of the credit period, the number of installments is also a factor, as it dictates the total funds the buyer must set aside for each payment. Here, we introduce an economic order quantity (EOQ) model offering an installment facility and its impact on the demand rate. The installment facility has two benefits to the seller: (1) it attracts more buyers who are financially low-abled, and (2) it can be considered as an alternative to the offer of price discount because it neither elicits price cutting from competitors nor does it introduce permanent price reductions. The effect of delay in payment on customer demand rate attracted Heydari et al., [5], Marchi et al., [6], and Bi et al., [7] in their recent research. The research was based on the concept that trade credit will increase buyers' purchase volume due to additional liquidity and attract new customers. However, the studies only examined the duration of the delay periods.

The selling price of a product or service is a crucial factor that impacts consumers' purchase intention. Therefore, sellers must select the optimal pricing to ensure a profit and attract customers. The effect of price on the EOQ model has been studied and considered in the demand function [8, 9, 10]. Other researchers combined the price and other factors such as stock level [11, 12], promotion effort [13, 14], and time for deteriorating products [15]. Numerous studies have examined the topic of increasing consumer purchases by providing a discount on the selling price, ensuring warranty, promoting products, etc., but analyses that target low-abled customers' liquidity have been a fascinating field of research at present. In this study, we introduce a new EOQ model offering an installment facility to make the products affordable for customers. Further, one can find the positive effect of the installment on the customer's demand. The installment facility draws the customer's attention to purchase products, i.e., promoting the business [16]. On the other hand, buyers with limited financial resources might utilize this installment facility to purchase their products [17]. In this approach, the major contributions of the proposed model are as follows:

a. We introduce an EOQ model with price and credit installments-dependent demand. The policy can stimulate customer demand while assisting financially low-abled customers by providing installment plans with a down payment.
b. The proposed model optimizes the selling price and positive inventory stock time to ensure that the seller maximizes the total profit. A solution algorithm is formulated for this purpose.
c. Two cases are studied, one with an installment facility and another without an installment facility. Further, numerical examples and sensitivity analysis illustrate the proposed model and its characteristics.

In the other parts of this paper, the next section presents the relevant literature review of this study. Then, the assumptions and notations are presented. According to the assumption and its solutions, the mathematical model formula is formulated and followed by the algorithm for solution procedures. The numerical results of the problems and the sensitivity analysis are presented respectively. A discussion of the results is presented and in the end, the conclusion with future scopes of research is provided.

LITERATURE REVIEW

Researchers have introduced various techniques to manage inventories. A classical EOQ model was introduced by Harris [18] to manage the stocks in which demand is considered constant. Then Resh et al. [19] exaggerate the EOQ model with variable demand rates. Further, Kim et al. [20] proposed the EOQ model with a price-dependent demand rate, which is more specific. Some researchers considered demand as price dependent in a different pattern [21, 22, 23]. Pando et al. [12] formulated such an inventory model in which demand is price and stock-dependent. Demand is also considered a function of price, promotion/advertisement, and time [13, 14]. Accordingly, demand is dependent on various market factors as addressed by van Steenbergen and Mes [24]. Therefore, the research on inventory management with variable demand is a continuous. In the same way, this work considers the variable demand pattern to make it more appropriate for natural business phenomena. Customers are pondering about the price of a product before making a purchase. Therefore, our study has included the effect of price on demand.

In this process, the EOQ model is being modified for the time being. A number of studies have addressed the flexibility of payment to attract buyers. Goyal [2] studied an EOQ model with a permissible payment delay. The issues of "delay in payment, as an incentive system, in the EOQ or EPQ models" were discussed by Glock et al., [25]. According to Molamohamadi et al., [26], payment delays can be classified into four types depending on the period of payment. Accordingly, Jain and Aggarwal [27] built an inventory model with a trade credit policy in which demand is inflation induced. Pal et al. [11] implemented permissible delay in payments in an inventory model in which demand is stock and price-dependent, whereas Jaggi et al., [28] discussed different trade credit periods with the EOQ model for allowing shortages. Geetha and Uthayakumar [29] allow permissible delay payments and partial backlogging, whereas, Ghoreishi et al., [30] included delay in payments and customer return policy to attract customers. Again, the EOQ model with trade credit, shortages, and inspection for defective items simultaneously appeared in Zhou et al. [31]. Mohanty et al., [32] extended trade credit studies by considering preservation technology investment. Mashud [9] considered trade credit and price-sensitive demand with fully backlogged shortages in an EOQ model. Recently, Ghosh et al., [17] studied the EOQ model with delay in payment by allowing multiple installments after order delivery. Meanwhile, Wang et al., [16] identified different purchase behavior of the customers when the seller offered installment payment services. Thus, delayed payment with installments is an appealing research area in inventory management. According to our best knowledge, this is the first integration of the effect of installment in demand for EOQ inventory model studies. Consequently, we addressed payment flexibility for setting the number of installments among sellers and buyers within a complete replenishment cycle in our present study.

Providing price discounts is another inventory management technique to attract more customers. Several studies on EOQ inventory models frequently use this technique. Makoena and Olufemi [33] and Hasan et al., [34] constructed an EOQ model with a price discount facility for deteriorating items. Alfares and Ghaithan [35]

produced a model with quantity discounts, and Lin [36] combined discounts and imperfect quantities with EOQ. More recently, Hasan et al., [10] introduced the price discount effect on demand, but the discount is not always compelling to financially low-abled customers. By considering the issues of financially low-abled customers, there is scope to launch other policies to serve them. This approach represents the installment policy to convince more customers to make purchases instead of offering price discounts.
The inventory shortage is a significant challenge for mitigating the customer's demand. Molamohamadi et al., [37] allowed shortages, later policy, and trade credit to consider the retailer's ordering decisions. Geetha and Uthayakumar [29] also allowed permissible delay payments in which shortage is partial backlogging. Shaikh et al., [38] studied an EOQ model with the discount facility and shortage challenge, which is partially backlogged. The challenge of the shortage of products is still within the research interest. As a result, we studied our proposed model under partially backlogged shortages. Our study proposes an EOQ model with partially backlogging shortages where the demand is price and installment-dependent. From the above literature review, it is evident that our current approach has never been studied.

METHODOLOGY

First, notations for defining the mathematical model and the assumptions for describing the environment of the proposed model are presented in this section. Notations are listed in Table 1 and are used throughout the paper.

Table 1
List of notations

Notations	Description
μ	Area parameter
η	Selling price coefficient
n	Number of installments
p_1	Amount of down payment
c_h	Holding cost per unit
c_l	Cost due to lost sales per unit
c_p	Purchase cost per unit
S	Maximum stock per cycle
R	Shortages quantity
Q	Total quantity sold
$TP(p,t_f)$	Total profit per unit of time
I_e	Interest rate
Decision variable	
t_f	Positive stock time
p	Selling price

Assumptions

The proposed inventory model works under the following assumptions:

a. A single type of product is purchased and all items are of good quality. Further, the rate of replenishment from the supplier is infinite while lead time is negligible similar to Mashud et al. [1] and San-José et al., [14]. Hence, the order quantity Q will arrive instantly as shown in Fig. 1.

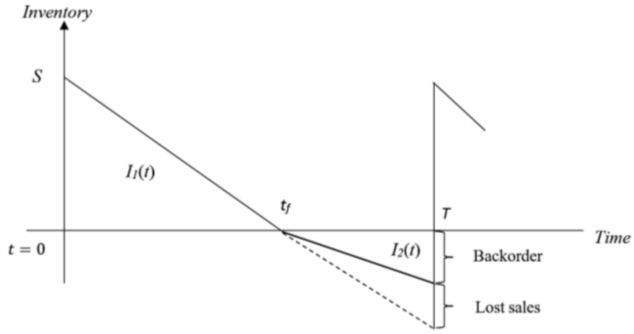

Fig. 1 The pictorial appearance of I(t): Inventory vs. time

b. To study the effect of installment on the customer's demand, there are two cases as follows:
   Case I: EOQ model with installment facility where interest should be charged.
   Case II: EOQ model without installment facility
c. The demand for a single product is considered. The demand rate is as follows:

$$D(p,n) = \begin{cases} \mu - \eta p + \delta p_1; & \text{for Case 1} \\ \mu - \eta p & ;\ \text{for Case 2} \end{cases}$$

where:
   $\mu > 0, \eta > 0, \delta = (\frac{n-1}{n})$ and $p > p_1$; p_1 is the down payment, which will be a part of the original price p ($\omega p_1 = p$; $\omega > 1$)), and n is the number of installments which is a positive integer.

d. For simplicity, we assumed $\omega = 2$ throughout the calculation to determine the formulae because the down payment is considered constant for the current arrangement.
e. Shortages are allowed for this arrangement and are partially backlogged with the rate of

$$B(t) = \frac{1}{1+\gamma(T-t)},$$

where:
   γ is the backlogging parameter $0 < \gamma \le 1$.
   Positive inventory level $I_1(t)$ is when $t \in [0, t_f]$ and inventory level $I_2(t)$ is negative when $t \in [t_f, T]$. The shortages are backlogged at the next arrival similar to Geetha and Uthayakumar [29] and Shaikh et al. [38].
f. There is no effect of product deterioration and inflation on costs.

In Fig. 1, the stocks in the warehouse are at a maximum S at $t = 0$. The inventory depletes due to demand during $[0, t_f]$ and becomes zero at $t = t_f$. Since full market demand is not satisfied after $t = t_f$, the shortage R is raised at the time $[t_f, T]$. The whole process will be replenished in the next cycle.
Therefore, the inventory system described by the resulting differential equations and considering demand $D = \mu - \eta p + \delta p_1$ are:

$$\frac{dI(t)}{dt} = \begin{cases} -D & ; 0 \le t \le t_f \\ -\frac{D}{1+\gamma(T-t)} & ; t_f \le t \le T \end{cases} \tag{1}$$

$$I(t) = \begin{cases} I_1(t) & ; 0 \le t \le t_f \\ I_2(t) & ; t_f \le t \le T \end{cases} \tag{2}$$

with conditions $I_1(t) = S$ at $t = 0, I_1(t) = I_2(t) = 0$ at $t = t_f$ and $I_2(T) = -R$.

Solving Eqs. (1) and (2), we have the following results:

$$I_1(t) = -Dt + Dt_f \tag{3}$$

$$I_2(t) = \frac{D}{\gamma}\log\{1 + \gamma(T - t)\} - \frac{D}{\gamma}\log\{1 + \gamma(T - t_f)\} \tag{4}$$

Maximum stock,

$$S = Dt_f \tag{5}$$

and shortage

$$R = \frac{D}{\gamma}\log\{1 + \gamma(T - t_f)\} \tag{6}$$

Sales Revenue

The sales revenue depends on customer demand, shortage, interest, selling price, and positive stocks. The following mathematical formula calculates the revenue for our arrangement.

Revenue :

$$SR = p\int_0^{t_f} Ddt + Rp + (p - p_1)I_e(R + S)$$
$$= \begin{bmatrix} \frac{p\left(\frac{p\delta}{2}-p\eta+\mu\right)\left(\gamma(T-t_f)+\frac{1}{2}\gamma^2(T-t_f)^2\right)}{\gamma} + p\left(\frac{p\delta}{2} - p\eta + \mu\right)t_f \\ +\frac{1}{2}pi_e\left(\frac{\left(\frac{p\delta}{2}-p\eta+\mu\right)(\gamma(T-t_f)+\frac{1}{2}\gamma^2(T-t_f)^2)}{\gamma} + \left(\frac{p\delta}{2} - p\eta + \mu\right)t_f\right) \end{bmatrix} \tag{7}$$

Inventory Costs

The ordering cost, inventory holding cost, purchase cost, and lost sale cost are calculated in this section, as follows: Every replenishment cycle needs to expense some due to processing the order from the supplier, termed as order cost (*OC*). Our inventory system considers *OC* is constant. After receiving the ordered inventories at the warehouse, they will be held before getting sold out. This period from receiving to depleted/sold is the holding period for each inventory. In this holding period, to manage the products at the warehouse, a cost named holding cost (*HC*) arises. If c_h is the per-unit holding cost for inventories $I_1(t)$, then the total holding cost is calculated by the following formula.

Holding cost:

$$\begin{aligned} HC &= c_h\left[\int_0^{t_f} I_1(t)dt\right] = c_h\left(-\frac{1}{2}Dt_f^2 + St_f\right) \\ &= \frac{1}{2}\left(\frac{p\delta}{2} - p\eta + \mu\right)c_h t_f^2 \end{aligned} \tag{8}$$

The cost due to purchase products is called the purchase cost (*PC*). Consider the unit purchase cost is c_p, then the total purchase cost is:

$$PC = c_p(S + R) \tag{9}$$

Due to shortages, retailers will lose some opportunities to sell. This loss is termed lost sale cost (*LC*), which is derived from the following formula:

$$\begin{aligned} LC &= c_l\left[\int_{t_f}^{T} D[1 - B(t)]dt\right] = c_l D\left[(T - t_f) + \frac{\ln\{1+\gamma(T-t_f)\}}{\gamma}\right. \\ &= \left(\frac{p\delta}{2} - p\eta + \mu\right)c_l\left(T + \frac{\gamma(T-t_f)+\frac{1}{2}\gamma^2(T-t_f)^2}{\gamma} - t_f\right) \end{aligned} \tag{10}$$

Profit

Total inventory profit X is derived by excluding all costs from the total sales revenue, i.e., *X* = sales revenue – ordering cost – holding cost – cost due to lost sale – purchase cost.

Then *X* = *SR* – *OC* – *HC* – *CL* – *PC*

(i) With installment facility

With an installment policy, customers will not pay the total value of products when they buy them. Customers only pay the down payment, and the rest will be paid through the installment process. This installment policy allows affording an expensive product with less expense. Therefore, low-abled customers will also be able to purchase a product that was not affordable to them without an installment facility. In this way, the seller can cover more customers in the market. The total profit per cycle is:

$$TP_1(p, t_f) = \frac{X}{T}$$

$$TP_1(p, t_f) = \frac{1}{T}[SR - OC - HC - CL - PC]$$

$$TP_1(p,t_f) = \begin{bmatrix} m_1 + pm_2 + p^2m_3 + pt_fm_4 + p^2t_fm_5 \\ +p^2t_f^2m_6 + t_fm_7 + t_f^2m_8 + pt_f^2m_9 \end{bmatrix} - \frac{1}{T}[OC] \tag{11}$$

where:

$m_1 = \frac{1}{T}\left[-2T\mu c_l - \frac{1}{2}T^2\gamma\mu c_l - T\mu c_p - \frac{1}{2}T^2\gamma\mu c_p\right]$

$m_2 = \frac{1}{T}\begin{bmatrix} T\mu + \frac{1}{2}T^2\gamma\mu + \frac{1}{2}T\mu i_e + \frac{1}{4}T^2\gamma\mu i_e - T\delta c_l - \frac{1}{4}T^2\gamma\delta c_l + 2T\eta c_l \\ +\frac{1}{2}T^2\gamma\eta c_l \\ -\frac{1}{2}T\delta c_p - \frac{1}{4}T^2\gamma\delta c_p + T\eta c_p + \frac{1}{2}T^2\gamma\eta c_p \end{bmatrix}$

$m_3 = \frac{1}{T}\begin{bmatrix} \frac{T\delta}{2} + \frac{1}{4}T^2\gamma\delta - T\eta - \frac{1}{2}T^2\gamma\eta + \frac{1}{4}T\delta i_e + \frac{1}{8}T^2\gamma\delta i_e \\ -\frac{1}{2}T\eta i_e - \frac{1}{4}T^2\gamma\eta i_e \end{bmatrix}$

$m_4 = \frac{1}{T}\begin{bmatrix} -T\gamma\mu - \frac{1}{2}T\gamma\mu i_e + \delta c_l + \frac{1}{2}T\gamma\delta c_l - 2\eta c_l - T\gamma\eta c_l \\ +\frac{1}{2}T\gamma\delta c_p - T\gamma\eta c_p \end{bmatrix}$

$m_5 = \frac{1}{T}\left[-\frac{1}{2}T\gamma\delta + T\gamma\eta - \frac{1}{4}T\gamma\delta i_e + \frac{1}{2}T\gamma\eta i_e\right]$

$m_6 = \frac{1}{T}\left[\frac{\gamma\delta}{4} - \frac{\gamma\eta}{2} + \frac{1}{8}\gamma\delta i_e - \frac{1}{4}\gamma\eta i_e\right]$

$m_7 = \frac{1}{T}\left[2\mu c_l + T\gamma\mu c_l + T\gamma\mu c_p\right]$

$m_8 = \frac{1}{T}\left[-\frac{\mu c_h}{2} - \frac{1}{2}\gamma\mu c_l - \frac{1}{2}\gamma\mu c_p\right]$

$m_9 = \frac{1}{T}\left[\frac{\gamma\mu}{2} + \frac{1}{4}\gamma\mu i_e - \frac{\delta c_h}{4} + \frac{\eta c_h}{2} - \frac{1}{4}\gamma\delta c_l + \frac{1}{2}\gamma\eta c_l - \frac{1}{4}\gamma\delta c_p + \frac{1}{2}\gamma\eta c_p\right]$

Lemma 1. The total profit $TP_1(p, t_f)$ has a maximum value when the optimal price and positive inventory time are given by:

$p \cdot (t_f) = \frac{\theta_2(t_f)}{2\theta_1(t_f)}$ and $t_f \cdot (p) = \frac{p\xi_4+p^2\xi_5+\xi_7}{2(p^2\xi_6+\xi_8+p\xi_9)}$

which satisfies the following inequality $K_1(t_f^*) > K_2(t_f^*)$.

Proof: See Appendix A.1

(ii) Without installment facility

Without an installment policy, customers are required to pay all cash at once, i.e., *n* = 1 or *δ* = 0. Therefore, the total profit for this case is:

$$TP_2(p,t_f) = [TP_1]_{\delta=0}$$

$$TP_2(p,t_f) = \begin{bmatrix} L_1 + pL_2 + p^2L_3 + pt_fL_4 + p^2t_fL_5 + p^2{t_f}^2L_6 + t_fL_7 \\ +{t_f}^2L_8 + p{t_f}^2L_9 \end{bmatrix} - \frac{1}{T}[OC] \quad (12)$$

where:

$L_1 = m_1; L_7 = m_7; L_8 = m_8$

$L_2 = \frac{1}{T}\left[T\mu + \frac{1}{2}T^2\gamma\mu + 2T\eta c_l + \frac{1}{2}T^2\gamma\eta c_l + T\eta c_p + \frac{1}{2}T^2\gamma\eta c_p\right]$

$L_3 = \frac{1}{T}\left[-T\eta - \frac{1}{2}T^2\gamma\eta\right]$

$L_4 = \frac{1}{T}\left[-T\gamma\mu - 2\eta c_l - T\gamma\eta c - T\gamma\eta c_{p_l}\right]$

$L_5 = \frac{1}{T}[T\gamma\eta]$

$L_6 = \frac{1}{T}\left[-\frac{\gamma\eta}{2}\right]$

$L_9 = \frac{1}{T}\left[\frac{\gamma\mu}{2} + \frac{\eta c_h}{2} + \frac{1}{2}\gamma\eta c_l + \frac{1}{2}\gamma\eta c_p\right]$

Lemma 2. The profit $TP_2(p,t_f)$ has a maximum value when the optimal price and positive inventory time are given by:

$$p*(t_f) = \frac{\sigma_2(t_f)}{2\sigma_1(t_f)} \text{ and } t_f{}^*(p) = \frac{p\omega_4 + p^2\omega_5 + \omega_7}{2(p^2\omega_6 + \omega_8 + p\omega_9)}$$

which satisfies the following inequality $F_1(t_f^*) > F_2(t_f^*)$.

Proof: See Appendix A.2

The following Proposition 1 discusses which case is more suitable for selling more quantity. From Proposition 1, the seller can sell the products more in the case of an installment facility than in the case without an installment facility. Hence, an installment facility will allow sellers to reach more customers.

Proposition 1. Seller sells more quantity in case of installment facility than the case of without installment facility.

$$Q_1 > Q_2$$

i.e., the total sold quantity for Case I is greater than for Case II.

Proof: See Appendix B.1

Proposition 2. The profit function $TP_1(p,t_f)$ will be greater than $TP_2(p,t_f)$ with the following possibilities

(i) If $A, B, C > 0$, then $n < L$; $c_p > M$; $i_e > N$

(ii) If $A, B > 0$ and $C \geq 0$, then $n < L$; $c_p > M$; $i_e \geq N$

(iii) If $A, C > 0$ and $B \geq 0$, then $n < L$; $c_p \geq M$; $i_e > N$

(iv) If $B, C > 0$ and $A \geq 0$, then $n \leq L$; $c_p > M$; $i_e > N$

(v) If $A > 0$ and $B, C \geq 0$, then $n < L$; $c_p \geq M$; $i_e \geq N$

(vi) If $B > 0$ and $A, C \geq 0$, then $n \leq L$; $c_p > M$; $i_e \geq N$

(vii) If $C > 0$ and $A, B \geq 0$, then $n \leq L$; $c_p \geq M$; $i_e > N$

where:

$A = \left(\gamma\left((\delta - 2\eta + 2\mu)i_e - 2\delta(-1 + c_l + c_p)\right)\right) > 0;$

$B = (4(-2+\gamma)\delta c_l + (-1+\gamma)(-2(\delta - 2\eta + 2\mu)i_e + 4\delta(-1+c_p))) > 0;$

$C = \gamma(\delta - 2\eta + 2\mu)i_e - 2\delta(c_h + (-4+\gamma)c_l + \gamma(-1+c_p)) > 0;$

$L = \frac{-2 - i_e + 2c_l + 2c_p}{-2 - i_e + 2\eta i_e - 2\mu i_e + 2c_l + 2c_p};$

$M = \frac{2(-1+\gamma)(\delta - 2\eta + 2\mu)i_e + 4\delta(-1+\gamma-(-2+\gamma)c_l)}{4(-1+\gamma)\delta};$

$N = \frac{2\delta(c_h + (-4+\gamma)c_l + \gamma(-1+c_p))}{\gamma(\delta - 2\eta + 2\mu)}.$

Proof: See Appendix B.2

From Proposition 2, the total profit for Case I is greater than Case II under seven possibilities.

Solution Algorithm

Next, an algorithm is depicted in Fig. 2 to describe the solution procedures of the proposed models.

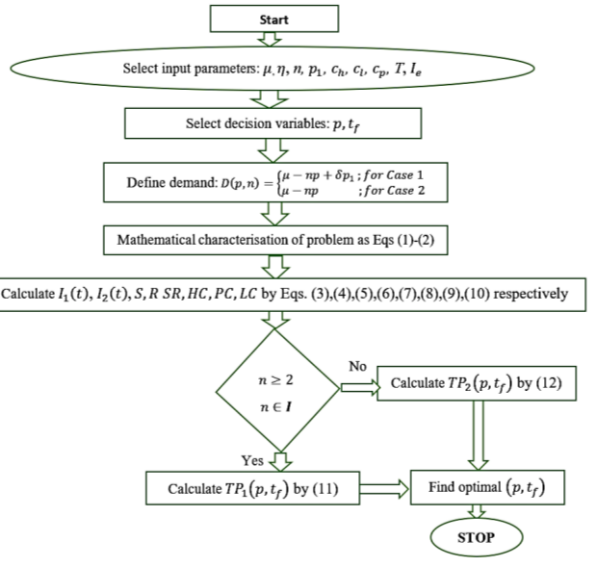

Fig. 2 Flowchart of the solution procedure

RESULTS

In this section, two numerical examples are provided to study the applicability and to gain some insights from the proposed model. The optimal results are derived and presented with corresponding figures to illustrate the model and its optimal solutions.

Example 1: (With installment policy) Consider the order cost $OC = \$100$, purchase cost per unit $c_p = \$6$, holding cost per unit $c_h = \$1$, lost sale cost $c_l = \$7$, interest rate $i_e = \$0.1$, cycle length T = 3 months, number of installments n = 5, and the values of parameters are $\mu = 200$, $\eta = 8, \gamma = 0.4$.

Our objective is to find the optimum values of TP_1, Q, p, t_f.

Example 2: (Without installment policy) All the values remain the same for this case except n = 1, $p = p_1$, i.e., the customer pays for the product at once and i_e = 0.

Our objective is to find the optimum values of TP_1, Q, p, t_f.

The optimum results of the examples are solved by LINGO 17 software. Table 2 represents the results for two cases.

Table 2

Optimum results of the models

Examples	Optimum results			
	Q*	**p***	**t_f***	**TP_i***
1	219.58	16.59	2.59	714.34
2	211.28	16.09	2.56	593.93

Table 2 shows that the total sold quantity and profit are higher for the case of installment facility than without installment facility though the price is higher for the first one. The concavity of the profit functions for different selling prices and positive stock time in both cases is examined by MATLAB R2019a in Fig. 3 and Fig. 4.

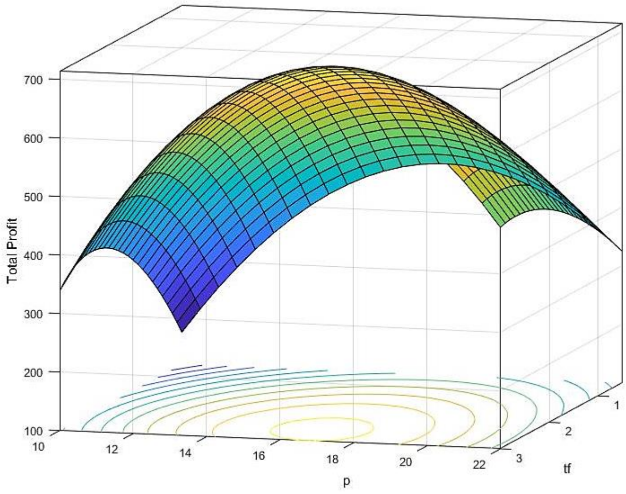

Fig. 3 TP_1 with regard to p, t_f

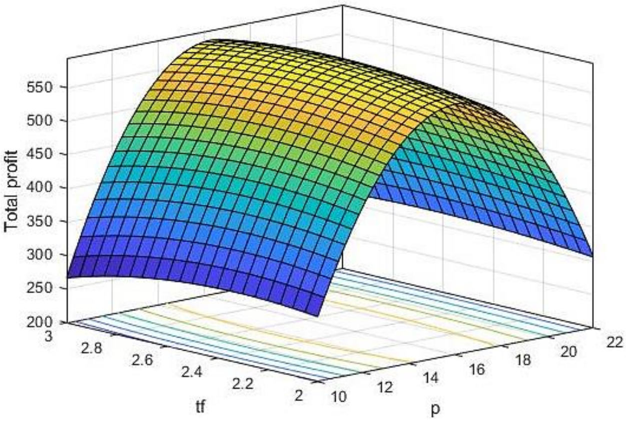

Fig. 4 TP_2 with regard to p, t_f

Both the figures above ensure the same optimal results in Table 2. Hence, it is confirmed numerically and graphically that the profit function has an optimal value.

Next, the sensitivity analysis is performed in this part. The total profit for the first case is derived by changing the parameters from -30% to 30%. But, for the number of installments, n runs as $n = 1, 2, 3, 4, 5, 6$ because for $n \in I$, I represents a set of positive integers.

(i) Analysis of area parameters

We study the parameter μ and the relationship between TP_1, Q, p, t_f, and μ. For the changing of the parameter μ, the effect on TP_1, p, Q, and t_f is shown in Table 3, and the effect in percentage is shown in Fig. 5.

Table 3

The impact on the profits for varying μ

Parameter	percentage	Variations of parameters	Variations of p	Q	t_f	TP_1
μ	-30%	140	12.62	130.20	2.47	231.19
	-20%	160	13.94	159.94	2.52	364.81
	-10%	180	15.26	189.74	2.56	525.84
	0%	200	16.59	219.58	2.59	714.34
	10%	220	17.91	249.45	2.62	930.34
	20%	240	19.23	279.34	2.64	1173.86
	30%	260	20.55	309.25	2.67	1444.92

Fig. 5 shows the percentage change according to Table 3. The profit, selling price, total quantity, and positive stock time change positively with the variation of parameters. In this case, the increasing rate of profit is highest compared to the others. The total profit varies from -67.64% to 102.27%.

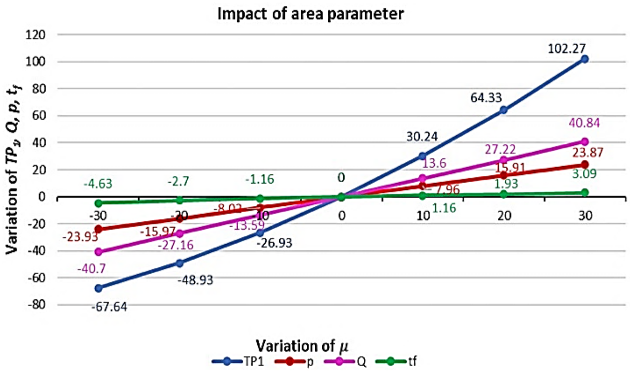

Fig. 5 The relationship between μ and TP_1, Q, p, t_f

(ii) Analysis of selling price coefficient

The effect of the selling price coefficient is shown in Table 4. The resulting percentage effect of changing η is shown in Figure 6 graphically.

Table 4

Changes in profits for varying η

Parameter	Percentage	Variations of parameters	Variations of p	Q	t_f	TP_1
η	-30%	5.6	22.69	244.67	2.70	1317.53
	-20%	6.4	20.11	236.27	2.66	1059.96
	-10%	7.2	18.14	227.91	2.62	865.56
	0%	8	16.59	219.58	2.59	714.34
	10%	8.8	15.33	211.28	2.56	593.93
	20%	9.6	14.29	203.02	2.53	496.28
	30%	10.4	13.41	194.79	2.50	415.89

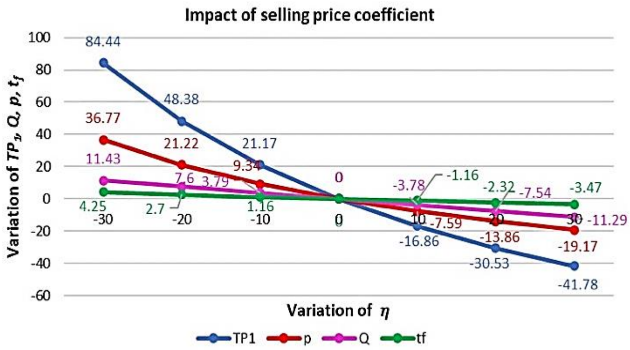

Fig. 6 The relationship between η and TP_1, Q, p, t_f

The total profit, total quantity, selling price, and positive stock time are inversely proportional to the parameter η. Though the parameter η has a negative effect on all the considerations, it is the highest in the percentage of total profit. The total profit varies from 84.44% to -41.78%. The relationship between η and TP_1, Q, p, t_f is shown in percentage in Fig. 6.

(iii) Analysis of cost parameters

We study the cost parameters c_p, c_h, c_l and the relationship between TP_1, Q, p, t_f, and the cost parameters. For changing the cost parameters, the effect on profit, selling price, total quantity, and positive stock time is shown in Table 5, Table 6, and Table 7, and their percentage is shown in Fig. 7, Fig. 8, and Fig. 9.

Table 5

Changes in profits for varying c_p

Parameter	Percentage	Variations of parameters	Variations of			
			p	Q	t_f	TP_1
c_p	-30%	4.2	15.73	239.03	2.61	851.92
	-20%	4.8	16.02	232.55	2.60	804.76
	-10%	5.4	16.30	226.06	2.60	758.90
	0%	6	16.59	219.58	2.59	714.34
	10%	6.6	16.87	213.10	2.58	671.07
	20%	7.2	17.16	206.62	2.58	629.10
	30%	7.8	17.44	200.14	2.57	588.43

Table 6

Changes in profits for varying c_h

Parameter	Percentage	Variations of parameters	Variations of			
			p	Q	t_f	TP_1
c_h	-30%	0.7	16.44	223.94	2.71	740.50
	-20%	0.8	16.49	222.46	2.67	731.47
	-10%	0.9	16.54	221.01	2.63	722.76
	0%	1	16.59	219.58	2.59	714.34
	10%	1.1	16.63	218.17	2.55	706.21
	20%	1.2	16.67	216.79	2.51	698.36
	30%	1.3	16.71	215.43	2.48	690.78

Table 7

Changes in profits for varying c_l

Parameter	Percentage	Variations of parameters	Variations of			
			p	Q	t_f	TP_1
c_l	-30%	4.9	16.56	219.55	2.54	716.10
	-20%	5.6	16.57	219.56	2.56	715.47
	-10%	6.3	16.58	219.57	2.57	714.88
	0%	7	16.59	219.58	2.59	714.34
	10%	7.7	16.59	219.58	2.61	713.84
	20%	8.4	16.60	219.58	2.62	713.37
	30%	9.1	16.61	219.58	2.63	712.93

The profit function, total quantity, and positive stock time are inversely proportional, while the selling price is proportional to the parameter c_p. The change in profit is greater due to the change of c_p. The total profit varies from 19.26% to -17.63%, which means the profit function decreases with the purchase cost increase. The relationship (in percentage) between c_p and TP_1, Q, p, t_f is shown in Fig. 7.

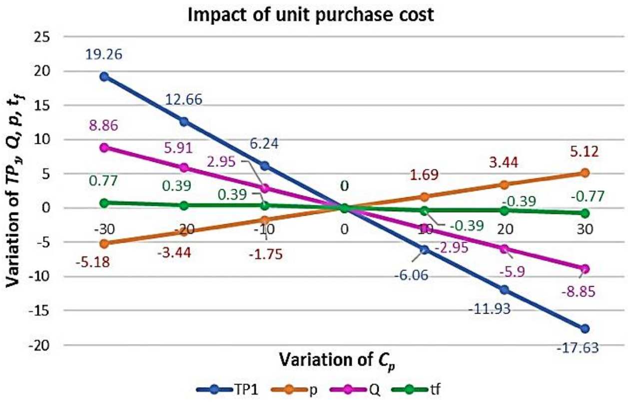

Fig. 7 The relationship between c_p and TP_1, Q, p, t_f

The profit function, total quantity, and positive stock time are inversely proportional, but the selling price is proportional to the parameter c_h and has the highest effect on t_f. For the variation of c_p from -30% to 30%, positive stock varies from 4.63% to -4.25%, whereas total profit varies from 3.66% to -3.3% only. That means profit decreases with the increase in holding costs. The percentage relationship between c_p and TP_1, Q, p, t_f is shown in Fig. 8.

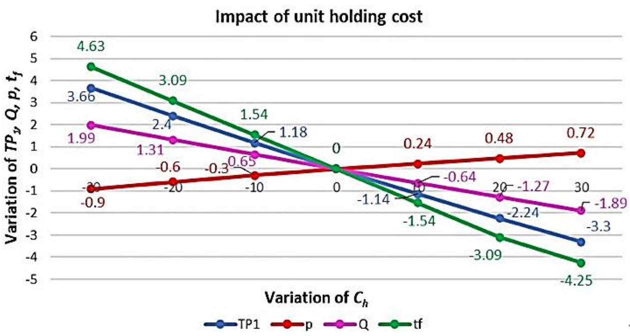

Fig. 8 The relationship between c_h and TP_1, Q, p, t_f

The percentage relationship between c_l and TP_1, Q, p, t_f is shown in Fig. 9. The profit function is inversely proportional to the lost sell cost c_l, while total quantity, selling price, and positive stock time are positively proportional. c_l causes the highest impact on t_f. For the variation of c_p from -30% to 30%, positive stock time varies from -1.93% to 1.54%, whereas total profit varies from 0.25 to -0.20 only. Therefore, with the increase of lost sell cost, the profit decreases.

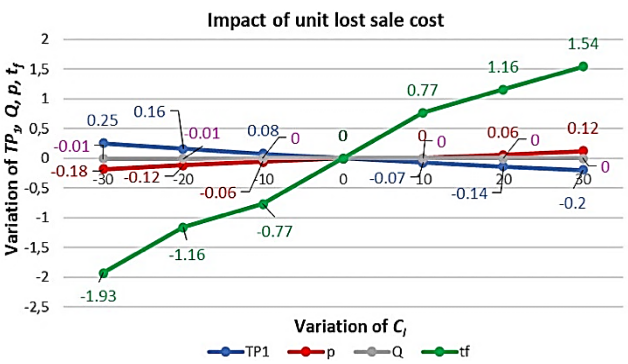

Fig. 9 The relationship between c_l and TP_1, Q, p, t_f

(iv) Analysis of the backlogging parameter γ

This section refers to the influence of backlogging parameter γ and the relationship between TP_1, Q, p, t_f, and the backlogging parameter γ. For changing the backlogging parameters, the effect on profit is shown in Table 8 and its percentage is in Fig. 10.

Table 8

Changes in profit TP_1 for varying γ

Parameter	Percentage	Variations of parameters	Variations of p	Q	t_f	TP_1
	-30%	0.28	16.54	220.01	2.45	719.37
	-20%	0.32	16.56	219.83	2.51	717.34
	-10%	0.36	16.57	219.69	2.55	715.70
γ	0%	0.4	16.59	219.58	2.59	714.34
	10%	0.44	16.60	219.49	2.62	713.20
	20%	0.48	16.61	219.41	2.65	712.23
	30%	0.52	16.62	219.34	2.67	711.39

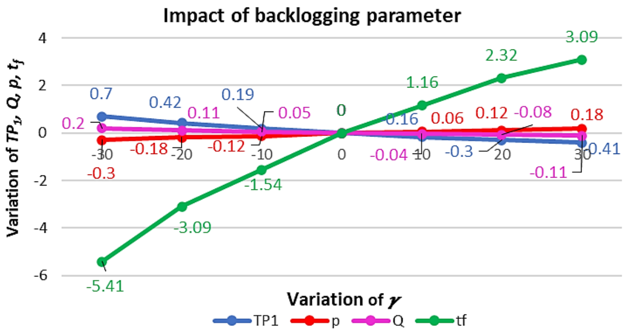

Fig. 10 The relationship between γ and TP_1, Q, p, t_f

The profit function and total quantity are inversely proportional to backlogging parameter γ, but selling price and positive stock time are positively proportional. γ causes the biggest change on t_f. For the variation of γ from -30% to 30%, positive stock time varies from -5.41% to 3.09% while total profit varies from 0.70% to -0.41%. With the increase of backlogging parameter γ, the profit decreases. The relationship in percentage between γ and TP_1, Q, p, t_f is shown in Fig. 10.

(v) Analysis of the number of installments n

In Table 9, the influence of installment is shown.

Table 9

Changes in profit TP_1 for varying n

Parameter	Percentage	Variations of parameters	Variations of p	Q	t_f	TP_1
	-30%	2	16.33	218.02	2.58	689.71
	-20%	3	16.47	218.89	2.59	703.26
	-10%	4	16.54	219.32	2.59	710.16
n	0%	5	16.59	219.58	2.59	714.34
	10%	6	16.62	219.75	2.59	717.14
	20%	7	16.64	219.88	2.59	719.15
	30%	8	16.65	219.97	2.59	720.66

The relationship between n and TP1, Q, and p, tf is graphically evaluated in Fig. 11.

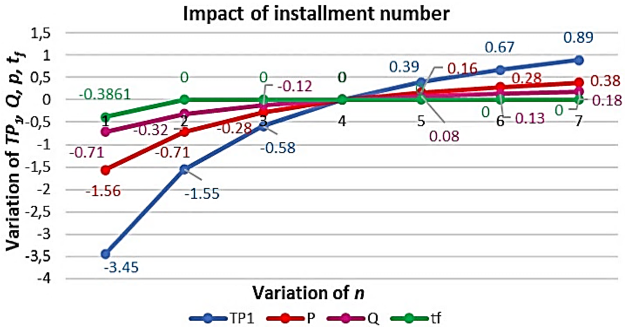

Fig. 11 The relationship between n and TP_1, Q, p, t_f

The profit function, selling price, and total quantity are positively proportional to the installment parameter n. Surprisingly positive stock time remains the same with the variation of installment parameter n. With the increase of the installment parameter, the profit increase is the highest. The total profit varies from -3.45% to 0.89%. The relationship in percentage between n and TP_1, Q, p, t_f is shown in Fig. 11.

(vi) Analysis of I_e

The interest rate I_e plays an important role in this study. The influence of I_e on profit is shown in Table 10, and the variation in percentage for the optimal decisions is evaluated in Fig. 12 graphically.

Table 10

Changes in profit TP_1 for varying I_e

Parameter	Percentage	Variations of parameters	Variations of p	Q	t_f	TP_1
	-30%	0.07	16.64	218.43	2.59	696.15
	-20%	0.08	16.59	218.82	2.59	702.21
	-10%	0.09	16.58	219.95	2.59	708.27
I_e	0%	0.1	16.59	219.58	2.59	714.34
	10%	0.11	16.54	219.95	2.59	720.41
	20%	0.12	16.53	220.32	2.59	726.49
	30%	0.13	16.54	220.69	2.60	732.57

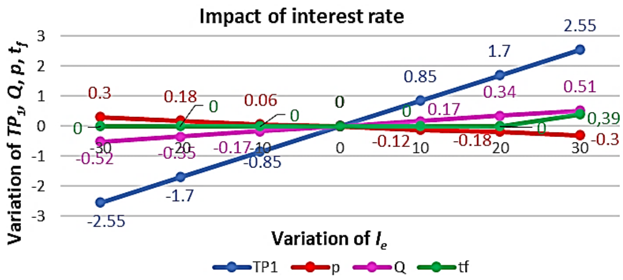

Fig. 12 The relationship between I_e and TP_1, Q, p, t_f

Interest plays a vital role in a business. The profit function, total stock, and positive stock time are proportional to the interest parameter I_e. However, the selling price is inversely proportional. With the increase in interest parameter I_e, the profit increase is highest. The total profit varies from -2.55% to 2.55%.

DISCUSSION

This study considers an installment policy to facilitate people who are unable to pay the cost as a one-time payment. We introduce a demand function that depends on the selling price and the number of installments. Some special scenarios can be addressed that will lead to some previous research:

a. If n = 1 (or δ = 0), the model reduces to an inventory EOQ model with price-dependent demand with a partially backlogged shortage (Sahoo et al. [22])
b. If γ = o and n = 1, then the model will be fully backlogged (Shaikh et al. [39])
c. If $\gamma = \infty$ and n = 1, the model reduces to an EOQ model with price-dependent demand without shortages (Teksan and Geunes [40])
d. If μ = 0, n = 1 and γ = 0, then the model becomes fully backlogged with constant demand [41].

The results of this study show that with an installment policy, the seller can sell more items and thereby obtain more profit compared to those without installments. Thus, the installment facility provides a chance to serve financially low-abled customers with regular customers in the market region. The total profit obtained from the model with an installment policy will be higher than the total profit obtained from the model without an installment policy. This result is in line with the finding of Mashud et al., [4]. The number of installments is an important factor as it dictates the total funds the buyer must set aside for each payment.

Further, there are some practical outlines for managers as follows:

a. Managers must pay attention to the purchase cost c_p, holding cost c_h, cost due to lost sales per unit c_l, and the down payment p_1, because the changes in these parameters significantly reduce total profit with their increasing values. Hence, the system maintains the minimum possible cost components per unit, and the amount of the down payment should be clearly stated.
b. Managers can handle installment policy planning at a high level, thinking about the number of more significant installments for a business's profit. It is clear from the sensitivity analysis that more installments provide an opportunity for more profit. Therefore, managers have the flexibility to set more installments within the replenishment period.
c. As the total profit obtained from the model with an installment policy will be higher than the total profit obtained from the model without an installment policy, there are some possibilities/conditions that can generate higher profit for Case I. Hence, managers should be aware of the possibilities and maintain at least one condition among seven to make their business profitable.

CONCLUSION

In a competitive business market, all sellers and retailers strive to serve more customers in order to increase their profits. Thus, the supplier implements multiple policies to attract more customers. Some customers are not able to purchase products for cash price when it is beyond their financial capacity. Hence, this study considers an installment policy for people who are unable to pay the cost as a one-time payment. We introduce a demand function that depends on the selling price and the number of installments. This study combined the EOQ inventory model with the installment facility and partially backlogged shortages. The proposed algorithm can solve the model by optimizing the selling price and positive stock time. Two cases were examined, and it was found that more customers can purchase with the installment option compared to the full payment policy. The result demonstrates that the installment plan is proportional to the entire profit. Installment-dependent demand shows that it has a positive impact on the demand of customers. Finally, our studied EOQ model successfully represents how the installment facility can serve the low-abled customers in the market.

The current study is limited in that it only examines the installment and price-dependent demand for non-deteriorating commodities, but, since some products are deteriorating too, the future study can incorporate the effect of the deterioration rate. Further research can be extended to using different demand patterns such as stock-dependent demand, time-dependent demand, promotion-dependent demand, and offering a discount on advance payment where shortages can be fully backlogged for the same condition. Furthermore, sustainable inventory management that considers environmental and social criteria can be analyzed using the proposed model in the current study. The effect of inflation also can be incorporated and a study for multi-products will give more directions for managers.

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Appendix A.1
Proof of Lemma 1
To prove the concavity, follow the steps:
Step 1: Firstly, we rearrange the total profit equation (11) again as,

$$TP_1(p,t_f)=\begin{bmatrix} m_1+pm_2+p^2m_3+pt_fm_4+p^2t_fm_5+p^2{t_f}^2m_6+t_fm_7 \\ +{t_f}^2m_8+p{t_f}^2m_9 \end{bmatrix} \tag{A.1.1}$$

$$TP_1(p,t_f)=p^2\varphi_1(t_f)+p\varphi_2(t_f)+\varphi_3(t_f)$$

For notational convenience, we define

$\varphi_1(t_f)=t_f^2m_6+t_fm_5+m_3$
$\varphi_2(t_f)=t_f^2m_9+t_fm_4+m_2$
$\varphi_3(t_f)=t_f^2m_8+t_fm_7+m_1$

Now find $\frac{\partial TP_1}{\partial p}$, $\frac{\partial TP_1(Q,G)}{\partial t_f}$ and set equal to zero.
Differentiating Eq. (A.1.1) with respect to p

$$\frac{\partial TP_1\,(\mathrm{p},t_f)}{\partial p}=2p\varphi_1(t_f)+\varphi_2(t_f)=0$$

$$\frac{\partial TP_1\,(\mathrm{p},t_f)}{\partial p}=$$

$$\left[\begin{array}{l} 2p\left\{\begin{array}{c} t_f^2\left(\frac{1}{T}\left[\frac{\gamma\delta}{4}-\frac{\gamma\eta}{2}+\frac{1}{8}\gamma\delta i_e-\frac{1}{4}\gamma\eta i_e\right]\right)+t_f\left(\frac{1}{T}\left[-\frac{1}{2}T\gamma\delta+T\gamma\eta-\frac{1}{4}T\gamma\delta i_e+\frac{1}{2}T\gamma\eta i_e\right]\right) \\ +\frac{1}{T}\left[\frac{T\delta}{2}+\frac{1}{4}T^2\gamma\delta-T\eta-\frac{1}{2}T^2\gamma\eta+\frac{1}{4}T\delta i_e+\frac{1}{8}T^2\gamma\delta i_e-\frac{1}{2}T\eta i_e-\frac{1}{4}T^2\gamma\eta i_e\right] \end{array}\right\} \\ -\left\{\begin{array}{l} t_f^2\left(\frac{1}{T}\left[\frac{\delta c_h}{4}+\frac{1}{4}\gamma\delta c_l+\frac{1}{4}\gamma\delta c_p-\frac{\gamma\mu}{2}-\frac{1}{4}\gamma\mu i_e-\frac{\eta c_h}{2}-\frac{1}{2}\gamma\eta c_l-\frac{1}{2}\gamma\eta c_p\right]\right) \\ +t_f\left(\frac{1}{T}\left[T\gamma\mu+\frac{1}{2}T\gamma\mu i_e+2\eta c_l+T\gamma\eta c_l+T\gamma\eta c_p-\delta c_l-\frac{1}{2}T\gamma\delta c_l-\frac{1}{2}T\gamma\delta c_p\right]\right) \\ +\frac{1}{T}\left[\begin{array}{c}\frac{1}{2}T\delta c_p+\frac{1}{4}T^2\gamma\delta c_p+T\delta c_l+\frac{1}{4}T^2\gamma\delta c_l-T\mu-\frac{1}{2}T^2\gamma\mu-\frac{1}{2}T\mu i_e-\frac{1}{4}T^2\gamma\mu i_e \\ -2T\eta c_l-\frac{1}{2}T^2\gamma\eta c_l-T\eta c_p-\frac{1}{2}T^2\gamma\eta c_p\end{array}\right] \end{array}\right\} \end{array}\right]$$

Let,

$$\Theta_1(t_f)=\begin{array}{c} t_f^2\left(\frac{1}{T}\left[\frac{\gamma\delta}{4}-\frac{\gamma\eta}{2}+\frac{1}{8}\gamma\delta i_e-\frac{1}{4}\gamma\eta i_e\right]\right)+t_f\left(\frac{1}{T}\left[-\frac{1}{2}T\gamma\delta+T\gamma\eta-\frac{1}{4}T\gamma\delta i_e+\frac{1}{2}T\gamma\eta i_e\right]\right) \\ +\frac{1}{T}\left[\frac{T\delta}{2}+\frac{1}{4}T^2\gamma\delta-T\eta-\frac{1}{2}T^2\gamma\eta+\frac{1}{4}T\delta i_e+\frac{1}{8}T^2\gamma\delta i_e-\frac{1}{2}T\eta i_e-\frac{1}{4}T^2\gamma\eta i_e\right] \end{array}$$

and,

$$\Theta_2(t_f)=\left\{\begin{array}{l} t_f^2\left(\frac{1}{T}\left[\frac{\delta c_h}{4}+\frac{1}{4}\gamma\delta c_l+\frac{1}{4}\gamma\delta c_p-\frac{\gamma\mu}{2}-\frac{1}{4}\gamma\mu i_e-\frac{\eta c_h}{2}-\frac{1}{2}\gamma\eta c_l-\frac{1}{2}\gamma\eta c_p\right]\right) \\ +t_f\left(\frac{1}{T}\left[T\gamma\mu+\frac{1}{2}T\gamma\mu i_e+2\eta c_l+T\gamma\eta c_l+T\gamma\eta c_p-\delta c_l-\frac{1}{2}T\gamma\delta c_l-\frac{1}{2}T\gamma\delta c_p\right]\right) \\ +\frac{1}{T}\left[\begin{array}{c}\frac{1}{2}T\delta c_p+\frac{1}{4}T^2\gamma\delta c_p+T\delta c_l+\frac{1}{4}T^2\gamma\delta c_l-T\mu-\frac{1}{2}T^2\gamma\mu-\frac{1}{2}T\mu i_e-\frac{1}{4}T^2\gamma\mu i_e \\ -2T\eta c_l-\frac{1}{2}T^2\gamma\eta c_l-T\eta c_p-\frac{1}{2}T^2\gamma\eta c_p\end{array}\right] \end{array}\right\} \tag{A.1.1}$$

$$\therefore \frac{\partial TP_1\,(\mathrm{p},t_f)}{\partial p}=2p\Theta_1(t_f)-\Theta_2(t_f)=0$$

Now differentiating Eq. (A.1.1) with respect to t_f

$$\frac{\partial TP_1\,(\mathrm{p},t_f)}{\partial t_f}=pm_4+p^2m_5+m_7+2p^2m_6t_f+2m_8t_f+2pm_9t_f=0$$

$$\Rightarrow\left[\begin{array}{l} -p\left(\frac{1}{T}\left[T\gamma\mu+\frac{1}{2}T\gamma\mu i_e-\delta c_l-\frac{1}{2}T\gamma\delta c_l+2\eta c_l+T\gamma\eta c_l-\frac{1}{2}T\gamma\delta c_p+T\gamma\eta c_p\right]\right) \\ -p^2\left(\frac{1}{T}\left[\frac{1}{2}T\gamma\delta-T\gamma\eta+\frac{1}{4}T\gamma\delta i_e-\frac{1}{2}T\gamma\eta i_e\right]\right)-\left(\frac{1}{T}\left[-2\mu c_l-T\gamma\mu c_l-T\gamma\mu c_p\right]\right) \\ 2t_f\left[\begin{array}{l} p^2\left(\frac{1}{T}\left[\frac{\gamma\delta}{4}-\frac{\gamma\eta}{2}+\frac{1}{8}\gamma\delta i_e-\frac{1}{4}\gamma\eta i_e\right]\right)+\frac{1}{T}\left[-\frac{\mu c_h}{2}-\frac{1}{2}\gamma\mu c_l-\frac{1}{2}\gamma\mu c_p\right] \\ +p\left(\frac{1}{T}\left[\frac{\gamma\mu}{2}+\frac{1}{4}\gamma\mu i_e-\frac{\delta c_h}{4}+\frac{\eta c_h}{2}-\frac{1}{4}\gamma\delta c_l+\frac{1}{2}\gamma\eta c_l-\frac{1}{4}\gamma\delta c_p+\frac{1}{2}\gamma\eta c_p\right]\right) \end{array}\right] \end{array}\right]=0$$

where:

$$\begin{aligned} &\xi_4=\frac{1}{T}\left[T\gamma\mu+\frac{1}{2}T\gamma\mu i_e-\delta c_l-\frac{1}{2}T\gamma\delta c_l+2\eta c_l+T\gamma\eta c_l-\frac{1}{2}T\gamma\delta c_p+T\gamma\eta c_p\right] \\ &\xi_5=\frac{1}{2}T\gamma\delta-T\gamma\eta+\frac{1}{4}T\gamma\delta i_e-\frac{1}{2}T\gamma\eta i_e \\ &\xi_7=\frac{1}{T}\left[-2\mu c_l-T\gamma\mu c_l-T\gamma\mu c_p\right] \\ &\xi_6=\frac{1}{T}\left[\frac{\gamma\delta}{4}-\frac{\gamma\eta}{2}+\frac{1}{8}\gamma\delta i_e-\frac{1}{4}\gamma\eta i_e\right] \\ &\xi_8=\frac{1}{T}\left[-\frac{\mu c_h}{2}-\frac{1}{2}\gamma\mu c_l-\frac{1}{2}\gamma\mu c_p\right] \\ &\xi_9=\frac{1}{T}\left[\frac{\gamma\mu}{2}+\frac{1}{4}\gamma\mu i_e-\frac{\delta c_h}{4}+\frac{\eta c_h}{2}-\frac{1}{4}\gamma\delta c_l+\frac{1}{2}\gamma\eta c_l-\frac{1}{4}\gamma\delta c_p+\frac{1}{2}\gamma\eta c_p\right] \end{aligned} \tag{A.1.3}$$

$$\therefore \frac{\partial TP_1\,(\mathrm{p},t_f)}{\partial t_f}=-p\xi_4-p^2\xi_5-\xi_7+2t_f(p^2\xi_6+\xi_8+p\xi_9)=0$$

Step 2: Find $p*(t_f)$ and $t_f{}^*(p)$ by using **step 1**
Solving equations (A.1.2) and (A.1.3) we get
$p*(t_f) = \frac{\Theta_2(t_f)}{2\Theta_1(t_f)}$
and
$t_f{}^*(p) = \frac{p\xi_4+p^2\xi_5+\xi_7}{2(p^2\xi_6+\xi_8+p\xi_9)}$
So, the profit function turns into,

$$TP_1(p,t_f) = \frac{\Theta_2{}^2(t_f)}{4\Theta_1{}^2(t_f)}\varphi_1(t_f) + \frac{\Theta_2(t_f)}{2\Theta_1(t_f)}\varphi_2(t_f) + \varphi_3(t_f) \qquad \text{(A.1.4)}$$

Putting the value of $\Theta_1(t_f)$ and $\Theta_2(t_f)$ we get,
$TP_1(p,t_f) =$

$$\left[\begin{array}{c}
\frac{\left\{\begin{array}{c} t_f^2\left(\frac{1}{T}\left[\frac{\delta c_h}{4}+\frac{1}{4}\gamma\delta c_l+\frac{1}{4}\gamma\delta c_p-\frac{\gamma\mu}{2}-\frac{1}{4}\gamma\mu i_e-\frac{\eta c_h}{2}-\frac{1}{2}\gamma\eta c_l-\frac{1}{2}\gamma\eta c_p\right]\right) \\ +t_f\left(\frac{1}{T}\left[T\gamma\mu+\frac{1}{2}T\gamma\mu i_e+2\eta c_l+T\gamma\eta c_l+T\gamma\eta c_p-\delta c_l-\frac{1}{2}T\gamma\delta c_l-\frac{1}{2}T\gamma\delta c_p\right]\right) \\ +\frac{1}{T}\left[\begin{array}{c}\frac{1}{2}T\delta c_p+\frac{1}{4}T^2\gamma\delta c_p+T\delta c_l+\frac{1}{4}T^2\gamma\delta c_l-T\mu-\frac{1}{2}T^2\gamma\mu-\frac{1}{2}T\mu i_e-\frac{1}{4}T^2\gamma\mu i_e \\ -2T\eta c_l-\frac{1}{2}T^2\gamma\eta c_l-T\eta c_p-\frac{1}{2}T^2\gamma\eta c_p\end{array}\right]\end{array}\right\}^2}{4\left\{\begin{array}{c} t_f^2\left(\frac{1}{T}\left[\frac{\gamma\delta}{4}-\frac{\gamma\eta}{2}+\frac{1}{8}\gamma\delta i_e-\frac{1}{4}\gamma\eta i_e\right]\right)+t_f\left(\frac{1}{T}\left[-\frac{1}{2}T\gamma\delta+T\gamma\eta-\frac{1}{4}T\gamma\delta i_e+\frac{1}{2}T\gamma\eta i_e\right]\right) \\ +\frac{1}{T}\left[\frac{T\delta}{2}+\frac{1}{4}T^2\gamma\delta-T\eta-\frac{1}{2}T^2\gamma\eta+\frac{1}{4}T\delta i_e+\frac{1}{8}T^2\gamma\delta i_e-\frac{1}{2}T\eta i_e-\frac{1}{4}T^2\gamma\eta i_e\right]\end{array}\right\}^2}\varphi_1(t_f) \\
+\frac{\left\{\begin{array}{c} t_f^2\left(\frac{1}{T}\left[\frac{\delta c_h}{4}+\frac{1}{4}\gamma\delta c_l+\frac{1}{4}\gamma\delta c_p-\frac{\gamma\mu}{2}-\frac{1}{4}\gamma\mu i_e-\frac{\eta c_h}{2}-\frac{1}{2}\gamma\eta c_l-\frac{1}{2}\gamma\eta c_p\right]\right) \\ +t_f\left(\frac{1}{T}\left[T\gamma\mu+\frac{1}{2}T\gamma\mu i_e+2\eta c_l+T\gamma\eta c_l+T\gamma\eta c_p-\delta c_l-\frac{1}{2}T\gamma\delta c_l-\frac{1}{2}T\gamma\delta c_p\right]\right) \\ +\frac{1}{T}\left[\begin{array}{c}\frac{1}{2}T\delta c_p+\frac{1}{4}T^2\gamma\delta c_p+T\delta c_l+\frac{1}{4}T^2\gamma\delta c_l-T\mu-\frac{1}{2}T^2\gamma\mu-\frac{1}{2}T\mu i_e-\frac{1}{4}T^2\gamma\mu i_e \\ -2T\eta c_l-\frac{1}{2}T^2\gamma\eta c_l-T\eta c_p-\frac{1}{2}T^2\gamma\eta c_p\end{array}\right]\end{array}\right\}}{2\left\{\begin{array}{c} t_f^2\left(\frac{1}{T}\left[\frac{\gamma\delta}{4}-\frac{\gamma\eta}{2}+\frac{1}{8}\gamma\delta i_e-\frac{1}{4}\gamma\eta i_e\right]\right)+t_f\left(\frac{1}{T}\left[-\frac{1}{2}T\gamma\delta+T\gamma\eta-\frac{1}{4}T\gamma\delta i_e+\frac{1}{2}T\gamma\eta i_e\right]\right) \\ +\frac{1}{T}\left[\frac{T\delta}{2}+\frac{1}{4}T^2\gamma\delta-T\eta-\frac{1}{2}T^2\gamma\eta+\frac{1}{4}T\delta i_e+\frac{1}{8}T^2\gamma\delta i_e-\frac{1}{2}T\eta i_e-\frac{1}{4}T^2\gamma\eta i_e\right]\end{array}\right\}}\varphi_2(t_f) \\
+\varphi_3(t_f)
\end{array}\right]$$

Using the value of $\varphi_1(t_f)$, $\varphi_2(t_f)$ and $\varphi_3(t_f)$ and simplifying we get,

$$TP_1\left(t_f,p^*(t_f)\right) = -\frac{\varphi_2{}^2[t_f]}{4*\varphi_1[t_f]} + \varphi_3[t_f] \qquad \text{(A.1.5)}$$

Step 3: Find $\frac{\partial TP_1\left(t_f,p^*(t_f)\right)}{\partial t_f}, \frac{\partial^2 TP_1\left(t_f,p^*(t_f)\right)}{\partial t_f{}^2}$
Differentiate (A.1.5) with respect to t_f,

$$\frac{\partial TP_1(t_f)}{\partial t_f} = \frac{\varphi_2^2[t_f]\varphi_1{}'[t_f]}{4\varphi_1[t_f]^2} - \frac{(\varphi_2^2)'[t_f]}{4\varphi_1[t_f]} + \varphi_3{}'[t_f]$$

$$\frac{\partial TP_1(t_f)}{\partial t_f} = \left[\begin{array}{c} m_7 + 2m_8 t_f - \frac{(m_4+2m_9t_f)(m_2+m_4t_f+m_9t_f^2)}{2\left(m_3+m_5t_f+m_6t_f^2\right)} \\ +\frac{(m_5+2m_6t_f)(m_2+m_4t_f+m_9t_f^2)^2}{4(m_3+m_5t_f+m_6t_f^2)^2}\end{array}\right]$$

Again, differentiating this equation with respect to t_f

$$\frac{\partial^2 TP_1(t_f)}{\partial t_f{}^2(t_f)} = \frac{\varphi_2^2[t_f]\varphi_1{}'[t_f]^2}{2\varphi_1[t_f]^3} + \frac{\varphi_1{}'[t_f](\varphi_2^2)'[t_f]}{2\varphi_1[t_f]^2} + \frac{\varphi_2^2[t_f]\varphi_1{}''[t_f]}{4\varphi_1[t_f]^2} - \frac{(\varphi_2^2)''[t_f]}{4\varphi_1[t_f]} + \varphi_3{}''[t_f]$$

$$\frac{\partial^2 TP_1(t_f)}{\partial t_f{}^2(t_f)} = \left[\begin{array}{c} \left(\begin{array}{c}\frac{(m_5+2m_6t_f)(m_4+2m_9t_f)(m_2+m_4t_f+m_9t_f^2)}{\left(m_3+m_5t_f+m_6t_f^2\right)^2} \\ +\frac{m_6\left(m_2+m_4t_f+m_9t_f^2\right)^2}{2\left(m_3+m_5t_f+m_6t_f^2\right)^2}+2m_8\end{array}\right) \\ -\left(\begin{array}{c}\frac{\left(m_4+2m_9t_f\right)^2}{2\left(m_3+m_5t_f+m_6t_f^2\right)}+\frac{\left(m_5+2m_6t_f\right)^2\left(m_2+m_4t_f+m_9t_f^2\right)^2}{2\left(m_3+m_5t_f+m_6t_f^2\right)^3} \\ +\frac{m_9\left(m_2+m_4t_f+m_9t_f^2\right)}{m_3+m_5t_f+m_6t_f^2}\end{array}\right)\end{array}\right] \qquad \text{(A.1.6)}$$

Step 4: Proof the conditions for the optimality of the objective function.
To ensure that the optimal profit function is concave, the following condition must be satisfied:
$\frac{\partial^2 TP_1(t_f)}{\partial t_f{}^2(t_f)} < 0$
Now simplifying (A.1.5), we get,

$$\frac{\partial^2 TP_1(t_f)}{\partial t_f{}^2(t_f)} = K_2(t_f) - K_1(t_f) \qquad \text{(A.1.7)}$$

where,

$$K_1(t_f) = \begin{bmatrix} \frac{(m_4+2m_9t_f)^2}{2(m_3+m_5t_f+m_6t_f^2)} + \frac{(m_5+2m_6t_f)^2(m_2+m_4t_f+m_9t_f^2)^2}{2(m_3+m_5t_f+m_6t_f^2)^3} \\ + \frac{m_9(m_2+m_4t_f+m_9t_f^2)}{m_3+m_5t_f+m_6t_f^2} \end{bmatrix}$$

$$K_2(t_f) = \begin{bmatrix} \frac{(m_5+2m_6t_f)(m_4+2m_9t_f)(m_2+m_4t_f+m_9t_f^2)}{(m_3+m_5t_f+m_6t_f^2)^2} \\ + \frac{m_6(m_2+m_4t_f+m_9t_f^2)^2}{2(m_3+m_5t_f+m_6t_f^2)^2} + 2m_8 \end{bmatrix}$$

$\frac{\partial^2 TP_1(t_f)}{\partial {t_f}^2(t_f)}$ will be less than zero at optimal t_f^* if $K_2(t_f^*) - K_1(t_f^*)$ is negative;

i.e., if $K_1(t_f^*) > K_2(t_f^*)$ (A.1.8)

Finally, if condition (A.1.8) is satisfied, the total profit is always concave.

Appendix A.2

Proof of Lemma 2

To prove the concavity, follow the steps:

Step 1: Firstly, we rearrange the total profit equation (12) again as,

$$TP_2(p,t_f) = \begin{bmatrix} m_1 + pL_2 + p^2L_3 + pt_fL_4 + p^2t_fL_5 + p^2{t_f}^2L_6 + t_fm_7 \\ +{t_f}^2m_8 + p{t_f}^2L_9 \end{bmatrix} \quad \text{(A.2.1)}$$

$$TP_2(p,t_f) = p^2\phi_1(t_f) + p\phi_2(t_f) + \phi_3(t_f)$$

For notational convenience, we define

$\phi_1(t_f) = t_f^2L_6 + t_fL_5 + L_3$

$\phi_2(t_f) = t_f^2L_9 + t_fL_4 + L_2$

$\phi_3(t_f) = t_f^2m_8 + t_fm_7 + m_1$

Now find $\frac{\partial TP_2}{\partial p}, \frac{\partial TP_2(p,t_f)}{\partial t_f}$ and set equal to zero.

Differentiating Eq. (A.2.1) with respect to p

$$\frac{\partial TP_2\,(\mathrm{p},t_f)}{\partial p} = 2p\phi_1(t_f) + \phi_2(t_f) = 0 \quad \text{(A.2.2)}$$

$$\frac{\partial TP_2\,(\mathrm{p},t_f)}{\partial p} =$$

$$\begin{bmatrix} 2p\left(t_f^2\left(\frac{1}{T}\left[-\frac{\gamma\eta}{2}\right]\right) + t_f\left(\frac{1}{T}[T\gamma\eta]\right) + \frac{1}{T}\left[-T\eta - \frac{1}{2}T^2\gamma\eta\right]\right) \\ -\begin{bmatrix} t_f^2\left(\frac{1}{T}\left[-\frac{\gamma\mu}{2} - \frac{\eta c_h}{2} - \frac{1}{2}\gamma\eta c_l - \frac{1}{2}\gamma\eta c_p\right]\right) + t_f\left(\frac{1}{T}[T\gamma\mu + 2\eta c_l + T\gamma\eta c_l + T\gamma\eta c_p]\right) \\ +\frac{1}{T}\left[-T\mu - \frac{1}{2}T^2\gamma\mu - 2T\eta c_l - \frac{1}{2}T^2\gamma\eta c_l - T\eta c_p - \frac{1}{2}T^2\gamma\eta c_p\right] \end{bmatrix} \end{bmatrix} = 0$$

Let,

$$\sigma_2(t_f) = \begin{bmatrix} t_f^2\left(\frac{1}{T}\left[-\frac{\gamma\mu}{2} - \frac{\eta c_h}{2} - \frac{1}{2}\gamma\eta c_l - \frac{1}{2}\gamma\eta c_p\right]\right) + t_f\left(\frac{1}{T}[T\gamma\mu + 2\eta c_l + T\gamma\eta c_l + T\gamma\eta c_p]\right) \\ +\frac{1}{T}\left[-T\mu - \frac{1}{2}T^2\gamma\mu - 2T\eta c_l - \frac{1}{2}T^2\gamma\eta c_l - T\eta c_p - \frac{1}{2}T^2\gamma\eta c_p\right] \end{bmatrix}$$

and

$$\sigma_1(t_f) = 2p\left(t_f^2\left(\frac{1}{T}\left[-\frac{\gamma\eta}{2}\right]\right) + t_f\left(\frac{1}{T}[T\gamma\eta]\right) + \frac{1}{T}\left[-T\eta - \frac{1}{2}T^2\gamma\eta\right]\right)$$

$$\therefore \frac{\partial TP_2\,(\mathrm{p},t_f)}{\partial p} = 2p\sigma_1(t_f) - \sigma_2(t_f) = 0 \quad \text{(A.2.3)}$$

Now differentiating Eq. (A.2.1) with respect to t_f

$$\frac{\partial TP_2\,(\mathrm{p},t_f)}{\partial t_f} = pL_4 + p^2L_5 + m_7 + 2p^2L_6t_f + 2m_8t_f + 2pL_9t_f = 0 \quad \text{(A.2.4)}$$

$$\Rightarrow \begin{bmatrix} -p\left(\frac{1}{T}[T\gamma\mu + 2\eta c_l + T\gamma\eta c_l + T\gamma\eta c_p]\right) - p^2\left(\frac{1}{T}[-T\gamma\eta]\right) \\ -\left(\frac{1}{T}[-2\mu c_l - T\gamma\mu c_l - T\gamma\mu c_p]\right) \\ +2t_f\begin{bmatrix} p^2\left(\frac{1}{T}\left[-\frac{\gamma\eta}{2}\right]\right) + \frac{1}{T}\left[-\frac{\mu c_h}{2} - \frac{1}{2}\gamma\mu c_l - \frac{1}{2}\gamma\mu c_p\right] \\ +p\left(\frac{1}{T}\left[\frac{\gamma\mu}{2} + \frac{\eta c_h}{2} + \frac{1}{2}\gamma\eta c_l + \frac{1}{2}\gamma\eta c_p\right]\right) \end{bmatrix} \end{bmatrix} = 0$$

where:

$$\begin{aligned}
\omega_4 &= -\frac{1}{T}[T\gamma\mu + 2\eta c_l + T\gamma\eta c_l + T\gamma\eta c_p] \\
\omega_5 &= -\frac{1}{T}[-T\gamma\eta] \\
\omega_7 &= -\frac{1}{T}\left[-2\mu c_l - T\gamma\mu c_l - T\gamma\mu c_p\right] \\
\omega_6 &= \frac{1}{T}\left[-\frac{\gamma\eta}{2}\right] \\
\omega_8 &= \frac{1}{T}\left[-\frac{\mu c_h}{2} - \frac{1}{2}\gamma\mu c_l - \frac{1}{2}\gamma\mu c_p\right] \\
\omega_9 &= \frac{1}{T}\left[\frac{\gamma\mu}{2} + \frac{\eta c_h}{2} + \frac{1}{2}\gamma\eta c_l + \frac{1}{2}\gamma\eta c_p\right] \\
\therefore \frac{\partial TP_2(\mathrm{p}, t_f)}{\partial t_f} &= -p\omega_4 - p^2\omega_5 - \omega_7 + 2t_f(p^2\omega_6 + \omega_8 + p\omega_9) = 0
\end{aligned} \tag{A.2.5}$$

Step 2: Find $p*(t_f)$ and $t_f{}^*(p)$ by using **step 1**

Solving equations (A.2.3) and (A.2.5) we get

$p*(t_f) = \frac{\sigma_2(t_f)}{2\sigma_1(t_f)}$

and

$t_f{}^*(p) = \frac{p\omega_4 + p^2\omega_5 + \omega_7}{2(p^2\omega_6 + \omega_8 + p\omega_9)}$

So, the profit function turns into

$$TP_2(t_f, p^*(t_f)) = -\frac{\phi_2{}^2}{4*\phi_1} + \phi_3 \tag{A.2.6}$$

Step 3: Find $\frac{\partial TP_2(t_f, p^*(t_f))}{\partial t_f}, \frac{\partial^2 TP_2(t_f, p^*(t_f))}{\partial t_f{}^2}$

Differentiate (A.2.6) with respect to t_f,

$$\frac{\partial TP_2(t_f)}{\partial t_f} = \frac{\phi_2^2[t_f]\phi_1{}'[t_f]}{4\phi_1[t_f]^2} - \frac{(\phi_2^2)'[t_f]}{4\phi_1[t_f]} + \phi_3{}'[t_f]$$

$$\frac{TP_2(t_f)}{\partial t_f} = \begin{bmatrix} m_7 + 2m_8 t_f - \frac{(L_4 + 2L_9 t_f)(L_2 + L_4 t_f + L_9 t_f^2)}{2(L_3 + L_5 t_f + L_6 t_f^2)} \\ + \frac{(L_5 + 2L_6 t_f)(L_2 + L_4 t_f + L_9 t_f^2)^2}{4(L_3 + L_5 t_f + L_6 t_f^2)^2} \end{bmatrix}$$

Again, differentiating this equation with respect to t_f

$$\frac{\partial^2 TP_2(t_f)}{\partial t_f{}^2(t_f)} = -\frac{\phi_2^2[t_f]\phi_1{}'[t_f]^2}{2\phi_1[t_f]^3} + \frac{\phi_1{}'[t_f](\phi_2^2)'[t_f]}{2\phi_1[t_f]^2} + \frac{\phi_2^2[t_f]\phi_1{}''[t_f]}{4\phi_1[t_f]^2} - \frac{(\phi_2^2)''[t_f]}{4\phi_1[t_f]} + \phi_3{}''[t_f]$$

$$\frac{\partial^2 TP_2(t_f)}{\partial t_f{}^2(t_f)} = \begin{bmatrix} -\left(\begin{matrix}\frac{(L_4 + 2L_9 t_f)^2}{2(m_3 + m_5 t_f + m_6 t_f^2)} + \frac{(L_5 + 2L_6 t_f)^2 (L_2 + L_4 t_f + L_9 t_f^2)^2}{2(m_3 + m_5 t_f + m_6 t_f^2)^3} \\ + \frac{L_9(L_2 + L_4 t_f + L_9 t_f^2)}{m_3 + m_5 t_f + m_6 t_f^2}\end{matrix}\right) \\ + \left(\frac{(L_5 + 2L_6 t_f)(L_4 + 2L_9 t_f)(L_2 + L_4 t_f + n_9 t_f^2)}{(m_3 + m_5 t_f + m_6 t_f^2)^2} + 2m_8 + \frac{L_6(L_2 + L_4 t_f + L_9 t_f^2)^2}{2(L_3 + L_5 t_f + L_6 t_f^2)^2}\right) \end{bmatrix} \tag{A.2.7}$$

Step 4: Proof the conditions for the optimality of the objective function.

To ensure that the optimal profit function is concave, the following condition must be satisfied:

$\frac{\partial^2 TP_2(t_f)}{\partial t_f{}^2(t_f)} < 0$

Now simplifying (A.2.7) we get,

$$\frac{\partial^2 TP_2(t_f)}{\partial t_f{}^2(t_f)} = F_2 - F_1 < 0 \tag{A.2.8}$$

where:

$$F_1 = \left[\left(\begin{matrix}\frac{(L_4 + 2L_9 t_f)^2}{2(m_3 + m_5 t_f + m_6 t_f^2)} + \frac{(L_5 + 2L_6 t_f)^2 (L_2 + L_4 t_f + L_9 t_f^2)^2}{2(m_3 + m_5 t_f + m_6 t_f^2)^3} \\ + \frac{L_9(L_2 + L_4 t_f + L_9 t_f^2)}{m_3 + m_5 t_f + m_6 t_f^2}\end{matrix}\right)\right]$$

$$F_2 = \frac{(L_5 + 2L_6 t_f)(L_4 + 2L_9 t_f)(L_2 + L_4 t_f + n_9 t_f^2)}{(L_3 + L_5 t_f + L_6 t_f^2)^2} + 2m_8 + \frac{L_6(L_2 + L_4 t_f + L_9 t_f^2)^2}{2(L_3 + L_5 t_f + L_6 t_f^2)^2}$$

$\frac{\partial^2 TP_2(t_f)}{\partial t_f{}^2(t_f)}$ will be less than zero if $F_2(t_f^*) - F_1(t_f^*)$ is negative at the optimal point,

$$\text{i.e., if } F_1(t_f^*) > F_2(t_f^*) \tag{A.2.9}$$

Finally, if condition (A.2.9) is satisfied, the total profit is always concave.

Appendix B.1
Proof of Proposition 1

Let Q_1 be the total quantity sold in a replenishment cycle for Case I.

Then $Q_1 = S + R = t_f + \frac{D}{\gamma}\log\{1+\gamma(T-t_f)\}$

$Q_1 = (\mu - \eta p + \delta p_1)\left\{t_f + \frac{1}{\gamma}\log\{1+\gamma(T-t_f)\}\right\}$

Again let Q_2 be the total quantity sold in a replenishment cycle for Case II.

Then $Q_2 = [Q_1]_{\delta=0}$

$Q_2 = (\mu - \eta p)\left\{t_f + \frac{1}{\gamma}\log\{1+\gamma(T-t_f)\}\right\}$

Now, $Q_1 - Q_2 = \delta p_1\left\{t_f + \frac{1}{\gamma}\log\{1+\gamma(T-t_f)\}\right\}$

Since $\delta p_1\left\{t_f + \frac{1}{\gamma}\log\{1+\gamma(T-t_f)\}\right\} > 0$ then $Q_1 - Q_2 > 0$

Therefore, $Q_1 > Q_2$

i.e., the total sold quantity for Case I is greater than for Case II.

Appendix B.2
Proof of Proposition 2

We have,

$$TP_1(p,t_f) = \begin{bmatrix} m_1 + pm_2 + p^2m_3 + pt_fm_4 + p^2t_fm_5 + p^2{t_f}^2m_6 + t_fm_7 \\ +{t_f}^2m_8 + p{t_f}^2m_9 \end{bmatrix} - \frac{1}{T}[OC] \tag{B.2.1}$$

$$TP_2(p,t_f) = \begin{bmatrix} m_1 + pL_2 + p^2L_3 + pt_fL_4 + p^2t_fL_5 + p^2{t_f}^2L_6 + t_fm_7 \\ +{t_f}^2m_8 + p{t_f}^2L_9 \end{bmatrix} - \frac{1}{T}[OC] \tag{B.2.2}$$

Subtracting (B.2.2) from (B.2.1) we get,

$$\begin{aligned} &TP_1(p,t_f) - TP_2(p,t_f) = \\ &\begin{bmatrix} pm_2 + p^2m_3 - pL_2 - p^2L_3 + pm_4t_f + p^2m_5t_f - pt_fL_4 - p^2t_fL_5 \\ +p^2m_6t_f^2 + pm_9t_f^2 - p^2{t_f}^2L_6 - p{t_f}^2L_9 \end{bmatrix} \\ &= \begin{bmatrix} p(m_2-L_2) + p^2(m_3-L_3) + pt_f(m_4-L_4) + p^2t_f(m_5-L_5) \\ +p^2t_f^2(m_6-L_6) + pt_f^2(m_9-L_9) \end{bmatrix} \end{aligned} \tag{B.2.3}$$

Since p and t_f are always positive then (B.2.3) will always positive if

$m_2 - L_2, m_3 - L_3, m_4 - L_4, m_5 - L_5, m_6 - L_6, m_9 - L_9 > 0$

which produces the following inequality:

$$\begin{aligned} &m_2 + m_3 + m_4 + m_5 + m_6 + m_9 > L_2 + L_3 + L_4 + L_5 + L_6 + L_9 \\ &=> m_2 + m_3 + m_4 + m_5 + m_6 + m_9 - (L_2 + L_3 + L_4 + L_5 + L_6 + L_9) > 0 \end{aligned} \tag{B.2.4}$$

Putting the values of $m_2, m_3, m_4, m_5, m_6, m_9, L_2, L_3, L_4, L_5, L_6$ and L_9 in the equation (B.2.4) we get,

$$= \begin{bmatrix} \begin{bmatrix} -\eta + \gamma\eta - \frac{\gamma\eta}{2T} - \frac{T\gamma\eta}{2} + \mu - \gamma\mu + \frac{\gamma\mu}{2T} + \frac{T\gamma\mu}{2} + (-\frac{\eta}{2} + \frac{\gamma\eta}{2} - \frac{\gamma\eta}{4T} - \frac{T\gamma\eta}{4} + \frac{\mu}{2} \\ -\frac{\gamma\mu}{2} + \frac{\gamma\mu}{4T} + \frac{T\gamma\mu}{4})i_e + \frac{\eta c_h}{2T} + 2\eta c_l - \frac{2\eta c_l}{T} - \gamma\eta c_l + \frac{\gamma\eta c_l}{2T} + \frac{1}{2}T\gamma\eta c_l + \eta c_p \\ -\gamma\eta c_p + \frac{\gamma\eta c_p}{2T} + \frac{1}{2}T\gamma\eta c_p + \delta(\frac{1}{2} - \frac{\gamma}{2} + \frac{\gamma}{4T} + \frac{T\gamma}{4} + +(\frac{1}{4} - \frac{\gamma}{4} + \frac{\gamma}{8T} + \frac{T\gamma}{8})i_e \\ -\frac{c_h}{4T} - c_l + \frac{c_l}{T} + \frac{\gamma c_l}{2} - \frac{\gamma c_l}{4T} - \frac{1}{4}T\gamma c_l - \frac{c_p}{2} + \frac{\gamma c_p}{2} - \frac{\gamma c_p}{4T} - \frac{1}{4}T\gamma c_p) \end{bmatrix} \\ -\begin{bmatrix} -\eta + \gamma\eta - \frac{\gamma\eta}{2T} - \frac{T\gamma\eta}{2} + \mu - \gamma\mu + \frac{\gamma\mu}{2T} + \frac{T\gamma\mu}{2} + \frac{\eta c_h}{2T} + 2\eta c_l - \frac{2\eta c_l}{T} \\ -\gamma\eta c_l + \frac{\gamma\eta c_l}{2T} + \frac{1}{2}T\gamma\eta c_l + \eta c_p - \gamma\eta c_p + \frac{\gamma\eta c_p}{2T} + \frac{1}{2}T\gamma\eta c_p \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \left(-\frac{\eta}{2} + \frac{\gamma\eta}{2} - \frac{\gamma\eta}{4T} - \frac{T\gamma\eta}{4} + \frac{\mu}{2} - \frac{\gamma\mu}{2} + \frac{\gamma\mu}{4T} + \frac{T\gamma\mu}{4}\right)i_e \\ +\delta\begin{Bmatrix} \frac{1}{2} - \frac{\gamma}{2} + \frac{\gamma}{4T} + \frac{T\gamma}{4} + \left(\frac{1}{4} - \frac{\gamma}{4} + \frac{\gamma}{8T} + \frac{T\gamma}{8}\right)i_e - \frac{c_h}{4T} \\ -c_l + \frac{c_l}{T} + \frac{\gamma c_l}{2} - \frac{\gamma c_l}{4T} - \frac{1}{4}T\gamma c_l - \frac{c_p}{2} + \frac{\gamma c_p}{2} - \frac{\gamma c_p}{4T} - \frac{1}{4}T\gamma c_p \end{Bmatrix} \end{bmatrix}$$

$$= \frac{1}{8T}\begin{bmatrix} T^2\big(\gamma((\delta - 2\eta + 2\mu)i_e - 2\delta(-1 + c_l + c_p))\big) \\ +T(4(-2+\gamma)\delta c_l + (-1+\gamma)(-2(\delta - 2\eta + 2\mu)i_e + 4\delta(-1 + c_p))) \\ +\gamma(\delta - 2\eta + 2\mu)i_e - 2\delta(c_h + (-4+\gamma)c_l + \gamma(-1 + c_p)) \end{bmatrix}$$

$$= \frac{AT^2 + BT + C}{8T}$$

where:

$A = \big(\gamma((\delta - 2\eta + 2\mu)i_e - 2\delta(-1 + c_l + c_p))\big)$

$B = (4(-2+\gamma)\delta c_l + (-1+\gamma)(-2(\delta - 2\eta + 2\mu)i_e + 4\delta(-1 + c_p)))$

$C = \gamma(\delta - 2\eta + 2\mu)i_e - 2\delta(c_h + (-4+\gamma)c_l + \gamma(-1 + c_p))$

Since T is always positive then $\frac{AT^2+BT+C}{8T} > 0$ if $AT^2 + BT + C$ is positive.
Again, $AT^2 + BT + C > 0$ under the following possibilities:
(i) $A, B, C > 0$, (ii) $A, B > 0$ and $C \geq 0$, (iii) $A, C > 0$ and $B \geq 0$, (iv) $B, C > 0$ and $A \geq 0$
(v) $A > 0$ and $B, C \geq 0$, (vi) $B > 0$ and $A, C \geq 0$, (vii) $C > 0$ and $A, B \geq 0$
Now (i) $A, B, C > 0$:
$A = \left(\gamma((\delta - 2\eta + 2\mu)i_e - 2\delta(-1 + c_l + c_p))\right) > 0$ which produce $n < \frac{-2-i_e+2c_l+2c_p}{-2-i_e+2\eta i_e-2\mu i_e+2c_l+2c_p}$
$B = (4(-2+\gamma)\delta c_l + (-1+\gamma)(-2(\delta - 2\eta + 2\mu)i_e + 4\delta(-1 + c_p))) > 0$ which produce
$c_p > \frac{2(-1+\gamma)(\delta-2\eta+2\mu)i_e+4\delta(-1+\gamma-(-2+\gamma)c_l)}{4(-1+\gamma)\delta}$
and
$C = \gamma(\delta - 2\eta + 2\mu)i_e - 2\delta(c_h + (-4+\gamma)c_l + \gamma(-1 + c_p)) > 0$ which produce
$i_e > \frac{2\delta(c_h+(-4+\gamma)c_l+\gamma(-1+c_p))}{\gamma(\delta-2\eta+2\mu)}$
Let,
$L = \frac{-2-i_e+2c_l+2c_p}{-2-i_e+2\eta i_e-2\mu i_e+2c_l+2c_p}$,
$M = \frac{2(-1+\gamma)(\delta-2\eta+2\mu)i_e+4\delta(-1+\gamma-(-2+\gamma)c_l)}{4(-1+\gamma)\delta}$
$N = \frac{2(-1+\gamma)(\delta-2\eta+2\mu)i_e+4\delta(-1+\gamma-(-2+\gamma)c_l)}{4(-1+\gamma)\delta}$
then $n < L; c_p > \mathrm{M}; i_e > N$
Similarly, other possibilities
(iv) If $A, B > 0$ *and* $C \geq 0$, then $n < L; c_p > M; i_e \geq N$
(v) If $A, C > 0$ *and* $B \geq 0$, then $n < L; c_p \geq M; i_e > N$
(vi) If $B, C > 0$ *and* $A \geq 0$, then $n \leq L; c_p > M; i_e > N$
(vii) If $A > 0$ *and* $B, C \geq 0$, then $n < L; c_p \geq M; i_e \geq N$
(viii) If $B > 0$ *and* $A, C \geq 0$, then $n \leq L; c_p > M; i_e \geq N$
(ix) If $C > 0$ *and* $A, B \geq 0$, then $n \leq \mathrm{L}; c_p \geq \mathrm{M}; i_e > N$
Hence total profit with an installment facility $TP_1(p, t_f)$ is always more profitable than $TP_2(p, t_f)$ under any possibilities among (i)-(vii).