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AN INVERSE PROBLEM FOR A DISSIPATIVE HYPERBOLIC EQUATION WITH DISCONTINUOUS COEFFICIENTS*

By

ROBERT J. KRUEGER
University of Nebraska, Lincoln

1. Introduction. In this paper we generalize results obtained by Weston [9] concerning the inverse problem for a particular type of hyperbolic partial differential equation. The motivation for our work comes from an electromagnetic scattering problem, and so we begin by briefly outlining such a problem.

Consider a slab of width L situated between the planes $z = 0$ and $z = L$, with conductivity $\sigma(z)$ and permittivity $\epsilon(z)$. Outside of the slab we assume $\sigma = 0$, $\epsilon = \epsilon_0$. The conductivity and permittivity of the slab are unknown, and we seek a method for determining these functions by measuring electromagnetic waves scattered from the slab. As in all inverse scattering problems, two questions arise immediately: What type of scattering data is needed and how are the data used to reconstruct the unknown functions $\sigma(z)$, $\epsilon(z)$?

The difference between this problem and the one considered by Weston is that we no longer require that the functions ϵ and σ be continuous at $z = 0$ and $z = L$. This generalization is clearly necessary since the interface between two different media is characterized by such discontinuities.

We use a time-dependent approach in our solution of the inverse problem, and show that the necessary scattering data can be obtained by illuminating the slab with a single plane wave and measuring a finite portion of the reflected wave and the wave transmitted out the far side of the slab. This result is in marked contrast to Weston's, in which two incident waves are needed (one from either side of the slab) and the resulting reflected and transmitted waves are measured.

2. Statement of the problem and results. We now present a general formulation of the inverse problem and outline our results. The equation we consider is

$$Lu = u_{xx} - u_{tt} + A(x)u_x + B(x)u_t + C(x)u = 0, \quad -\infty < x < \infty, \quad -\infty < t < \infty \tag{2.1}$$

where

(a) Support $A, B, C \subseteq [0, l]$.

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(b) A and B are C^1 functions on $(0, l)$ with jump discontinuities at $x = 0$, $x = l$.
(c) C is a continuous function on $(0, l)$ with jump discontinuities at $x = 0$, $x = l$.
(d) The solution $u(x, t)$ is everywhere continuous and piecewise C^2.
(e) There is a jump discontinuity in u_x across $x = 0$, $x = l$, with

$$c_0 u_x(0-, t) = u_x(0+, t), \qquad c_l u_x(l+, t) = u_x(l-, t)$$

where c_0 and c_l are constants.

For the problem we are considering, the coefficients A, B, C of Eq. (2.1) are unknown on $(0, l)$. In this paper we present a method for constructing the coefficient $B(x)$ and the combination of coefficients $C(x) - A'(x)/2 - A^2(x)/4$ by using a certain set of scattering data. To understand the nature of this data, let us establish some notation. We refer to the region $(0, l)$ as the "slab". Now consider a plane wave $u_+^i(x - t)$ propagating in the $+x$ direction, incident normally on the slab. This gives rise to a wave u_+^r reflected off the slab and a wave u_+^t transmitted through the slab. Thus, exterior to the slab we have the following solution of (2.1):

$$\begin{aligned} u(x, t) &= u_+^i(x - t) + u_+^r(x + t), \qquad & x < 0 \\ &= u_+^t(x - t), & x > l. \end{aligned} \tag{2.2}$$

We also require that $u_+^i(\eta) = u_+^t(\eta) = 0$ for $\eta > 0$ and $u_+^r(\xi) = 0$ for $\xi < 0$. (Similarly, we could use an incident wave $u_-^i(x + t)$ propagating in the $-x$ direction. This generates a solution of (2.1)

$$\begin{aligned} u(x, t) &= u_-^i(x + t) + u_-^r(x - t), \qquad & x > l \\ &= u_-^t(x + t), & x < 0 \end{aligned} \tag{2.3}$$

where $u_-^i(\xi) = u_-^t(\xi) = 0$ for $\xi < 0$ and $u_-^r(\eta) = 0$ for $\eta > 2l$.) Notice that in formulating the problem (2.1) we required that the solution u be continuous and piecewise C^2. In particular, we require that $u_\pm^i$, u_+^r, $u_\pm^t$ have these properties and to be such that $u(x, t)$ satisfies conditions (d) and (e) given above.

To reconstruct the coefficients we need the following data:

$$u_+^i(\eta), \quad -4l < \eta < 0; \quad u_+^r(\xi), \quad 0 < \xi < 4l; \quad u_+^t(\eta), \quad -2l - < \eta < 0.$$

Let us describe briefly how this data is used in solving the inverse problem. In Sec. 4 we establish the following identities (scattering operators):

$$u_+^t(\eta) = b_0^{-1} u_+^i(\eta) + b_0^{-1} g u_+^i(2l + \eta) + \int_\eta^0 T_+(\eta - s) u_+^i(s)\, ds, \quad -4l < \eta < 0, \tag{2.4}$$

$$u_+^r(\xi) = f_0 u_+^i(-\xi) - j_0 u_+^i(2l - \xi) + \int_{-\xi}^0 R_+(\xi + s) u_+^i(s)\, ds, \qquad 0 < \xi < 4l, \tag{2.5}$$

where the constants b_0, f_0, g, j_0 are defined below. The kernels R_+, T_+ of these operators are called the reflection and transmission kernels respectively. These are piecewise C^2 functions and are, under the hypotheses given above, independent of the choice of incident wave used in the scattering experiment.

Now by measuring an incident, reflected and transmitted wave, it is possible to determine the kernels R_+, T_+ and constants b_0, f_0, g, j_0. We show in Sec. 5 that these kernels and constants can be used to derive an integral equation which is a generalization of the Gelfand-Levitan equation. The solution of this equation enables us to easily construct the coefficient B and the combination $C - A'/2 - A^2/4$. We also show how our results are related to those obtained by Weston. In Sec. 6 we illustrate this technique by solving an inverse problem for a transmission line with line losses taken into account. We conclude with an appendix which describes a procedure for solving the inverse problem when the unknown quantities are constant functions.

We should mention that the hypotheses on $u_+^{\,i}$, $u_+^{\,r}$, $u_+^{\,t}$ given above are sufficient to determine the solution of the inverse problem provided we can determine the scattering kernels and constants in (2.4) and (2.5). (The sufficiency of these conditions is demonstrated in Secs. 3, 4 and 5.) If we also require $u_+^{\,i\prime}(0-) \neq 0$, then it can be shown that these kernels and constants can be uniquely determined. For example, differentiating (2.4) and bearing in mind that $u_+^{\,i}(0) = 0$ yields

$$u_+^{\,t\prime}(\eta) = b_0^{-1}u_+^{\,i\prime}(\eta) + b_0^{-1}gu_+^{\,i\prime}(2l + \eta) + \int_\eta^0 T_+(s)u_+^{\,i\prime}(\eta - s)\,ds$$

which is valid for all η between $-4l$ and 0 where $u_+^{\,i\prime}$ is defined. Setting $\eta = 0-$ gives

$$u_+^{\,t\prime}(0-) = b_0^{-1}u_+^{\,i\prime}(0-)$$

and so b_0 is determined. Similarly

$$u_+^{\,t\prime}(-2l+) - u_+^{\,t\prime}(-2l-) = b_0^{-1}\{u_+^{\,i\prime}(-2l+) - u_+^{\,i\prime}(-2l-)\} - b_0^{-1}gu_+^{\,i\prime}(0-)$$

which yields g. Differentiating a second time produces

$$u_+^{\,t\prime\prime}(\eta) = b_0^{-1}u_+^{\,i\prime\prime}(\eta) + b_0^{-1}gu_+^{\,i\prime\prime}(2l + \eta) - T_+(\eta)u_+^{\,i\prime}(0-) + \int_\eta^0 T_+(s)u_+^{\,i\prime\prime}(\eta - s)\,ds$$

which is a Volterra equation of the second kind for $T_+(\eta)$, so T_+ is easily determined.

To show the motivation for formulating problem (2.1) in the above manner, consider again the example of electromagnetic scattering cited in Sec. 1. An electromagnetic wave propagating along the z-axis normal to the slab situated between $z = 0$ and $z = L$ has transverse electric field $E(z, t)$ satisfying

$$E_{zz} - \epsilon(z)\mu_0 E_{tt} - \sigma(z)\mu_0 E_t = 0, \qquad -\infty < z < \infty, \qquad -\infty < t < \infty \tag{2.6}$$

where the permeability μ_0 is constant. The change of variable

$$x = \int_0^z \{\epsilon(s)\mu_0\}^{1/2}\,ds, \qquad x(L) = l, \qquad u(x, t) = E(z, t)$$

reduces (2.6) to (2.1) with

$$A(x) = -\frac{d}{dz}\{\epsilon(z)\mu_0\}^{-1/2}, \qquad z \neq 0, L,$$

$$B(x) = -\sigma(z)/\epsilon(z), \qquad C(x) = 0.$$

The continuity of E and E_z at $z = 0$, $z = L$ implies that u is continuous at $x = 0$, $x = l$ and that u_x satisfies

$$\{\epsilon_0/\epsilon(0+)\}^{1/2}u_x(0-, t) = u_x(0+, t)$$

$$\{\epsilon_0/\epsilon(L-)\}^{1/2}u_x(l+, t) = u_x(l-, t)$$

which corresponds to (e) above. Hence, by measuring an incident wave and the resulting scattered waves we can determine $B(x)$ and $A(x)$ (see Sec. 5). In [9] Weston shows how to recover ϵ and σ once A and B are known.

Observe that if $B \equiv 0$ in (2.1) then we are dealing with a nondissipative problem. If we further assume A is continuous at $x = 0$, $x = l$, then the transformation

$$v(x, t) = u(x, t) \exp\left\{\frac{1}{2}\int^x A\right\}$$

reduces our equation to a selfadjoint one which is amenable to a time-dependent (see Kay [3]) or steady-state (see Kay and Moses [5], [6]) inverse scattering technique. In this case only a reflected wave or set of reflection coefficients is needed to reconstruct the combination $C - A'/2 - A^2/4$.

Although our main interest is the fact that we are dealing with a dissipative medium, our results do have applications to some unresolved problems for nondissipative media. For example, Moses and deRidder [7] have shown that the permittivity of a dielectric slab can be obtained by a steady-state approach in two ways: (1) by considering a wave of fixed frequence and varying the angle of incidence on the slab, or (2) by considering a fixed angle of incidence and varying the frequency. In the first case it is not necessary to assume a continuous change in permittivity at the faces of the slab, but in the second case the continuity is essential. Similarly, it has been pointed out by several authors ([2, 4, 7, 8]) that the inverse problem for lossless transmission lines can only be solved for the case where the capacitance and inductance are continuous functions. The time-dependent approach which we present eliminates the continuity assumptions in the above problems.

Let us establish some notation to be used throughout this paper. It is convenient to denote $A(0+)$ by $A(0)$ and $A(l-)$ by $(A(l)$, with a similar convention for B. Also, to denote differences in function values, we write

$$[f]^Q{}_P = f(Q) - f(P).$$

Finally, let $G(x) = \exp\{-\int_0^x B(s)\,ds\}$, and let the function $\theta(s)$ be such that $\theta(s) = 0$ if $s \leq 0$, $\theta(s) = 1$ if $s > 0$.

We list here constants which will be used at various points in this work:

$$\begin{aligned}
a_\pm &= \exp\left\{\frac{1}{2}\int_0^l (A(s) \pm B(s))\,ds\right\} \\
b_0 &= \frac{1}{4c_0}(c_0 + 1)(c_l + 1)a_- \\
b_l &= \frac{1}{4c_l}(c_0 + 1)(c_l + 1)a_+^{-1} \\
f_0 &= (c_0 - 1)/(c_0 + 1) \\
f_l &= (c_l - 1)/(c_l + 1) \\
g &= f_0 f_l G(l)^{-1} \\
h_0 &= b_0 f_l G(l)^{-1} \\
h_l &= b_l f_0 G(l)^{-1} \\
j_0 &= (c_l - 1)/(c_0 + 1)b_0^{-1}a_+ \\
j_l &= (c_0 - 1)/(c_l + 1)b_l^{-1}a_-^{-1}.
\end{aligned}$$

3. The weak Riemann function. We begin by developing an expression for the solution of (2.1) in terms of waves (2.2), (2.3). This is done by solving the Cauchy problem for (2.1) with the data being $u(x, t)$, $u_x(x, t)$ where $x = 0+$ or $x = l-$. We present a general approach to the solution of this Cauchy problem, using a weak Riemann function. Such a function should prove helpful in future studies relating to this type of problem.

Set $\xi = x + t$, $\eta = x - t$ and let $g(\xi, \eta; \xi_0, \eta_0)$ denote the weak Riemann function. We require that g have the following properties:

(a) $L^*g = 0$ for $\xi + \eta \neq 0, 2l$ (where L^* is the formal adjoint of L).

(b) g is a continuous function of ξ, η, ξ_0, η_0.

(c) $$g(\xi_0, \eta; \xi_0, \eta_0) = \exp\left\{\frac{1}{2}\int_{(\xi_0+\eta_0)/2}^{(\xi_0+\eta)/2} (A(s) + B(s))\, ds\right\}.$$

(d) $$g(\xi, \eta_0; \xi_0, \eta_0) = \exp\left\{\frac{1}{2}\int_{(\xi_0+\eta_0)/2}^{(\xi+\eta_0)/2} (A(s) - B(s))\, ds\right\}.$$

(e) $[g_\xi + g_\eta]^{\xi+\eta=0+}{}_{\xi+\eta=0-} = [A]^{0+}{}_{0-}\, g|_{\xi+\eta=0}$.

(f) $[g_\xi + g_\eta]^{\xi+\eta=2l+}{}_{\xi+\eta=2l-} = [A]^{l+}{}_{l-}\, g|_{\xi+\eta=2l}$.

It turns out that the case $\xi_0 + \eta_0 > 2l$ is only of minor interest and so in the remainder of this paper we consider $\xi_0 + \eta_0 < 2l$ and, where appropriate, merely state results for the case $\xi_0 + \eta_0 > 2l$. Now several remarks are in order:

(a′) It is clear that g exists and is unique.

(b′) If $0 < \xi + \eta < 2l$ and $0 < \xi_0 + \eta_0 < 2l$, then g coincides with the classical Riemann function.

(c′) If $\xi_0 + \eta_0 < 0$, condition (e) becomes $g_\xi + g_\eta = A(0)$ on $\xi + \eta = 0+$.

(d′) If $\xi_0 + \eta_0 < 0$, then from the propagation of discontinuities (see [1, vol. II, p. 618])

$$[g_\xi]^{\xi=-\eta_0+}{}_{\xi=-\eta_0-} = -\tfrac{1}{4}(A + B)|_0 \exp\left\{\frac{1}{2}\int_0^{(\eta-\eta_0)/2} (A(s) + B(s))\, ds\right\}$$
$$\text{for} \quad \eta_0 < \eta < 2l + \eta_0, \tag{3.1}$$

$$[g_\eta]^{\eta=-\xi_0+}{}_{\eta=-\xi_0-} = -\tfrac{1}{4}(A - B)|_0 \exp\left\{\frac{1}{2}\int_0^{(\xi-\xi_0)/2} (A(s) - B(s))\, ds\right\}$$
$$\text{for} \quad \xi_0 < \xi < 2l + \xi_0. \tag{3.2}$$

(See Fig. 1.)

We now express the solution u of (2.1) in terms of Cauchy data on $x = l-$.

LEMMA 1. For $x_0 < l$ and any t_0, we have

$$2u(x_0, t_0) + (c_0 - 1)\theta(-x_0)\int_{-x_0}^{x_0} u_x(0-, t_0 + s)\, ds$$
$$= gu|_P + gu|_Q + \int_P^Q \{u(g_\xi + g_\eta) - u_x g - A(l)ug\}\, ds$$

where the integral on the right-hand side is along the line $x = l-$ and P, Q have space-time coordinates $(l-, x_0 + t_0 - l)$, $(l-, t_0 - x_0 + l)$ respectively.

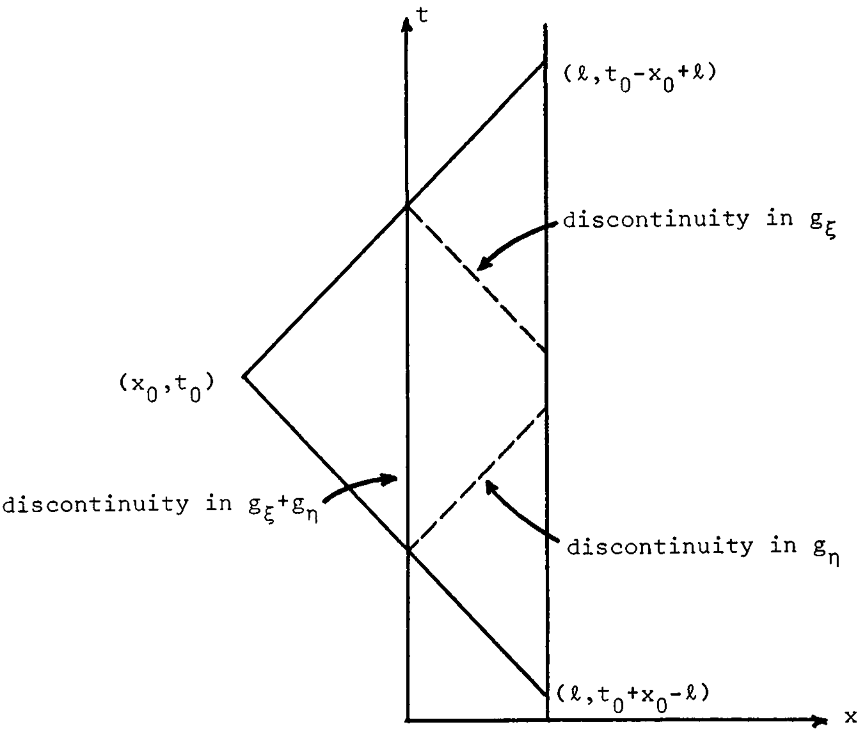

FIG. 1.

Proof: If $x_0 \geq 0$ the above formula follows as in the classical case. If $x_0 < 0$ integrate the expression $gLu - uL^*g = 0$ over the regions $\xi + \eta < 0$, $\xi \geq \xi_0$, $\eta \geq \eta_0$ and $0 < \xi + \eta \leq 2l$, $\xi \geq \xi_0$, $\eta \geq \eta_0$. Add the resulting quantities, using the fact that $g_\xi - g_\eta$ is continuous across $x = 0$ and also using property (e), and the lemma follows.

In a similar manner we get

LEMMA 2. For $x_0 > 0$ and any t_0,

$$2u(x_0, t_0) + (c_l - 1)\theta(x_0 - l) \int_{2l-x_0}^{x_0} u_x(l+, s + t_0 - l)\, ds$$

$$= gu|_P + gu|_Q + \int_P^Q \{u(g_\xi + g_\eta) - u_x g - A(0)ug\}\, ds$$

where the integral on the right-hand side is along the line $x = 0+$ and P, Q have space-time coordinates $(0+, t_0 + x_0)$, $(0+, t_0 - x_0)$ respectively.

A simplified expression for g and its derivatives as they appear under the integral signs in Lemmas 1 and 2 can be obtained due to the fact that the coefficients of (2.1) are independent of time. This result is expressed in the next lemma.

LEMMA 3. The functions K_+ and K_- defined by

$$K_+(x_0, y, n) \exp\left\{\frac{1}{2}\int_{x_0}^{n} (A(s) - B(s))\, ds\right\}$$

$$= g_\eta(2n - y + t_0, y - t_0; x_0 + t_0, x_0 - t_0), \qquad (3.3)$$

$$K_-(x_0\,,y,n)\exp\left\{\frac{1}{2}\int_{x_0}^{n}(A(s)+B(s))\,ds\right\}$$
$$= g_\xi(y+t_0\,,2n-y-t_0\,;x_0+t_0\,,x_0-t_0) \tag{3.4}$$

(where $n = 0$ or $n = l$) satisfy the following conditions:

$$\left\{\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\pm B(x)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)+D_\pm(x)\right\}K_\pm(x,y,n)=0, \qquad x\neq 0,\,l, \tag{3.5}$$

$$K_\pm(x,2n-x,n)=\tfrac{1}{4}(A\pm B)|_n\exp\left\{\pm\int_x^n B(s)\,ds\right\}, \tag{3.6}$$

$$K_\pm(x,x,n)=-\frac{1}{2}\int_x^n D_\pm(s)\,ds+\tfrac{1}{4}(A\pm B)|_n$$
$$+\tfrac{1}{4}\theta(\{x-l\}\{l-n\}-xn)(A\mp B)|_{l-n} \tag{3.7}$$

where

$$D_\pm(x)=C(x)-\tfrac{1}{2}A'(x)\pm\tfrac{1}{2}B'(x)+\tfrac{1}{4}(B^2(x)-A^2(x)), \qquad x\neq 0,\,l.$$

Proof: We consider the case $n = l$ and verify formulas (3.4)–(3.7). The other cases follow in a similar manner.

Let $L_{(\xi,\eta)}$ denote the operator L in characteristic coordinates. It can be shown that as a function of $(\xi_0\,,\eta_0)$, $g(\xi,\eta;\xi_0\,,\eta_0)$ satisfies $L_{(\xi_0,\eta_0)}g = 0$. It follows that $L_{(\xi_0,\eta_0)}g_\xi = 0$. (The method of successive approximations [1, vol. II, p. 462] shows the continuity of the mixed partial derivatives which are involved.) A boundary condition for g_ξ on $\eta = \eta_0$ is obtained from item (d) above, and a condition on $\xi = \xi_0$ is obtained from (a) and (c). We thus have a characteristic initial value problem for g_ξ. The change of variable $\alpha = \xi_0 - \xi$, $\beta = \eta_0 + \xi$ and $g_\xi(\xi, 2l - \xi; \alpha + \xi, \beta - \xi) = f(\xi, \alpha, \beta, l)$ converts this problem into one which is independent of ξ. Now set $\alpha = x_0 - y$, $\beta = x_0 + y$ and

$$f(\xi,\alpha,\beta,l)=K_-(x_0\,,y,l)\exp\left\{\frac{1}{2}\int_{x_0}^{l}(A(s)+B(s))\,ds\right\}$$

and our result follows.

In the same way as in Lemma 3, we can prove

LEMMA 4. The functions L_+ and L_- defined by

$$L_+(x_0\,,y,n)\exp\left\{\frac{1}{2}\int_{x_0}^{n}(A(s)-B(s))\,ds\right\}$$
$$= g(2n-y+t_0\,,y-t_0\,;x_0+t_0\,,x_0-t_0), \tag{3.8}$$

$$L_-(x_0\,,y,n)\exp\left\{\frac{1}{2}\int_{x_0}^{n}(A(s)+B(s))\,ds\right\}$$
$$= g(y+t_0\,,2n-y-t_0\,;x_0+t_0\,,x_0-t_0) \tag{3.9}$$

satisfy Eq. (3.5) and have boundary values

$$L_\pm(x,2n-x,n)=\exp\left\{\pm\int_x^n B(s)\,ds\right\} \tag{3.10}$$

$$L_+(x,x,n)=1. \tag{3.11}$$

We are now in a position to express the solution of the Cauchy problem in terms of waves outside of the slab.

THEOREM 1. Assume the Cauchy data for Eq. (2.1) is

$$u(n, t) = v(n - t) + w(n + t), \qquad u_x(n, t) = c_n v'(n - t) + c_n w'(n + t) \quad (3.12)$$

where $n = 0$ or $n = l$ and v, w are continuous, piecewise C^2 functions. Then the solution u of (2.1) is given by

$$\begin{aligned} &2u(x, t) + (c_0 - 1)\theta(-x) \int_{-x}^{x} u_x(0-, s + t)\, ds + (c_l - 1)\theta(x - l) \\ &\quad \cdot \int_{2l-x}^{x} u_x(l+, s + t - l)\, ds = \exp\left\{\frac{1}{2}\int_x^n (A(s) - B(s))\, ds\right\}\Big\{(c_n + 1)v(x - t) \\ &\quad - (c_n - 1)v(2n - x - t) \exp\left\{\int_x^n B(s)\, ds\right\} + \int_x^{2n-x} v(y - t)N_+(x, y, n)\, dy\Big\} \\ &\quad + \exp\left\{\frac{1}{2}\int_x^n (A(s) + B(s))\, ds\right\}\Big\{(c_n + 1)w(x + t) \\ &\quad - (c_n - 1)w(2n - x + t) \exp\left\{-\int_x^n B(s)\, ds\right\} + \int_x^{2n-x} w(y + t)N_-(x, y, n)\, dy\Big\} \end{aligned} \quad (3.13)$$

where

$$\begin{aligned} N_\pm(x, y, n) &= (c_n + 1)K_\pm(x, y, n) \\ &\quad - (c_n - 1)K_\mp(x, 2n - y, n) \exp\left\{\pm\int_x^n B(s)\, ds\right\} - A(n)L_\pm(x, y, n). \end{aligned} \quad (3.14)$$

Proof: In Lemmas 1 and 2 write the u_x term in the expression $\int_P^Q u_x g\, ds$ in terms of the waves (3.12) and then integrate by parts. The theorem now follows from Lemmas 1–4.

4. The scattering operators. The scattering operators for Eq. (2.1) map an incident wave into the corresponding reflected and transmitted waves. Before constructing these operators we first notice that by using Lemmas 3 and 4 and Eq. (3.14), we can write

$$N_+(x, y, l) = V_+^1(x + y) + W_+^1(x - y)$$

for $x < 0$, $x < y < 2l - x$, where V_+^1, W_+^1 are piecewise C^2. We can also assume that $V_+^1(s) = 0$ if $s < 0$. From (3.2), (3.3) and (3.14), we then deduce that

$$[V_+^1(s)]^{s=0+}_{s=0-} = V_+^1(0+) = -\tfrac{1}{4}(c_l + 1)(A - B)|_0 . \quad (4.1)$$

Similarly, (3.1), (3.4) and (3.14) imply that

$$[W_+^1(s)]^{s=(-2l)-}_{s=(-2l)+} = \tfrac{1}{4}(c_l - 1)G(l)^{-1}(A + B)|_0 . \quad (4.2)$$

We now establish a result which expresses the incident and reflected waves in terms of the transmitted wave.

THEOREM 2. There exist unique piecewise C^2 functions V_+ and W_+ such that

$$u_+^r(\xi) = b_0 f_0 u_+^t(-\xi) - h_0 u_+^t(2l - \xi) + \int_{-\xi}^{0} V_+(\xi + s)u_+^t(s)\, ds, \qquad \xi > 0, \quad (4.3)$$

$$u_+^i(\eta) = b_0 u_+^t(\eta) - b_0 g u_+^t(2l + \eta) + \int_\eta^0 W_+(\eta - s)u_+^t(s)\, ds, \qquad \eta < 0. \quad (4.4)$$

Proof: If such functions exist, it is clear that they are unique. Now consider an incident wave $u_+^i(\eta)$ propagating in the $+x$ direction. In Theorem 1 set $n = l$ (hence $v = u_+^t$ and $w \equiv 0$) and $x < 0$. The integral on the left-hand side of Eq. (3.13) can be written

$$\int_{-x}^{x} u_x(0-, s + t)\, ds = u_+^i(\eta) - u_+^i(-\xi) - u_+^r(-\eta) + u_+^r(\xi).$$

Assuming $\xi < 0$, we then get

$$(c_0 + 1)u_+^i(\eta) - (c_0 - 1)u_+^r(-\eta) = a_-\left\{(c_l + 1)u_+^t(\eta) + \int_\eta^0 W_+^1(\eta - s)u_+^t(s)\, ds\right\} \tag{4.5}$$

for all η. Setting $\xi > 0$ in Theorem 1 and using (4.5) yields

$$(c_0 + 1)u_+^r(\xi) - (c_0 - 1)u_+^i(-\xi) = a_-\left\{-(c_l - 1)u_+^t(2l - \xi) + \int_0^{2l-\xi} W_+^1(\eta - s)u_+^t(s)\, ds + \int_{-\xi}^{2l-\xi} V_+^1(\xi + s)u_+^t(s)\, ds\right\}.$$

Since $W_+^1(\eta)$ is constant for $\eta < -2l$, we define

$$V_+^1(\xi) = -W_+^1(-\xi) \tag{4.6}$$

for $\xi > 2l$ and get

$$(c_0 + 1)u_+^r(\xi) - (c_0 - 1)u_+^i(-\xi) = a_-\left\{-(c_l - 1)u_+^t(2l - \xi) + \int_{-\xi}^0 V_+^1(\xi + s)u_+^t(s)\, ds\right\} \tag{4.7}$$

for all ξ.

The theorem now follows from (4.5) and (4.7) with

$$W_+(\eta) = (a_-/4c_0)\{(c_0 + 1)W_+^1(\eta) + (c_0 - 1)V_+^1(-\eta)\}, \qquad n < 0 \tag{4.8}$$

$$V_+(\xi) = (a_-/4c_0)\{(c_0 - 1)W_+^1(-\xi) + (c_0 + 1)V_+^1(\xi)\}, \qquad \xi > 0. \tag{4.9}$$

If we consider $u_+^t(\eta)$ as an unknown, then (4.4) is a Volterra integral equation involving a delayed argument. Inverting this equation will give an expression for the transmitted wave in terms of the incident wave. To accomplish this, first consider $-2l < \eta < 0$ and invert in the usual way (see [1, vol. I, p. 140]) using the resolvent kernel T_+ to get

$$u_+^t(\eta) = b_0^{-1}u_+^i(\eta) + \int_\eta^0 T_+(\eta - s)u_+^i(s)\, ds, \qquad -2l < \eta < 0$$

where

$$b_0T_+(\eta) + b_0^{-1}W_+(\eta) + \int_\eta^0 W_+(\eta - s)T_+(s)\, ds = 0, \qquad -2l < \eta < 0.$$

Using this result we can write the forward scattering operator (or transmission operator) by inverting (4.4):

$$u_+^t(\eta) = b_0^{-1}u_+^i(\eta) + b_0^{-1}gu_+^i(2l + \eta) + \int_\eta^0 T_+(\eta - s)u_+^i(s)\, ds, \qquad -4l < \eta < 0 \tag{4.10}$$

where

$$b_0T_+(\eta) + b_0^{-1}W_+(\eta) - b_0gT_+(2l + \eta) + b_0^{-1}gW_+(2l + \eta) + \int_\eta^0 W_+(\eta - s)T_+(s)\, ds, \qquad -4l < \eta < 0 \tag{4.11}$$

and by definition

$$W_+(\eta) = T_+(\eta) = 0 \quad \text{for} \quad \eta > 0.$$

Clearly the kernel T_+ is unique. Notice also that in Eq. (4.10) the term $u_+^i(2l + \eta)$ corresponds to an internal reflection in the slab. We would pick up additional reflections by inverting (4.4) on a domain larger than $-4l < \eta < 0$.

Combining (4.3) and (4.10) we get the back scattering (reflection) operator

$$u_+^r(\xi) = f_0u_+^i(-\xi) - j_0u_+^i(2l - \xi) + \int_{-\xi}^0 R_+(\xi + s)u_+^i(s)\, ds, \qquad 0 < \xi < 4l \tag{4.12}$$

where

$$R_+(\xi) = b_0f_0T_+(-\xi) + b_0^{-1}V_+(\xi) - h_0T_+(2l - \xi) + b_0^{-1}gV_+(\xi - 2l) + \int_{-\xi}^0 V_+(\xi + s)T_+(s)\, ds, \qquad 0 < \xi < 4l \tag{4.13}$$

and by definition $V_+(\xi) = 0$ for $\xi < 0$. Using (4.11), we also get

$$V_+(\xi) = b_0R_+(\xi) - b_0^2f_0T_+(-\xi) - b_0gR_+(\xi - 2l) + 2\frac{c_0^2 + 1}{(c_0 + 1)^2}b_0h_0T_+(2l - \xi) + \int_0^\xi W_+(s - \xi)\{R_+(s) - b_0f_0T_+(-s) + h_0T_+(2l - s)\}\, ds. \tag{4.14}$$

We now establish a crucial identity in N_+ and N_- to be used in Sec. 5.

LEMMA 5. For $0 < x < l$ and $-x < y < x$, we have

$$N_+(x, y, 0) = G(x)\{(c_0 + 1)R_+(x + y) - f_0N_-(x, -y, 0) - \int_{-y}^x R_+(y + s)N_-(x, s, 0)\, ds\}\cdot \tag{4.15}$$

Proof: For an incident wave $u_+^i(\eta)$ it follows by causality that for any x the solution u of (2.1) satisfies $u(x, t) = 0$ for $t < x$. Using such a value of t, set $n = 0$ in Theorem 1. Then the left-hand side of Eq. (3.13) vanishes, and on the right-hand side we have $v = u_+^i$ and $w = u_+^r$. Using the reflection operator to express u_+^r in terms of u_+^i, we obtain our result by virtue of the fact that u_+^i is arbitrary.

LEMMA 6. For $0 < x < l$, we have

$$G(x)^{-1} = U(2x) - \frac{1}{c_0 + 1}\int_{-x}^x U(x + y)N_-(x, y, 0)\, dy \tag{4.16}$$

where $U(y) = 1 + (c_0 + 1)/(2c_0) \int_0^y R_+(s)\, ds$.

Proof: Let $n = 0$ in Theorem 1. Now setting $v \equiv 1$ and $w \equiv 0$ produces the same value $u(x, t)$ on the left-hand side of Eq. (3.13) as does setting $v \equiv 0$ and $w \equiv 1$. Equate the resulting quantities to obtain

$$2x_0 - \int_{-x}^{x} N_+(x, y, 0)\, dy = G(x)\left\{2c_0 - \int_{-x}^{x} N_-(x, y, 0)\, dy\right\}. \tag{4.17}$$

Our result follows by substituting into Eq. (4.17) the expression for N_+ obtained in Lemma 5.

Thus far we have considered an incident wave $u_+^i(x - t)$. In an analogous manner, we can derive results for an incident wave $u_-^i(x + t)$.

THEOREM 3. There exist unique piecewise C^2 functions V_- and W_- such that

$$u_-^r(\eta) = b_l f_l u_-^t(2l - \eta) - h_l u_-^t(-\eta) + \int_0^{2l-\eta} V_-(\eta + s) u_-^t(s)\, ds, \qquad \eta < 2l$$

$$u_-^i(\xi) = b_l u_-^t(\xi) - b_l g u_-^t(\xi - 2l) + \int_0^{\xi} W_-(\xi - s) u_-^t(s)\, ds, \qquad \xi > 0. \tag{4.18}$$

Inverting (4.18), we get the transmission operator

$$u_-^t(\xi) = b_l^{-1} u_-^i(\xi) + b_l^{-1} g u_-^i(\xi - 2l) + \int_0^{\xi} T_-(\xi - s) u_-^i(s)\, ds, \qquad 0 < \xi < 4l$$

where

$$b_l T_-(\xi) + b_l^{-1} W_-(\xi) - b_l g T_-(\xi - 2l) + b_l^{-1} g W_-(\xi - 2l) + \int_0^{\xi} W_-(\xi - s) T_-(s)\, ds = 0, \qquad 0 < \xi < 4l$$

and by definition, $W_-(\xi) = T_-(\xi) = 0$ for $\xi < 0$. The reflection operator is

$$u_-^r(\eta) = f_l u_-^i(2l - \eta) - j_l u_-^i(-\eta) + \int_0^{2l+\eta} R_-(\eta + s) u_-^i(s)\, ds, \qquad -2l < \eta < 2l$$

where

$$R_-(\eta) = b_l f_l T_-(2l - \eta) + b_l^{-1} V_-(\eta) - h_l T_-(-\eta) + b_l^{-1} g V_-(2l + \eta) + \int_0^{2l+\eta} V_-(\eta + s) T_-(s)\, ds, \qquad -2l < \eta < 2l$$

and, by definition, $V_-(\eta) = 0$ for $\eta > 2l$. As before, we get

$$V_-(\eta) = b_l R_-(\eta) - b_l^2 f_l T_-(2l - \eta) - b_l g R_-(2l + \eta) + 2\,\frac{c_l^2 + 1}{(c_l + 1)^2}\, b_l h_l T_-(-\eta) + \int_\eta^{2l} W_-(s - \eta)\{R_-(s) - b_l f_l T_-(2l - s) + h_l T_-(-s)\}\, ds.$$

Finally, causality yields

LEMMA 7. For $0 < x < l$ and $x < y < 2l - x$, we have

$$G(x) N_-(x, y, l) = -G(l)\left\{(c_l + 1) R_-(x + y) + f_l N_+(x, 2l - y, l) + \int_x^{2l-y} R_-(y + s) N_+(x, s, l)\, ds\right\}.$$

5. The inverse problem. Our strategy in solving the inverse problem is to derive an integral equation whose solution is the function $N_-(x, y, 0)$ for $0 < x < l$, $-x < y < x$. Using this equation to determine N_-, we can then easily find $G(x)$ (and therefore $B(x)$) from Lemma 6. Finally, since the boundary values $N_-(x, x, 0)$ can be expressed in terms of the coefficients of Eq. (2.1), we can then determine the quantity $C - A'/2 - A^2/4$.

We show in this section that the integral equation for N_- can be constructed from the following data:

(a) Reflection kernel $R_+(s)$, $0 < s < 4l$

(b) Transmission kernel $T_+(s)$, $-2l < s < 0$

(c) Constants c_0, b_0, g and $(c_l - 1)a_+$.

As indicated in Sec. 2, this data can be determined by measuring incident and scattered waves.

For future reference, we write down several useful identities (for $0 < x < l$):

$$2N_-(x, x, 0) = -(A + c_0B)|_0 + (c_0 + 1) \int_0^x D_-(s)\, ds \tag{5.1}$$

$$2G(x)N_-(x, -x, 0) = -(A + c_0B)|_0 - (c_0 - 1) \int_0^x D_+(s)\, ds.$$

By using (4.13), (4.11), (4.9), (4.8) and (4.1) in succession, it can be established that

$$R_+(0+) = -\frac{c_0}{(c_0 + 1)^2} (A - B)|_0 . \tag{5.2}$$

Similarly, by using (4.13), (4.11), (4.9), (4.8), (4.6), (4.2) and (4.1) in succession, it can also be shown that

$$[R_+(s)]^{s=2l+}_{s=2l-} = \frac{2c_0G(l)^{-1}}{(c_0 + 1)^2(c_l + 1)} \left\{\frac{2}{c_l + 1} (A - c_lB)|_l + \frac{c_l - 1}{c_0 + 1} (2A + (c_0 - 1)B)|_0 - (c_l - 1) \int_0^l (D_+(s) + D_-(s))\, ds\right\}. \tag{5.3}$$

We now derive a system of integral equations satisfied by N_+ and N_-.

THEOREM 4. For $0 < x < l$ we have

$$(c_0 + 1)F_+(x - y) + (c_0 + 1)G(x)S_+(x, x, y) - G(x) \int_{-x}^x N_-(x, s, 0)S_+(s, x, y)\, ds$$

$$= (c_0 + 1)a_-N_+(x, y, l), \qquad x < y < 2l - x \tag{5.4}$$

$$= -(c_l - 1)a_+G(x)N_-(x, 2l - y, 0), \qquad 2l - x < y < 2l + x \tag{5.5}$$

$$= 0, \qquad 2l + x < y \tag{5.6}$$

where $F_+(y) = (c_0 + 1)W_+(y) - (c_0 - 1)V_+(-y)$ and (for $-x < s < x$)

$$S_+(s, x, y) = (c_0 + 1)b_0R_+(s + y) - (c_l - 1)b_0^{-1}a_+W_+(2l - s - y) - (c_0 + 1) \int_y^x W_+(z - y)R_+(z + s)\, dz, \qquad y < 2l + x,$$

$$= (c_0 + 1)V_+(s + y) - (c_0 - 1)W_+(-s - y) - (c_0 + 1) \int_{-s}^x W_+(z - y)R_+(z + s)\, dz, \qquad 2l + x < y.$$

Proof: Consider a fixed point (x, t) where $0 < x < l$ and an incident wave u^{+i}.

Fig. 2 illustrates how Theorem 1 can be used in two different ways to represent the solution u of (2.1) at the point (x, t). (The dotted lines in that figure are characteristics.) We equate these two expressions for $u(x, t)$, but in doing so, find it necessary to distinguish three cases. In the first case we require $x < t < 2l - x$ (region 1 in Fig. 2) so that $u(x, t)$ is unaffected by internal reflections. Using Theorem 2, we then write u_+^i and u_+^r in terms of u_+^t and thus obtain an expression involving an integral of the form $\int_{-x}^{x} (\cdot)u_+^t(y - t)\, dy$ and an integral $\int_x^t (\cdot)u_+^t(y - t)\, dy$. By virtue of the identities (4.11), (4.14) and (4.15) it can be shown that the integrand in the first integral vanishes. The same identities applied to the second integrand along with the fact that u_+^t is arbitrary (implying the integrand must vanish) yields Eq. (5.4).

Now set $2l - x < t < 2l + x$ (region 2 in Fig. 2) which corresponds to the case in which $u(x, t)$ is influenced by reflections off the interface at $x = l$. Repeating the above procedure, we again obtain integrals $\int_{-x}^{x} (\cdot)u_+^t(y - t)\, dy$, $\int_x^{2l-x} (\cdot)u_+^t(y - t)\, dy$ and $\int_{2l-x}^{t} (\cdot)u_+^t(y - t)\, dy$. As before, the first integral vanishes, and Eq. (5.4) is used to eliminate the second integral. The internal reflection produces additional terms in the third integrand which were not present in the second integrand, and so we finally obtain Eq. (5.5).

Setting $t > 2l + x$, we pick up multiple internal reflections, and in the same manner as above, arrive at Eq. (5.6). This completes the proof.

By considering an incident wave u_-^i we obtain

THEOREM 5. For $0 < x < l$ we have

$$(c_l + 1)G(x)F(l)^{-1}F_-(x - y) + (c_l + 1)S_-(x, x, y) + \int_x^{2l-x} N_+(x, s, l)S_-(s, x, y)\, ds$$

$$= -(c_l + 1)a_+^{-1}G(x)G(l)^{-1}N_-(x, y, 0), \qquad -x < y < x \tag{5.7}$$

$$= (c_0 - 1)a_-^{-1}N_+(x, -y, l), \qquad x - 2l < y < -x \tag{5.8}$$

$$= 0, \qquad y < x - 2l \tag{5.9}$$

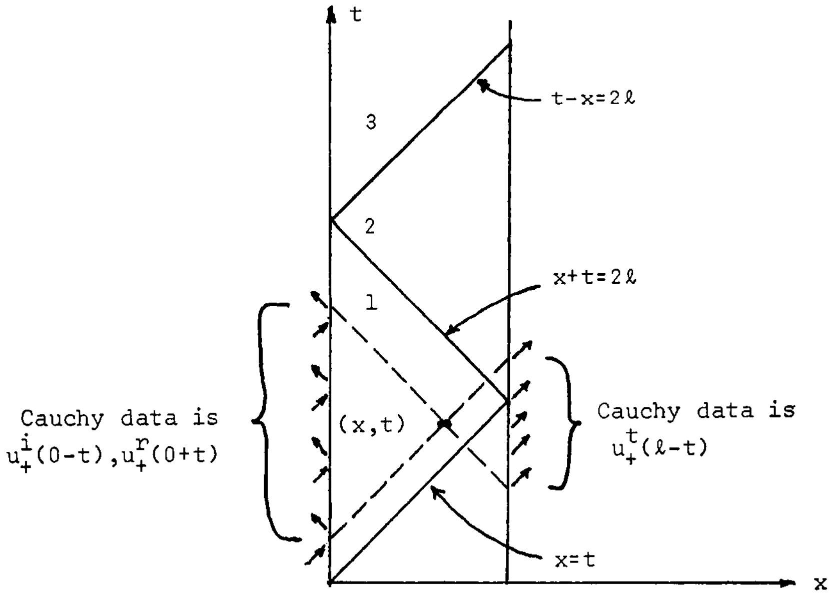

FIG. 2.

where $F_-(y) = (c_l + 1)W_-(y) - (c_l - 1)V_-(2l - y)$ and (for $x < s < 2l - x$)

$$S_-(s, x, y) = (c_l + 1)b_l R_-(s + y) - (c_0 - 1)b_l^{-1}a_-^{-1}W_-(-s - y) + (c_l + 1)\int_y^x W_-(z - y)R_-(z + s)\,dz, \qquad x - 2l < y,$$

$$= (c_l + 1)V_-(s + y) - (c_l - 1)W_-(2l - s - y) - (c_l + 1)\int_x^{2l-s} W_-(z - y)R_-(z + s)\,dz, \qquad y < x - 2l.$$

We observe that Eq. (5.5) contains two unknown functions, namely, $G(x)$ and $N_-(x, y, 0)$. However, by using Lemma 6 we can eliminate $G(x)^{-1}$ from (5.5) and obtain (after a change of variable)

$$(c_0 + 1)X(x, x, y) - \int_{-x}^{x} N_-(x, s, 0)X(s, x, y)\,ds = -(c_l - 1)a_+ N_-(x, y, 0), \qquad -x < y < x \tag{5.10}$$

where

$$X(s, x, y) = F_+(x + y - 2l)U(s + x) + S_+(s, x, 2l - y).$$

If $c_l \neq 1$, Eq. (5.10) is for fixed x a Fredholm equation of the second kind. In a sense it is a generalization of the Gelfand-Levitan equation since the kernel X depends on both the reflection and transmission operators. Upon solving this equation for $N_-(s, y, 0)$, we use (4.16) to obtain $G(x)$ and thence $B(x)$ from the relation $B = -G'/G$. Finally, the combination $C - A'/2 - A^2/4$ is found by means of the identity

$$2\frac{d}{dx}N_-(x, x, 0) = (c_0 + 1)D_-(x) \tag{5.11}$$

which follows from (5.1).

In the electromagnetic problem mentioned in Sec. 2 we saw that $C \equiv 0$. To find $A(x)$ we need to know $A(0)$. This is found by using (5.2) or by just setting $x = 0+$ in (5.1).

If $c_l = 1$ the above procedure does not yield a Fredholm equation of the second kind. However, if $c_0 \neq 1$ this problem is circumvented by using an incident wave propagating in the opposite direction.

In the special case $c_0 = c_l = 1$, the right-hand sides of (5.5) and (5.8) vanish and so we are in a situation similar to that considered by Weston [9] although we still allow A, B, C to be discontinuous at $x = 0$, $x = l$. The method of solution given above can no longer be used since we cannot obtain a Fredholm equation of the second kind. In other words, the lack of a hard reflection at the interface between the surrounding medium and the side of the slab opposite the incident wave makes the slab "indistinguishable" if only one incident wave is used. As in Weston's case, we need two incident waves from opposite directions in order to solve the problem. These are used to obtain the following data:

(a) Reflection kernels $R_+(s)$, $R_-(s)$, $0 < s < 2l$
(b) Transmission kernels $T_+(-s)$, $T_-(s)$, $0 < s < 2l$
(c) Constants a_+, a_-.

(In a manner similar to Weston's, it is readily shown that knowledge of $R_\pm$ and T_+

(or T_-) enables one to determine T_- (or T_+).) Now combining (5.4) and (5.7) and using Lemma 6 to eliminate $G(x)^{-1}$, we obtain for $0 < x < l$

$$2X_1(x, x, y) + a_+F_-(x - y) - \int_{-x}^{x} N_-(x, s, 0)X_1(s, x, y)\, ds$$
$$= -N_-(x, y, 0), \qquad -x < y < x \qquad (5.12)$$

where

$$X_1(s, x, y) = \tfrac{1}{2}a_-U(s + x)S_-(x, x, y)$$
$$+ \frac{1}{4}\int_x^{2l-x} S_-(z, x, y)\{U(s + x)F_+(x - z) + S_+(s, x, z)\}\, dz.$$

We now use (5.12) to determine N_- and then proceed as before to reconstruct the coefficients of (2.1).

6. Example: a transmission line with losses. We now illustrate our technique for solving the inverse problem by considering an example involving a transmission line. Instead of using actual scattering data, we shall (for the sake of simplicity) begin with a partial differential equation with known coefficients and determine the scattering operators analytically. These operators then serve as the starting point for the inverse problem and are used in reconstructing the original partial differential equation.

Consider a transmission line of length l_1 lying between $z = 0$ and $z = l_1$. We assume that the shunt capacitance and conductance are constants, C and G respectively, and that the resistance and inductance are given by $R(z)$ and $L(z)$. It is also assumed that R is C^1 and L is C^2. In this case the model equation for the current $I(z, t)$ is

$$I_{zz} - L(z)CI_{tt} - (R(z)C + L(z)G)I_t - R(z)GI = 0, \qquad 0 < z < l_1. \qquad (6.1)$$

We assume the line is terminated in such a way that we have

$$I_{zz} - L_0C_0I_{tt} = 0 \quad \text{for} \quad z < 0 \quad \text{or} \quad z > l_1 \qquad (6.2)$$

where L_0 and C_0 are constants. The change of variables

$$\begin{aligned} x &= \{L_0C_0\}^{1/2}z, && z < 0 \\ &= \int_0^z \{L(s)C\}^{1/2}\, ds, && 0 < z < l_1 \qquad (6.3) \\ &= l + \{L_0C_0\}^{1/2}(z - l_1), && z > l_1 \end{aligned}$$

where $l = \int_0^{l_1} \{L(s)C\}^{1/2}\, ds$ and $u(x, t) = I(z, t)$, transforms (6.1) and (6.2) into

$$u_{xx} - u_{tt} + A(x)u_x + B(x)u_t + C_1(x)u = 0, \qquad 0 < x < l \qquad (6.4)$$

$$u_{xx} - u_{tt} = 0, \qquad x < 0 \quad \text{or} \quad x > l \qquad (6.5)$$

where

$$A(x) = -\frac{d}{dz}\{L(z)C\}^{-1/2}, \qquad (6.6)$$

$$B(x) = -(R(z)C + L(z)G)/L(z)C, \qquad (6.7)$$

$$C_1(x) = -R(z)G/L(z)C. \qquad (6.8)$$

Furthermore, we have the jump conditions

$$c_0 u_x(0-, t) = u_x(0+, t), \qquad c_l u_x(l+, t) = u_x(l-, t)$$

where

$$c_0 = \{L_0 C_0 / L(0+)C\}^{1/2}, \qquad c_l = \{L_0 C_0 / L(l_1 -)C\}^{1/2}. \tag{6.9}$$

We now proceed to determine the scattering operators. To do this, we need to specify the coefficients of (6.1) and (6.2). For simplicity, we set R and L constant and require $L = C = a$, $L_0 = C_0 = a_0$ and $R = G = r$, with $a \neq a_0$. Then Eq. (6.4) becomes

$$u_{xx} - u_{tt} - 2bu_t - b^2 u = 0, \qquad 0 < x < l \tag{6.10}$$

where $b = a/r$, and we also get

$$c_0 = c_l = a_0/a = c.$$

For a wave incident from the left the solution of (6.9) is

$$u(x, t) = \{f(x + t) + g(x - t)\} \exp(-bt), \qquad 0 < x < l$$

with (2.2) holding elsewhere. Notice that $f(\xi) = 0$ for $0 < \xi < 2l$. Using the continuity of u at $x = 0$ and the jump condition on u_x, we can obtain a first-order linear differential equation for $u_+^{\,r}$ with a nonhomogeneous term involving $u_+^{\,i}$. Solving this, we can express $u_+^{\,r}(\xi)$ in terms of $u_+^{\,i}(-\xi)$ for $0 < \xi < 2l$ and thence obtain

$$R_+(\xi) = \frac{-2bc}{(c+1)^2} \exp(-k\xi), \qquad 0 < \xi < 2l$$

where $k = b/(c + 1)$. A similar procedure across $x = l$ yields

$$T_+(\eta) = \frac{4bc}{(c+1)^4} \{bc\eta + c^2 - 1\} \exp(k\eta - bl), \qquad -2l < \eta < 0. \tag{6.11}$$

Now write $f(\xi)$ (for $2l < \xi < 4l$) in terms of $u_+^{\,i}(-\xi)$ (for $0 < \xi < 2l$) and consider u and u_x at $x = 0$, $2l < t < 4l$. We can again express $u_+^{\,r}(\xi)$ in terms of $u_+^{\,i}(-\xi)$ for $2l < \xi < 4l$ and so arrive at

$$R_+(\xi) = \frac{-2bc}{(c+1)^2} \exp(-k\xi)\{1 + d_3\theta(\xi - 2l)\{d_2(2l - \xi)^2 + d_1(2l - \xi) + d_0\}\}, \qquad 0 < \xi < 4l \tag{6.12}$$

where

$$d_3 = \frac{2}{(c+1)^3} \exp(-2ckl), \qquad d_2 = bc^2 k,$$

$$d_1 = 3bc(c - 1), \qquad d_0 = (c + 1)(c^2 - 4c + 1).$$

We now turn to the inverse problem. We seek to determine the constants C, G and functions $R(z)$, $L(z)$ in Eq. (6.1) by using scattering data. It is assumed that the constants L_0, C_0 in Eq. (6.2) are known and that $L_0 = C_0 = a_0$. At this point a scattering experiment is performed in which an incident wave produces reflected and transmitted waves which are measured in the regions $z < 0$, $z > l_1$. These waves are easily converted into x coordinates since for $z < 0$ and $z > l_1$ the transformation (6.3) is known once l is determined. But since we seek to reduce our problem to one with unit velocity of propagation, we see that l is just equal to the time required for the transmitted wave to emerge from the line. Thus, we express the incident and scattered waves in x coordinates, and then use (4.10) and (4.12) to determine the constants c_0, b_0, g, $(c_l - 1)a_+$ and the kernels R_+, T_+. We obtain

$$c_0 = c, \qquad b_0 = \frac{(c+1)^2}{4c} \exp(bl),$$

$$g = \frac{(c-1)^2}{(c+1)^2} \exp(2bl), \qquad (c_l - 1)a_+ = (c-1)\exp(-bl)$$

and R_+, T_+ as given in (6.11), (6.12).

We begin our solution by finding $W_+(\eta)$ for $-2l < \eta < 0$. Taking the Laplace transform of Eq. (4.11), we are able to obtain

$$W_+(\eta) = \frac{b}{4c} \exp(bl)\{1 - c^2 \exp(b\eta)\}, \qquad -2l < \eta < 0.$$

From (4.14) we get

$$V_+(\xi) = -\frac{b}{4c} \exp(bl)\{1 + c^2 \exp(-b\xi)\}, \qquad 0 < \xi < 2l.$$

It follows that

$$F_+(s) = \frac{b}{2} \exp(bl)\{1 - c\exp(bs)\},$$

$$U(s) = \exp(-ks),$$

$$S_+(s, x, y) = \frac{b}{2} \exp(bl - k(s+y))\left\{-\exp(k(y-x))(1 + c\exp(b(x-y))) + \theta(s + y - 2l)\frac{4c}{c+1}\exp(-2ckl)\right\},$$

and so

$$X(s, x, y) = -bc\exp(b(x + y - l) - k(x+s)) + 2ck\exp(-bl + k(y - s))\theta(s - y).$$

Substituting this last expression into Eq. (5.10) we get (for x fixed)

$$N_-(x, y, 0) = f_1(y) + Mf_2(y) + g(y)f_3(y) \tag{6.13}$$

where

$$\begin{aligned} f_1(y) &\quad \frac{1}{c-1}\{bc(c+1)\exp((b - 2k)x + by) - 2bc\exp(k(y-x))\} \\ f_2(y) &= -\frac{bc}{c-1}\exp(ckx + by) \\ f_3(y) &= \frac{2ck}{c-1}\exp(ky) \\ g(y) &= \int_y^x N_-(x, s, 0)\exp(-ks)\,ds \\ M &= g(-x). \end{aligned} \tag{6.14}$$

From (6.13) and (6.14) it follows that

$$g(y) \exp(my) = \exp(ckx)\{(c+1)\exp(-kx) - M\}\{\exp((m+ck)x) - \exp((m+ck)y\}$$
$$- (c+1)\exp(-kx)\{\exp(mx) - \exp(my)\} \qquad (6.15)$$

where $m = 2ck/(c-1)$. Setting $y = -x$ in (6.15) we get

$$M = (c+1)\exp(-kx)(1 - \exp(-2ckx))$$

and so

$$g(y) = (c+1)(\exp(-kx) - \exp(-bx + cky)).$$

Finally,

$$N_-(x, y, 0) = bc \exp(b(y-x)).$$

Using (4.16) and $B = -G'/G$ we get

$$B(x) = -2b, \qquad 0 < x < l \qquad (6.16)$$

and from (5.11),

$$C_1(x) - A'(x)/2 - A^2(x)/4 = -b^2, \qquad 0 < x < l. \qquad (6.17)$$

Now from (6.7) and (6.16) we see that $R(z)$ is just a constant multiple of $L(z)$, and so it follows from (6.8) that $C_1(x)$ is constant. Furthermore, from (5.2) we determine that

$$A(0) = 0 \qquad (6.18)$$

and from (5.3) we get

$$A(l) = 0. \qquad (6.19)$$

Since C_1 is constant, it now follows from (6.17), (6.18), (6.19) that $A(x) \equiv 0$. Using (6.6) we then conclude that $L(z)$ (and therefore $R(z)$) is constant for $0 < z < l_1$.

Thus, we have that

$$\frac{R}{L} + \frac{G}{C} = 2b$$

and

$$GR/LC = b^2$$

from which it follows that

$$\frac{R}{L} = \frac{G}{C} = b.$$

Furthermore, since the constants c and a_0 are known, it follows from (6.9) that the product LC is known,

$$LC = a^2.$$

With no other information given to us, this is the most we can determine about the coefficients of (6.1). However, if we can obtain by other means that, for example, $C = a$, then we find

$$L = a, \qquad R = ab = r, \qquad G = ab = r.$$

Appendix: The constant coefficient case. If it is known a priori that the coefficients A, B, C of (2.1) are constant on $(0, l)$, then it is a simple matter to determine these constants. In doing so it is no longer necessary to determine the transmission kernel. We outline the reconstruction of the constants A, B, C under the assumption that $c_0 = c_l = c \neq 1$. The required data is

(a) $R_+(0+)$

(b) $[R_+(s)]^{s=2l+}_{s=2l-}$

(c) Constants c and $G(l)^{-1}$. (These are found from the constants f_0 and j_0 in Eq. (4.12).)

We first determine B from the relation

$$B = -\frac{1}{l} \ln G(l)$$

and then A is found from (5.2). Now for our case, Eq. (5.3) reduces to

$$[R_+(s)]^{s=2l+}_{s=2l-} = \frac{2cG(l)^{-1}}{(c+1)^4} \{2(A - cB) + (c-1)(2A + (c-1)B) - (c^2 - 1)(2C + \tfrac{1}{2}(B^2 - A^2))l\}$$

from which C is easily found.

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