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An investigation of temporal moments of stochastic waves

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It is proposed to describe the temporal characteristics of a wave propagating in a random medium in terms of its temporal moments. The first two moments are related to the mean arrival time and the mean pulse width. It is shown that the one-position two-frequency mutual coherence function enters in the formulation naturally. The form of the expression suggests expanding the mutual coherence function in a narrow-band expansion whose coefficients can be solved exactly from the parabolic equation that takes into account all multiple scattering effects except the backscattering. A brief survey of the literature shows that the irregularity spectrum, under various conditions, has a power-law dependence. In order to conform to this observation a Bessel function spectrum proposed by Shkarofsky is found convenient to use since it not only reduces to the desired power-law form in the proper range of wavenumber space, but also has all the finite moments. Exact expressions for the mean arrival time and mean square pulse width are obtained; some numerical examples are given. Finally, the effect of noise on these moments is discussed.

1. INTRODUCTION

There has lately been increased interest in investigating the effects of propagation in a random medium on pulses [*Erukhimov et al.*, 1973; *Liu et al.*, 1974; *Lee and Jokipii*, 1975; *Munk and Zachariasen*, 1976]. This interest is partially supported by applications in precise ranging measurements and in high data rate communication. In various investigations, the background medium can be nondispersive (such as electromagnetic waves in the atmosphere and sound waves in the ocean) or dispersive (such as electromagnetic waves in the ionosphere, interplanetary, and interstellar media). In terms of geometry, the wave may propagate entirely in the turbulent medium or the turbulence is confined only to a slab so that on exiting from the slab only phase mixing through diffraction will take place. The precise computation of pulse shape is sometimes mathematically demanding and in many applications such precision is not needed. In this paper the concept of temporal moments is introduced and it is shown that the first two moments are related to the mean arrival time of the pulse and the mean pulse width. The mathematics required to obtain these temporal moments is much simplified and general solutions can be found when the background medium is either nondispersive or dispersive, or when the path geometry is completely submerged in the turbulence or when the turbulence is confined to a slab geometry outside of which wave diffraction occurs.

The problem of wave propagation in a random medium is usually formulated in terms of the moments of the wave function $u(z,\omega)\exp j(\omega t - kz)$ for a wave propagating in the z direction. Here the wave number $k(\omega)$ corresponds to that of the background medium and is allowed to be an arbitrary function of circular frequency ω such as is the case in a dispersive medium. The quantity u is often called the complex amplitude and is for convenience assumed to be unity at the boundary, i.e., $u(0,\omega) = 1$ at $z = 0$ at which the wave enters the medium and propagates in the space $z > 0$. A real pulse $p(z,t)$ is obtained by superposing these plane waves, i.e.,

$$p(z,t) = \int_{-\infty}^{\infty} f(\omega)u(z,\omega)\, e^{j(\omega t - kz)} d\omega \tag{1}$$

where the amplitude spectrum $f(\omega)$ is introduced to take into account the possibilities that (i) the impressed signal at $z = 0$ is modulated, for which $f(\omega)$ is the spectrum of impressed signal; (ii) the broad-band signal is detected with a narrow-band receiver, for which $f(\omega)$ represents the receiver transfer function; and (iii) both of the above, for which $f(\omega)$ is the product of the impressed spectrum function and the receiver transfer function. For real $p(z,t)$, both $f(\omega)$ and $u(z,\omega)$ are required to be even in ω, which we assume. Rewrite (1) in the form

$$p(z,t) = \text{Re}\, A(z,t) \exp j(\omega_c t - k_c z) \tag{2}$$

Equation (2) represents a wave of (carrier) frequency ω_c, wave number $k_c = k(\omega_c)$, and slowly varying complex envelope A given by

$$A(z,t) = \int_{-\infty}^{\infty} F(\Omega) U(z,\Omega)\, e^{j[\Omega t - (k - k_c)z]}\, d\Omega \tag{3}$$

where $F(\Omega) = f(\omega_c + \Omega)$, $U(z,\Omega) = u(z,\omega_c + \Omega)$ and $\Omega = \omega - \omega_c$. In a random medium A is random since u is random.

Define the nth temporal moment by the equation

$$\langle\langle t^n(z) \rangle\rangle \equiv \int_{-\infty}^{\infty} \langle A^*(z,t) t^n A(z,t) \rangle\, dt, \quad n = 0, 1, 2, \ldots \tag{4}$$

Definitions of this kind have been used in computations of quantum mechanical packets [*Baird*, 1972], in studies of wave dispersion [*Anderson and Askne*, 1974], and in calculating the spatial fluctuations of a light beam [*Kon et al.*, 1974]. Because A is random, the ensemble average denoted by $\langle\ \rangle$ is also introduced in (4). Insertion of (3) in (4) leads immediately to

$$\langle\langle t^n(z) \rangle\rangle = \iiint_{-\infty}^{\infty} F^*(\Omega_2) F(\Omega_1) \Gamma\, t^n \exp\{j[(\Omega_1 - \Omega_2)t - (k_1 - k_2)z]\} d\Omega_1\, d\Omega_2\, dt \tag{5}$$

where $k_1 \equiv k(\omega_1) = k(\omega_c + \Omega_1)$ and $k_2 \equiv k(\omega_2) = k(\omega_c + \Omega_2)$. As can be seen in (5) the two-frequency and one-position mutual coherence function

$$\Gamma = \langle U(z,\Omega_1) U^*(z,\Omega_2) \rangle = \langle u(z,\omega_1) u^*(z,\omega_2) \rangle \tag{6}$$

enters in the formulation naturally. A more thorough discussion of Γ will be postponed until the next section; here it is useful to point out that for plane waves $\Gamma(\Omega_1,\Omega_2,z = 0) = \Gamma(\Omega_1 = \Omega_2, z > 0) = 1$ under the forward scatter assumption.

In order to interpret (4) physically we need to impose two conditions. They are (i) normalization condition $\langle\langle t^0(z) \rangle\rangle = 1$, and (ii) time origin condition $\langle\langle t(0) \rangle\rangle = 0$. The normalization condition implies, from (5),

$$2\pi \int_{-\infty}^{\infty} F^*(\Omega) F(\Omega)\, d\Omega = 1 \tag{7}$$

If the amplitude spectrum of the impressed signal $F(\Omega)$ is not normalized according to (7), the right-hand side of (4) must be divided by the left-hand side of (7) so that $\langle\langle t^n(z) \rangle\rangle$ can be interpreted as temporal moments. The time origin condition is useful because it provides a convenient time reference that the mean time of arrival of the signal at $z = 0$ is 0. But, from (5),

$$\langle\langle t(0) \rangle\rangle = 2\pi j \int_{-\infty}^{\infty} F^*(dF/d\Omega) d\Omega \tag{8}$$

which in general does not vanish. If the impressed signal has a real symmetric envelope, i.e., $A(0,t) = A(0,-t) = A^*(0,t)$, $F(\Omega)$ must then be real and even in Ω, which implies $\langle\langle t(0) \rangle\rangle = 0$ and the time origin condition is automatically established. For convenience, such a symmetric envelope will be assumed, unless otherwise stated. For general pulses one must then subtract the right-hand side of (8) in order to establish the time origin.

Integrate (5) with respect to t by using the relation

$$\int_{-\infty}^{\infty} t^n e^{j(\Omega_1 - \Omega_2)t} dt = 2\pi(-j)^n \delta^{(n)}(\Omega_1 - \Omega_2) \tag{9}$$

where $\delta^{(n)}(\Omega_1 - \Omega_2)$ is the nth-order Dirac delta function. Subsequent integration with respect to Ω_1 is made easy by the presence of the Dirac delta function, yielding

$$\langle\langle t^n(z) \rangle\rangle = 2\pi(j)^n \int_{-\infty}^{\infty} F^*(\Omega_2) e^{jk_2 z} [\partial^n F(\Omega_1)\Gamma \cdot e^{-jk_1 z}/\partial\Omega_1^n]_{\Omega_1 = \Omega_2} d\Omega_2 \tag{10}$$

Further integration of (10) requires knowledge in Γ, which is given in the next section. The form of (10) suggests an expansion of Γ in the series

$$\Gamma(k_1, k_2) = \Gamma_0 + \Gamma_1 \eta + \Gamma_2 \eta^2 + \ldots \tag{11}$$

where $\eta = (k_2 - k_1)/k_2$, $\Gamma_0 = \Gamma(k_2,k_2)$, and

$$\Gamma_n = [(-1)^n k_2^n / n!] \partial^n \Gamma / \partial k_1^n \Big|_{k_1 = k_2}$$

Note that in expanding (11), the independent variables k_1 and k_2 have been chosen instead of Ω_1 and Ω_2. This is done for later convenience (see section 2.2). As can be seen in (10), the nth temporal moment depends only on Γ_0, Γ_1, ..., Γ_n and not on Γ_{n+1} and beyond. This property has an important implication in that we only need to compute the beginning few coefficients in the expansion (11) since in practice the most important temporal moments are the beginning few. As is shown in the next section, these coefficients can be computed exactly even for cases where the random fluctuations of the properties of the media are large, thus allowing us to study the behavior of the temporal moments of the signal propagating in strongly turbulent media.

Physical interpretations are easy for the first few moments given in (10). The case $n = 1$ can be interpreted as the mean arrival time at $z > 0$. The case $n = 2$ is also useful as $\langle\langle(t - \langle\langle t\rangle\rangle)^2\rangle\rangle$ can be interpreted as the mean square pulse width centered at the mean arrival time. The higher-order moments are related to temporal distortion of the pulse, but in a manner less easily interpreted [*Baird*, 1972].

2. TWO-FREQUENCY MUTUAL COHERENCE FUNCTION

The mutual coherence function needed in calculating the temporal moments in (10) satisfies a transport equation which can be derived in various ways [*Chernov*, 1970; *Tatarskii*, 1971; *Brown*, 1972; *Lee*, 1974]. The reader should be cautioned that most early publications are concerned with mutual coherence functions at the same frequency while what we need in (10) is the two-frequency mutual coherence function. The two-frequency mutual coherence function arises from considerations of pulse distortion [*Erukhimov et al.*, 1973; *Liu et al.*, 1974; *Lee and Jokipii*, 1975; *Sreenivasiah et al.*, 1976] and they can be derived by following similar mathematical reasoning. The general starting point is the Helmholtz wave equation for the wave function Ψ

$$\nabla^2\Psi + k^2(1 + \beta\xi)\Psi = 0 \qquad (12)$$

where the time dependence $\exp(j\omega t)$ is understood. The random function ξ is assumed to be a homogeneous random field and is independent of frequency; the frequency dependence, if any, is taken care of by the factor β. The validity of (12) for various waves has been discussed by *Tatarskii* [1971]; in general it requires that the typical turbulent scale be large compared with the wavelength, which we assume. This assumption is especially essential for electromagnetic waves because otherwise depolarization effects must be taken into account. As examples, the quantities β and ξ are listed below for three cases.

(i) Electromagnetic waves propagating in a nondispersive turbulent atmosphere with fluctuating dielectric permittivity $\Delta\epsilon$:

$$\beta = 1, \quad \xi = \Delta\epsilon/\epsilon$$

(ii) Electromagnetic waves propagating in a turbulent plasma such as ionosphere, interplanetary, or interstellar media with fluctuating electron density ΔN:

$$\beta = -\omega_p^2/(\omega^2 - \omega_p^2), \quad \xi = \Delta N/N$$

where ω_p is the circular plasma frequency of the background.

(iii) Sound propagation in an ocean with fluctuating sound speed ΔC:

$$\beta = -2, \quad \xi = \Delta C/C$$

Under the quasi-optics forward scattering assumption (better known as the parabolic equation approximation in the Russian literature), the complex amplitude u, given by $\Psi = u\exp(-jkz)$, is found to satisfy

$$\nabla_T^2 u - 2jk\partial u/\partial z + k^2\beta(\omega)\xi(\vec{\rho},z)\,u = 0 \qquad (13)$$

Here $\nabla_T^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ and $\vec{\rho} = (x,y)$. From (13), under the Markov approximation, the equation for the two-frequency, two-position mutual coherence function

$$\Gamma \equiv \langle u(\vec{\rho}_1, z, \omega_1)\,u^*(\vec{\rho}_2, z, \omega_2)\rangle$$

can be derived [*Tatarskii*, 1971; *Liu et al.*, 1974].

$$\partial\Gamma/\partial z + (j/2k_1k_2)(k_2\nabla_{T_1}^2 - k_1\nabla_{T_2}^2)\Gamma - (1/2)\,A_p(\vec{\rho}_2 - \vec{\rho}_1)\Gamma = 0 \qquad (14)$$

The function A_p is related to the medium parameters in the following way. Let

$$B_\xi(\vec{\rho},z) = \langle\xi(\vec{\rho}\,',z')\xi(\vec{\rho}\,' + \vec{\rho},z' + z)\rangle$$

be the correlation function of the homogeneous random field ξ. Define a two-dimensional correlation as

$$A_\xi(\vec\rho) \equiv \int_{-\infty}^{\infty} B_\xi(\vec\rho,z)dz \tag{15}$$

Then

$$A_p(\vec\rho_2 - \vec\rho_1) = -(1/4)[(k_1^2\beta_1^2 + k_2^2\beta_2^2)\, A_\xi(0) - 2k_1\beta_1 k_2\beta_2 A_\xi(\vec\rho_2 - \vec\rho_1)] \tag{16}$$

where $\beta_1 \equiv \beta(\omega_1)$, $\beta_2 \equiv \beta(\omega_2)$.

If the impressed wave at $z = 0$ is plane, Γ is a function of $\vec\rho_2 - \vec\rho_1 \equiv \vec\rho$ for which (14) reduces to

$$\frac{\partial\Gamma}{\partial z} + \frac{j\Delta k}{2k_1k_2}\nabla_T^2\Gamma - \frac{1}{2}A_p(\vec\rho)\Gamma = 0 \tag{17}$$

where $\Delta k = k_2 - k_1$. The "initial" condition for (17) is $\Gamma = 1$ at $z = 0$. Our aim is to solve for Γ in (17). Past attempts at solving this equation have been carried out for specific media with various approximations and assumptions; we choose to approach it with calculation of temporal moments in mind.

2.1. *Approximate solutions.* Introduce the transformation

$$\Gamma = \exp(\psi + \phi) \tag{18}$$

where $\psi = -(k_1^2\beta_1^2 + k_2^2\beta_2^2)A_\xi(0)z/8$. The equation for ϕ is obtained when (18) is inserted in (17), yielding

$$\partial\phi/\partial z + (j\Delta k/2k_1k_2)[\nabla_T^2\phi + (\nabla_T\phi)^2] = (1/4)\, k_1\beta_1 k_2\beta_2 A_\xi(\vec\rho) \tag{19}$$

with the "initial" condition $\phi = 0$ at $z = 0$. Equation (19) is nonlinear and cannot be solved analytically in its general form. In a weakly random medium, the nonlinear term $(\nabla_T\phi)^2$ can be neglected, resulting in the Rytov solution [*Liu et al.*, 1974].

$$\phi \cong \frac{j\pi k_1^2k_2^2\beta_1\beta_2}{\Delta k}\iint_{-\infty}^{\infty} \Phi_\xi(\vec\kappa_T,0)[1 - \exp(j\Delta k\kappa_T^2 z \div 2k_1k_2)]\exp(-j\vec\kappa_T\cdot\vec\rho)\, d^2\kappa_T/\kappa_T^2 \tag{20}$$

where $\Phi_\xi(\vec\kappa_T,\kappa_z)$ is the spatial spectrum of the random field ξ and is the Fourier transform of the correlation function $B_\xi(\vec\rho,z)$. If the turbulence is confined to a slab from $z = 0$ to $z = L$, the solution (20) is valid up to $z = L$, beyond which there will be no scattering and only diffraction. The problem for $z \geq L$ can be formulated similarly by setting $A_\xi = 0$ in (19), but its "initial" condition at $z = L$ is obtained by solving the problem inside the slab. The final result is, for $z \geq L$,

$$\phi \cong \frac{j\pi k_1^2k_2^2\beta_1\beta_2}{\Delta k}\iint_{-\infty}^{\infty} \Phi_\xi(\vec\kappa_T,0)[\exp(j\Delta k\kappa_T^2/2k_1k_2)\cdot(z - L) - \exp(j\Delta k\kappa_T^2/2k_1k_2)z]\exp(-j\vec\kappa_T\cdot\vec\rho)\cdot d^2\kappa_T/\kappa_T^2 \tag{21}$$

For narrow-band applications, we may expand ϕ and β in the form

$$\phi = \phi_0 + \phi_1\eta + \phi_2\eta^2 + \ldots \tag{22}$$

$$\beta_1 = \beta_2 + b_1\eta + b_2\eta^2 + \ldots \tag{23}$$

where $\eta \equiv \Delta k/k_2$. If, further, we expand $A_\xi(\vec\rho)$ for an isotropic random field in the form

$$A_\xi(\vec\rho) = A_0 + A_2\rho^2 + A_4\rho^4 + \ldots \tag{24}$$

it is possible to obtain from (21)

$$\phi_0(\rho = 0) = k_2^2\beta_2^2A_0L/4 \tag{25}$$

$$\phi_1(\rho = 0) = (-j/4)k_2\beta_2^2A_2L(2z - L) + (1/4)k_2^2\beta_2(b_1 - \beta_2)A_0L \tag{26}$$

$$\phi_2(\rho = 0) = (1/4)k_2^2\beta_2(b_2 - b_1)A_0L - (j/4)k_2\beta_2b_1A_2L(2z - L) - (2/3)\beta_2^2A_4L\cdot(L^2 - 3Lz + 3z^2) \tag{27}$$

These expressions will be useful in computing the temporal moments according to (10). In a strongly random medium, the Rytov solution (21) is no longer valid. One has to look for other approximate solutions. Fortunately, what is needed in (10) is the one-position, two-frequency mutual coherence function. This suggests the expansion

$$\phi = \tilde\phi_0 + \tilde\phi_2\rho^2 + \tilde\phi_4\rho^4 + \ldots \tag{28}$$

Substituting (28) and (24) in (19), one gets a hierarchy of equations. In case of strong multiple scattering, the transverse correlation distance of ϕ is small in comparison with that of ξ. This suggests approximating $A_\xi(\vec\rho)$ by $A_0 + A_2\rho^2$ as done by *Shishov* [1974], *Lee and Jokipii* [1975], and *Sreenivasiah et al.* [1976]. The hierarchy is then reduced to two equations:

$$\frac{\partial\tilde\phi_0}{\partial z} + \frac{j2\Delta k}{k_1k_2}\tilde\phi_2 = \frac{1}{4}k_1\beta_1k_2\beta_2A_0 \tag{29}$$

$$\frac{\partial\tilde{\phi}_2}{\partial z} + \frac{j2\Delta k}{k_1 k_2}\tilde{\phi}_2{}^2 = \frac{1}{4} k_1\beta_1 k_2\beta_2 A_2 \qquad (30)$$

This set is now closed even though nonlinear and can be integrated to give

$$\tilde{\phi}_0(z) = k_1\beta_1 k_2\beta_2 A_0 z/4 - \ln[\cos(\beta_1\beta_2 A_2\Delta k/2j)^{1/2} z] \qquad (31)$$

$$\tilde{\phi}_2(z) = (k_1 k_2/2)(j\beta_1\beta_2 A_2/2\Delta k)^{1/2} \cdot \tan[(\beta_1\beta_2 A_2\Delta k/2j)^{1/2} z] \qquad (32)$$

For the case of scattering from a slab of turbulent irregularities confined to $0 \le z \le L$, the solution can be similarly obtained by using (31) and (32) at $z = L$ as initial conditions for the problem (29) and (30) with $A_0 = A_2 = 0$. The result is, for $z \ge L$,

$$\phi = \tilde{\phi}_0(L) - \ln[1 + j2\eta(z-L)\tilde{\phi}_2(L)/k_1] + \tilde{\phi}_2(L)\rho^2/[1 + j2\eta(z-L)\tilde{\phi}_2(L)/k_1] \qquad (33)$$

If one applies the narrow-band expansion (22) to (33), one can show that $\phi_0(\rho = 0)$ is identical to (25) and $\phi_1(\rho = 0)$ is identical to (26). In place of (27), one gets

$$\phi_2(\rho = 0) = (1/4)k_2^2\beta_2(b_2 - b_1)A_0 L - (j/4)k_2\beta_2 b_1 A_2 L(2z - L) - (1/48)k_2^2\beta_2^4 A_2^2 L^2 \cdot (6z^2 - 8Lz + 3L^2) \qquad (34)$$

A comparison of (27) and (34) shows that the last term of (27) is missing in (34) because this term is proportional to A_4, the fourth derivative of A_ξ at $\rho = 0$ which is ignored in the derivation of (34) by truncation. This comparison also shows that the last term of (34) is missing in (27) because the Rytov approximation ignores the nonlinear term $(\nabla_T\phi)^2$ in (19). Hence the source of discrepancy in (27) and (34) can be completely explained in terms of the nature of approximations.

2.2. *Narrow-band expansion.* For the purpose of computing temporal moments according to (10), it is not necessary to first obtain Γ by solving (17). The form of (10) suggests an expansion of Γ according to (11). In this vein we let

$$\Gamma = W\exp\psi \qquad (35)$$

and expanding W in the form

$$W = W_0 + W_1\eta + W_2\eta^2 + \dots \qquad (36)$$

When (35) and (36) are substituted in (17), the following set of equations is obtained.

$$\partial W_0/\partial z - k_2^2\beta_2^2 A_\xi(\vec{\rho})W_0/4 = 0$$

$$\partial W_1/\partial z - k_2^2\beta_2^2 A_\xi(\vec{\rho})W_1/4 = g_1 W_0$$

$$\dots$$

$$\partial W_n/\partial z - k_2^2\beta_2^2 A_\xi(\vec{\rho})W_n/4 = g_n W_0 \qquad (37)$$

where the expressions for the right-hand side are given by

$$g_1 W_0 = (-j/2k_2)\nabla_T^2 W_0 + k_2^2\beta_2(b_1 - \beta_2)A_\xi(\vec{\rho})W_0/4$$

$$g_2 W_0 = (-j/2k_2)\nabla_T^2(W_0 + W_1) + k_2^2\beta_2 A_\xi(\vec{\rho}) \cdot [(b_2 - b_1)W_0 + (b_1 - \beta_2)W_1]/4$$

$$\dots \qquad (38)$$

Integrating (37) by noting the "initial" conditions $W_0 = 1$, $W_1 = W_2 = \dots = 0$ at $z = 0$, we obtain

$$W_0 = \exp(k_2^2\beta_2^2 A_\xi(\vec{\rho})z/4) \qquad (39)$$

$$W_n = W_0\int_0^z g_n(\zeta)d\zeta, \quad n = 1, 2, \dots$$

Our solution is then

$$\Gamma = \left[1 + \sum_{n=1}^{\infty}\eta^n\int_0^z g_n(\zeta)d\zeta\right] W_0\exp\psi \qquad (40)$$

A similar approach can be applied to the problem of scattering from a turbulent slab confined within $z = 0$ and $z = L$. For $z \ge L$, one obtains

$$\Gamma = \left\{1 + \sum_{n=1}^{\infty}\eta^n\left[\left[\int_0^L g_n(\zeta)d\zeta + \int_L^z g_n'(\zeta)d\zeta\right]\right]\right\} W_0(L)\exp\psi(L) \qquad (41)$$

where g_n' and W_n' are obtained by solving (37) with $A(\rho) = 0$ and the boundary conditions $W_n'(L) = W_n(L)$. It is interesting to compare the solution just obtained with those given in section 2.1. For these solutions to be identical it is necessary that

$$W_1(z)/W_0(L)\Big|_{\rho=0} = \phi_1(\rho = 0) \qquad (42)$$

$$W_2(z)/W_0(L)\Big|_{\rho=0} = \phi_2(\rho = 0) + (1/2)[\phi_1(\rho = 0)]^2 \qquad (43)$$

. . .

After some rather tedious algebraic manipulations, it is possible to show that (42) is exactly satisfied for ϕ_1 given by (26), and (43) is satisfied if the last term of (34) is included in (27), i.e., the missing term in (27) or (34) is restored. With the missing terms restored in the Rytov solution or using the solution (41), one can compute the coefficients in the expansion (11) when $\rho = 0$, obtaining for $z \geq L$:

$$\Gamma_0 = 1$$

$$\Gamma_1 = (-j/4)k_2\beta_2^2 A_2 L(2z - L)$$

$$\Gamma_2 = (-1/8)k_2^2(\beta_2 - b_1)^2 A_0 L - (j/4)k_2\beta_2 b_1 A_2 L \cdot (2z - L) - (2/3)\beta_2^2 A_4 L(L^2 - 3Lz + 3z^2) - (1/96)k_2^2\beta_2^4 A_2^2 L^2(24z^2 - 28zL + 9L^2) \quad (44)$$

The coefficients Γ_n for $n \geq 3$ can be computed in practice if desired in a similar manner, except that the algebra involved will be rather tedious.

3. IRREGULARITY SPECTRUM

It is seen in the previous section that the propagation of waves in random media is intimately related to the correlation, or equivalently to the spectrum, of the random field ξ. There exist in the literature many theories and many measurements, some measurements made *in situ* and some made by inference, which seem to suggest a power-law spectrum for various situations, i.e., the power spectrum has the spatial wave number dependence $1/\kappa^p$. Examples can be found for the turbulent atmosphere given by the Kolmogorov-Obukhov theory in the inertial range (summarized by *Tatarskii* [1971] who also gives some experimental results), for the turbulent ocean [*Garrett and Munk,* 1975], for the turbulent ionosphere [*Elkins and Papagiannis,* 1969; *Dyson et al.,* 1974; *Phelps and Sagalyn,* 1976], for the turbulent interplanetary medium [*Jokipii and Hollweg,* 1970], for the plasma irregularities behind the comet's tail [*Lee,* 1976], and for the interstellar turbulence [*Lee and Jokipii,* 1976]. In each case, the power index p may differ in numerical values, but the power-law dependence seems to be valid nearly always. There are cases in which a more complex spectrum is inferred [*Wernik and Liu,* 1974; *Lotova,* 1975], but such a possibility is ignored in the following.

A three-dimensional power spectrum of the form $1/\kappa^p$ has several difficulties. First, for $p > 2$ its associated correlation will not exist. Second, for any finite value of p some spectral moments will always fail to exist. In order to remedy this situation, *Tatarskii* [1971] introduces an outer scale l_0 and writes the spectrum as

$$\Phi_\xi(\kappa) = \frac{\Gamma(p/2)}{\pi^{3/2}\Gamma[(p-3)/2]} \frac{l_0^3\sigma_\xi^{\,2}}{(1+\kappa^2 l_0^2)^{p/2}} \quad (45)$$

where Γ represents the gamma function and $\sigma_\xi^2 = \langle\xi^2\rangle$. This spectrum has been criticized by *Shkarofsky* [1968] who introduces in addition an inner scale r_0 by writing the spectrum as

$$\Phi_\xi(\kappa) = (\kappa_0 r_0)^{(p-3)/2} r_0^3/(2\pi)^{3/2} K_{(p-3)/2}(\kappa_0 r_0) \cdot K_{p/2}[r_0(\kappa^2+\kappa_0^2)^{1/2}]\sigma_\xi^2/[r_0(\kappa^2+\kappa_0^2)^{1/2}]^{p/2} \quad (46)$$

where $\kappa_0 = 1/l_0$ and K is a Hankel function of imaginary argument. The spectrum (46) is more convenient than the one introduced by *Lee and Jokipii* [1976] because all the related correlation functions can be evaluated analytically. For example, the corresponding three-dimensional correlation to (46) is

$$B_\xi(r) = [\kappa_0(r^2+r_0^2)^{1/2}]^{(p-3)/2} \cdot K_{(p-3)/2}[\kappa_0(r^2+r_0^2)^{1/2}]\sigma_\xi^2/(\kappa_0 r_0)^{(p-3)/2} \cdot K_{(p-3)/2}(\kappa_0 r_0) \quad (47)$$

The corresponding two-dimensional correlation defined by (15) is

$$A_\xi(\rho) = \sqrt{2\pi}\,[\kappa_0(\rho^2+r_0^2)^{1/2}]^{(p-2)/2} \cdot K_{(p-2)/2}[\kappa_0(\rho^2+r_0^2)^{1/2}]\sigma_\xi^2/\kappa_0(\kappa_0 r_0)^{(p-3)/2} \cdot K_{(p-3)/2}(\kappa_0 r_0) \quad (48)$$

Upon expanding $A_\xi(\rho)$ in ρ in the form (24), it is possible to find

$$A_0 = (2\pi r_0/\kappa_0)^{1/2}\sigma_\xi^2 K_{(p-2)/2}(\kappa_0 r_0)/K_{(p-3)/2}(\kappa_0 r_0) \quad (49)$$

$$A_2 = -(\pi\kappa_0/2r_0)^{1/2}\sigma_\xi^2 K_{(p-4)/2}(\kappa_0 r_0)/K_{(p-3)/2}(\kappa_0 r_0) \quad (50)$$

$$A_4 = (\pi\kappa_0^3/2^5 r_0^3)^{1/2}\sigma_\xi^2 K_{(p-6)/2}(\kappa_0 r_0)/K_{(p-3)/2}(\kappa_0 r_0) \quad (51)$$

These expressions are needed in (44).

In usual applications, $r_0 \ll l_0$; it is therefore possible to find a range of κ values between $1/l_0$ and $1/r_0$ for which the spectrum (29) reduces to the Tatarskii spectrum (45). The behavior of the spectrum, including the three-dimensional as well as the one-dimensional correlations, has been given by *Shkarofsky* [1968] and will not be repeated here. It is sufficient to point out that all spectral moments exist as well as derivatives of $A_\xi(\rho)$ at $\rho = 0$.

4. MEAN ARRIVAL TIME AND MEAN SQUARE PULSE WIDTH

The mean time of arrival of a pulse for an observer at z is given by (10) with $n = 1$. The needed expression in (10) is

$$(\partial/\partial\Omega_1)\, F(\Omega_1)\Gamma e^{-jk_1 z}\Big|_{\Omega_1=\Omega_2}$$
$$= [F'(\Omega_2) - F\Gamma_1 k_2'/k_2 - jk_2' F z]e^{-jk_2 z}$$

where $F'(\Omega_2) = \partial F(\Omega_2)/\partial\Omega_2$, $k_2' = \partial k_2/\partial\omega_2$, and Γ_1 is given by (44). Inserting the above expression in (10) and dropping the subscript 2 on Ω, the mean arrival time becomes

$$\langle\langle t(z)\rangle\rangle = 2\pi j \int_{-\infty}^{\infty} F^*(F' - F\Gamma_1 k'/k - jk' Fz)d\Omega \tag{52}$$

The first term in (52) is just $\langle\langle t(0)\rangle\rangle$ according to (8) and it vanishes identically for pulses with real symmetric envelopes. The evaluation of the remaining terms in (52) requires the specification of the $F(\Omega)$ and the medium. If $F(\Omega)$ is sharply peaked at $\Omega = 0$, i.e., $f(\omega)$ is sharply peaked at ω_c, we may approximate k by the sum

$$k(\omega) \cong k_c + k_c'\Omega + (1/2)k_c''\Omega^2 + (1/6)k_c'''\Omega^3 \tag{53}$$

where $k_c = k(\omega_c)$. This narrow-band approximation reduces (52) to

$$\langle\langle t(z)\rangle\rangle = (z/v_g)[1 + (v_g'/v_g - v_g''/2v_g)\overline{\Omega^2}$$
$$+ L(2z - L)/zz_s] \tag{54}$$

for the case of slab geometry. The result when the receiver is inside the random medium can be obtained by setting $L = z$ in (54). In (54) v_g is the group velocity at the carrier frequency and the prime on v_g denotes differentiation with respect to ω_c. The contributions to the mean arrival time in (54) are contained in three terms. The first term is well known since a pulse propagating over a distance z with group velocity v_g has its time of arrival equal to z/v_g. The second term is a correction to the first term due to finite bandwidth in the signal. This second term is proportional to the mean square bandwidth defined by

$$\overline{\Omega^2} \equiv 2\pi \int_{-\infty}^{\infty} F^* F\Omega^2 d\Omega \tag{55}$$

and the dispersive characteristics of the medium. It can be called a term of higher-order dispersion. In a nondispersive medium this second term vanishes. The third term in (54) results from random scattering and diffraction. The distance $z_s \equiv -4/A_2\beta_c^2$ is that distance of propagation in the turbulence medium (i.e., $z = L = z_s$), at the end of which the contribution to arrival time from scattering is equal to z/v_g. The quantity A_2 for the power-law spectrum is given by (50). If we take $p = 4$, the distance z_s can be easily expressed as

$$z_s = -4/\kappa_0\sigma_\xi^2\beta_c^2 \ln(\kappa_0 r_0) \tag{56}$$

The distance z_s is usually quite large. But even when $z \ll z_s$, there may be occasions when scattering effects should be taken into account. This may be so, for example, in precise ranging measurements. As a numerical example, we take a case of transionospheric propagation. Assuming an irregularity layer of 200 km thickness ($L = 200$ km) at a height $z = 500$ km with a 20% density fluctuation from its background ($f_p = 10$ MHz), with $\kappa_0 = 10^{-4}$ m^{-1} and $r_0 = 10^{-1}$ m, a ranging signal at 250 MHz with $(\overline{\Omega^2})^{1/2}/2\pi = 10$ MHz will experience a time delay due to random scattering of the order of 20 nsec, while the contribution from the higher-order dispersion is negligibly small.

The second temporal moment is just (10) with $n = 2$ and it can be evaluated similarly, except that many more terms will result. When the square of the mean arrival time is subtracted out from the second moment, the resulting expression is the mean square pulse width. Under the narrow-band approximation where only leading terms are retained, one obtains for the slab geometry

$$\langle\langle t^2(z)\rangle\rangle - \langle\langle t(z)\rangle\rangle^2 = \langle\langle t^2(0)\rangle\rangle + z^2(v_g'^2/v_g^4)\overline{\Omega^2}$$
$$+ (1/4)A_0 L(\beta_c/v_g + \beta_c' k_c)^2 + (4/3)(\beta_c^2/k_c^2 v_g^2)$$

$$\cdot A_4 L(L^2 - 2Lz + 3z^2) + L^2(6L^2 - 16Lz + 12z^2)$$
$$\div 3z_s^2 v_g^2 + [2v_g'^2 zL(2z - L)/v_g^4 z_s]\,\overline{\Omega^2} \quad (57)$$

The first term on the right-hand side of (57) is the mean square pulse width of the impressed signal at $z = 0$. The second term on the right-hand side arises from dispersive characteristics of the medium and it vanishes in a nondispersive medium. The remaining terms are all related to scattering and diffraction as well as dispersion. The relative importance of each term depends on the parameters of the medium and the signal. For example, the ratio between the term involving A_0 and the term involving A_4 is proportional to the quantity $[(r_0/\kappa_0)^{1/2}(\lambda z)^{1/2}]$ to the fourth power. This quantity is the ratio of the geometric mean of the inner and outer scale sizes to the Fresnel radius. Using available data for the atmospheric turbulence ($\kappa_0 = 10^{-1}\text{m}^{-1}$, $r_0 = 10^{-3}$m), for visible light pulse propagation in the atmosphere, the contribution to the pulse lengthening due to the A_0 term is dominant. For the ionospheric case, however, the situation is quite different. For all possible combinations of the outer and inner scale sizes ($\kappa_0 = 10^{-3} \sim 10^{-5}\text{m}^{-1}$, $r_0 = 10^{-1} \sim 10^{-2}$m), the term involving A_4 seems to be the most important one for carrier frequencies up to GHz range. As an example, we take $L = 200$ km, $z = 500$ km, $f_p = 10$ MHz, $\langle(\Delta N/N_0)^2\rangle^{1/2} = 1\%$, $\kappa_0 = 10^{-4}\text{m}^{-1}$, $r_0 = 10^{-2}$m. For a pulse with $(\overline{\Omega^2})^{1/2} = 10^7$ at the carrier frequency of 1 GHz, the most important pulse broadening term is due to the A_4 term and is equal to 10^{-15} sec^2, which is approximately equal to $\langle\langle t^2(0)\rangle\rangle$, the original mean square pulse width. That is, under these circumstances, the pulse length is approximately doubled even at a frequency as high as one gigahertz.

Higher-order temporal moments can be similarly computed at least in principle, although owing to the large number of terms the computations will be algebraically tedious. These higher-order moments show the skewness and other higher-order properties of the pulse.

5. DISCUSSION AND CONCLUSION

We have proposed to describe the temporal signal characteristics after propagating through a random medium in terms of their temporal moments defined by (4). In order to interpret the physical meaning of these moments, two conditions are imposed: the normalization condition and the time origin condition. With the imposition of these two conditions, the first moment can be interpreted as the mean arrival time, and the second moment minus the square of the first moment can be interpreted as the mean square pulse width. In the process of derivation it is shown that the one-position, two-frequency mutual coherence function enters in the formulation naturally. The form of the expression suggests expanding the mutual coherence function in the form (11) where the coefficients Γ_0, Γ_1, Γ_2, ... can be solved exactly, as is done in section 2, from the parabolic equation (17). The first three coefficients have been derived and are given in (44). Higher-order coefficients can be similarly obtained if one is sufficiently patient. All of these coefficients are of course dependent on the statistical properties of the medium such as the irregularity spectrum. Current data suggest strongly that under a variety of conditions, the random irregularities in nature seem to follow a power-law spectrum for which we need to introduce both an inner scale and an outer scale. This is discussed in section 3. In this section the Bessel spectrum introduced by *Shkarofsky* [1968] is especially convenient to use because its associated analytic expressions exist for the three- two- and one-dimensional correlation functions and because a range of spatial wavenumber can be found within which the Bessel spectrum reduces to the desired power-law spectrum. These expressions are used in section 4 where the mean arrival time and mean square pulse width are derived.

In all the above discussion the possible effects caused by noise are completely ignored. In practice the signal given by (2) must compete against the noise. The statistical properties of a deterministic sinusoidal signal in a narrow-band gaussian noise are well known [*Rice*, 1944, 1945]. Specifically, the resultant envelope has a Ricean distribution and the total power is equal to the sum of the signal power and the noise power. However, our signal is not deterministic because it has propagated through the random medium. If we assume that the random signal is statistically independent of the noise, a slight generalization of these earlier results will give

$$\langle R^2\rangle = \langle A^2\rangle + P_N \quad (58)$$

where R is the envelope of the resultant and A

is the envelope of the signal. Since the observer can measure only R and not A, the nth temporal moment is then

$$\langle\langle t^n(z)\rangle\rangle_N = \int_{-\infty}^{\infty} \langle R^2\rangle t^n \, dt$$
$$= \langle\langle t^n(z)\rangle\rangle + P_N \int_{-\infty}^{\infty} t^n \, dt \qquad (59)$$

where subscript N is used to denote the temporal moments in the presence of noise as contrasted with those in the noiseless environment when no subscript is used. In practical signal detection the temporal signal is usually gated for the duration T centered at some instant t_0 which is close to the actual arrival time. If we assume that the signal is of finite duration and is not affected by gating, (59) reduces to, for $n \neq 0$,

$$\langle\langle t^n(z)\rangle\rangle_N = \langle\langle t^n(z)\rangle\rangle + N t_0^n \Bigg\{ 1 + \sum_{p=1}^{n} \frac{n(n-1)\ldots[n-(2p-1)]}{(2p+1)!} \left(\frac{T}{2t_0}\right)^{2p} \Bigg\} \qquad (60)$$

Here we have defined $N = P_N T$ as the noise energy. The zeroth-order moment is, from (59),

$$\langle\langle t^0(z)\rangle\rangle_N = \langle\langle t^0(z)\rangle\rangle \, (1 + N/S) \qquad (61)$$

which will be used to normalize the moment in order to interpret the moments physically. The quantity $S = \langle\langle t^0\rangle\rangle \int A^2 \, dt$ is the signal energy which was conveniently taken as unity in earlier discussions. The mean arrival time in the presence of noise to the first order in N/S is

$$\langle\langle t\rangle\rangle_N / \langle\langle t^0\rangle\rangle_N = t_a + (t_0 - t_a) \, N/S \qquad (62)$$

where $t_a = \langle\langle t\rangle\rangle / \langle\langle t^0\rangle\rangle$ is the arrival time in the absence of noise. Equation (62) shows that if the signal is gated at a time very close to t_a, the presence of noise will have minimum effect on measuring the arrival time. By using (60) it is also possible to compute the mean square pulse width in the noisy environment as

$$\frac{\langle\langle t^2\rangle\rangle_N}{\langle\langle t^0\rangle\rangle_N} - \left(\frac{\langle\langle t\rangle\rangle_N}{\langle\langle t^0\rangle\rangle_N}\right)^2 = \frac{\langle\langle t^2\rangle\rangle}{\langle\langle t^0\rangle\rangle} - t_a^2 + \left[t_0^2 - \frac{\langle\langle t^2\rangle\rangle}{\langle\langle t^0\rangle\rangle} + \frac{T^2}{12} - 2t_a(t_0 - t_a) \right] \frac{N}{S} \qquad (63)$$

again to the first order in N/S. On the right-hand side of (63) the first two terms give the mean square pulse width in the absence of noise and the last term arises from noise. Equations (62) and (63) show that in practice the noise may affect the determination of the mean arrival time and the mean square pulse width of the signal.

In applications to communication, the lengthening of pulse by propagation effects discussed in this paper is very undesirable because of its possible cause on intersymbol interference. One way to overcome this difficulty is to design a pulse so that such propagation effects will be minimized. This problem is being looked into and will be a topic for future communication.

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