AN IQML TYPE ALGORITHM FOR AR PARAMETER ESTIMATION FROM NOISY COVARIANCE SEQUENCES

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ABSTRACT

In this paper, we deal with the problem of AR parameter estimation from noisy covariance sequences. An iterative quadratic maximum likelihood (IQML) type of algorithm is proposed to solve the problem. The convergence and consistency properties of the method are studied by carrying out several numerical simulations.

1. INTRODUCTION

The problem of parameter estimation of an ARMA or an MA process from its sample covariance sequence has been studied quite extensively in the literature [1–4]. For example [1], solves the problem by fitting the sample covariance sequence to the true sequence of a specific order MA process. Similarly [4], deals with ARMA parameter estimation with the additional problem of missing data. In this paper, we deal with the problem of parameter estimation of an AR process of specific order from its noisy covariance sequence.

The problem of identifying the parameters of an AR model from noisy measurements of the autocovariance model is motivated by a classical geophysical inverse problem and also by its counterpart in vocal tract estimation. Physically, one has normal-incidence wave propagation in a stratified medium comprising homogeneous equal travel-time layers (the so-called Goupillaud model). The interface between each pair of adjacent layers is described by a reflection coefficient whose magnitude is less than one; at the bottom of the deepest (M-th) layer the reflection coefficient and all deep reflection coefficients are equal to zero. Consider the application of an impulsive pressure pulse at the free surface: the resulting discrete-time reflection response, sampled at one-half of the one-way travel time in each layer, can be shown to equal one side of the autocovariance sequence that is associated with an M-order AR spectrum [5]. The reflection coefficients (a.k.a partial correlation coefficients or Schur parameters) are identical to the physical reflection coefficients. Given the noise-free reflection response, a straightforward application of the Levinson algorithm will perform the inversion. However the noisy reflection response may not be positive-definite, and in any case the application of the Levinson algorithm is certainly suboptimal. The methods like Yule-Walker (YLW) and total least squares (TLS) are not applicable here, as they are applicable only for AR parameter estimation from noisy sample data, and to our knowledge there are no methods available for AR parameter estimation from noisy covariance sequences. In this paper, we will propose an IQML type method for AR parameter estimation from noisy covariance sequences, and study its convergence and consistency.

In section II, we state the problem, and section III presents the IQML type algorithm and its brief theoretical study. Section IV contains two numerical examples and section V draws some conclusions.

2. PROBLEM FORMULATION

Let $\{\hat{r}_k\}_{k=0}^{T}$ denote the given noisy covariance sequence and $\{r_k\}_{k=0}^{T}$ denote the true covariance of an AR(M) process, where M denotes the AR order and is considered to be known and less than T. The noise in the sequence is assumed to be white Gaussian with zero mean and variance σ^2, and independent of the process:

$$\hat{r}_k = r_k + \varepsilon_k \Leftrightarrow \hat{r} = r + \varepsilon, \qquad (1)$$

where

$$r = [r_0 \cdots r_T]^T, \qquad \hat{r} = [\hat{r}_0 \cdots \hat{r}_T]^T,$$
$$\varepsilon = [\varepsilon_0 \cdots \varepsilon_T]^T, \qquad E(\varepsilon\varepsilon^T) = \sigma^2 I. \qquad (2)$$

Let $a = [a_0 \cdots a_M]^T$ denote the parameters of the AR(M) process. The problem is: given $\{\hat{r}_k\}_{k=0}^{T}$, find a, the parameters of AR(M).

3. IQML TYPE METHOD

Let us introduce a matrix A of size $T-M+1$ by $T+1$, which is made from a as shown below:

$$A^T = \begin{bmatrix} a_M & \cdots & a_0 & & 0 \\ & \ddots & & \ddots & \\ 0 & & a_M & \cdots & a_0 \end{bmatrix}. \qquad (3)$$

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The true covariance sequence r lies in the nullspace of matrix A^T, that is

$$A^T r = 0. \tag{4}$$

This follows from the fact that

$$a_0 r_k + a_1 r_{k-1} + \cdots + a_M r_{k-M} = 0, \quad k = M, \ldots, T. \tag{5}$$

By premultiplying (1) with A^T on both sides and using (4), we get

$$A^T \hat{r} = A^T \varepsilon. \tag{6}$$

This observation implies that asymptotically (for $T \gg M$), the following minimization problem will yield the maximum likelihood estimate of a:

$$\min_{\{a\}} \{ f(a) \triangleq \hat{r}^T A (A^T A)^{-1} A^T \hat{r} \}. \tag{7}$$

The above mentioned problem is highly nonlinear in a. And also the minimizer is not unique, because for any $\rho \neq 0$, the cost function will have the same value for ρa and a. The nonuniqueness can be tackled by imposing the constraint that $a_0 = 1$, but it will still be hard to handle the nonlinearity. However in the literature, this kind of problems are handled effectively by an algorithm called the iterative quadratic maximum likelihood (IQML). The idea of IQML was introduced in [6] to handle a similar minimization problem in DOA estimation. According to the IQML principle, the minimization in (7) is solved iteratively as follows:

1. Initialize $A^T A = I$, and solve the minimization problem to obtain $\hat{a}_0$:

$$\hat{a}_0 = \min_{\{a\}} \left|\left| A^T \hat{r} \right|\right|_2^2 \quad \text{s.t.} \quad u^T a = 1, \tag{8}$$

   where $u = [1 \;\; 0 \cdots 0]^T$ constraints the first element of a to be 1, in order to avoid the trivial estimate $a = 0$.
2. For $k = 1, 2, \ldots,$ end, solve the following minimization problem:

$$\hat{a}_k = \min_{\{a\}} \left|\left| A^T \hat{r} \right|\right|_W^2 \quad \text{s.t.} \quad u^T a = 1, \tag{9}$$

   where $W = (A_{k-1}^T A_{k-1})^{-1}$ and A_{k-1}^T is obtained from $\hat{a}_{k-1}$ as in (3).

The iteration can be stopped when the change in a is negligibly small, that is $||\hat{a}_k - \hat{a}_{k-1}||^2 / M < \varepsilon$, where ε can be chosen around 10^{-8}. The constrained minimization in (8) and (9) can be carried out easily as follows. At any iteration the cost function in (9) can be rewritten as

$$\left|\left| A^T \hat{r} \right|\right|_W^2 \Leftrightarrow \left|\left| \hat{R}^T a \right|\right|_W^2 \tag{10}$$

where $\hat{R}$ is given by

$$\hat{R}^T \;\; = \;\; \begin{bmatrix} \hat{r}_M & \hat{r}_{M-1} & \cdots & \hat{r}_0 \\ \hat{r}_{M+1} & \hat{r}_M & \cdots & \hat{r}_1 \\ \vdots & \vdots & \vdots & \vdots \\ \hat{r}_T & \cdots & \cdots & \hat{r}_{T-M} \end{bmatrix}. \tag{11}$$

So the constrained minimization in (9) can be rewritten as

$$\min_{\{a\}} \left|\left| \hat{R}^T a \right|\right|_W^2 \quad \text{s.t.} \quad u^T a = 1. \tag{12}$$

The above problem is quadratic in a with a linear constraint; its solution can be written analytically as follows:

$$\hat{a} = \frac{Q^{-1} u}{u^T Q^{-1} u}, \tag{13}$$

where $Q = \hat{R} W \hat{R}^T$ (see, e.g., [7]).

Instead of constraining the first element of a to be equal to one, the norm of a can be constrained, such that $a^T a = 1$, in which case the solution equals the eigenvector corresponding to the minimum eigenvalue of Q. In general, the AR parameters obtained via the above mentioned procedure may lead to unstable systems for which the zeros of the polynomial, $a(z) = a_0 + a_1 z^{-1} + \cdots + a_M z^{-M}$, lie outside the unit circle. However, we have observed empirically that even at low values of signal to noise ratio (SNR) and T, the zeros of $a(z)$ lie inside the unit circle. For $T >> M$, the above mentioned method requires about $O(2T^3 - 4MT^2 + 3M^2 T)$ flops to compute the solution at each iteration. In the next section, the convergence and consistency of the IQML type algorithm are discussed.

4. CONVERGENCE AND CONSISTENCY ANALYSIS

Convergence and consistency of IQML type algorithms in the array processing context are discussed in detail in [8, 9]. The IQML algorithm generally converges, but the convergence point might not be a stationary point of $f(a)$ in (7). In general, the proposed IQML type method will converge, but not necessarily to a local minimum. This claim can be verified by carrying out a simple test as follows:

1. Obtain a_c, the convergence point of IQML type method, as discussed in equations (8) and (9).
2. Minimize the cost function, $f(a) = \hat{r}^T A (A^T A)^{-1} A^T \hat{r}$, via a nonlinear unconstrained minimization algorithm, initialized with a_c, to obtain the local minimum a_{loc}. For example the function "fminsearch" in MATLAB can be used to obtain a_{loc}.
3. Find the normalized error, $\frac{|f(a_c) - f(a_{loc})|}{f(a_{loc})}$.
4. If the normalized error is "zero", then IQML has converged to a local minimum, else the convergence point a_c is not a local minimum.

As shown in the simulations in the next section, most of the time the IQML type method does not converge to a local minimum.

Next in this section, we will discuss the consistency (as T, the number of covariance lags, tends to infinity) of the algorithm. At moderate and low SNRs, the first step in the IQML algorithm will always give biased estimates. However it is important to note that the consistency of the IQML algorithm does not depend on the bias in $\hat{a}_0$, the estimate

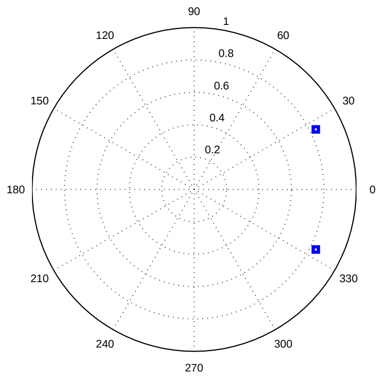

Figure 1: True pole locations of AR(2) process.

from the first step of the IQML algorithm.

Let us rewrite the cost function in (8) as

$$f(a) = \hat{r}^T AA^T \hat{r}. \tag{14}$$

Substituting (1) into (8), we get

$$f(a) = r^T AA^T r + 2r^T AA^T \varepsilon + \varepsilon^T AA^T \varepsilon. \tag{15}$$

Taking expectation and using a trace property yield:

$$\begin{aligned} E(f(a)) = r^T AA^T r + 2tr(AA^T E(\varepsilon) r^T) \\ + tr(AA^T E(\varepsilon\varepsilon^T)). \end{aligned} \tag{16}$$

Because the noise in the covariance sequence is white with zero mean, we get

$$E(f(a)) = r^T AA^T r + \sigma^2 tr(AA^T). \tag{17}$$

Using (10) and (3), the above expression can be rewritten as

$$E(f(a)) = a^T RR^T a + \sigma^2 (T - M + 1)(a^T a). \tag{18}$$

Dividing the above expression by $\frac{1}{T}$ on both sides and applying the limit $T \longrightarrow \infty$, we get

$$\begin{aligned} \lim_{T\to\infty} \frac{1}{T} E(f(a)) &= \lim_{T\to\infty} \frac{1}{T} a^T RR^T a + \lim_{T\to\infty} \frac{\sigma^2 (T - M + 1)}{T} (a^T a) \\ &= \lim_{T\to\infty} \frac{1}{T} a^T RR^T a + \sigma^2 (a^T a). \end{aligned} \tag{19}$$

The first term in the right hand side of the equation (19) will attain its minimum value zero only at a_{true}, the true value of the a. So at high SNRs (low σ^2), the first step of the IQML algorithm will give a good estimate of a. However at low SNRs (high σ^2), the second term in (19) dominates the first

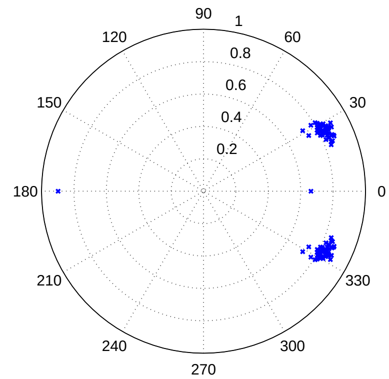

Figure 2: IQML estimated pole locations of AR(2) process for 50 trials, SNR = 5dB, $T = 50$.

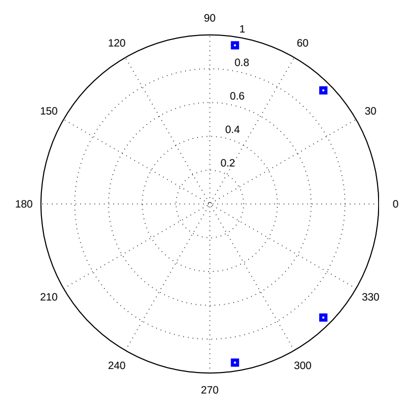

Figure 3: True pole locations of AR(4) process.

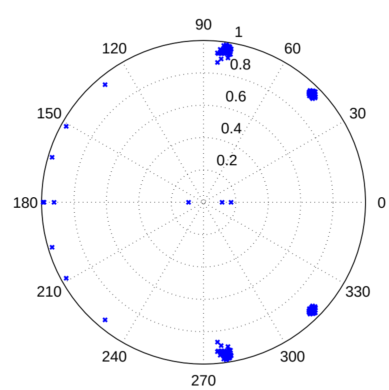

Figure 4: IQML estimated pole locations of AR(4) process for 50 trials, SNR = 5dB, $T = 50$.

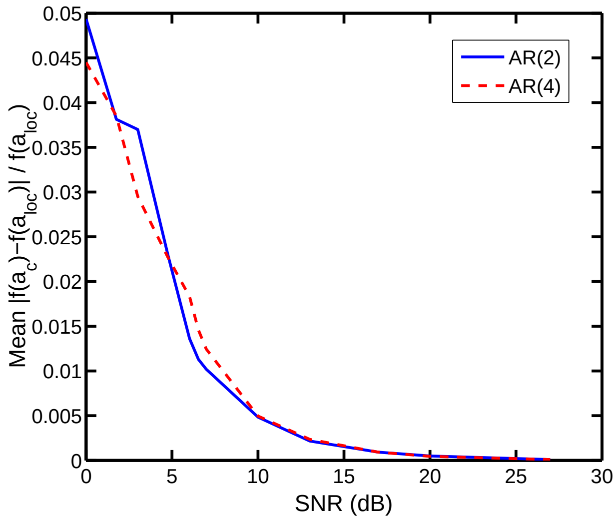

Figure 5: $E\left(\frac{|f(a_c)-f(a_{loc})|}{f(a_{loc})}\right)$ vs SNR, $T=50$.

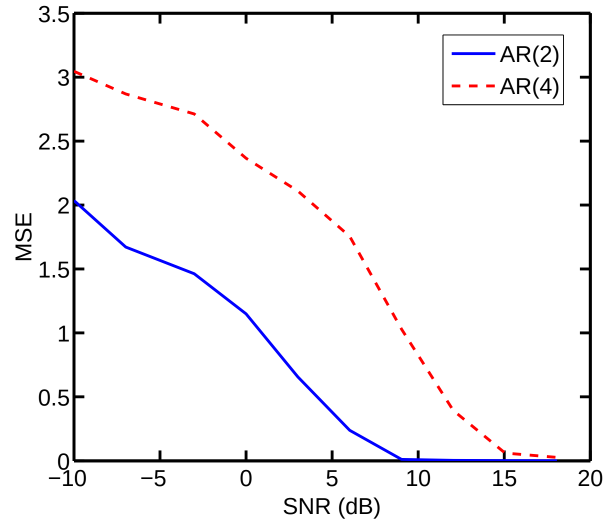

Figure 6: MSE vs SNR, $T=50$.

term in the cost function, so the estimate of a obtained by minimizing that cost function will be biased towards zero.

Regarding the consistency analysis of step 2 (9) of IQML, it is more complicated to analyze the iterations. In [8, 9], the authors have derived the necessary condition for consistency for the same sort of algorithm. By following the derivation in [8], the necessary condition for consistency of step 2 (9), is given by

$$V^T P a_{true} = 0, \quad P = U^T (I \otimes (A_{true}^T A_{true})^{-1}) U, \tag{20}$$

where U and V are given by

$$vec(A^T) = Ua \quad I = [u \;\; V]. \tag{21}$$

From equation (20), it is readily seen that for the IQML algorithm to be consistent, Pa_{true} should lie in the nullspace of V^T, which is too restrictive to hold in general. The inconsistency of the IQML algorithm can also be explained simply as follows: for higher values of T, the true covariance sequence of the finite order AR process will be close to zero, and the sample covariance sequence will be made only of noise, so the performance of IQML algorithm (or of any other algorithm, for that matter) will not improve with the increasing of value of T.

5. NUMERICAL RESULTS

In this section, we will use the IQML type method in two examples, a second order and a fourth order AR process with coefficients:

1. $a = [1 \;\; -1.5 \;\; 0.7]^T$.
2. $a = [1 \;\; -1.6408 \;\; 2.2044 \;\; -1.4808 \;\; 0.8145]^T$.

For each example, 500 Monte Carlo simulations are carried out on the noisy covariance sequence generated from the true covariance sequence and white Gaussian noise of zero mean and a certain variance, and the following results are shown:

1. The estimated pole locations of the AR process.
2. Mean square error (MSE) of estimated coefficients versus signal to noise ratio (SNR) and T.
3. Convergence of IQML estimate to a local minimum (we will show the averaged value of $\frac{|f(a_c)-f(a_{loc})|}{f(a_{loc})}$ versus SNR).

The MSE analysis has been carried out to analyze the consistency of the IQML estimate and the convergence analysis has been carried out to show that the IQML will not always converge to a local minimum.

The plots in Fig. 1, Fig. 3, Fig. 2 and Fig. 4 show the true and the IQML estimates of the pole locations of the two AR processes. As indicated in the discussion in section III about the stability of the system, it is clearly seen that all the estimated poles lie inside the unit circle. Furthermore, the estimated pole locations are quite accurate in most realizations. Regarding the convergence of the IQML, the plot in Fig. 5 of $E\left(\frac{|f(a_c)-f(a_{loc})|}{f(a_{loc})}\right)$ vs SNR, shows that indeed the IQML type method does not necessarily converge to a local minimum. It shows that only at high SNRs, the IQML type method converges to or very near to a local minimum.

The plot in Fig. 6 shows the variation of MSE vs SNR; it is clearly seen from the figure that the MSE decreases with increasing SNR. Fig. 7 shows the variation of MSE vs SNR for the estimates from the first and the final step of the IQML; it is seen from the figure that the estimates from the final step are more accurate than the estimates from the first step

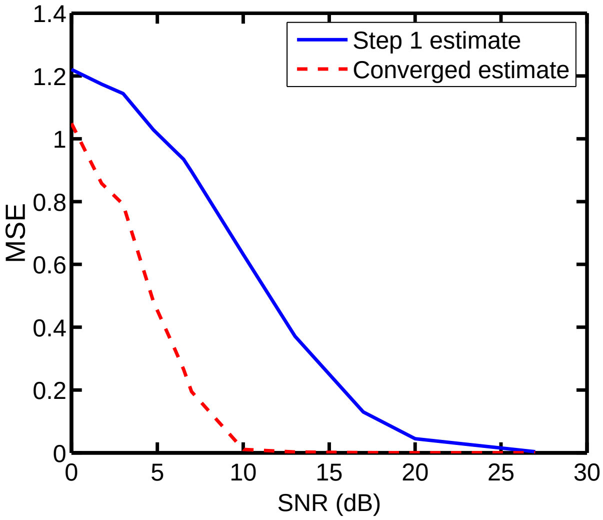

Figure 7: MSE vs SNR for AR(2) process, $T = 50$

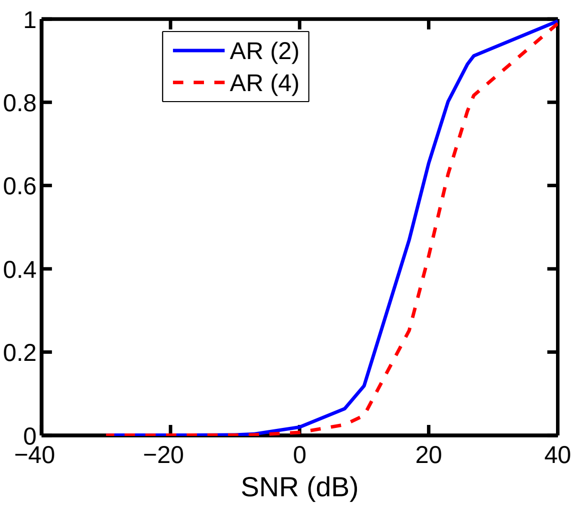

Figure 8: $E\left(\frac{\|u-\hat{a}_0\|^2}{\|u-a_{true}\|^2}\right)$ vs SNR, $T = 10^3$.

of the IQML. Finally numerical simulations are carried out to analyze the consistency of the IQML in the number of samples (T). Fig. 8 plots the mean value of $\left(\frac{\|u-\hat{a}_0\|^2}{\|u-a_{true}\|^2}\right)$ vs SNR, where $u = [1\,0\cdots0]^T$, and $\hat{a}_0$ represents the estimate from the first step of the IQML; it is seen from the figure that at low SNRs, the estimates from the first step of the algorithm are biased towards zero. Fig. 9 shows the variation of MSE vs T for AR(2) process at an SNR = 0 dB; it shows that, as T increases, the MSE decreases but does not quite converge to zero.

6. CONCLUSIONS

The problem of AR parameter estimation from noisy covariance sequences has been considered for an application in the field of geophysics. An IQML type method has been proposed to solve the problem, and its convergence and consistency have also been studied. Finally numerical simulations were carried out to back the theoretical analysis.

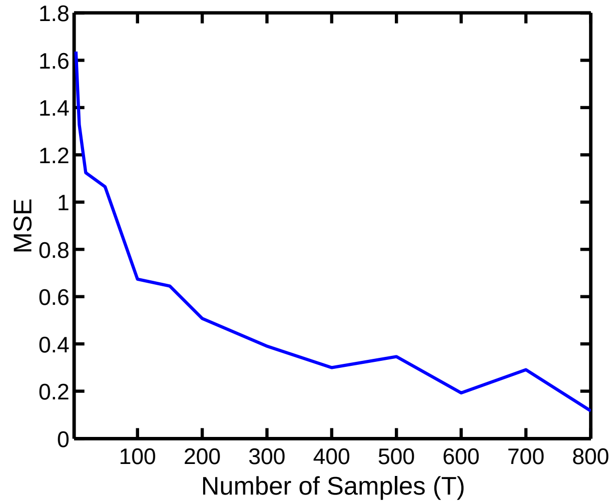

Figure 9: MSE vs T for AR(2) process, SNR=0dB.

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