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An Invitation to C-semigroups — Source link

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RESEARCH ARTICLE

An Invitation to C-Semigroups

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Abstract

Semigroups with an additional unary operation called a (right) closure are investigated. These "closure semigroups" may be viewed as (not necessarily regular) generalisations of inverse semigroups, and several powerful structural aspects of inverse semigroup theory are shown to extend naturally to some important classes of closure semigroups. These include representations as partial transformations on sets, natural partial orders that are multiplication (and closure) respecting, and simple descriptions of some important congruences.

1. Introduction

An important and fundamental concept in mathematics is that of an (*algebraic*) *closure operator* on a set X, that is, an operator $C : 2^X \to 2^X$ satisfying (for $A, B \subseteq X$) $C(C(A)) = C(A)$, $A \subseteq C(A)$, $A \subseteq B \Rightarrow C(A) \subseteq C(B)$ (see [4] for example). This is easily generalised to partially ordered sets in the obvious manner, and in particular to semilattices. It is this last case which we generalise here. The generalisation is motivated in part by a number of examples, including partial functions, direct products of monoids, and inverse semigroups, and we can often characterise such families of examples algebraically.

Let S be a semigroup with an additional unary operation C satisfying

1. $aC(a) = a$;
2. $C(a)C(b) = C(b)C(a)$;
3. $C(C(a)) = C(a)$; and
4. $C(ab)C(b) = C(ab)$

for all $a, b \in S$. Then C is a *right closure* on S and S is a *right closure semigroup*, or an *RC-semigroup*.

Note that if S is a semilattice then the usual definition of a closure operation on a semilattice is recovered.

Left closure semigroups, or *LC-semigroups*, are defined in a dual manner. A semigroup which is both an LC- and an RC-semigroup is *two-sided*. All definitions and results to follow have obvious left-sided analogs.

Let S be an RC-semigroup. Let $C(S) = \{C(a) \mid a \in S\}$, and let $R(S) = \{b \in S \mid C(a)b = bC(a) \text{ for all } a \in S\}$, the *centralizer of* S. Evidently $R(S)$ is a subsemigroup of S, and because $C(S) \subseteq R(S)$, it is an *RC-subsemigroup*—a subsemigroup closed under C and therefore itself an RC-semigroup. We say S is *central* if $R(S) = S$. The class of central RC-semigroups is an RC-semigroup variety, that is, a variety of unary semigroups in which the unary operation C is a right closure.

Note that if S is central, the second rule in the definition is superfluous and it follows from the fourth that

$$C(ab) = C(abC(a)) = C(abC(a))C(C(a)) = C(abC(a))C(a) = C(ab)C(a),$$

so central RC-semigroups may be viewed as being two-sided RC-semigroups in which the left and right closures are the same. Indeed the converse holds, and we have

Proposition 1.1. *Let A be an RC-semigroup, with right closure C. Then A is central if and only if C is also a left closure on A.*

Proof. It remains to show that if C is both a left and right closure on A, then A is central. For all $a, b \in A$,

$$\begin{aligned} C(a)b &= C(a)bC(C(a)b) \text{ by axiom 1} \\ &= C(a)bC(C(a)b)C(C(a))C(b) \text{ by axiom 4 and its right dual} \\ &= C(a)bC(a)C(b) \text{ by axioms 1 and 3} \\ &= C(a)bC(b)C(a) \text{ by axiom 2} \\ &= C(a)bC(a) \text{ by axiom 1.} \end{aligned}$$

Similarly, $bC(a) = C(a)bC(a)$, so in fact $C(a)b = bC(a)$. Hence A is central. ∎

If the semigroup S has a right identity e then S can be given a RC-semigroup structure: let $C(a) = e$ for all $a \in S$. If S is a monoid then it is a central RC-semigroup in the same way.

In any semilattice S, define $a \leq b$ whenever $a = ab$; then $\leq$ is a partial order (the natural partial order on S) with $ab = glb(a, b)$ for all $a, b \in S$.

Proposition 1.2. *If S is an RC-semigroup, then $C(S)$ is a subsemigroup of S which is a semilattice, and $C(a) = \min\{e \in C(S) \mid ae = a\}$ under the natural partial order on $C(S)$.*

Proof. For $a, b \in S$,

$$C(a)C(b) = C(a)C(b)C(C(a)C(b))$$

$$
\begin{aligned}
&= [C(C(a)C(b))C(C(b))]C(a) \\
&= C(C(a)C(b))C(a) \\
&= C(C(b)C(a))C(C(a)) \\
&= C(C(b)C(a)) \\
&= C(C(a)C(b)),
\end{aligned}
$$

so $C(S)$ is a subsemigroup of S. Moreover, for $a \in S$, $C(a)C(a) = C(a)C(C(a)) = C(a)$, and because $C(a)C(b) = C(b)C(a)$ for all $a, b \in S$, it follows that $C(S)$ is a semilattice.

Finally, $aC(a) = a$, and therefore if $ae = a$ for some $e \in C(S)$, then $C(a)e = C(a)C(e) = C(ae)C(e) = C(ae) = C(a)$, and so $C(a) = \min\{e \in C(S) \mid ae = a\}$. ∎

It follows that the fourth rule in the RC-semigroup definition simply says that $C(ab) \leq C(b)$ for all $a, b \in S$ under the semilattice ordering of $C(S)$. In an LC-semigroup, $C(ab) \leq C(a)$; while in a two-sided C-semigroup in which the two closures are equal, $C(ab) \leq C(a)C(b)$.

The natural converse to Proposition 1.2 is true also.

Proposition 1.3. *Suppose the semigroup S has a subsemigroup L which is a semilattice for which $C(a) = \min\{e \in L | ae = a\}$ exists for all $a \in S$. Then C is a right closure on S for which $C(S) = L$.*

Proof. All the axioms are immediate except the last one. If $be = b$ for $b \in S$ and $e \in L$ then $(ab)e = a(be) = ab$ so $e \geq C(ab)$ for all $a \in S$; in particular $C(b) \geq C(ab)$ and so $C(ab)C(b) = C(ab)$. ∎

In view of these last two propositions, an RC-semigroup S is exactly a right $C(S)$-semiadequate semigroup, in the terminology of [14] or a type SL γ-semigroup in the terminology of [1].

Recall Green's equivalence relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ and $\mathcal{D}$, defined on any semigroup S. L_a will denote the $\mathcal{L}$-class containing $a \in S$, and similarly for R_a, H_a and D_a in terms of $\mathcal{R}$, $\mathcal{H}$ and $\mathcal{D}$ respectively.

On any C-semigroup S we introduce a fifth relation $\widetilde{\mathcal{L}}$, defined as follows: for $a, b \in S$, $a \,\widetilde{\mathcal{L}}\, b$ if and only if $C(a) = C(b)$. Evidently $\widetilde{\mathcal{L}}$ is an equivalence relation; let $\widetilde{\mathcal{L}}_a$ denote the $\widetilde{\mathcal{L}}$-class containing $a \in S$.

We say that S satisfies the *right congruence condition* if $\widetilde{\mathcal{L}}$ is a right congruence on the semigroup S.

Proposition 1.4. *A (left or right) C-semigroup S satisfies the right congruence condition if and only if $C(C(a)b) = C(ab)$ for all $a, b \in S$.*

Proof. If $C(C(a)b)=C(ab)$ for all $a,b\in S$ and $C(x)=C(y)$ then $C(xz)=C(C(x)z)=C(C(y)z)=C(yz)$, so $\widetilde{\mathcal{L}}$ is a right congruence. Conversely, if $\widetilde{\mathcal{L}}$ is a right congruence then $C(x)\,\widetilde{\mathcal{L}}\,x$ since $C(C(x))=C(x)$, so $C(x)y\,\widetilde{\mathcal{L}}\,xy$, that is, $C(C(x)y)=C(xy)$. ∎

Thus the right congruence condition defines an RC-semigroup variety.

For all a,b in the RC-semigroup S, define $a\leq_r b$ to mean that $a=bC(d)$ for some $d\in S$. The proof that this is a partial order on S is straightforward. Clearly it extends the natural semilattice partial order on $C(S)$.

Proposition 1.5. *For all a,b in the RC-semigroup S, $a\leq_r b$ if and only if $a=bC(a)$.*

Proof. If $a\leq_r b$ then $a=be$ for some $e\in C(S)$, so

$$\begin{aligned} bC(a) &= bC(be)\\ &= bC(be)C(e)\\ &= beC(be)\\ &= be\\ &= a. \end{aligned}$$

The converse is immediate. ∎

Let S be an RC-semigroup. We say that $a\in S$ is *translucent* if for all $e\in C(S)$ there exists $f\in C(S)$ for which $ea=af$. Thus a is translucent if and only if $ea\leq_r a$, which by the previous result is equivalent to the condition that $ea=aC(ea)$. Thus every element of S is translucent if and only if S satisfies the *right ample* condition in the sense of [9] (there x^* is used instead of $C(x)$). Hence the right ample condition also defines a variety of RC-semigroups, given by $\{C(x)y=yC(C(x)y)\}$, and evidently containing the variety of central RC-semigroups. In fact the class of structures studied in [9] satisfying the right ample condition (called *right type A semigroups*) also satisfy the stronger law $C(x)y=yC(xy)$. This law will play a central role in Section 3. If an RC-semigroup has all its elements translucent, we say it is *translucent*.

Much of the usefulness of the natural partial order on an inverse semigroup stems from the fact that it respects multiplication. This is also true for translucent RC-semigroups. Indeed the right closure is also respected in such cases.

Proposition 1.6. *If S is a translucent RC-semigroup, with $a,b,a',b'\in S$, such that $a\leq a'$ and $b\leq b'$, then $ab\leq a'b'$ and $C(a)\leq C(a')$.*

Proof. Suppose $a = a'e$ and $b = b'f$ for some $e, f \in C(S)$. Then $ab = a'eb'f = a'b'gf$ for some $g \in C(S)$, and so $ab \leq a'b'$. Further, $aC(a') = a'eC(a') = a'C(a')e = a'e = a$, so that $C(a) \leq C(a')$ as required. ■

There are many other generalisations of the natural partial order on an inverse semigroup to broader classes of semigroups but these rarely respect multiplication (see Section 1.4 of [11] for a discussion of some of these).

Suppose S is a semigroup with a subsemigroup L which is also a semilattice. It is easily checked that for all $a \in S$, the set $F_a = \{e \in L \mid ae = a\}$ is a filter of L (that is, a subsemilattice F of L in which $a \in F$ and $a \leq b$ imply $b \in F$), and it is immediate from the above results that S is an RC-semigroup in which $C(S) = L$ if and only if F_a is principal for all $a \in S$ (that is, F_a consists of all $e \in L$ greater than or equal to a given element, in this case $C(a)$).

We conclude this section with two important families of examples.

Example 1.7 *Relations and closure operators.* Let X be a set equipped with a closure operator C. Then any intersection of closed subsets of X under C is itself closed. The collection B_X of all binary relations on X, here most conveniently viewed as multiply defined partial functions $X \to X$, is a semigroup under composition. We denote by fg the (generalised) map obtained by applying first f and then g. Define L_X to be the set of restrictions of the identity map to closed subsets of X: L_X is clearly a semilattice and for all $f \in B_X$, $C(f) = \min\{e \in L_X \mid fe = f\}$ exists and is the restriction of the identity map to the closure of the domain of definition of f, and so C is a right closure on B_X. (Similarly B_X has a left closure where the closure of a given multiply defined partial map f is the restriction of the identity map to closure of the range of f.) Important RC-subsemigroups of B_X include the collection P_X of all partial transformations on X and the collection I_X of all one-to-one partial transformations on X: clearly $L_X \subseteq I_X \subseteq P_X \subseteq B_X$.

Example 1.8 *Direct products and closure operators.* Let X be a set with closure operator C. Let S_x, $x \in X$, be a collection of semigroups each having distinguished right identity 1 and right zero 0, let $S = \Pi_{x \in X} S_x$ and let L be all elements of S which take the value 1 on a closed subset of X and 0 elsewhere. Then L is a semilattice of idempotents in S, central if $0, 1$ are two-sided, and $C(a) = \min\{e \in L \mid ae = a\}$ exists for all $a \in S$ and gives the "characteristic function" of the closure of the support of a. Hence S equipped with C is an RC-semigroup, with $C(S) = L$.

In connection with I_X of Example 1.7, Section 2 concerns the relationship between RC-semigroups and inverse semigroups. Several aspects of inverse semigroup theory naturally extend to some quite general classes of RC-semigroups, and a particularly well behaved example of such a class (motivated by P_X of Example 1.7) is introduced in Section 3. In Section 4 we turn to an examination of congruences on RC-semigroups and in Section 5 we describe all semigroup varieties consisting of RC-semigroups with particular properties and briefly examine some RC-semigroup varieties.

2. Inverse C-semigroups

Let S be a semigroup, $E(S)$ the collection of idempotents of S. An element $a \in S$ is *regular* if there exists $b \in S$ for which $aba = a$. The semigroup S is *regular* if every $a \in S$ is regular, and is *inverse* if there is a unary operation $^{-1}$ definable on S such that $aa^{-1}a = a$, $(a^{-1})^{-1} = a$, and $(aa^{-1})(bb^{-1}) = (bb^{-1})(aa^{-1})$ for all $a, b \in S$. Clearly any inverse semigroup is regular; indeed a regular semigroup is inverse if and only if any two idempotents commute. If S is inverse, the following can also be shown: a^{-1} is the inverse of $a \in S$, in the sense that $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$, and a^{-1} is unique with these properties; $E(S) = \{aa^{-1} \mid a \in S\}$ is a subsemigroup of S which is a semilattice; $e^{-1} = e$ for all $e \in E(S)$; and semigroup congruences on S automatically respect $^{-1}$. For these and other fundamental facts concerning semigroups the reader is referred to [13].

Proposition 2.1. *Every inverse semigroup S is an RC-semigroup if one defines $C(a) = a^{-1}a$ for all $a \in S$.*

Proof. For all $a \in S$, $a^{-1}a \in E(S)$ and $a(a^{-1}a) = a$; moreover if $e \in E(S)$ satisfies $ae = a$ then $(a^{-1}a)e = a^{-1}(ae) = a^{-1}a$, so $a^{-1}a = \min\{e \in E(S) \mid ae = a\}$. Hence defining $C(a) = a^{-1}a$ makes C a right closure on S. ∎

This also follows by noting that on a regular semigroup $\widetilde{\mathcal{L}} = \mathcal{L}$ (see [8] for example).

Let us call C as in Proposition 2.1 the *standard right closure* on the inverse semigroup S and the resulting RC-semigroup the *standard RC-semigroup structure on S*.

In any inverse semigroup S, it is possible to define a partial order as follows: $a \leq b$ means $b = ae$, where e is an idempotent. In other words if C is the standard right closure on S then $a \leq b$ if and only if $b = aC(d)$ for some $d \in S$ and so $\leq$ is exactly the partial order $\leq_r$ defined earlier. Thus Proposition 1.5 generalises a familiar fact about inverse semigroups.

In an arbitrary inverse RC-semigroup S, the right closure is wholly determined by its action on $E(S)$, as we see in the following

Proposition 2.2. *Let S be an inverse RC-semigroup. If C is a right closure on S, then $C(a) = C(a^{-1}a)$ for all $a \in S$. Moreover any closure on $E(S)$ can be extended via this rule to a right closure on S.*

Proof. For an element a of an inverse RC-semigroup S, $C(a) = C(aa^{-1}a) \leq C(a^{-1}a) \leq C(a)$, so $C(a) = C(a^{-1}a)$.

Now if S is an inverse semigroup and C is a right closure on $E(S)$, let $C(a) = C(a^{-1}a)$ for all $a \in S$. Then for all $a, b \in S$: $aC(a) = aC(a^{-1}a) = aa^{-1}aC(a^{-1}a) = aa^{-1}a = a$; $C(a)C(b) = C(b)C(a)$ and $C(C(a)) = C(a)$ are immediate; finally, $((ab)^{-1}ab)(b^{-1}b) = b^{-1}a^{-1}abb^{-1}b = b^{-1}a^{-1}ab = (ab)^{-1}ab$ so $(ab)^{-1}ab \leq b^{-1}b$ in $E(S)$ and therefore

$$C(ab)C(b) = C((ab)^{-1}ab)C(b^{-1}b) = C((ab)^{-1}ab) = C(ab)$$

since C is order-preserving on $E(S)$. ∎

The standard RC-semigroup structure on an inverse semigroup S arises in this way by setting $C(e) = e$ for all $e \in E(S)$. If X is a set with C a closure operator on X, then I_X as defined in Example 1.7 is an inverse RC-semigroup in which $L_X = C(I_X)$; again, it is the action of C on $E(I_X)$, or equivalently the action of the original closure operator on X itself, which determines the action of C on the whole of I_X.

Such an inverse RC-semigroup is generic, at least in the finite case.

Theorem 2.3. *Every finite inverse RC-semigroup is embeddable in one of the form I_X (X a set equipped with a closure operator) as in Example* 1.7.

Proof. In the inverse RC-semigroup I_X of all one-to-one partial maps on a set X equipped with a closure operator as in Example 1.7, and for $S \subseteq X$, let 1_S be the restriction of the identity map to S. Then $L_X = \{1_S \mid C(S) = S\}$, which is isomorphic as a semilattice under composition to the $\cap$-semilattice of closed subsets of X.

Now let S be a finite inverse RC-semigroup. For any $a \in S$, define $\psi_a : a^{-1}aS \to aS$, by setting $\psi_a(x) = ax$ for all $x \in a^{-1}aS$. Now define $\theta : S \to I_S$ by setting $\theta(a) = \psi_a$ for all $a \in S$. We show θ is an RC-semigroup embedding.

The usual Vagner-Preston representation establishes that θ is an inverse semigroup embedding, in which the idempotent elements of S are mapped to partial identity maps on X and the inverse of an element in S is mapped to its inverse in I_X. It remains to define an appropriate closure operator on the subsets of S and to show that the induced RC-semigroup structure on I_S embeds that on S.

Note that if $eS \subseteq fS$ for $e, f \in E(S)$, then $e^2 = fx$ for some $x \in S$, and so $fe = f(fx) = fx = e$, so $e \leq f$, and so if also $fS \subseteq eS$ then $f = e$. Thus we can define C_1 on the subsets of S of the form eS (where $e \in E(S)$) by setting $C_1(eS) = C(e)S$.

Now if $eS \subseteq fS$ then as just shown, $e = ef$ and so

$$C_1(eS) = C(e)S = C(ef)S = C(ef)C(f)S = C(f)C(ef)S \subseteq C(f)S = C_1(fS),$$
$$C_1(C_1(eS)) = C_1(C(e)S) = C(C(e))S = C(e)S = C_1(eS),$$

and
$$eS = eC(e)S = C(e)eS \subseteq C(e)S = C_1(eS).$$

Thus C_1 is a closure operator.

Now it is easily shown that $eS \cap fS = efS$, so the subsets of the form eS are closed under finite intersections (hence arbitrary intersections, since S is finite). Thus we can extend C_1 to arbitrary subsets of S by setting $C_1(T) = C(\cap\{eS \mid e \in E(S), T \subseteq eS\})$, giving a closure operator on all subsets of S which restricts to the original C_1. Hence we can define C_2 on all of $I(S)$ using C_1 as in the example. We now show that $\theta(C(a)) = C_2(\theta(a))$ for all $a \in S$, thereby completing the proof.

Now for any $a \in S$, $\theta(C(a)) = \psi_{C(a)}$, which has domain $C(a)^{-1}C(a)S = C(a)S$, and $\psi_{C(a)}(x) = C(a)x = x$ for all $x \in C(a)S$, so $\psi_{C(a)}$ is a restriction of the identity to $C(a)S$, By Proposition 2.2, $C(a) = C(a^{-1}a)$, and so

$$\psi_{C(a)} = 1_{C(a)S} = 1_{C(a^{-1}a)S} = 1_{C_1(a^{-1}aS)} = C_2(\psi_a) = C_2(\theta(a)),$$

as required. ∎

It is possible to extend this result to arbitrary inverse RC-semigroups, although the closure operator C_1 in the proof cannot be extended to arbitrary subsets of S and the result loses much of its simplicity.

An important property possessed by an inverse semigroup S endowed with its standard right closure is that for every $a \in S$ there exists $a^{-1} \in S$ for which $a^{-1}a = C(a)$. In general if S is an RC-semigroup such that for every $a \in S$ there exists $b \in S$ for which $ba = C(a)$, then we say S is a *strong RC-semigroup.* This condition implies regularity of S since for all $a \in S$ there exists b such that $aba = aC(a) = a$.

In the case of strong RC-semigroups, some of the RC-semigroup axioms are redundant.

Proposition 2.4. *Let S be a semigroup equipped with a unary operation C satisfying the following rules:*

1. $aC(a) = a$*;*
2. $C(a)C(b) = C(b)C(a)$*; and*
3. *for all $a \in S$ there exists $b \in S$ for which $ba = C(a)$.*

Then S is a (strong) RC-semigroup.

Proof. Let $a, b \in S$. Suppose $a', b', c \in S$ are such that $a'a = C(a)$, $b'b = C(b)$ and $cab = C(ab)$. Then $C(ab)C(b) = cabC(b) = cab = C(ab)$, and so

$$C(b) = C(b)C(C(b)) = C(C(b))C(b) = C(b'b)C(b) = C(b'b) = C(C(b)).$$

Thus all the RC-semigroup rules are satisfied by S. ∎

A second property possessed by an inverse semigroup with its standard right closure is as follows. If the semigroup S is such that there is a right closure C on S for which $C(S) = E(S)$, we say S is *right full* or just *full*. (Similarly we define left full and two-sided full semigroups.) Thus an inverse RC-semigroup is full if and only if C is the standard right closure.

Proposition 2.5. *If S is a regular full RC-semigroup then S is inverse with the standard closure.*

Proof. Suppose S is a regular full RC-semigroup. Then $E(S) = C(S)$ and so any two idempotents commute by Proposition 1.3. Hence S is inverse since it is regular. Furthermore, Proposition 1.3 implies that the right closure on S is the standard right closure. ∎

The standard right closure on an inverse semigroup is both full and strong. Conversely we have

Proposition 2.6. *If the semigroup S is a full and strong RC-semigroup, then S is inverse and the right closure on S is the standard right closure.*

Proof. A strong RC-semigroup is regular and so Proposition 2.5 applies. ∎

In particular every (right) full and strong RC-semigroup is two-sided, and indeed left full and left strong.

Full semigroups need not be inverse: for instance, in the additive monoid of non-negative integers N, $E(N) = \{0\}$, so N is a RC-semigroup in which $C(N) = \{0\}$ and is full. However N is not inverse and so cannot be strong either. Likewise a left zero semigroup S (that is, a semigroup satisfying $xy = x$) admits a strong right closure but is not inverse since idempotents do not commute: select an element $a \in S$ and define $C(b) = a$ for all $b \in S$.

For RC-semigroups, any two of "full", "strong" and "inverse" implies the third. One case was dealt with in Proposition 2.6. If S is full and inverse then $C(a) = a^{-1}a$ for all $a \in S$, so S is evidently strong. The third case is established by the following

Proposition 2.7. *If an RC-semigroup is inverse and strong then it is full.*

Proof. Suppose S is an inverse RC-semigroup. Then in $E(A)$, $C(a) \geq a^{-1}a$ for all $a \in S$ (since $a^{-1}a$ is the least $e \in E(S)$ for which $ae = a$, and $aC(a) = a$ with $C(a) \in E(S)$). Now S is strong, so for all $a \in S$ there exists $b \in S$ for which $ba = C(a)$, so $a^{-1}a \leq ba$ in $E(S)$, and so $a^{-1}a = baa^{-1}a = ba = C(a)$. Thus S is full. ∎

Proofs of the following useful facts concerning Green's relations can be found in most semigroup texts (see [13] for example).

Lemma 2.8. *Let S be a semigroup.*

1. *Every $\mathcal{H}$-class contains at most one idempotent.*
2. *If e is idempotent then the maximal subgroup of S containing e is H_e.*
3. *If a is a regular element of S then every element of D_a is regular.*
4. *If every element of a $\mathcal{D}$-class D is regular then every $\mathcal{L}$-class and every $\mathcal{R}$-class of D contains at least one idempotent.*
5. *If D is a regular $\mathcal{D}$-class of a semigroup S and $a, b \in D$ then $ab \in R_a \cap L_b$ if and only if $L_a \cap R_b$ contains an idempotent.*

In many of the cases we will be considering, the relations $\mathcal{D}$ and $\mathcal{L}$ coincide.

We now show that strong RC-semigroups are one-sided generalisations of inverse semigroups.

Lemma 2.9. *If S is a strong RC-semigroup then $a \,\mathcal{L}\, C(a)$ for all $a \in S$.*

Proof. Since S is an RC-semigroup, $aC(a) = a$, and since the right closure on S is strong, there is an element b such that $ba = C(a)$. ∎

Thus every $\mathcal{L}$-class of a strong RC-semigroup S contains an element of $C(S)$. Note that the condition $a \,\mathcal{L}\, C(a)$ characterises strong RC-semigroups since if $a \,\mathcal{L}\, C(a)$ then by definition there exists b such that $ba = C(a)$.

If $a \,\mathcal{L}\, e$ and $e^2 = e$ then there exists b such that $be = a$ and therefore $ae = bee = be = a$, that is, e is a right identity for every element in L_e. Thus two $\mathcal{L}$-related idempotents e and f commute if and only if $e = f$. By the dual nature of the relations $\mathcal{L}$ and $\mathcal{R}$ we have the same for $\mathcal{R}$-related idempotents. Thus by the definition of C and Lemma 2.9 we have

Lemma 2.10. *If S is a strong RC-semigroup then every $\mathcal{L}$-class of S contains exactly one element of $C(S)$ and every $\mathcal{R}$-class contains at most one element of $C(S)$.*

Lemma 2.11. *If S is a strong RC-semigroup then $C(a)$ is the only idempotent in the $\mathcal{R}$-class $R_{C(a)}$.*

Proof. Suppose $C(a)\,\mathcal{R}\,e$ for some idempotent e. Now $e\,\mathcal{L}\,C(e)$ by Lemma 2.9 so $L_{C(e)} \cap R_{C(a)}$ contains the idempotent e. Thus by Lemma 2.8, $C(e)C(a) \in L_{C(a)} \cap R_{C(e)}$ and so $C(e)C(a)\,\mathcal{L}\,C(a)$. Therefore by Lemma 2.10, $C(e)C(a) = C(a)$, but then $C(a) \in L_{C(a)} \cap R_{C(e)}$, and by Lemma 2.10, $C(a) = C(e)\,\mathcal{H}\,e$. Since $C(a)$ is idempotent, $C(a) = e$. ■

Theorem 2.12. *Let C_l and C_r be left and right closures on a semigroup. If S is a strong LC- and RC-semigroup with respect to C_l and C_r respectively then S is inverse with $C_l(a) = aa^{-1}$, $C_r(a) = a^{-1}a$.*

Proof. Combining Lemmas 2.10 and 2.11 with their left duals shows that every $\mathcal{L}$-class and every $\mathcal{R}$-class of S contains exactly one idempotent. It is well known that S must be inverse in this case (see [13] for example). Now $a^{-1}a$ is an idempotent in the $\mathcal{L}$-class L_a, so $C_r(a) = a^{-1}a$. Likewise $C_l(a) = aa^{-1}$. ■

Combining with Proposition 2.1, we have characterised inverse semigroups as strong LC- and RC-semigroups: fullness follows as a corollary. Note that if S is a strong central RC-semigroup, then Proposition 1.1 and Theorem 2.12 imply that $a^{-1}a = aa^{-1}$ and so $\mathcal{D} = \mathcal{L} = \mathcal{R} = \mathcal{H}$. In this case S is known as a Clifford semigroup.

Let a *true inverse* of an element a in a RC-semigroup be an element a' such that $a'a = C(a)$ and $aa' = C(a')$. Clearly not every element in an RC-semigroup need have a true inverse and in fact we have

Corollary 2.13. *If S is an RC-semigroup in which every $a \in S$ has a true inverse a', then S is an inverse semigroup with C the standard right closure, and $a^{-1} = a'$ for all $a \in S$.*

Proof. Let S satisfy the conditions of the theorem and let a' and a^* be two true inverses of an element a. Then $a^* = a^*aa^* = C(a)a^* = a'aa^* = a'C(a^*)$. That is $a^* \leq a'$ and by symmetry, $a' \leq a^*$. Therefore it follows that $a' = a^*$ and every element a in S has a unique true inverse a^{-1}. Define $C_l(a) = aa^{-1}$, $C_r(a) = a^{-1}a$; then $C_l(a) = C(a^{-1})$ and $C_r(a) = C(a)$. Then C_r is a strong right closure on S. Furthermore, for $a \in S$, $C_l(a)a = a^{-1}a = aC(a) = a$, and $C_l(a)C_l(b) = C_l(b)C_l(a)$, so C_l is a strong left closure on S by the left dual of by Proposition 2.4. Hence by Theorem 2.12, S is an inverse RC-semigroup and the unique true inverse a^{-1} of a is the unique inverse of a. ■

Two of the many well-known generalisations of inverse semigroups within the class of regular semigroups are the classes of orthodox semigroups (regular semigroups whose idempotents form a subsemigroup) and locally inverse semigroups (regular semigroups S with the property that for any idempotent $e \in S$, the subsemigroup eSe is inverse). As noted above a left zero semigroup S always admits a strong right closure: for some fixed $a \in S$, let $C(b) = a$ for all $b \in S$. On the other hand, a non-trivial right zero semigroup (that is, a semigroup satisfying the identity $xy = y$) never admits a strong right closure or even a right closure: if C were a right closure on such a semigroup, then $a = aC(a) = C(a)$ and therefore for all a, b, $a = C(a) = C(b)C(a) = C(a)C(b) = C(b) = b$. Left and right zero semigroups are both orthodox and locally inverse, so neither of these classes is contained in the class of semigroups on which a strong right closure can be defined.

Conversely, the class of semigroups admitting a strong right closure is not a subset of either of these two generalisations of inverse semigroups. Let G be a group, I and Λ non-empty sets and $P = (p_{\lambda i})$ a $\Lambda \times I$ matrix over $G \cup \{0\}$ for which every row and every column contains at least one non-zero entry. We define a multiplication on the set $(I \times G \times \Lambda) \cup \{0\}$ by letting $(i, g, \lambda)(j, h, \nu) = (i, gp_{\lambda j}h, \nu)$ if $p_{\lambda j}$ is not zero, with all other products zero. Semigroups constructed in this way are called Rees matrix semigroups with 0 and are denoted by $\mathcal{M}^0[G, I, \Lambda, P]$ (see [13] for further details). If we exclude the element 0 from this definition then the resulting semigroups are called simply Rees matrix semigroups and are denoted by $\mathcal{M}[G, I, \Lambda, P]$ for suitable matrices P.

Example 2.14. The semigroup M^1 given by adjoining an identity element to the Rees matrix semigroup $\mathcal{M}^0[\{e\}, I, \Lambda, P]$ where $\{e\}$ is the trivial group, I and Λ are three and two element sets respectively and P is the matrix $\begin{pmatrix} e & 0 & e \\ 0 & e & e \end{pmatrix}$ admits a strong right closure but is neither locally inverse nor orthodox.

The right closure can be defined on M^1 as follows: $C((i, e, \lambda)) = (\lambda, e, \lambda)$, $C(0) = 0$ and $C(1) = 1$. It is not hard to verify that S is a strong RC-semigroup. However M^1 cannot be an orthodox semigroup since $(1, e, 1)$ and $(3, e, 2)$ are idempotent but $(1, e, 1)(3, e, 2) = (1, e, 2)$ is not. It cannot be locally inverse either since clearly a monoid is locally inverse if and only if it is inverse.

The definitions of orthodox and locally inverse semigroups do not have the one-sided nature of the strong RC-semigroup definition. However there are generalisations of inverse semigroups which do: a regular semigroup S is said to be R-unipotent if $efe = ef$ for all idempotents $e, f \in S$ (or equivalently if every $\mathcal{R}$-class of S contains exactly one idempotent).[1.] Example 2.14 is not

R-unipotent since $(1,e,1)$ and $(3,e,2)$ are idempotent but $(1,e,1)(3,e,2) = (1,e,2) \neq 0 = (1,e,1)(3,e,2)(1,e,1)$. Furthermore we have

Example 2.15. The four element band (that is, idempotent semigroup) defined by the Cayley table:

$\cdot$	1	2	3	4
1	1	1	1	1
2	2	2	2	2
3	1	1	3	1
4	2	2	2	4

is R-unipotent but does not admit a right closure.

The first claim follows since the product of any two distinct idempotents e and f is a left zero element and so $efe = ef$. It has no right closure since the property $aC(a) = a$ implies that $C(3) = 3$ and $C(4) = 4$ whence $C(3)C(4) \neq C(4)C(3)$, contradicting the commutativity of closed elements.

3. Twisted C-semigroups

We have seen that RC-semigroups can be viewed as generalisations of inverse semigroups. In this section we refine our RC-semigroup definition to a smaller class, still containing all inverse semigroups with the standard right closure but also containing many non-regular members. Several important tools associated with inverse semigroup theory extend to this class. While there are many well known generalisations of inverse semigroups within the class of regular semigroups (we discussed some in the previous section), there have been noticeably fewer attempts to generalise inverse semigroup theory outside of regular semigroups (some examples are the adequate semigroups of Fountain [7] as well as the examples mentioned above: the semiabundant semigroups of Lawson [14]; the right type A semigroups of [9]; and the SL γ-semigroups of [1]).

Let S be an RC-semigroup and suppose $a,b \in S$ are such that $C(b)a = ae$ for some $e \in C(S)$. Thus $bae = bC(b)a = ba$, so $e \geq C(ba)$, although in general $C(b)a \neq aC(ba)$.

Definition 3.1. *If $C(b)a = aC(ba)$ for all elements a,b in an RC-semigroup S then we say S is a* twisted *RC-semigroup.*

Thus the class of twisted RC-semigroups is a variety of RC-semigroups, contained within the variety of translucent RC-semigroups (since every element of a twisted RC-semigroup is plainly translucent).

Recall that an RC-semigroup S is exactly a $C(S)$-semiadequate semigroup.

Proposition 3.2. *A twisted RC-semigroup S is exactly a right $C(S)$-semiadequate semigroup, satisfying the right congruence condition and the right ample condition.*

Proof. Let S be right $C(S)$-semiadequate with $\widetilde{\mathcal{L}}$ a right congruence and the right ample condition holding. Let $a, b \in S$. Then $C(b)a = aC(C(b)a)$ from the right ample condition. By the right congruence condition, $C(C(b)a) = C(ba)$, so that $C(b)a = aC(ba)$ and S is twisted.

Conversely, suppose S is twisted. Let $a, b, c \in S$ with $C(a) = C(b)$. Then by the twisted condition,

$$C(ac) = C(aC(a)c) = C(aC(b)c) = C(acC(bc)) = C(acC(bc))C(bc).$$

Thus $C(ac) \leq C(bc)$, and similarly we obtain the reverse inequality, so that $C(ac) = C(bc)$, so that the right congruence condition holds. Now for any $a \in S$ and $e \in C(S)$, $ea = C(e)a = aC(ea)$ so that the right ample condition holds. ∎

Hence from Proposition 1.4, we have

Corollary 3.3. *The variety of twisted RC-semigroups is defined within the variety of RC-semigroups by the equations $\{C(x)y = yC(C(x)y), C(xy) = C(C(x)y)\}$.*

Every monoid M is a twisted RC-semigroup if one defines $C(a) = 1$ for all $a \in M$. Indeed we can characterise monoids in terms of twisted RC-semigroups.

Proposition 3.4. *Suppose S is an RC-semigroup. Then S is a twisted RC-semigroup with $|C(S)| = 1$ if and only if S is a monoid with $C(a) = 1$ for all a.*

Proof. If $C(S) = \{e\}$ then $ea = ae = aC(a) = a$ for all $a \in S$. The other direction is trivial. ∎

We have the following immediate

Corollary 3.5. *The class of monoids (viewed as RC-semigroups by setting $C(a) = 1$ for all a) is a subvariety of the variety of twisted RC-semigroups, specified by the equation $C(x) = C(y)$.*

Standard RC-semigroup structures may be characterised amongst twisted RC-semigroups by the strong property.

Theorem 3.6. *An RC-semigroup is the standard RC-semigroup structure on an inverse semigroup if and only if it is strong and twisted.*

Proof. Suppose S is an inverse semigroup, C the standard right closure on S. Then for all $a, b \in S$,

$$bC(ab) = b(ab)^{-1}(ab) = bb^{-1}a^{-1}ab = a^{-1}abb^{-1}b = a^{-1}ab = C(a)b,$$

so S is twisted; moreover S is strong.

Conversely, assume that the RC-semigroup S is strong and twisted. We show that the right closure on S is full. Let e be an idempotent. Since S is strong, there exists e' such that $e'e = C(e)$. Therefore

$$e = eC(e) = eC(ee) = C(e)e = e'ee = e'e = C(e).$$

By Proposition 2.6, S is an inverse RC-semigroup with the standard right closure. ∎

Clearly the set $C(S)$ of closed elements of an RC-semigroup is a subsemilattice of S which is an inverse RC-semigroup, and by the comments preceding Proposition 2.5, it has the standard right closure. In a twisted RC-semigroup this idea can be extended further.

If S is twisted, let $I(S) = \{s \in S \mid s \text{ has a true inverse in } S\}$.

Theorem 3.7. *If S is a twisted RC-semigroup then $I(S)$ is the maximum inverse RC-subsemigroup of S with the standard right closure.*

Proof. Clearly for all $s \in S$, $C(s) \in I(S)$ since $C(s)C(s) = C(s) = C(C(s))$ and therefore $C(S) \subseteq I(S)$. Also if $s \in I(S)$ then every true inverse of s is contained in $I(S)$. Now if $s, t \in I(S)$ have true inverses s' and t' respectively then

$$t's'st = t'C(s)t = t'tC(st) = C(t)C(st) = C(st)$$

and likewise

$$stt's' = sC(t')s' = ss'C(t's') = C(s')C(t's') = C(t's'),$$

so $I(S)$ is an RC-subsemigroup. It is inverse with the standard right closure by Corollary 2.13.

That $I(S)$ is the maximum inverse RC-subsemigroup of S with the standard right closure follows immediately from the fact that every element in an inverse RC-semigroup with the standard right closure has a true inverse. ∎

An immediate corollary is that the twisted RC-semigroup S is inverse with the standard right closure if and only if $I(S) = S$.

As a special case, if S is a monoid equipped with the right closure C given by $C(a) = 1$ for all $a \in S$ (a subvariety of the variety of twisted RC-semigroups given by the identity $xC(y) = C(y)x = x$), then $I(S)$ is the group of all invertible elements in S.

This idea can be extended to a more general class using the following lemma.

Lemma 3.8. *If S is an RC-semigroup satisfying the right congruence condition, then $W(S) = \{x \in S \mid C(a)x = xC(C(a)x) \text{ for all } C(a) \in C(S)\}$ is the maximum twisted RC-subsemigroup of S containing $C(S)$.*

Proof. First, any RC-subsemigroup W containing $C(S)$ but which is not contained in $W(S)$ cannot be a twisted RC-subsemigroup since the identity $C(a)x = xC(C(a)x)$ must fail on it. This means $W(S)$ contains all twisted RC-subsemigroups of S that contain $C(S)$. Now we show that $W(S)$ is an RC-semigroup.

Consider a, $b \in W(S)$. Now for all $x \in S$, we have from Proposition 1.4 that $C(x)ab = aC(C(x)a)b = aC(xa)b = abC(C(xa)b) = abC(xab) = abC(C(x)ab)$; that is, $ab \in W(S)$. Finally, clearly $C(S) \subseteq W(S)$ so $W(S)$ is closed under taking right closures. ∎

For example, if we take $C(x)$ to be x^* (in the notation of [7]) the right adequate semigroups of [7] are full RC-semigroups satisfying the right congruence condition ([7], Proposition 1.6). Thus every right adequate semigroup contains a maximal right type A subsemigroup which in turn contains a maximal inverse subsemigroup, all sharing the same idempotents.

A further useful property of twisted RC-semigroups is that they admit a simple representation that is a natural variant of the Vagner-Preston representation of inverse semigroups. Let P_X be the semigroup under composition of all partial maps $X \to X$, one-to-one or not, with $C(f)$ defined to be the restriction of the identity map to the domain of $f \in P_X$. This is a RC-subsemigroup of B_X as in Example 1.7 in the case where the closure operator on X is the identity. Moreover, restricting the range of g to $\mathrm{dom}(f)$ gives the same as restricting the domain of g to $\mathrm{dom}(fg)$. Thus P_X is twisted: $C(f)g = gC(fg)$ for all $f, g \in P_X$. As an example of Theorem 3.7, note that the maximal inverse RC-subsemigroup with the standard right closure is I_X, as is easily seen.

The twisted RC-semigroup P_X turns out to be generic.

Theorem 3.9. *Every twisted RC-semigroup S is embeddable as a RC-subsemigroup of P_S with S given the identity closure operator. Moreover $I(S)$ consists of exactly those elements corresponding to maps with inverses contained in this embedding.*

Proof. The approach is a generalisation of the Vagner-Preston representation of inverse semigroups.

With any $a \in S$ associate the partial mapping $\psi_a : C(a)S \to aS$, defined by setting $\psi_a(x) = ax$ for all $x \in C(a)S = \{c \in S \mid C(a)c = c\}$. Now define $\theta : S \to P_S$ by setting $\theta(a) = \psi_a$ for all $a \in S$. We show θ isan RC-semigroup embedding.

Clearly $\psi_{C(a)}$ is the identity map on its domain, and $\mathrm{dom}(\psi_a) = \mathrm{dom}(\psi_{C(a)})$, so $C(\psi_a) = \psi_{C(a)}$, that is, $\theta(C(a)) = C(\theta(a))$. Thus θ respects C.

For $a, b \in S$, if $(\psi_a \circ \psi_b)(x)$ and $\psi_{ab}(x)$ are both defined for some $x \in S$, then each is equal to abx. We need to show that one is defined if and only if the other is. Now $\psi_{ab}(x)$ exists if and only if $x = C(ab)x$, while $(\psi_a \circ \psi_b)(x)$ exists if and only if $x = C(b)x$ and $bx = C(a)bx$.

Suppose $x = C(ab)x$. Then

$$C(b)x = C(b)C(ab)x = C(ab)x = x,\ bx = bC(ab)x = C(a)bx.$$

Thus $\mathrm{dom}(\psi_{ab}) \subseteq \mathrm{dom}(\psi_a \circ \psi_b)$.

Conversely, suppose $x = C(b)x$ and $bx = C(a)bx$. Then since S is twisted, $x = xC(bx)$ and $bx = bxC(abx)$. Hence by Proposition 1.2, $C(bx) \geq C(x)$ and $C(abx) \geq C(bx)$, so $C(abx) \geq C(x)$ and so $x = xC(abx)$, that is, $x = C(ab)x$. Hence $\mathrm{dom}(\psi_a \circ \psi_b) \subseteq \mathrm{dom}(\psi_{ab})$, and so the two sets are equal.

Thus $\theta(ab) = \theta(a)\theta(b)$, so θ respects multiplication. All that remains is to show that θ is one-to-one.

Suppose $\psi_a = \psi_b$. Then $\mathrm{dom}(\psi_a) = \mathrm{dom}(\psi_b)$. Obviously $C(a) \in \mathrm{dom}(\psi_a)$, so $C(a) \in \mathrm{dom}(\psi_b)$, and so $C(a) = C(b)C(a)$ so $C(a) \leq C(b)$. Similarly we obtain the opposite inequality, so $C(a) = C(b)$. Hence $a = aC(a) = \psi_a(C(a)) = \psi_b(C(a)) = bC(a) = bC(b) = b$ as required.

Finally, if $a \in I(S)$ then for all $C(a)x \in C(a)S$, $\psi_{a^{-1}}(\psi_a(C(a)x)) = a^{-1}aC(a)x = a^{-1}ax = C(a)x$, and similarly $\psi_a(\psi_{a^{-1}}(C(a^{-1})x)) = C(a^{-1})x$ for all $C(a^{-1})x \in C(a^{-1})S$, so ψ_a has inverse $\psi_{a^{-1}}$. Conversely, if ψ_a has inverse ψ_b, then $\psi_b(\psi_a(C(a)x)) = C(a)x$, so $bax = C(a)x$ for all x, and letting $x = C(a)$, we have $ba = C(a)$; similarly, $ab = C(b)$, and it follows that b must be the unique true inverse a^{-1} of a. Thus $a \in I(S)$ if and only if ψ_a has an inverse map of the same form. ∎

The case in which $I(S) = S$ gives the usual Vagner-Preston representation for inverse semigroups. On the other hand, the case in which $|C(S)| = 1$ leads to a representation in which every ψ_a is a full mapping, and the usual left regular representation of a monoid is recovered. In general, the bigger $C(S)$ is, the smaller and more varied the domains can be.

It is obvious that not all twisted RC-semigroups are central (consider for example any inverse semigroup not satisfying $xx^{-1}x = x^{-1}xx$). To show the converse, let $S = \mathbf{Z}_4$ under multiplication, with C defined by setting $C(0) = 0$

and $C(a) = 1$ if $a \neq 0$; then $C(2)2 = 1 \times 2 = 2 \neq 0 = 2C(0) = 2C(2 \times 2)$, so S is not twisted although it is obviously central.

The following result leads to a representation theorem along the lines of Theorem 3.9 for twisted RC-semigroups which are also central.

Proposition 3.10. *Consider the following three RC-semigroup equations.*

1. $C(x)C(y) = C(xy)$.
2. $C(x)y = yC(xy)$ *(twisted property).*
3. $C(x)y = yC(x)$ *(central property).*

If any two hold on an RC-semigroup S, so does the third.

Proof. If the first two equations hold, then $C(x)y = yC(xy) = yC(x)C(y) = yC(x)$, so the third holds. If the second and third hold, then by Corollary 3.3,

$$C(x)C(y) = C(y)C(x) = C(x)C(yC(x)) = C(yC(x)) = C(C(x)y) = C(xy),$$

and so the first holds. Finally, if the first and third equations hold, then $yC(xy) = yC(x)C(y) = yC(x) = C(x)y$ and the second holds. ∎

Let X be a set equipped with an equivalence relation ρ. Let P_X^ρ be all partial maps on X that are defined on entire ρ-classes and are such that $a \,\rho\, f(a)$ whenever $f(a)$ is defined. (Thus "modulo ρ" these maps are just restrictions of the identity, and if ρ is the universal relation then P_X^ρ is all maps on X.) For any $f \in P_X^\rho$, define $C(f)$ to be the identity map restricted to the domain of f; clearly $C(f) \in P_X^\rho$. Now P_X^ρ is an RC-subsemigroup of the twisted RC-semigroup P_X of all partial maps on X, and hence is itself twisted; moreover it is central as is easily seen.

Conversely, we have

Corollary 3.11. *Every twisted central RC-semigroup S is embeddable as an RC-subsemigroup of P_S^ρ, where $\rho = \widetilde{\mathcal{L}}$.*

Proof. Using the same mapping θ as in Theorem 3.9, we need only show that S maps into P_S^ρ. Suppose $a \in S$. Assume $\psi_a(b)$ exists for some $b \in S$. Then $b = C(a)b$ and so, using the previous proposition, $C(\psi_a(b)) = C(ab) = C(a)C(b) = C(C(a)b) = C(b)$ so $b \,\rho\, \psi_a(b)$. Moreover if $b \,\rho\, c$ then $C(b) = C(c)$, and so, using the just established fact that $C(b) = C(a)C(b)$, it follows that $c = cC(c) = C(c)c = C(b)c = C(a)C(b)c = C(a)C(c)c = C(a)c$, so $\psi_a(c)$ exists also. Hence $\psi_a \in P_S^\rho$. ∎

The case in which S is a monoid with $C(x) = 1$ is central as well as twisted, and corresponds to the case in which ρ is the universal relation. At the other extreme, in which ρ is equality, $C(x) = x$ holds, and S is (isomorphic to) a semigroup of restrictions of the identity, or equivalently a semilattice of sets under intersection.

4. Congruences on C-semigroups

In the case of groups it is well known that every congruence is determined by a corresponding normal subgroup, that is, the set of elements congruent to the identity element. This situation generalises to inverse semigroups where every congruence is determined by a corresponding congruence on the idempotents and the set of elements congruent to an idempotent (see [13] for example). A description in this style does not appear possible for RC-semigroups in general—for example it is easily verified that any semigroup congruence θ on a semigroup S can be extended with no significant changes to an RC-semigroup congruence on the (twisted and central) RC-semigroup obtained from S by adjoining a new identity element 1 and letting $C(x) = 1$ for all $x \in S \cup \{1\}$. Thus an RC-semigroup congruence on $S \cup \{1\}$ is not determined at all by the set of elements of $S \cup \{1\}$ which are congruent to an element in $C(S \cup \{1\})$. In this section we show that despite this, two particularly important kinds of inverse semigroup congruences carry over to twisted RC-semigroups. On the other hand we show that the class of finite congruence free RC-semigroups is likely to be significantly more complicated than for the corresponding class of semigroups.

Following standard notation for similar inverse semigroup concepts (see [13] for example), we say that if S is a twisted RC-semigroup and ρ is a congruence on $C(S)$ then ρ is a *normal congruence* on $C(S)$ if $C(a)\,\rho\,C(b) \Rightarrow C(ax)\,\rho\,C(bx)$ for all $x \in S$. The *trace* of a congruence θ on a RC-semigroup S is the restriction of θ to $C(S)$, a congruence on $C(S)$.

Note that if S is a twisted RC-semigroup and θ is a RC-semigroup congruence on S then the trace of θ is a normal congruence on $C(S)$, since if $C(a)\,\theta\,C(b)$ then for all x, $C(a)x\,\theta\,C(b)x$ and so $C(ax) = C(C(a)x)\,\theta\,C(C(b)x) = C(bx)$. The following two propositions provide converses to this fact and generalise two well known and powerful descriptions from inverse semigroup theory.

In the RC-semigroup S, define

$$\begin{aligned}\rho_{\min} = \{(a,b) \mid{}& C(a)\,\rho\,C(b) \\ &\text{and there exists } e \in C(S) \cap C(a)/\rho \text{ such that } ae = be\}.\end{aligned}$$

In [9], Fountain defines this relation for right type A semigroups.

Proposition 4.1. *Let S be a twisted RC-semigroup and let ρ be a normal congruence on $C(S)$. Then $\rho_{\min}$ is the minimum congruence on S with trace ρ.*

Proof. First, $\rho_{\min}$ is clearly symmetric and reflexive. Now suppose (a,b), $(b,d) \in \rho_{\min}$. Then $C(a)\,\rho\, C(b)\,\rho\, C(d)$ and there are $e \in C(S) \cap C(a)/\rho$ and $f \in C(S) \cap C(b)/\rho$ for which $ae = be$ and $bf = df$. Therefore $aef = bef = bfe = dfe = def$ and since $e\,\rho\, C(a)\,\rho\, C(b)\,\rho\, f$ the relation $\rho_{\min}$ is transitive too. Hence $\rho_{\min}$ is an equivalence relation.

Now we show that $\rho_{\min}$ is a congruence. By definition if $(a,b) \in \rho_{\min}$ then there is $e \in C(S) \cap C(a)/\rho$ such that $ae = be$. Thus $C(a)e = C(C(a)e) = C(ae) = C(be) = C(C(b)e) = C(b)e$ by Corollary 3.3, so $C(a)\,\rho_{\min}\, C(b)$ and $\rho_{\min}$ respects C.

Now suppose $(a,b) \in \rho_{\min}$, and let $e \in C(S)$ be such that $e \in C(a)/\rho$ and $ae = be$. Now by the normality of ρ, $C(ex)\,\rho\, C(C(a)x) = C(ax)$ for all $x \in S$, by Corollary 3.3, so $C(ex) \in C(ax)/\rho$. However we also have that $axC(ex) = aC(e)x = aex = bex = bC(e)x = bxC(ex)$. Therefore $(ax, bx) \in \rho_{\min}$. Furthermore, for all $x \in S$, $C(xa) = C(xa)C(a)\,\rho\, C(xa)e = eC(xae)$, and similarly $C(xb)\,\rho\, eC(xbe)$, which equals $eC(xae)$ since $ae = be$, so $C(xa)\,\rho\, C(xb)$. Also, $xaeC(xa) = xbeC(xa)$ where $eC(xa)\,\rho\, C(a)C(xa) = C(xa)C(a) = C(xa)$, so $(xa, xb) \in \rho_{\min}$. Hence $\rho_{\min}$ is a congruence.

Next, we show that the trace of $\rho_{\min}$ is ρ. If $C(a)\,\rho\, C(b)$ then

$$C(a)[C(a)C(b)] = C(a)C(b) = C(a)C(b)C(b) = C(b)[C(a)C(b)]$$

so that $C(a)\,\rho_{\min}\, C(b)$. Conversely if $C(a)\,\rho_{\min}\, C(b)$ then by definition $C(a)\,\rho\, C(b)$ as required.

To complete the proof it remains to show that $\rho_{\min}$ is the minimal congruence with trace ρ. Let θ be a congruence with trace ρ and let $a\,\rho_{\min}\, b$. Then $C(a)\,\rho_{\min}\, C(b)$, and since $\rho_{\min}$ and θ share the same trace, $C(a)\,\theta\, C(b)$. Also there is an element $e \in C(a)/\rho$ for which $ae = be$, so $a = aC(a)\,\theta\, ae = be\,\theta\, bC(b) = b$, that is, $a\,\theta\, b$. ∎

Likewise we have the following result (which does not depend on the right ample condition).

Proposition 4.2. *If S is an RC-semigroup satisfying the right congruence condition and ρ is a normal congruence on $C(S)$ then*

$$\rho_{\max} = \{(a,b) \mid C(a)\,\rho\, C(b) \text{ and } C(ea)\,\rho\, C(eb) \text{ for all } e \in C(S)\}$$

is the maximum congruence on S with trace ρ.

Proof. That $\rho_{\max}$ is an equivalence relation respecting the right closure follows immediately from its definition.

Now suppose $(a,b) \in \rho_{\max}$. Since $C(ea)\,\rho\, C(eb)$ for all $e \in C(S)$, by the normality of ρ we have that for every $x \in S$, $C(eax) = C(ebx)$, so

$(ax, bx) \in \rho_{\max}$ if $C(ax)\,\rho\, C(bx)$, which is immediate from the normality of ρ. Also by the right congruence condition, we have that for all $e \in C(S)$ and $x \in S$, $C(exa) = C(C(ex)a)\,\rho\, C(C(ex)b) = C(exb)$ as required, so $(xa, xb) \in \rho_{\max}$ follows if $C(xa)\,\rho\, C(xb)$. Now $C(xa) = C(C(x)a)\,\rho\, C(C(x)b) = C(xb)$ since $a\,\rho_{\max}\, b$. Therefore $\rho_{\max}$ is a congruence.

Now we must show that the trace of $\rho_{\max}$ is ρ. If $C(a)\,\rho\, C(b)$ then for all $e \in C(S)$, $eC(a)\,\rho\, eC(b)$ since ρ is a congruence on $C(S)$, so certainly $C(a)\,\rho_{\max}\, C(b)$. Conversely by the first condition in the definition of $\rho_{\max}$, it follows that the trace of $\rho_{\max}$ is contained in ρ, so the trace of $\rho_{\max}$ is ρ.

Finally we show that if θ is a congruence with trace equal to ρ then $a\,\theta\, b$ implies $a\,\rho_{\max}\, b$. Now $a\,\theta\, b$ implies that $C(a)\,\theta\, C(b)$ or equivalently that $C(a)\,\rho\, C(b)$. Furthermore for every $e \in C(S)$, $ea\,\theta\, eb$ and therefore $C(ea)\,\rho\, C(eb)$, and so $a\,\rho_{\max}\, b$. Thus $\theta \subseteq \rho_{\max}$. ∎

By the definitions of $\rho_{\min}$ and $\rho_{\max}$ we also have the following.

Corollary 4.3. *If T is a subsemigroup of a twisted RC-semigroup S with $C(S) = C(T)$ and ρ is a normal congruence on $C(S)$ then the restrictions of $\rho_{\min}$ and $\rho_{\max}$ to T are the minimum and maximum congruences respectively on T with trace ρ.*

Two of the most important congruences on $C(S)$, the universal (∇) and diagonal (Δ) relations, are obviously normal congruences on $C(S)$. The corresponding congruences on the twisted RC-semigroup S obtained from the above two propositions are

$$\sigma = \nabla_{\min} = \{(a, b) \mid ae = be \text{ for some } e \in C(S)\}$$

and

$$\mu = \Delta_{\max} = \{(a, b) \mid C(a) = C(b) \text{ and for all } x, C(xa) = C(xb)\},$$

giving the minimum congruence σ for which S/σ lies in the RC-semigroup variety given by $C(x) = C(y)$ (in the case of inverse semigroups with the standard right closure, this is the minimum group congruence) and the maximum congruence, μ, separating $C(S)$. The existence of the congruence μ generalises a corresponding result of [7] (see Proposition 2.3 of that paper) relating to right adequate semigroups (which necessarily satisfy the right congruence condition).

For example, Corollary 4.3 implies the following

Corollary 4.4. *If μ is trivial on the twisted RC-semigroup S, then μ is trivial on the inverse semigroup $I(S)$.*

An inverse RC-semigroup S with the standard right closure for which μ is trivial is called *fundamental*. A famous result due to Munn [16] shows that an inverse semigroup with idempotents E is fundamental if and only if it is isomorphic to a subsemigroup of the so-called Munn semigroup $\mathcal{T}_E$ for E (for details see [13]). The existence of such an inverse semigroup establishes an upper bound on the cardinality of a fundamental inverse semigroup with idempotents E: it follows using the description of the Munn semigroup in [13] for example that $\mathcal{T}_E$ is a subinverse semigroup of I_E, the inverse semigroup of all partial one to one functions on the set E. We now show a similar bound exists for the more general class of twisted RC-semigroups.

Let a twisted RC-semigroup S be called *fundamental* if μ is trivial.

Proposition 4.5. *If S is a fundamental twisted RC-semigroup then $|S| \leq |C(S)|\ 2^{|C(S)|}$.*

Proof. We first show that there is a semigroup homomorphism from S into the semigroup $T_{C(S)}$ of all transformations of the set $C(S)$ and then obtain a bound on the cardinality of the corresponding congruence classes of S. (Note that $T_{C(S)}$ is different from the Munn semigroup $\mathcal{T}_{C(S)}$ of the semilattice $C(S)$.)

For $s \in S$, let $\alpha_s : E \to E$ be the map defined by $x\alpha_s = C(xs)$. Therefore for $s, t \in S$ we have $x\alpha_s\alpha_t = C(xs)\alpha_t = C(C(xs)t) = C(xst) = x\alpha_{st}$. Then the map $\iota : S \to T_E$ defined by $\iota(s) = \alpha_s$ is a semigroup homomorphism since for $s, t \in S$, $\iota(s)\iota(t) = \alpha_s\alpha_t = \alpha_{st} = \iota(st)$. Note that this homomorphism is essentially the same as that used in the proof of Lemma 2.1 of [7] for right type A semigroups.

For $s, t \in S$, if $\iota(s) = \iota(t)$, then for all $x \in C(S)$ it must be the case that $C(xs) = C(xt)$. It follows from the above description of the congruence μ and the assumption that S is fundamental that $s \neq t$ implies that $C(s) \neq C(t)$. Now for any element $s \in S$ there are at most $|C(S)|$ choices for the value of $C(S)$ and therefore there is a bound of $|C(S)|$ on the cardinality of each congruence class of S with respect to the congruence $ker(\iota)$ associated with ι. Since $|T_{C(S)}| = 2^{|C(S)|}$ the result follows. ■

Next we obtain further characterisations of central twisted RC-semigroups, generalising a well known result of Howie concerning inverse semigroups with central idempotents [12] and of Fountain concerning right type A semigroups with central idempotents [7].

Corollary 4.6. *Let S be a twisted RC-semigroup. The following are equivalent.*

1. *S is central;*
2. *$S/\mu \cong C(S)$;*
3. *for every $b \in S$, $\widetilde{\mathcal{L}}_b$ is a subsemigroup of S.*

Proof. If a twisted RC-semigroup S is central then for all $x \in S$, $C(xa) = C(x)C(a) = C(C(x)C(a)) = C(xC(a))$ by Proposition 3.10, and since $C(a) = C(C(a))$, we have that $a\mu C(a)$. Since μ is $C(S)$-separating, it follows that $S/\mu \cong C(S)$. Conversely if $S/\mu \cong C(S)$ then because $C(x)\mu C(y)$ only if $C(x) = C(y)$ and $C(a)$ is the minimum element in $C(S)$ for which $aC(a) = a$, we must have $a\mu C(a)$ for all $a \in S$, and therefore $C(xa) = C(xC(a)) = C(xC(a))C(a) = C(a)C(xC(a)) = C(x)C(a)$ for every $x \in S$. It now follows from Proposition 3.10 that S is central.

Equivalence of the second and third conditions follows immediately since every element in $C(S)$ is idempotent and a congruence class is idempotent if and only if its elements form a subsemigroup. Clearly such subsemigroups are also RC-subsemigroups satisfying $C(x) = C(y)$. ■

Of course μ here is just $\widetilde{\mathcal{L}}$ as in Corollary 3.11.

We now examine the class of finite congruence free RC-semigroups. Aside from the finite simple groups, there are in fact relatively few different forms of finite congruence free semigroups. There are, up to isomorphism, four non-group two element semigroups, each of which is a congruence free semigroup. All other finite congruence free semigroups are isomorphic to a Rees matrix semigroup with zero of the form $\mathcal{M}^0[\{e\}, I, \Lambda, P]$ where P is a matrix whose entries are either 0 or e and has the property that no two columns and no two rows are identical (see [13] for example). Given this description it is easy to describe which finite congruence free semigroups are congruence free RC-semigroups.

Proposition 4.7. *If S is a finite congruence free semigroup and also a congruence free RC-semigroup, then S is isomorphic to either a finite simple group, a two element semilattice with closure $C(x) = 1$, a two element semilattice with closure $x = C(x)$, a two element left zero semigroup or for some n and $m \geq n$ a Rees matrix semigroup over the trivial group with an $n \times m$ sandwich matrix consisting of an $n \times n$ identity submatrix and then the remaining columns such that no two columns are identical and each has at least two non-zero entries. All these semigroups admit strong right closures.*

Proof. Since the two element null semigroup and the two element right zero semigroup do not admit right closures, these cannot be congruence free RC-semigroups. On the other hand, all simple groups and all other two element semigroups admit right closures as described above and so must be congruence free as RC-semigroups. Now consider a congruence free Rees matrix semigroup M with zero that admits a right closure and let a be a non-zero element of M. Assume further that the sandwich matrix P of M is an $n \times m$ matrix, that is that M contains n non-zero $\mathcal{L}$-classes and m non-zero $\mathcal{R}$-classes. Now since $aC(a) = a$, the element $C(a)$ cannot be 0. It now follows from the structure of a Rees matrix semigroup with zero and from the fact that $aC(a) = a$ that

$a\,\mathcal{L}\,C(a)$ (which implies that the right closure on M is strong). It is clear that two distinct $\mathcal{D}$-related idempotents commute in a Rees matrix semigroup if and only if their product is zero. Since every $\mathcal{L}$-class contains a closed element and closed elements commute, it follows that there is an $n \times n$ submatrix of P in which every row and every column contains exactly one non-zero element. Up to isomorphism, rearranging the columns (or the rows) of the matrix P leaves the original Rees matrix semigroup with zero unchanged and so we may assume that there is a submatrix of P that is an $n \times n$ identity matrix. Since every row and every column of P must be different, M is of the described form.

Conversely, let M be a Rees matrix semigroup of the described form and Q an $n \times n$ identity submatrix of the $n \times m$ matrix P. It is then easily verified that the unary operation defined by $C(0) = 0$ and $C((i,e,j)) = (j,e,j)$ is a (strong) right closure as required. ∎

The path from congruence free semigroups to congruence free RC-semigroups is simplified by the fact that an RC-semigroup congruence on an RC-semigroup defines a semigroup congruence on the underlying semigroup. However the converse is certainly not true and in general the situation for finite congruence free RC-semigroups appears to be much more complicated than the corresponding semigroup case. In particular while every congruence free semigroup consists of just one non-zero $\mathcal{J}$-class, for any n there exists a (finite) congruence free RC-semigroup with n distinct $\mathcal{J}$-classes. We describe such an example for any even natural number n but it is obvious that an analogous construction exists for odd numbers as well.

Proposition 4.8. *Let n be an even natural number and let D be the set*

$$\{(i,j)_k \mid i,j \in \{1,2\},\ 1 \le k \le n\} \cup \{0,1\}$$

with the multiplication $0a = a0 = 0$, $1a = a1 = a$ for every a, and

$$(i,j)_k(i',j')_{k'} = \begin{cases} (i,j')_{\min(k,k')}, & \text{if } j = i', \\ 0, & \text{otherwise.} \end{cases}$$

Define a right closure on D by the following rules: $C(0) = 0$, $C(1) = 1$,

$$C((j,1)_i) = \begin{cases} (1,1)_{i+1}, & \text{if } i \text{ is even and } i < n, \\ (1,1)_i, & \text{if } i \text{ is odd}, \\ 1, & \text{if } i = n, \end{cases}$$

and

$$C((j,2)_i) = \begin{cases} (2,2)_i, & \text{if } i \text{ is even}, \\ (2,2)_{i+1}, & \text{if } i \text{ is odd}. \end{cases}$$

Then D is a finite congruence free RC-semigroup with a chain of $n+2$ distinct $\mathcal{J}$-classes.

Proof. It is convenient to first introduce some notation. Let the *index* of an element $(i,j)_k$ be the number k and let the indices of 0 and 1 be 0 and $n+1$ respectively. Let the *left* (or *right*) *coefficient* of $(i,j)_k$ be the number i (or j, respectively).

First, we show D is a semigroup. It is clear that $(i_1,j_1)_{k_1}[(i_2,j_2)_{k_2}(i_3,j_3)_{k_3}] = 0$ if and only if $[(i_1,j_1)_{k_1}(i_2,j_2)_{k_2}](i_3,j_3)_{k_3} = 0$. Consider the case when these products are non-zero. Then $j_1 = i_2$, $j_2 = i_3$ and both sides equal $(i_1,j_3)_{\min(k_1,k_2,k_3)}$, so the multiplication is associative and D is a semigroup.

Secondly, we show that D is an RC-semigroup. The closed elements are easily seen to form a subsemilattice of D. Furthermore for any element x, the element $C(x)$ is the smallest element of $C(D)$ for which $xC(x) = x$ and so by Proposition 1.3, D is an RC-semigroup.

Thirdly, we show that D is a congruence-free RC-semigroup with $n+2$ $\mathcal{J}$-classes. It is easy to verify that the $\mathcal{J}$-classes of D are the sets of equal index and therefore that there are exactly $n+2$ of them. Now consider an RC-semigroup congruence θ on D that is neither the diagonal nor the universal relation. If $(i_1,j_1)_k\,\theta\,(i_2,j_2)_k$ (where $(i_1,j_1) \neq (i_2,j_2)$) with k maximal, then it follows from the semigroup properties of D that $(1,1)_k\,\theta\,(2,2)_k$. However, $(1,1)_k(2,2)_k = 0$ and $(1,1)_k(1,1)_k = (1,1)_k$. Therefore, for any $i,j \in \{1,2\}$, $(i,j)_k\,\theta\,0$. This is not an RC-semigroup congruence since we may assume without serious loss of generality that either $C((1,1)_k) = (1,1)_{k+1}$ or $C((1,1)_k) = 1$ (the remaining case is when $C((1,1)_k) = (1,1)_k$, in which case the same arguments apply except using $(2,2)_k$ instead of $(1,1)_k$). In the first case we then have $(1,1)_{k+1}\,\theta\,0$ and therefore $(i,j)_{k+1}\,\theta\,0$ for all $i,j \in \{1,2\}$, contradicting the maximality of k. In the second case we have that $1\,\theta\,0$ and θ is the universal relation. Thus the congruence classes of θ contain at most one element of each index.

Now let k be the maximum index of any element $(i,j)_k$ in a given non-trivial congruence class of θ. Since $\mathcal{R}$ and $\mathcal{L}$ are left and right congruences respectively it follows that any element $\mathcal{R}$-related to $(i,j)_k$ also lies in a non-trivial congruence class of θ. As above we can assume without loss of generality that $C((1,1)_k)$ has index $k+1$. Then by the above arguments $(1,1)_k\,\theta\,(i,j)_{k'}$ for some $k' < k$. Hence $C((i,j)_{k'})$ has index at most k and therefore the fact that $C((1,1)_k)$ and $C((i,j)_{k'})$ must be θ-related either contradicts the maximality of k or implies that $1\,\theta\,C((i,j)_{k'})$. In this second case, since $C((i,j)_{k'}) \neq 1$, it easily follows that θ is the universal relation, contradicting the choice of θ. Thus no such congruence exists, that is, D is congruence free. ∎

This example depends on the fact that the right closure of an element x is capable of lying in a separate $\mathcal{J}$-class to that of x. In the case of strong RC-semigroups however, $C(x)$ is $\mathcal{L}$-related to x, meaning that the usual notions of semigroup ideal and Rees quotient apply. In particular this means that if a strong RC-semigroup S has a proper nonzero ideal I then the Rees congruence on the multiplicative part of S also respects the closure operation and so S/I is a proper RC-semigroup quotient of S. Thus a strong, congruence free finite RC-semigroup is isomorphic to a Rees matrix semigroup with or without zero element and with a strong right closure operation. As well as more complicated examples, this class includes all the finite simple abelian groups with the identity element as the unique closed element (which are isomorphic as semigroups to Rees matrix semigroups over themselves with a 1×1 sandwich matrix) the two element semilattice with every element closed (which is isomorphic as a semigroup to a Rees matrix semigroup with zero over the trivial group and with 1×1 sandwich matrix) and the two element left zero semigroup with a single closed element. The two element semilattice with only one closed element is not a strong RC-semigroup since a zero element, 0 say, cannot be $\mathcal{L}$-related to any other element, while every element is $\mathcal{L}$-related to its closure in a strong RC-semigroup. Descriptions of the possible semigroup congruences on Rees matrix semigroup (with or without a zero element) are well known (see [13] for example) and it is easily seen that all respect any possible right closure operation. Thus we have the following corollary to Proposition 4.7.

Corollary 4.9. *An RC-semigroup is a strong congruence free RC-semigroup if and only if it is of one of the forms described in Proposition* 4.7 *other than the semilattice with* $C(x) = 1$.

5. C-semigroups and varieties

It is of interest to determine all semigroup varieties consisting of semigroups admitting an RC-semigroup structure.

Theorem 5.1. *Let V be a semigroup variety. The following are equivalent:*

1. *every semigroup in V admits a right closure;*
2. *every semigroup in V admits a strong right closure;*
3. *for some n, V is a subvariety of either the semigroup variety defined by $\{x = x^{n+1}, x^n y^n = y^n x^n\}$ or the semigroup variety defined by $\{x = x^{n+1}, x^n y^n = x^n\}$.*

Proof. Suppose V satisfies (1). If V does not satisfy an identity of the form $x^{n+p} = x^n$ then the V-free semigroup generated by one element must be infinite and so is isomorphic to the natural numbers (without zero) under addition. However since the additive semigroup of natural numbers without zero does not admit a right closure (for example there is no positive number x such that $1 + x = 1$) this contradicts (1). Similarly if the number n in the identity $x^{n+m} = x^n$ cannot be chosen to be 1 then V contains a semigroup S in which

$a \neq a^{m+1}$ for some element $a \in S$. In the subsemigroup A of S generated by a (which is contained in V) there is no element b such that $ab = a$. Thus no closure operation can be defined on A, again contradicting (1). Therefore V satisfies $x = x^{n+1}$ for some n and it follows that every semigroup in V is a union of periodic groups of exponent dividing n, that is, V is a subvariety of the completely regular variety given by $x = x^{n+1}$. By a famous theorem of Clifford, every semigroup in V is a semilattice of completely simple semigroups (see [13] for example).

Now it is evident that the variety V cannot contain a right zero semigroup since we have seen earlier that such a semigroup does not admit a right closure. Furthermore, on any completely simple semigroup the $\mathcal{H}$ relation is a congruence which collapses each individual subgroup to an idempotent. The corresponding quotient is a completely simple band. The only such bands which do not contain right zero semigroups as subsemigroups are left zero semigroups. Therefore each completely simple semigroup in the variety V is a left zero semigroup of groups, called a *left group*. It is easily verified that the $\mathcal{R}$ relation in such a semigroup is equal to the $\mathcal{H}$ relation and also that $\mathcal{L} = \mathcal{J} = \mathcal{D}$. Thus every semigroup in V is a semilattice of left groups.

Now we show that for some n, V is a subvariety of the variety given by $\{x = x^{n+1}, (xy)^n(yx)^n = (xy)^n\}$. Let S be a semigroup in V and let a, b be elements of S. Now since S is a semilattice of completely simple semigroups, ab and ba lie within the same completely simple subsemigroup of S (the $\mathcal{J}$-class containing ab). By the above arguments, this is a left group. Now $(ab)^n$ and $(ba)^n$ are idempotents in this same left group and since (by the above arguments) $(ab)^n \,\mathcal{L}\, (ba)^n$ we have that $(ab)^n(ba)^n = (ab)^n$ as required.

This enables us to show that the idempotents in any semigroup in V form a band (that is, V is a variety of orthodox semigroups). Let S be a semigroup from V. Any idempotent in S is of the form a^n for some $a \in S$. Now

$$\begin{aligned}
a^n b^n &= a^n b^n (a^n b^n)^n \\
&= a^n b^n (a^n b^n)^n (b^n a^n)^n \\
&= a^n (b^n a^n)^n b^n (b^n a^n)^n \\
&= a^n (b^n a^n)^n (b^n a^n)^n \\
&= a^n (b^n a^n)^n \\
&= a^n (b^n a^n)^n (a^n b^n)^n \\
&= (a^n b^n)^n a^n (a^n b^n)^n \\
&= (a^n b^n)^n (a^n b^n)^n \\
&= (a^n b^n)^n,
\end{aligned}$$

which is an idempotent as required.

A band in the variety defined by $\{(xy)^n(yx)^n = (xy)^n, x = x^{n+1}\}$ satisfies $xyyx = xy$ which for bands is equal to $xyx = xy$. The lattice of all band varieties has been completely described by several authors ([3], [6], and [10]) and this variety of bands has exactly four proper subvarieties: the trivial variety, the variety SL of semilattices, the variety LZ of left zero semigroups and the variety LN of left normal bands (defined within the variety of bands by the identity $xyz = xzy$ or equivalently by taking the join $LZ \vee SL$). However it is easily verified that the semigroup in Example 2.15 is contained in the variety LN. Since this semigroup does not admit any right closure, every band in the variety V must be contained in one of the varieties SL or LZ. In the first case it follows that for some n, V is a subvariety of the variety given by the first set of identities in point (3) of Theorem 5.1 (the variety of semilattices of groups of exponent dividing n) and in the second case it follows that for some n, V is a subvariety of the variety defined by the second set of identities (the variety of left groups of exponent dividing n). Thus we have shown (1)$\Rightarrow$(3).

For (3)$\Rightarrow$(1), it is easy to define a right closure on any semigroup from one of the varieties in (3). For the first case we may let the right closure of an element a be the element a^n. In the second case we can choose an arbitrary idempotent e and define $C(x) = e$ for all x.

The equivalence of conditions (2) and (3) follows immediately since in both of the cases the right closures we defined were strong. ∎

Note that this result means that taking the join of two semigroup varieties of RC-semigroups does not necessarily result in a semigroup variety of RC-semigroups.

The simplicity of the varieties described in this theorem also leads to similar descriptions for restricted classes of RC-semigroups. For example, the two element left zero semigroup does not admit a right closure that is full, or twisted, or central (or indeed translucent). It also is not an inverse semigroup. On the other hand any subvariety of the semigroup variety defined by the first set of identities is a variety of Clifford semigroups – inverse semigroups which are semilattices of groups. The standard right closure on any such inverse semigroup satisfies all of the properties just described and therefore we have the following

Corollary 5.2. *Let V be a variety of semigroups. The following are equivalent:*

1. *for some n, V is contained in the semigroup variety defined by $\{x^{n+1} = x, x^n y^n = y^n x^n\}$;*
2. *every member of V is a Clifford semigroup (that is, a semilattice of groups);*
3. *every semigroup in V admits a full RC-semigroup structure;*
4. *every semigroup in V admits a full twisted RC-semigroup structure;*

5. *every semigroup in* V *admits a twisted RC-semigroup structure;*

6. *every semigroup in* V *admits a translucent RC-semigroup structure;*

7. *every semigroup in* V *admits a central RC-semigroup structure;*

8. *every semigroup in* V *is an inverse semigroup.*

We note that a description of the semigroup varieties admitting a full (in the same sense as a full right closure) unary operation I satisfying the identities

$$\{I(x)I(y) = I(y)I(x), xI(x) = I(x), I(I(x)) = I(x), I(xy)I(y) = I(x)I(y)\}$$

was obtained in [15]. Here $I(a) = \max\{e \in L_S \mid ae = e\}$, where L_S is the semilattice $\{I(a) \mid a \in S\}$ − analogs of Propositions 1.2 and 1.3 apply. The corresponding semigroup varieties are all subvarieties of the semigroup varieties given by the identities $\{x^{n+1} = x^n, x^n y^n = y^n x^n\}$ for some n, an almost "dual" set of identities to that of Corollary 5.2!

Theorem 5.1 and Corollary 5.2 show that the classification of RC-semigroups by semigroup varieties is unlikely to be useful. An alternative approach is to view RC-semigroups as algebras having one unary and one binary operation. As we have noted, the classes of translucent, central and twisted RC-semigroups are all subvarieties in this sense. Note that the free twisted RC-semigroups have been described in [1] and [9]. Indeed, the variety of RC-semigroups is exactly the variety generated by the quasivariety of all right type A semigroups (see [9] and [2]).

The atoms in the lattice of semigroup varieties are all varieties generated by a cyclic group of prime order, the varieties generated by the two element left and right zero semigroups, the variety generated by the two element null semigroup and the variety generated by the two element semilattice (see [5] for example). We now give a description of the atoms in the lattice of RC-semigroup varieties.

Theorem 5.3. *The atoms in the lattice of RC-semigroup varieties are: all varieties generated by a cyclic group of prime order with* $C(x) = 1$*; the variety generated by the two element left zero RC-semigroup (satisfying* $C(x) = C(y)$*); the variety generated by the two element semilattice with right closure satisfying* $x = C(x)$*; the variety generated by the two element semilattice with right closure satisfying* $C(x) = C(y)$*.*

Proof. If an RC-semigroup S contains more than one closed element then the variety generated by the subsemilattice of closed elements contains the two element semilattice satisfying $x = C(x)$. Otherwise, consider the case in which there is only one closed element. Suppose there is another idempotent e in

S. Then $C(e)eC(e)e = C(e)ee = C(e)e$. If $C(e)e \in \{e, C(e)\}$, then the RC-subsemigroup $\{e, C(e)\}$ is either the two element left zero RC-semigroup or the two element semilattice with one closed element. Otherwise, $\{C(e), C(e)e\}$ is the two element semilattice with one closed element. Thus the variety generated by S is one of those in the statement of the theorem.

Now suppose that $C(b)$ is the unique idempotent in S. Let a be any other element. If some power of a is equal to $C(a)$ then since $aC(a) = a$, a generates a cyclic group with right closure. In this case a cyclic group of prime order is contained in the variety generated by the RC-subsemigroup generated by a. Thus assume that no power of a equals $C(a)$ and consider the RC-subsemigroup $\langle a \rangle$ generated by a and $C(a)$. Since $C(a)a^iC(a)a^j = C(a)a^{i+j}$, $C(a)^2 = C(a)$ and $a^iC(a) = a^i$, every element in this subsemigroup can be written in the form a^i, or $C(a)a^j$, for $i > 0$ and $j \geq 0$ (here a^0 denotes the empty word). Since $C(a)$ is the unique idempotent in $\langle a \rangle$, there is no number $i > 0$ such that $C(a)a^i = C(a)$ (for otherwise $a^{i+1} = aC(a)a^i = aC(a) = a$, contradicting the assumption that no power of a is $C(a)$). Thus the equivalence on $\langle a \rangle$ with equivalence classes $\{C(a)\}$ and $\langle a \rangle \backslash \{C(a)\}$ is a congruence and the resulting quotient is isomorphic to the two element semilattice with right closure satisfying $C(x) = C(y)$.

Thus every RC-semigroup variety contains one of the RC-semigroups described in the theorem and it is easily established that no one of them is contained in the variety generated by any of the others. It follows that these are the atoms in the lattice of RC-semigroup varieties. ■

We note that while there is a single semigroup variety of semilattices, the above theorem shows that there are two distinct atoms in the variety of RC-semigroups consisting of semilattices, both of which are twisted. However, while a C-semilattice is obviously central, it need not be twisted: if we take the Boolean algebra of subsets of a two element set $\{a, b\}$ with closed sets $\{a, b\}$ and $\emptyset$ then $C(\{a\})\{b\} = \{b\}$ while $\{b\}C(\{a\}\{b\}) = \{b\}\emptyset = \emptyset$.

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References

[1] Batbedat, A., *γ-demi-groups, demi-modules, produit demi-directs*, Semigroups, Proceedings, Oberwolfalch, Germany 1978, Lecture Notes in Mathematics 855, Springer-Verlag 1981, 1–18.

[2] Batbedat, A. and J. B. Fountain, *Connections between left adequate semigroups and γ-semigroups*, Semigroup Forum **22** (1981), 59–65.

[3] Birjukov, A. P., *Varieties of idempotent semigroups*, Algebra i Logika **9** (1970), 255–273 (Russian).

[4] Burris, S. and H. P. Sankappanavar, “A Course in Universal Algebra”, Graduate Texts in Mathematics **78**, Springer Verlag, 1980.

[5] Evans, T., *The lattice of semigroup varieties*, Semigroup Forum **2** (1971), 1–43.

[6] Fennemore, C. F., *All varieties of bands I, II*, Math. Nachr. **48** (1971), 237–252.

[7] Fountain, J., *Adequate semigroups*, Proc. Edinburgh Math Soc. **22** (1979), 113–125.

[8] Fountain, J., *Abundant semigroups*, Proc. London Math. Soc. **44** (1982), 103–129.

[9] Fountain, J., *Free right type A semigroups*, Glasgow Math. Journal **33** (1991), 135–148.

[10] Gerhard, J. A., *The lattice of equational classes of idempotent semigroups*, J. Algebra **15** (1970), 195–224.

[11] Higgins, P. M., “Techniques of Semigroup Theory”, Oxford University Press, 1992.

[12] Howie, J. M., *The maximum idempotent-separating congruence on an inverse semigroup*, Proc. Edinburgh Math. Soc. **14** (1964), 71–79.

[13] Howie, J. M., “Fundamentals of Semigroup Theory”, London Mathematical Society Monographs, Oxford University Press, 2nd ed., 1995.

[14] Lawson, M. V., *Semigroups of ordered categories I. The reduced case*, J. Algebra **141** (1991), 422–462.

[15] Kelarev, A. V. and Stokes, T., *Interior algebras and varieties* to appear in Comm. Alg.

[16] Munn, W. D., *Uniform semilattices and bisimple inverse semigroups* Quart. J. Math. Oxford **17**(2) (1966), 151–159.

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