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An investigation of fractional Bagley-Torvik equation

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Abstract: In this paper the authors prove the existence as well as approximations of the solutions for the Bagley-Torvik equation admitting only the existence of a lower (coupled lower and upper) solution. Our results rely on an appropriate fixed point theorem in partially ordered normed linear spaces. Illustrative examples are included to demonstrate the validity and applicability of our technique.

Keywords: Bagley-Torvik equation, Fractional calculus, Partially fixed point, Mixed monotone operator, Existence, Uniqueness, Approximation

MSC: 26A33, 34A08, 34A12

1 Introduction

Fractional differential equations appear naturally in a number of fields such as physics, geophysics, polymer rheology, regular variation in thermodynamics, biophysics, blood flow phenomena, aerodynamics, electro-dynamics of complex medium, viscoelasticity, Bode's analysis of feedback amplifiers, capacitor theory, electrical circuits, electron-analytical chemistry, biology, control theory, fitting of experimental data, nonlinear oscillation of earthquake, the fluid-dynamic traffic model, etc. For more details and applications, we refer the reader to the books [1–6] and references [7–14].

Fractional differential equations also serve as an excellent tool for the description of hereditary properties of various materials and processes [15, 16]. The presence of memory term in such models not only takes into account the history of the process involved but also carries its impact to present and future development of the process. Fractional differential equations are also regarded as an alternative model to nonlinear differential equations [17]. In consequence, the subject of fractional differential equations is gaining much importance and attention.

The Bagley-Torvik equation is a prototype fractional differential equation which was proposed by Bagley and Torvik as an application of fractional calculus to the theory of viscoelasticity [18–20]. The governing equation is given by the fractional differential equation

$$\left(MD^2 + 2S\sqrt{\mu\rho} D^{3/2} + K \right) x(t) = f(t), \quad 0 < t \leq 1, \quad (1.1)$$

subject to initial conditions

$$x(0) = x_0, \quad x'(0) = x'_0, \quad (1.2)$$

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where $x(t)$ represents the displacement of the plate of mass M and surface area S . Furthermore, μ and ρ are the viscosity and density, respectively, of the fluid in which the plate is immersed, and K is the stiffness of the spring to which the plate is attached. Finally, $f(x)$ represents the loading force.

In the current paper we investigate the existence and uniqueness as well as approximations of the solutions for the Bagley-Torvik equation admitting only the existence of a lower (coupled lower and upper) solution. Our governing equation is a generalization of (1.1) to an arbitrary α with $1 < \alpha < 2$ in the fractional derivative term. In fact, we consider the following initial value equation

$$\left(D^2 + AD^\alpha + B\right)x(t) = f(t), \quad 0 < t \leq 1, \quad 1 < \alpha < 2, \quad (1.3)$$

subject to initial conditions

$$x(0) = a, \quad x'(0) = b, \quad (1.4)$$

where D^α is the Caputo fractional derivative of order α , $f : [0, 1] \rightarrow \mathbb{R}$ is a given function and a, b, A and B are real numbers.

For various applications in engineering and applied sciences fields, the Bagley-Torvik equation is extensively studied in literature both from numerical and theoretical point of view [3, 21–30]. Several numerical methods have been proposed for approximate solutions of this type equations, such as successive approximation method [3, Section 8.3], Adams predictor and corrector method [21], Taylor collocation method [22], hybridizable discontinuous Galerkin method [23], Discrete spline method [24] and others [25–28]. Also, Svatoslav Staněk [29] investigate the existence and uniqueness of solutions for generalized Bagley-Torvik fractional differential equation subject to the boundary conditions. In [30], the authors investigate the general solution of the Bagley-Torvik equation with $1/2$ -order derivative or $3/2$ -order derivative. Furthermore, they show that the general solution of the Bagley-Torvik equation involves actually two free constants only, and it can be determined fully by the initial displacement and initial velocity.

Our main aim is to prove the existence and uniqueness as well as construct an approximate solution for (1.3)-(1.4). This is done in Section 3. Our main tools are some applicable partially fixed point theorems which are applied in the suitable partially ordered sets as well as iterative methods, whose description can be found in [31–35]. The advantage and importance of this method arises from the fact that it is a constructive method that yields sequences that converge to the unique solution of (1.3)-(1.4) admitting only the existence of a lower (coupled lower and upper) solution.

2 Auxiliary facts and results

Here, we recall several known definitions and properties from fractional calculus theory. For details, see [1–3, 36]. Throughout the paper $AC^n[0, 1]$, $n \in \mathbb{N}$, denotes the set of functions having absolutely continuous n -th derivative on $[0, 1]$, and $AC[0, 1]$ is the set of absolutely continuous functions on $[0, 1]$. It is known that $x \in AC[0, 1]$ if and only if there exists a function $\varphi \in L^1[0, 1]$ such that $x(t) = c + \int_0^t \varphi(\tau) d\tau$.

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $x : [0, 1] \rightarrow \mathbb{R}$ is defined as

$$I^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} x(\tau) d\tau, \quad 0 \leq t \leq 1.$$

provided that the integral exists. For $\alpha = 0$, we set $I^\alpha := I$, the identity operator.

Definition 2.2. The Caputo fractional derivative of order $\alpha > 0$ of a function $x : [0, 1] \rightarrow \mathbb{R}$ is defined as

$$D^\alpha x(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_0^t (t - \tau)^{n-\alpha-1} \left(x(\tau) - \sum_{k=0}^{n-1} \frac{x^{(k)}(0)}{k!} \tau^k \right) d\tau$$

$$= D^n I^{n-\alpha} \left(x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(0)}{k!} t^k \right),$$

where $n - 1 < \alpha < n$ and $n \in \mathbb{N}$, provided the right side is pointwise defined on $[0, 1]$. We notice that the Caputo derivative of a constant is zero. Note that if $n - 1 < \alpha < n$ and $x \in AC^{n-1}[0, 1]$, then

$$D^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} x^{(n)}(\tau) d\tau = I^{n-\alpha} x^{(n)}(t).$$

Lemma 2.1. Let $\alpha, \beta \geq 0$. If $x \in L^1[0, 1]$, then $I^\alpha I^\beta x = I^{\alpha+\beta} x$.

Lemma 2.2. For $0 < \alpha < 1$, I^α is linear and continuous from $L^1[0, 1]$ to $L^p[0, 1]$ where $1 \leq p < \frac{1}{1-\alpha}$.

Lemma 2.3. For $\alpha > 0$, I^α is linear and continuous from $AC[0, 1]$ to $AC[0, 1]$.

Lemma 2.4. Let $\alpha \geq 1$. If $x \in L^1[0, 1]$, then $I^\alpha x \in AC[0, 1]$.

Proof. This is an immediate consequence of Lemma 2.1 and Lemma 2.2, because $I^\alpha x = I^{[\alpha]} I^{\alpha-[\alpha]} x$ where $[\alpha] = \max\{n \in \mathbb{N}, n \leq \alpha\}$. □

Lemma 2.5. Let $\alpha > 0$, $a \in \mathbb{R}$ and $x \in L^1[0, 1]$. If there exists $f \in AC[0, 1]$ such that $x(t) - aI^\alpha x(t) = f(t)$ on $[0, 1]$, then $x \in AC[0, 1]$.

Proof. If $a = 0$ we are done, and so we henceforth assume $a \neq 0$. Given $\alpha > 0$, choose $n_0 \in \mathbb{N}$ so that $n_0 \alpha > 1$. Now, by applying the operator I^α to both sides of $x(t) = aI^\alpha x(t) + f(t)$, we have

$$I^\alpha x(t) = aI^{2\alpha} x(t) + I^\alpha f(t),$$

or equivalently,

$$x(t) = a^2 I^{2\alpha} x(t) + aI^\alpha f(t) + f(t).$$

Continuing this process to the n_0 -th step, we get

$$x(t) = a^{(n_0+1)} I^{(n_0+1)\alpha} x(t) + \sum_{k=0}^{n_0} a^k I^{k\alpha} f(t),$$

for $t \in [0, 1]$. The desired result is therefore a consequence of Lemma 2.4. □

Definition 2.3. A function $x \in AC^1[0, 1]$ is a solution of (1.3)-(1.4) if it satisfies the initial conditions (1.4) and (1.3) holds for almost everywhere on $[0, 1]$.

Lemma 2.6. $x(t)$ is a solution of the problem (1.3)-(1.4) if and only if it is a solution of the following integral equation

$$x(t) = a + b \left(t + \frac{A}{\Gamma(4-\alpha)} t^{3-\alpha} \right) - AI^{3-\alpha} x'(t) + I^2 [f(t) - Bx(t)], \tag{2.1}$$

in the set $C^1[0, 1]$.

Proof. Let us note that this result is mainly proved in [29]. Let $x(t)$ be a solution of the problem (1.3)-(1.4). Then $x \in AC^1[0, 1]$ and the equality

$$x''(t) + AI^{2-\alpha} x''(t) + Bx(t) = f(t), \tag{2.2}$$

holds almost everywhere on $[0, 1]$. Applying the integral operator I^2 to both sides of (2.2) and using Lemma 2.1, we deduce

$$\begin{aligned}
 x(t) &= a + bt - AI^{4-\alpha}x''(t) + I^2 [f(t) - Bx(t)] \\
 &= a + bt - AI^{3-\alpha}I^1x''(t) + I^2 [f(t) - Bx(t)] \\
 &= a + bt - AI^{3-\alpha} [x'(t) - b] + I^2 [f(t) - Bx(t)] \\
 &= a + bt + \frac{Ab}{\Gamma(4-\alpha)}t^{3-\alpha} - AI^{3-\alpha}x'(t) + I^2 [f(t) - Bx(t)] \\
 &= a + b \left(t + \frac{A}{\Gamma(4-\alpha)}t^{3-\alpha} \right) - AI^{3-\alpha}x'(t) + I^2 [f(t) - Bx(t)].
 \end{aligned}
 \tag{2.3}$$

Note that $I^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}t^{\alpha+\beta}$, $\alpha \geq 0, \beta > -1$. Therefore, $x \in C^1[0, 1]$ is a solution of the integral equation (2.1). Now, we suppose that $x \in C^1[0, 1]$ is a solution of the integral equation (2.1). It is obvious that $x(0) = a$ and $x'(0) = b$. By calculations similar to those in (2.3), we establish that

$$\begin{aligned}
 x(t) &= a + b \left(t + \frac{A}{\Gamma(4-\alpha)}t^{3-\alpha} \right) - AI^{3-\alpha}x'(t) + I^2 [f(t) - Bx(t)] \\
 &= a + bt - AI^{3-\alpha}[x'(t) - b] + I^2 [f(t) - Bx(t)] \\
 &= a + bt - AI^{2-\alpha}[x(t) - a - bt] + I^2 [f(t) - Bx(t)] \\
 &= a + bt - AI^{2-\alpha}[x(t) - x(0) - x'(0)t] + I^2 [f(t) - Bx(t)].
 \end{aligned}
 \tag{2.4}$$

Differentiating twice, we get

$$x''(t) = -AD^2I^{2-\alpha}[x(t) - x(0) - x'(0)t] + f(t) - Bx(t),
 \tag{2.5}$$

and so

$$x''(t) + AD^\alpha x(t) + Bx(t) = f(t).
 \tag{2.6}$$

Now it suffices to show that $x \in AC^1[0, 1]$. By differentiating both sides of (2.1), we have

$$x'(t) + AI^{2-\alpha}x'(t) = b \left(1 + \frac{A}{\Gamma(3-\alpha)}t^{2-\alpha} \right) + I^1 [f(t) - Bx(t)].
 \tag{2.7}$$

Since $b \left(1 + \frac{A}{\Gamma(3-\alpha)}t^{2-\alpha} \right) + I^1 [f(t) - Bx(t)] \in AC[0, 1]$, the desired result is therefore a consequence of Lemma 2.5. □

3 Main results

Before continuing our investigation, we introduce a few fundamental concepts related to the required spaces and provide some partial order on them to serve as a background for the materials to be illustrated in the later sections.

By $C^1[0, 1]$ we denote the class of continuously differentiable functions on a finite interval $[0, 1]$ with the standard norm $\|x\|_{C^1[0,1]} = \max\{\|x\|_{C[0,1]}, \|x'\|_{C[0,1]}\}$ where $\|x\|_{C[0,1]} = \sup_{t \in [0,1]} |x(t)|$. Obviously, $C^1[0, 1]$ is a Banach space. Now, we define an appropriate partial order on $C^1[0, 1]$ and prove some essential properties in this partially ordered Banach space.

Definition 3.1. We define the following order relation for $C^1[0, 1]$,

$$x \preceq y \iff x(t) \leq y(t), \quad x'(t) \leq y'(t), \quad t \in [0, 1].$$

Lemma 3.1. $(C^1[0, 1], \preceq)$ is a partially ordered set and every pair of elements has a lower bound and an upper bound.

Proof. It is easy to see that $(C^1[0, 1], \preceq)$ is a partially ordered set. Now we prove that every pair of elements in $C^1[0, 1]$ has a lower bound and an upper bound. Let $x, y \in C^1[0, 1]$ and define

$$\underline{x}(t) = \int_0^t \min\{x'(\cdot), y'(\cdot)\}(\tau) d\tau + \min\{x(0), y(0)\},$$

and

$$\bar{x}(t) = \int_0^t \max\{x'(\cdot), y'(\cdot)\}(\tau) d\tau + \max\{x(0), y(0)\}.$$

So a simple calculation shows that the functions \underline{x} and \bar{x} are in $C^1[0, 1]$ and are the lower and upper bounds of $\{x, y\}$, respectively. □

Dividing by A, B (distinguish the cases $A \cdot B > 0$ and $A \cdot B < 0$), we will consider two cases separately.

3.1 Investigation in the case $A \cdot B < 0$

In this subsection we consider the case in which $A \cdot B < 0$. We consider only the case that $A > 0$ and $B < 0$. The other case is completely similar. Before continuing, we need to introduce the coupled fixed point theorems which play main role in our discussion. For complete details, see [34].

Definition 3.2. Let (X, \preceq) be a partially ordered set and $\mathcal{G} : X \times X \rightarrow X$. We say that \mathcal{G} has the mixed monotone property if $\mathcal{G}(x, y)$ is monotone non-decreasing in x and is monotone non-increasing in y .

Definition 3.3. We call an element $(x, y) \in X \times X$ a coupled fixed point of the mapping \mathcal{G} if

$$\mathcal{G}(x, y) = x, \quad \text{and} \quad \mathcal{G}(y, x) = y.$$

Theorem 3.1. Let (X, \preceq) be a partially ordered set and suppose there exists a metric d on X such that (X, d) is a complete metric space. Let $\mathcal{G} : X \times X \rightarrow X$ be a mapping having the mixed monotone property on X . Assume that there exists a $k \in [0, 1)$ with

$$d(\mathcal{G}(x, y), \mathcal{G}(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)], \quad \text{for each } x \succeq u \text{ and } y \preceq v.$$

Suppose either \mathcal{G} is continuous or X has the following property:

- (i) if a non-decreasing sequence $\{x_n\} \rightarrow x$, then $x_n \preceq x$ for all n ,
- (ii) if a non-increasing sequence $\{y_n\} \rightarrow y$, then $y \preceq y_n$ for all n .

If there exist $x_0, y_0 \in X$ such that $x_0 \preceq \mathcal{G}(x_0, y_0)$ and $y_0 \succeq \mathcal{G}(y_0, x_0)$, then \mathcal{G} has a coupled fixed point $(x^*, y^*) \in X \times X$.

We define the following partial order on the product space $X \times X$:

$$(x, y), (\tilde{x}, \tilde{y}) \in X \times X, \quad (x, y) \preceq (\tilde{x}, \tilde{y}) \Leftrightarrow x \preceq \tilde{x}, \tilde{y} \preceq y.$$

Theorem 3.2. In addition to the hypothesis of Theorem 3.1, suppose that for every $(x, y), (\tilde{x}, \tilde{y}) \in X \times X$, there exists an element $(u, v) \in X \times X$ that is comparable to (x, y) and (\tilde{x}, \tilde{y}) , then \mathcal{G} has a unique coupled fixed point (x^*, y^*) .

Theorem 3.3. In addition to the hypothesis of Theorem 3.2, suppose that every pair of elements of X has an upper bound or a lower bound in X . Then $x^* = y^*$. Moreover,

$$\lim_{n \rightarrow \infty} \mathcal{G}^n(x_0, y_0) = x^*,$$

where $\mathcal{G}^n(x_0, y_0) = \mathcal{G}(\mathcal{G}^{n-1}(x_0, y_0), \mathcal{G}^{n-1}(y_0, x_0))$.

In view of Lemma 2.6, we transform problem (1.3)-(1.4) as the following integral equation

$$\begin{aligned}
 x(t) = & a + b \left(t + \frac{A}{\Gamma(4-\alpha)} t^{3-\alpha} \right) \\
 & - \frac{A}{\Gamma(3-\alpha)} \int_0^t (t-\tau)^{2-\alpha} x'(\tau) d\tau + \int_0^t (t-\tau) [f(\tau) - Bx(\tau)] d\tau,
 \end{aligned} \tag{3.1}$$

in the set $C^1[0, 1]$.

Definition 3.4. An element $(x_0, y_0) \in C^1[0, 1] \times C^1[0, 1]$ is called a coupled lower and upper solution of problem (1.3)-(1.4) if

$$\begin{cases}
 x'_0(t) \leq b \left(1 + \frac{A}{\Gamma(3-\alpha)} t^{2-\alpha} \right) - \frac{A}{\Gamma(2-\alpha)} \int_0^t (t-\tau)^{1-\alpha} y'_0(\tau) d\tau + \int_0^t [f(\tau) + |B|x_0(\tau)] d\tau, \\
 x_0(0) \leq a,
 \end{cases} \tag{3.2}$$

and

$$\begin{cases}
 y'_0(t) \geq b \left(1 + \frac{A}{\Gamma(3-\alpha)} t^{2-\alpha} \right) - \frac{A}{\Gamma(2-\alpha)} \int_0^t (t-\tau)^{1-\alpha} x'_0(\tau) d\tau + \int_0^t [f(\tau) + |B|y_0(\tau)] d\tau, \\
 y_0(0) \geq a,
 \end{cases} \tag{3.3}$$

for all $t \in [0, 1]$.

Theorem 3.4. Assume that $(x_0, y_0) \in C^1[0, 1] \times C^1[0, 1]$ is a coupled lower and upper solution of problem (1.3)-(1.4) and $k = \max\{2|B|, \frac{2A}{\Gamma(3-\alpha)}\} < 1$.

- (i) Then the initial value problem (1.3)-(1.4) has a unique solution $x^* \in C^1[0, 1]$.
- (ii) Moreover, there exist two monotone iterative sequences $\{x_n\}$ and $\{y_n\}$ such that both sequences converge to x^* in $C^1[0, 1]$.
- (iii) In addition, the following error estimates hold,

$$\|x_n - x^*\|_{C^1[0,1]} \leq \frac{1}{2} \frac{k^n}{1-k} (\|x_1 - x_0\|_{C^1[0,1]} + \|y_1 - y_0\|_{C^1[0,1]}), \tag{3.4}$$

$$\|y_n - x^*\|_{C^1[0,1]} \leq \frac{1}{2} \frac{k^n}{1-k} (\|x_1 - x_0\|_{C^1[0,1]} + \|y_1 - y_0\|_{C^1[0,1]}), \tag{3.5}$$

$$\|x_n - y_n\|_{C^1[0,1]} \leq \frac{k^n}{1-k} (\|x_1 - x_0\|_{C^1[0,1]} + \|y_1 - y_0\|_{C^1[0,1]}). \tag{3.6}$$

Proof. In view of (3.1), we define the operator $\mathcal{G} : C^1[0, 1] \times C^1[0, 1] \rightarrow C^1[0, 1]$ by

$$\mathcal{G}(x, y)(t) = a + b \left(t + \frac{A}{\Gamma(4-\alpha)} t^{3-\alpha} \right) - \frac{A}{\Gamma(3-\alpha)} \int_0^t (t-\tau)^{2-\alpha} y'(\tau) d\tau + \int_0^t (t-\tau) [f(\tau) + |B|x(\tau)] d\tau. \tag{3.7}$$

Obviously, for any $x, y \in C^1[0, 1]$, we have $\mathcal{G}(x, y) \in C[0, 1]$. On the other hand,

$$\mathcal{G}'(x, y)(t) = b \left(1 + \frac{A}{\Gamma(3-\alpha)} t^{2-\alpha} \right) - \frac{A}{\Gamma(2-\alpha)} \int_0^t (t-\tau)^{1-\alpha} y'(\tau) d\tau + \int_0^t (f(\tau) + |B|x(\tau)) d\tau,$$

and so the operator \mathcal{G} is well defined.

Now we shall show that \mathcal{G} has the mixed monotone property. Let $x, \bar{x} \in C^1[0, 1]$ with $x \leq \bar{x}$. Using the monotonicity of integral operator, we have

$$\mathcal{G}(x, y)(t) - \mathcal{G}(\bar{x}, y)(t) = |B| \int_0^t (t-\tau) [x(\tau) - \bar{x}(\tau)] d\tau \leq 0,$$

and

$$\begin{aligned} \mathcal{G}'(x, y)(t) - \mathcal{G}'(\bar{x}, y)(t) &= |B| \int_0^t [x(\tau) - \bar{x}(\tau)] \, d\tau \\ &\leq 0, \end{aligned}$$

for every $t \in [0, 1]$. Hence, $\mathcal{G}(x, y)(t) \leq \mathcal{G}(\bar{x}, y)(t)$ and $\mathcal{G}'(x, y)(t) \leq \mathcal{G}'(\bar{x}, y)(t)$ for every $t \in [0, 1]$, that is, $\mathcal{G}(x, y) \preceq \mathcal{G}(\bar{x}, y)$. Similarly, let $y, \bar{y} \in C^1[0, 1]$ with $y \preceq \bar{y}$. From the monotonicity of Riemann-Liouville fractional integral operator, we have

$$\begin{aligned} \mathcal{G}(x, y)(t) - \mathcal{G}(x, \bar{y})(t) &= -\frac{A}{\Gamma(3-\alpha)} \int_0^t (t-\tau)^{2-\alpha} [y'(\tau) - \bar{y}'(\tau)] \, d\tau \\ &\geq 0. \end{aligned}$$

and

$$\begin{aligned} \mathcal{G}'(x, y)(t) - \mathcal{G}'(x, \bar{y})(t) &= -\frac{A}{\Gamma(2-\alpha)} \int_0^t (t-\tau)^{1-\alpha} [y'(\tau) - \bar{y}'(\tau)] \, d\tau \\ &\geq 0. \end{aligned}$$

for every $t \in [0, 1]$. Hence, $\mathcal{G}(x, y)(t) \geq \mathcal{G}(x, \bar{y})(t)$ and $\mathcal{G}'(x, y)(t) \geq \mathcal{G}'(x, \bar{y})(t)$ for every $t \in [0, 1]$, that is, $\mathcal{G}(x, y) \succeq \mathcal{G}(x, \bar{y})$. Thus, $\mathcal{G}(x, y)$ is monotone non-decreasing in x and monotone non-increasing in y .

Now, for $x, y, \bar{x}, \bar{y} \in C^1[0, 1]$ with $x \succeq \bar{x}, y \preceq \bar{y}$, we have

$$\begin{aligned} \left| \mathcal{G}(x, y)(t) - \mathcal{G}(\bar{x}, \bar{y})(t) \right| &= \left| |B| \int_0^t (t-\tau) [x(\tau) - \bar{x}(\tau)] \, d\tau - \frac{A}{\Gamma(3-\alpha)} \int_0^t (t-\tau)^{2-\alpha} [y'(\tau) - \bar{y}'(\tau)] \, d\tau \right| \\ &\leq \frac{|B|}{2} \sup_{t \in [0,1]} |x(t) - \bar{x}(t)| + \frac{A}{\Gamma(4-\alpha)} \sup_{t \in [0,1]} |y'(t) - \bar{y}'(t)|, \\ &\leq \frac{|B|}{2} \|x - \bar{x}\|_{C^1[0,1]} + \frac{A}{\Gamma(4-\alpha)} \|y - \bar{y}\|_{C^1[0,1]}, \end{aligned}$$

and

$$\begin{aligned} \left| \mathcal{G}'(x, y)(t) - \mathcal{G}'(\bar{x}, \bar{y})(t) \right| &= \left| |B| \int_0^t [x(\tau) - \bar{x}(\tau)] \, d\tau - \frac{A}{\Gamma(2-\alpha)} \int_0^t (t-\tau)^{1-\alpha} [y'(\tau) - \bar{y}'(\tau)] \, d\tau \right| \\ &\leq |B| \sup_{t \in [0,1]} |x(t) - \bar{x}(t)| + \frac{A}{\Gamma(3-\alpha)} \sup_{t \in [0,1]} |y'(t) - \bar{y}'(t)|. \\ &\leq |B| \|x - \bar{x}\|_{C^1[0,1]} + \frac{A}{\Gamma(3-\alpha)} \|y - \bar{y}\|_{C^1[0,1]}. \end{aligned}$$

Therefore,

$$\begin{aligned} \|\mathcal{G}(x, y) - \mathcal{G}(\bar{x}, \bar{y})\|_{C^1[0,1]} &\leq |B| \|x - \bar{x}\|_{C^1[0,1]} + \frac{A}{\Gamma(3-\alpha)} \|y - \bar{y}\|_{C^1[0,1]} \\ &\leq \frac{k}{2} \left[\|x - \bar{x}\|_{C^1[0,1]} + \|y - \bar{y}\|_{C^1[0,1]} \right]. \end{aligned} \tag{3.8}$$

Furthermore, it is easy to see that, if $\{x_n\}$ is a monotone non-decreasing sequence in $C^1[0, 1]$ that converges to $x \in C^1[0, 1]$ and $\{y_n\}$ is a monotone non-increasing sequence in $C^1[0, 1]$ that converges to $y \in C^1[0, 1]$, then $x_n \leq x$ and $y \leq y_n$, for all n .

On the other hand, from (3.2), we have

$$\begin{aligned}
 x'_0(t) &\leq b \left(1 + \frac{A}{\Gamma(3-\alpha)} t^{2-\alpha} \right) - \frac{A}{\Gamma(2-\alpha)} \int_0^t (t-\tau)^{1-\alpha} y'_0(\tau) d\tau + \int_0^t [f(\tau) + |B|x(\tau)] d\tau \\
 &= \frac{d}{dt} \left[a + b \left(t + \frac{A}{\Gamma(4-\alpha)} t^{3-\alpha} \right) \right. \\
 &\quad \left. - \frac{A}{\Gamma(3-\alpha)} \int_0^t (t-\tau)^{2-\alpha} y'_0(\tau) d\tau + \int_0^t (t-\tau) [f(\tau) + |B|x_0(\tau)] d\tau \right] \\
 &= \mathcal{G}'(x_0, y_0)(t),
 \end{aligned} \tag{3.9}$$

for every $t \in [0, 1]$. Now by applying the integral operator I^1 on both sides of inequality (3.9) and using $x(0) \leq a$, we deduce

$$\begin{aligned}
 x_0(t) &\leq x_0(0) + b \left(t + \frac{A}{\Gamma(4-\alpha)} t^{3-\alpha} \right) - \frac{A}{\Gamma(3-\alpha)} \int_0^t (t-\tau)^{2-\alpha} y'_0(\tau) d\tau + \int_0^t (t-\tau) [f(\tau) + |B|x_0(\tau)] d\tau \\
 &\leq a + b \left(t + \frac{A}{\Gamma(4-\alpha)} t^{3-\alpha} \right) - \frac{A}{\Gamma(3-\alpha)} \int_0^t (t-\tau)^{2-\alpha} y'_0(\tau) d\tau + \int_0^t (t-\tau) [f(\tau) + |B|x_0(\tau)] d\tau \\
 &= \mathcal{G}(x_0, y_0)(t),
 \end{aligned}$$

for every $t \in [0, 1]$. Furthermore, using (3.3) and applying similar calculation, we get $y'_0(t) \geq \mathcal{G}'(y_0, x_0)(t)$ and $y_0(t) \geq \mathcal{G}(y_0, x_0)(t)$ for every $t \in [0, 1]$. Therefore,

$$x_0 \leq \mathcal{G}(x_0, y_0) \quad \text{and} \quad y_0 \geq \mathcal{G}(y_0, x_0). \tag{3.10}$$

Consequently, Theorem 3.1 yields the existence of coupled fixed point $(x^*, y^*) \in C^1[0, 1] \times C^1[0, 1]$ for the operator \mathcal{G} .

Also, $C^1[0, 1] \times C^1[0, 1]$ is a partially ordered set if we define the following order relation in $C^1[0, 1] \times C^1[0, 1]$:

$$(x, y) \preceq (\bar{x}, \bar{y}) \iff x \preceq \bar{x}, \bar{y} \preceq y.$$

Now, if for every $(x, y), (\bar{x}, \bar{y}) \in C^1[0, 1] \times C^1[0, 1]$, we define

$$\tilde{x}(t) = \int_0^t \max\{x'(\cdot), \bar{x}'(\cdot)\}(\tau) d\tau + \max\{x(0), \bar{x}(0)\},$$

and

$$\tilde{y}(t) = \int_0^t \min\{y'(\cdot), \bar{y}'(\cdot)\}(\tau) d\tau + \min\{y(0), \bar{y}(0)\}.$$

then $(\tilde{x}, \tilde{y}) \in C^1[0, 1] \times C^1[0, 1]$ is comparable to (x, y) and (\bar{x}, \bar{y}) . The uniqueness of the coupled fixed point (x^*, y^*) therefore follows from Theorem 3.2. Finally, an application of Theorem 3.3, together with Lemma 3.1, yields $x^* = y^*$. This establishes the first assertion.

For the second assertion, we define iterative sequences $\{x_n\}$ and $\{y_n\}$ as follows

$$x_n = \mathcal{G}(x_{n-1}, y_{n-1}), \quad y_n = \mathcal{G}(y_{n-1}, x_{n-1}), \quad n = 1, 2, \dots, \tag{3.11}$$

where x_0 and y_0 are the coupled lower and upper solutions of problem (1.3)-(1.4). We intend to prove by induction on n that $\{x_n\}$ and $\{y_n\}$ are non-decreasing and non-increasing sequences, respectively. The case $n = 1$ being immediate from (3.10). Now we take an arbitrary positive integer n and we assume that $x_{n-1} \preceq x_n$ and $y_n \preceq y_{n-1}$. Then, using the mixed monotone property of \mathcal{G} , we have

$$x_n = \mathcal{G}(x_{n-1}, y_{n-1}) \preceq \mathcal{G}(x_n, y_{n-1}) \preceq \mathcal{G}(x_n, y_n) = x_{n+1},$$

and

$$y_{n+1} = \mathcal{G}(y_n, x_n) \preceq \mathcal{G}(y_n, x_{n-1}) \preceq \mathcal{G}(y_{n-1}, x_{n-1}) = y_n.$$

Moreover, both sequences $\{x_n\}$ and $\{y_n\}$ converge to x^* which follow from Theorem 3.3. This proves assertion (ii).

Now we prove the error estimates. From (3.8) it follows that

$$\|x_2 - x_1\|_{C^1[0,1]} \leq \frac{k}{2} (\|x_1 - x_0\|_{C^1[0,1]} + \|y_1 - y_0\|_{C^1[0,1]}), \tag{3.12}$$

$$\|y_2 - y_1\|_{C^1[0,1]} \leq \frac{k}{2} (\|x_1 - x_0\|_{C^1[0,1]} + \|y_1 - y_0\|_{C^1[0,1]}). \tag{3.13}$$

Again employing (3.8) and using (3.12) and (3.13), we deduce

$$\|x_3 - x_2\|_{C^1[0,1]} \leq \frac{k^2}{2} (\|x_1 - x_0\|_{C^1[0,1]} + \|y_1 - y_0\|_{C^1[0,1]}),$$

$$\|y_3 - y_2\|_{C^1[0,1]} \leq \frac{k^2}{2} (\|x_1 - x_0\|_{C^1[0,1]} + \|y_1 - y_0\|_{C^1[0,1]}).$$

By a mathematical induction, we obtain

$$\|x_{n+1} - x_n\|_{C^1[0,1]} \leq \frac{k^n}{2} (\|x_1 - x_0\|_{C^1[0,1]} + \|y_1 - y_0\|_{C^1[0,1]}), \tag{3.14}$$

$$\|y_{n+1} - y_n\|_{C^1[0,1]} \leq \frac{k^n}{2} (\|x_1 - x_0\|_{C^1[0,1]} + \|y_1 - y_0\|_{C^1[0,1]}). \tag{3.15}$$

Then for any $m \geq n \geq 1$,

$$\begin{aligned} \|x_m - x_n\|_{C^1[0,1]} &\leq \sum_{j=0}^{m-n-1} \|x_{n+j+1} - x_{n+j}\|_{C^1[0,1]} \\ &\leq \sum_{j=0}^{m-n-1} \frac{k^{n+j}}{2} (\|x_1 - x_0\|_{C^1[0,1]} + \|y_1 - y_0\|_{C^1[0,1]}) \\ &= \frac{1}{2} \frac{k^n - k^m}{1 - k} (\|x_1 - x_0\|_{C^1[0,1]} + \|y_1 - y_0\|_{C^1[0,1]}). \end{aligned} \tag{3.16}$$

Letting $m \rightarrow \infty$ in both sides of (3.16), we can obtain the error estimate (3.4). A similar argument can also be used to prove error estimate (3.5). Finally, (3.6) follows immediately from (3.4) and (3.5). \square

Remark 3.1. In view of Theorem 3.4, the sequences defined by (3.11) generate a sequence $\{(x_n, y_n)\}$ which each of its elements is a coupled lower and upper solution of problem (1.3)-(1.4).

Remark 3.2. Let (x_0, y_0) be a coupled lower and upper solution of problem (1.3)-(1.4) such that $x_0 \preceq y_0$ and let $\{x_n\}$ and $\{y_n\}$ be the sequences defined by (3.11) such that $\|x_n - x^*\|_{C^1[0,1]} \rightarrow 0$ and $\|y_n - x^*\|_{C^1[0,1]} \rightarrow 0$. Then

$$x_0 \preceq x_1 \preceq \dots \preceq x_n \preceq \dots \preceq x^* \preceq \dots \preceq y_n \preceq \dots \preceq y_1 \preceq y_0. \tag{3.17}$$

Example 3.1. Let us consider the following problem

$$\begin{cases} x''(t) + \frac{2}{5}D^{\frac{3}{2}}x(t) - \frac{1}{4}x(t) = f(t), & 0 < t < 1, \\ x(0) = 0, \quad x'(0) = 1, \end{cases} \tag{3.18}$$

where $f(t) = \frac{1}{4}t^2 - \frac{1}{4}t - \frac{8}{5} \frac{\sqrt{t}}{\sqrt{\pi}} - 2$ and the exact solution is $x(t) = t(1 - t)$. Observe that, we have $k = \max\{2|B|, \frac{2A}{\Gamma(3-\alpha)}\} = 0.90267 < 1$. Now, define $\mathcal{G} : C^1[0, 1] \times C^1[0, 1] \rightarrow C^1[0, 1]$ by setting

$$\begin{aligned} \mathcal{G}(x, y)(t) &= \left(t + \frac{2}{5\Gamma(4-\alpha)}t^{3-\alpha} \right) - \frac{2}{5\Gamma(3-\alpha)} \int_0^t (t-\tau)^{2-\alpha}y'(\tau) d\tau \\ &\quad + \int_0^t (t-\tau) \left(\frac{1}{4}\tau^2 - \frac{1}{4}\tau - \frac{8}{5} \frac{\sqrt{\tau}}{\sqrt{\pi}} - 2 + \frac{1}{4}x(\tau) \right) d\tau. \end{aligned} \tag{3.19}$$

A relatively simple calculation, with the help of Maple, shows that $(x_0(t), y_0(t)) = (-3t, 3t)$ is a coupled lower and upper solution of problem (3.18). Therefore, all the assumptions of Theorem 3.4 hold and consequently, problem (3.18) has a unique solution in $C^1[0, 1]$. Moreover, the unique solution of (3.18) can be obtained as $\lim_{n \rightarrow \infty} \mathcal{G}^n(x_0, y_0)$ where $\mathcal{G}^n(x_0, y_0) = \mathcal{G}(\mathcal{G}^{n-1}(x_0, y_0), \mathcal{G}^{n-1}(y_0, x_0))$. For simplicity, we set $x_n = \mathcal{G}(x_{n-1}, y_{n-1})$ and $y_n = \mathcal{G}(y_{n-1}, x_{n-1})$. Using simple calculation, with the help of Maple, we can now form the first few successive approximations as follows

$$\begin{aligned}x_1 &= \mathcal{G}(x_0, y_0) = t - \frac{16}{15} \frac{t^{\frac{3}{2}}}{\sqrt{\pi}} - 0.24072 t^{\frac{5}{2}} + 0.02083 t^4 - 0.16667 t^3 - t^2, \\y_1 &= \mathcal{G}(y_0, x_0) = t + \frac{32}{15} \frac{t^{\frac{3}{2}}}{\sqrt{\pi}} - 0.24072 t^{\frac{5}{2}} + 0.02083 t^4 + 0.08333 t^3 - t^2.\end{aligned}$$

Therefore, we have

$$\begin{aligned}x_2 &= \mathcal{G}^2(x_0, y_0) = \mathcal{G}(x_1, y_1) \\&= t - 0.00001 t^{\frac{5}{2}} - 0.03438 t^{\frac{7}{2}} - 0.00764 t^{\frac{9}{2}} + 0.00017 t^6 - 0.00208 t^5 \\&\quad + 0.05333 t^3 - 1.32 t^2\end{aligned}$$

and

$$\begin{aligned}y_2 &= \mathcal{G}^2(y_0, x_0) = \mathcal{G}(y_1, x_1) \\&= t + 0.00001 t^{\frac{5}{2}} + 0.06877 t^{\frac{7}{2}} - 0.00764 t^{\frac{9}{2}} + 0.00017 t^6 + 0.00104 t^5 \\&\quad + 0.05333 t^3 - 0.83996 t^2.\end{aligned}$$

Similarly,

$$\begin{aligned}x_3 &= \mathcal{G}^3(x_0, y_0) = \mathcal{G}(x_2, y_2) \\&= t - 0.03851 t^{\frac{5}{2}} - 0.011 t^{\frac{7}{2}} - 0.00052 t^{\frac{11}{2}} - 0.00008 t^{\frac{13}{2}} - 0.00001 t^7 + 0.002 t^5 - 0.02 t^4 - t^2,\end{aligned}$$

and

$$\begin{aligned}y_3 &= \mathcal{G}^3(y_0, x_0) = \mathcal{G}(y_2, x_2) \\&= t + 0.07703 t^{\frac{5}{2}} - 0.011 t^{\frac{7}{2}} - 0.00104 t^{\frac{11}{2}} - 0.00008 t^{\frac{13}{2}} + 0.002 t^5 + 0.01 t^4 - t^2.\end{aligned}$$

It is interesting to point out that $(x_n, y_n) = (\mathcal{G}^n(x_0, y_0), \mathcal{G}^n(y_0, x_0))$, $n = 1, 2, 3$ serve as an approximation to the unique coupled fixed point of \mathcal{G} of increasing accuracy as $n \rightarrow \infty$. On the other hand, from Theorem 3.3, the unique solution of (3.18) can be obtained as

$$x^* = \lim_{n \rightarrow \infty} \mathcal{G}^n(x_0, y_0) = \lim_{n \rightarrow \infty} \mathcal{G}^n(y_0, x_0).$$

The graphs of x_n and y_n , for $n = 0, 1, 6$ are shown in Figure 1. Furthermore, the graphs of x'_n and y'_n , for $n = 0, 1, 6$ are shown in Figure 2.

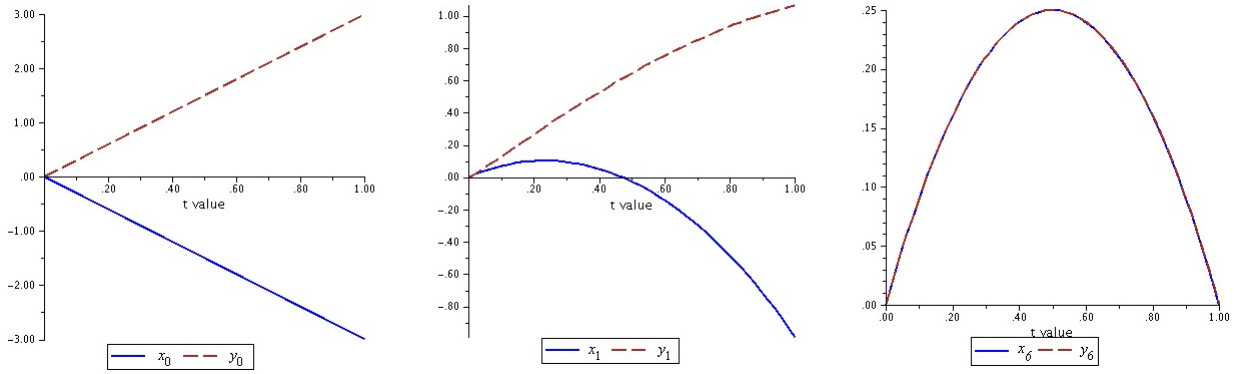


Figure 1: Graphs of x_n and y_n

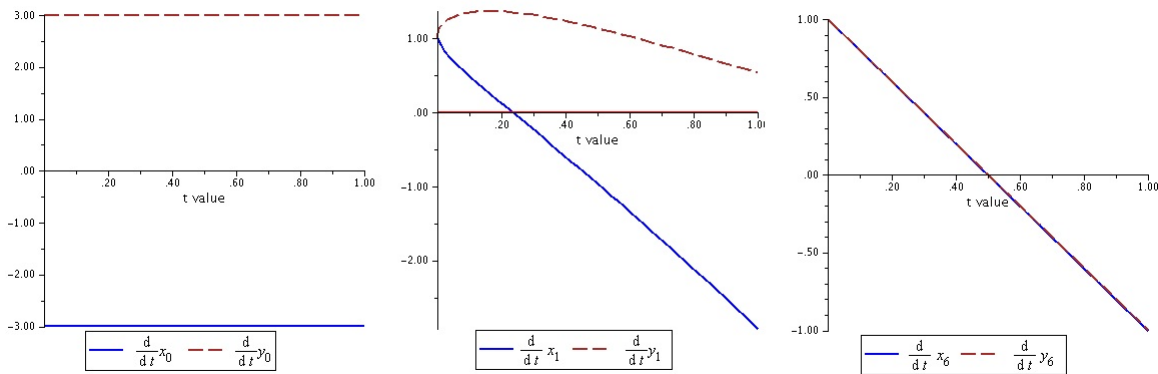


Figure 2: Graphs of x'_n and y'_n

Remark 3.3. To derive the same results for the case in which $A < 0$ and $B > 0$, we can apply the same technique as mentioned above. The main difference is the structure of the function \mathcal{G} and consequently the definition of coupled lower and upper solution of the problem (1.3)-(1.4). We could carry out a similar argument to prove the existence, uniqueness and approximation results.

3.2 Investigation in the case $A \cdot B > 0$

In this subsection we consider the case in which $A \cdot B > 0$. Before continuing, we need to introduce the fixed point theorem which play main role in our discussion. For complete details, see [31–33].

Theorem 3.5. Let (X, \preceq) be a partially ordered set such that every pair $x, y \in X$ has a lower bound and an upper bound. Furthermore, let d be a metric on X such that (X, d) is a complete metric space and \mathcal{G} is monotone (i.e., either order-preserving or order-reversing) map from X into X such that

$$\begin{aligned} \exists 0 \leq k < 1 : d(\mathcal{G}(x), \mathcal{G}(y)) &\leq kd(x, y), \quad \forall x \succeq y, \\ \exists x_0 \in X : x_0 &\preceq \mathcal{G}(x_0) \text{ or } x_0 \succeq \mathcal{G}(x_0). \end{aligned}$$

Suppose also that either \mathcal{G} is continuous or X is such that if $x_n \rightarrow x$ is a sequence in X whose consecutive terms are comparable, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that every term is comparable to the limit x . Then \mathcal{G} has a unique fixed point x^* . Moreover, for every $x \in X$, $\lim_{n \rightarrow \infty} \mathcal{G}^n(x) = x^*$.

In view of Lemma 2.6, we transform problem (1.3)-(1.4) as $x = \mathcal{G}(x)$ where $\mathcal{G} : C^1[0, 1] \rightarrow C^1[0, 1]$ is defined by

$$\mathcal{G}(x)(t) = a + b \left(t + \frac{A}{\Gamma(4-\alpha)} t^{3-\alpha} \right) - \frac{A}{\Gamma(3-\alpha)} \int_0^t (t-\tau)^{2-\alpha} x'(\tau) d\tau + \int_0^t (t-\tau) [f(\tau) - Bx(\tau)] d\tau.$$

Definition 3.5. An element $x_0 \in C^1[0, 1]$ is called a lower solution of problem (1.3)-(1.4) if

$$\begin{cases} x'_0(t) \leq b \left(1 + \frac{A}{\Gamma(3-\alpha)} t^{2-\alpha} \right) - \frac{A}{\Gamma(2-\alpha)} \int_0^t (t-\tau)^{1-\alpha} x'_0(\tau) d\tau + \int_0^t [f(\tau) - Bx_0(\tau)] d\tau, \\ x_0(0) \leq a, \end{cases} \tag{3.20}$$

for all $t \in [0, 1]$ and it is an upper solution of (1.3)-(1.4) if the above inequalities are reversed.

Theorem 3.6. Assume that $x_0 \in C^1[0, 1]$ is a lower solution of problem (1.3)-(1.4) and $k = |B| + \frac{|A|}{\Gamma(3-\alpha)} < 1$.

- (i) Then the initial value problem (1.3)-(1.4) has a unique solution $x^* \in C^1[0, 1]$.
- (ii) Moreover, the iterative sequence $\{x_n\}$ defined by

$$\begin{aligned} x_n(t) = a + b \left(t + \frac{A}{\Gamma(4-\alpha)} t^{3-\alpha} \right) \\ - \frac{A}{\Gamma(3-\alpha)} \int_0^t (t-\tau)^{2-\alpha} x'_{n-1}(\tau) d\tau + \int_0^t (t-\tau) [f(\tau) - Bx_{n-1}(\tau)] d\tau, \end{aligned}$$

converges to x^* in $C^1[0, 1]$.

- (iii) In addition, the following error estimates hold,

$$\begin{aligned} \|x_n - x^*\|_{C^1[0,1]} &\leq \frac{k^n}{1-k} \|x_1 - x_0\|_{C^1[0,1]}, \\ \|x_{n+1} - x_n\|_{C^1[0,1]} &\leq k^n \|x_1 - x_0\|_{C^1[0,1]}. \end{aligned}$$

Proof. The proof is similar to that for Theorem 3.4. It suffices to define $\mathcal{G} : C^1[0, 1] \rightarrow C^1[0, 1]$ by

$$\mathcal{G}(x)(t) = a + b \left(t + \frac{A}{\Gamma(4-\alpha)} t^{3-\alpha} \right) - \frac{A}{\Gamma(3-\alpha)} \int_0^t (t-\tau)^{2-\alpha} x'(\tau) d\tau + \int_0^t (t-\tau) [f(\tau) - Bx(\tau)] d\tau,$$

and to apply a similar arguments as in the proof of Theorem 3.4 correspondig to Theorem 3.5. □

Example 3.2. Let us consider the following problem

$$\begin{cases} x''(t) - \frac{2}{5} D^{\frac{3}{2}} x(t) - \frac{1}{2} x(t) = f(t), & 0 < t \leq 1, \\ x(0) = 0, \quad x'(0) = \frac{9}{16}, \end{cases} \tag{3.21}$$

where $f(t) = -\frac{1}{2}t^3 + \frac{3}{4}t^2 + \frac{183}{32}t - 3 - \frac{4}{5} \frac{\sqrt{t(-3+4t)}}{\sqrt{\pi}}$ and the exact solution is $x(t) = t^3 - \frac{3}{2}t^2 + \frac{9}{16}t$.

Here $A = -\frac{2}{5}$ and $B = -\frac{1}{2}$. Observe that $k = |B| + \frac{|A|}{\Gamma(3-\alpha)} = 0.95135 < 1$. Now, define $\mathcal{G} : C^1[0, 1] \rightarrow C^1[0, 1]$ by

$$\begin{aligned} \mathcal{G}(x)(t) = \frac{9}{16} \left(t - \frac{2}{5\Gamma(4-\alpha)} t^{3-\alpha} \right) + \frac{2}{5\Gamma(3-\alpha)} \int_0^t (t-\tau)^{2-\alpha} x'(\tau) d\tau \\ + \int_0^t (t-\tau) \left(-\frac{1}{2}\tau^3 + \frac{3}{4}\tau^2 + \frac{183}{32}\tau - 3 - \frac{4}{5} \frac{\sqrt{\tau(-3+4\tau)}}{\sqrt{\pi}} + \frac{1}{2}x(\tau) \right) d\tau. \end{aligned}$$

A relatively simple calculation, with the help of Maple, shows that $x_0(t) = -\frac{1}{2}t$ is a lower solution of problem (3.21). Therefore, all the assumption of Theorem 3.6 hold and consequently, problem (3.21) has a unique solution in $C^1[0, 1]$. Moreover, the unique solution of (3.21) can be obtained as $\lim_{n \rightarrow \infty} x_n$ where $x_n = \mathcal{G}(x_{n-1})$. The graphs of x_n and y_n , for $n = 0, 1, 6$ are shown in Figure 3. Furthermore, the graphs of x'_n and y'_n , for $n = 0, 1, 6$ are shown in Figure 4.

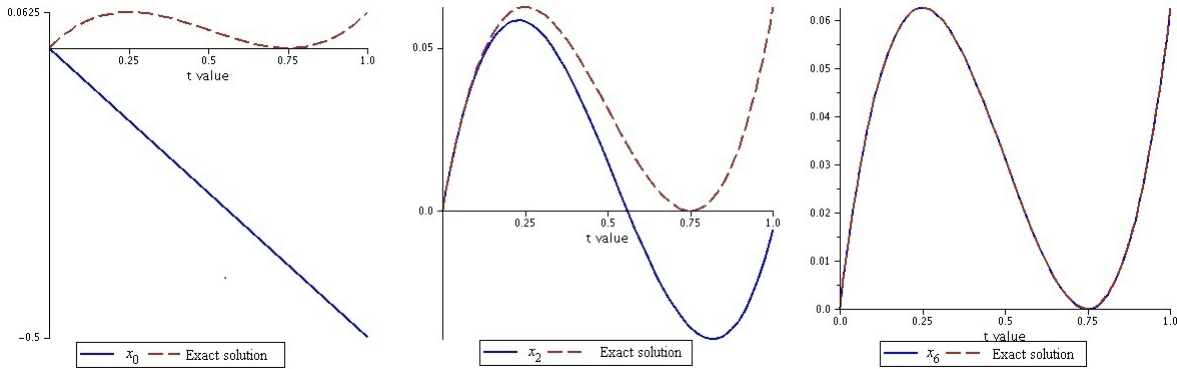


Figure 3: Graphs of x_n and exact solution

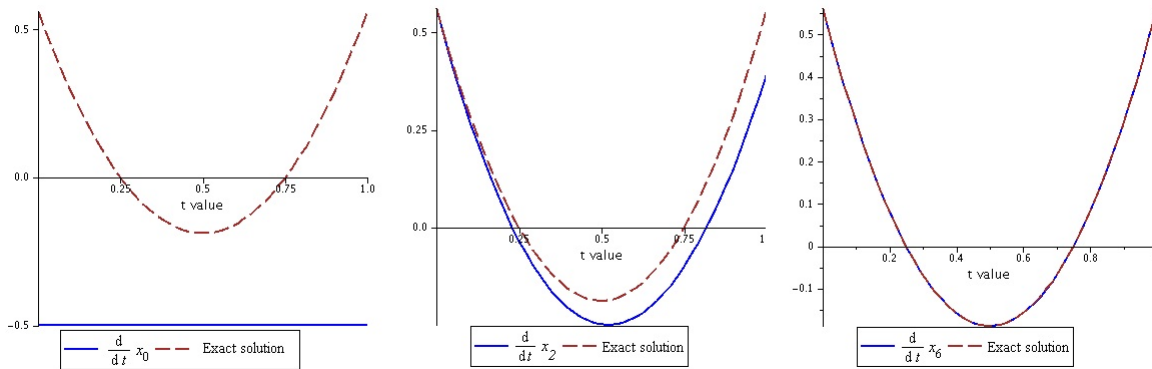


Figure 4: Graphs of x'_n and derivative of exact solution

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