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An Ising Ferromagnet with Discontinuous Long-Range Order

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Abstract. An infinite one-dimensional Ising ferromagnet M with long-range interactions is constructed and proved to have the following properties. (1) M has an order-disorder phase transition at a finite temperature. (2) Any Ising ferromagnet of the same structure as M, but with interactions tending to zero with distance more rapidly than those of M, cannot have a phase-transition. (3) The long-range-order parameter (thermal average of the spin-spin correlation at infinite distance) jumps discontinuously from zero in the disordered phase to a finite value in the ordered phase. All three properties have been conjectured by Anderson and Thouless to hold for a particular Ising ferromagnet which is relevant to the theory of the Kondo effect. Although M is not identical to Anderson's model, the results proved for M support the validity of the physical arguments of Anderson and Thouless.

I. Introduction

Anderson and his colleagues [1–4] have discussed the Ising ferromagnet with Hamiltonian

$$H = -J \sum_{n<m}\sum (m-n)^{-2} \mu_m \mu_n , \qquad J>0 , \tag{1.1}$$

an infinite one-dimensional linear chain of spins whose state is specified by the dichotomic variables $\mu_n = \pm 1$. Using asymptotic estimates which are probably correct although not rigorous, Anderson concludes that the system (1.1), which we shall call the "Anderson Model," has an order-disorder phase transition at a temperature T_c given approximately by

$$\beta_c J = 0 \cdot 63 , \qquad \beta_c = (kT_c)^{-1} . \tag{1.2}$$

It is known (Dyson [5]; Ruelle [6]) that the model (1.1) with α replacing 2 in the exponent has a phase-transition for $1<\alpha<2$ but not for $\alpha>2$ or $\alpha \leqq 1$. The Anderson model is thus a delicate border-line case, and a rigorous proof that it has a phase-transition is much to be desired. In this paper we do not supply such a proof. The best we can do in this direction is to prove

Theorem 1. *The linear Ising ferromagnet with Hamiltonian*

$$H = -J \sum_{n<m}\sum (m-n)^{-2} \log\log(m-n+3)\, \mu_m \mu_n , \qquad J>0 , \tag{1.3}$$

has a phase-transition.

But this obviously ephemeral result is not our main concern.

Thouless [7] has shown by an elegant physical argument, again not rigorous but probably correct, that the phase-transition of the Anderson model must be of an unusual kind if it exists. Thouless shows that the long-range-order parameter

$$m^2 = \operatorname*{Lim}_{N\to\infty} N^{-2} \left\langle \left(\sum_1^N \mu_n \right)^2 \right\rangle \tag{1.4}$$

cannot be a continuous function of temperature at $T = T_c$. If a transition occurs, then m^2 must jump discontinuously, so that

$$m^2 \geqq m_c^2 > 0 \quad \text{for } T < T_c , \tag{1.5}$$

$$m^2 = 0 \qquad \text{for } T > T_c . \tag{1.6}$$

This discontinuity of m^2 we shall call the "Thouless Effect." The numerical estimates of Anderson and Yuval [4] give

$$m_c^2 = 0.79 , \tag{1.7}$$

but a rigorous proof of Eq. (1.5) for the Anderson model is still lacking.

The purpose of this paper is to exhibit a one-dimensional Ising ferromagnet for which a Thouless effect can be rigorously demonstrated. We consider the "hierarchical model" M_N defined in Dyson [5]. This is a finite chain of 2^N spins $\mu_j = \pm 1$ with the Hamiltonian

$$H_N = - \sum_{p=0}^{N-1} 2^{-1-2p} b_{p+1} \sum_{r=1}^{2^{N-p-1}} (S_{p,2r-1} S_{p,2r}) , \tag{1.8}$$

$$S_{p,r} = \Sigma \mu_j , \qquad (r-1)\,2^p + 1 \leqq j \leqq r 2^p . \tag{1.9}$$

The positive coefficients b_p specify the dependence of the interaction between two spins upon their separation. The hierarchical model is obtained from an ordinary linear ferromagnet by lumping together interactions between blocks of 2^p spins. Thus the Anderson model (1.1) corresponds roughly to a hierarchical model with $b_p = J$, and the linear model (1.3) corresponds to a hierarchical model with $b_p = J \log p$. An infinite hierarchical model M is defined by letting $N \to \infty$ in Eq. (1.8). For technical reasons the definition of the b_p in Eq. (1.8) is not exactly the same as in Dyson [5]. The definition adopted here simplifies the analysis without changing anything essential.

The long-range-order parameter of the model M is defined by

$$m^2 = \lim_{p\to\infty} f(p) , \tag{1.10}$$

$$f(p) = \lim_{N\to\infty} f_N(p) , \tag{1.11}$$

$$f_N(p) = 2^{-2p} \langle (S_{p,r})^2 \rangle_N , \tag{1.12}$$

the average in Eq. (1.12) being taken in the finite model M_N. Both limits (1.10) and (1.11) exist, since $f(p)$ is a decreasing function of p and $f_N(p)$ is an increasing function of N. To avoid trivial complications, we assume that the series

$$R = \sum_{q=1}^{\infty} 2^{-q} b_q \tag{1.13}$$

converges. Then

$$m^2 = 0 \quad \text{for} \quad \beta R < 1, \tag{1.14}$$

so that there is no long-range order at high temperatures (Dyson [5]). Our main results for the model M are

Theorem 2. *If*

$$\operatorname*{Lim}_{p \to \infty} (b_p / \log p) = 0, \tag{1.15}$$

then $m^2 = 0$ for all β and there is no phase-transition.

Theorem 3. *If*

$$b_p \geqq J \log p, \quad J > 0, \tag{1.16}$$

then $m^2 > \frac{1}{4}$ for

$$\beta J > 8, \tag{1.17}$$

and there is a phase-transition with

$$\frac{1}{8} J < kT_c < R. \tag{1.18}$$

Theorem 4. *If*

$$b_p = J \log p, \quad J > 0, \tag{1.19}$$

then there is a Thouless effect at the transition, and Eqs. (1.5), (1.6) *hold with*

$$m_c \geqq [kT_c/(kT_c + J)] \geqq \frac{1}{9}. \tag{1.20}$$

Theorems 2 and 3 show that $b_p = J \log p$ is the borderline case between models with and without a phase-transition. They supersede the weaker Theorems 5 and 6 of Dyson [5], which left a gap of undecided cases on both sides of $J \log p$. Theorem 4 shows that the Thouless effect exists exactly at the borderline. The intuitive argument of Thouless [7] indicates that an analogous situation occurs for linear models with Hamiltonian

$$H = -\sum_{n<m}\sum f(m-n)\, \mu_m \mu_n, \tag{1.21}$$

the Anderson model being here the borderline case.

Namely, Eq. (1.21) should give no transition when

$$\lim_{n\to\infty} n^2 f(n) = 0\,, \tag{1.22}$$

a transition when

$$f(n) \geqq J n^{-2}, \quad J > 0, \quad \Sigma f(n) < \infty\,, \tag{1.23}$$

and a Thouless effect when

$$f(n) = J n^{-2}, \quad J > 0\,. \tag{1.24}$$

It is tempting to make a conjecture generalizing these results to a much wider class of circumstances.

Conjecture. *Suppose an infinite Ising ferromagnet of any kind has a Hamiltonian depending continuously upon a real parameter α, with a critical value α_0 such that an order-disorder transition occurs for $\alpha \geqq \alpha_0$ but not for $\alpha < \alpha_0$. Then a Thouless effect will occur at the transition for the borderline model $\alpha = \alpha_0$.*

It is unlikely that a conjecture of this sort would be valid for all possible ferromagnets. The problem is to establish a precise definition of the class of ferromagnets for which it holds, together with a specification of the topology in which the Hamiltonian is to be assumed continuous.

The remainder of this paper is concerned with the proof of Theorems 2, 3, 4. Theorem 1 is an immediate corollary of Theorem 3. In Section III we digress briefly to prove the analog of Theorem 3 for a classical Heisenberg ferromagnet, and to explain why no Thouless effect is to be expected in a Heisenberg model.

II. Proof of Theorem 3

The proof of Theorem 3 exploits in a straightforward way the hierarchical structure of the model M_N, which consists of two identical models M_{N-1} with their total spins coupled together. We make use of a lemma which may be stated in physical terms as follows. Suppose a finite ferromagnet F is composed from two ferromagnets F_1 and F_2 by coupling their total spins together. Let the root-mean-square spin of F_1 and F_2 separately be R_1 and R_2. Let F' be the two-spin ferromagnet obtained by coupling two Ising spins of magnitude R_1 and R_2 with the same coupling by which F_1 and F_2 are coupled in F. Then the mean-square spin of F is at least as large as the mean-square spin of F'.

The lemma can be stated more concisely in probability-theoretic terms.

Lemma 1. *Let x be a random variable with a probability distribution which is bounded and symmetrical about zero. Then for any positive real α*

$$[\langle x \exp(\alpha x)\rangle / \langle \exp(\alpha x)\rangle] \geqq \hat{x} \tanh(\alpha \hat{x}), \tag{2.1}$$

with

$$\hat{x} = [\langle x^2 \rangle]^{\frac{1}{2}}. \tag{2.2}$$

The lemma is proved at the end of this section. We now proceed with the deduction from it of Theorem 3.

Consider the model M_N, with Hamiltonian (1.8), and coefficients b_p satisfying Eq. (1.16). We write

$$s = S_{N-1,1}, \qquad t = S_{N-1,2}, \tag{2.3}$$

with $S_{p,r}$ defined by Eq. (1.9). Then

$$H_N = H_s + H_t - 2^{1-2N} b_N st, \tag{2.4}$$

where H_s, H_t are functions of the μ_j internal to the blocks whose sums are s, t respectively. H_s is the Hamiltonian of the model M_{N-1}. The partition-function of M_N is then

$$Z_N = \sum_s \sum_t Y(s)\, Y(t) \exp(\alpha\, st), \tag{2.5}$$

$$\alpha = 2^{1-2N} \beta\, b_N, \tag{2.6}$$

where $Y(s)$ is the part of the partition-function of the model M_{N-1} derived from states with

$$\Sigma\, \mu_j = s. \tag{2.7}$$

$Y(s)$ is positive and is an even function of s. The definition (1.12) gives

$$f_N(N) = 2^{-2N} \langle (s+t)^2 \rangle_N$$

$$= \frac{1}{2} f_N(N-1) + 2^{1-2N} (Z_N)^{-1} \sum_s \sum_t Y(s)\, Y(t) \exp(\alpha\, st)\, st. \tag{2.8}$$

We apply Lemma 1 to Eq. (2.8) with

$$x = st, \tag{2.9}$$

and the probability distribution

$$p(x) = \Big(\sum_{st=x}\sum Y(s)\, Y(t)\Big) \Big/ \Big(\sum_s Y(s)\Big)^2. \tag{2.10}$$

The lemma gives

$$f_N(N) \geqq \frac{1}{2} f_N(N-1) + 2^{1-2N} \hat{x} \tanh(\alpha \hat{x}), \tag{2.11}$$

with

$$\hat{x} = (\Sigma\, s^{\,2} Y(s))/(\Sigma\ Y(s)) = \langle s^2 \rangle_{N-1}\,. \tag{2.12}$$

By Eq. (1.12),

$$\hat{x} = 2^{2N-2} f_{N-1}(N-1)\,. \tag{2.13}$$

Since also

$$f_N(N-1) \geqq f_{N-1}(N-1)\,, \tag{2.14}$$

Eq. (2.11) implies

$$\begin{aligned} f_N(N) &\geqq f_{N-1}(N-1)\,\tfrac{1}{2}\,[1+\tanh(\tfrac{1}{2}\,\beta\, b_N f_{N-1}(N-1))] \\ &= f_{N-1}(N-1)\,[1+\exp(-\beta\, b_N f_{N-1}(N-1))]^{-1}\,. \end{aligned} \tag{2.15}$$

Now we use the hypothesis that b_N and β satisfy Eqs. (1.16), (1.17). Let

$$\mu_N = (N/(N+1))\, f_N(N) < f_N(N)\,. \tag{2.16}$$

Then Eqs. (1.16) and (2.15) give

$$(\mu_N/\mu_{N-1}) \geqq (1-N^{-2})^{-1}\,(1+N^{-\beta J \mu_{N-1}})^{-1}\,. \tag{2.17}$$

By the definition (1.12),

$$\mu_1 = \tfrac{1}{2} f_1(1) \geqq \tfrac{1}{4}\,. \tag{2.18}$$

If $\mu_{N-1} \geqq \frac{1}{4}$ for any value of N, then Eq. (1.17) and (2.17) give

$$\mu_N \geqq \tfrac{1}{4}(1-N^{-2})^{-1}(1+N^{-2})^{-1} \geqq \tfrac{1}{4}\,. \tag{2.19}$$

Therefore Eq. (2.18) implies $\mu_N \geqq \frac{1}{4}$ for all $N \geqq 1$. But then

$$f_N(p) \geqq f_p(p) \geqq \tfrac{1}{4} \tag{2.20}$$

for all $N \geqq p \geqq 1$, and Eqs. (1.10) and (1.11) imply

$$m^2 \geqq \tfrac{1}{4}\,. \tag{2.21}$$

This completes the proof of Theorem 3.

To deduce Theorem 1 from Theorem 3, it is only necessary to observe that the interactions in the linear model (1.3) are everywhere stronger than those of the hierarchical model with $b_p = \frac{1}{3} J \log p$. The model (1.3) therefore has long-range order at least for

$$\beta J > 24\,, \tag{2.23}$$

by virtue of Griffiths [8].

Proof of Lemma 1. The function

$$\varphi(\alpha) = (\langle x \exp(\alpha x)\rangle / \langle \exp(\alpha x)\rangle) \tag{2.24}$$

is continuous and differentiable for all real α. Therefore

$$\begin{aligned}(\partial\varphi/\partial\alpha) &= (\langle x^2 \exp\alpha x\rangle/\langle\exp(\alpha x)\rangle) - \varphi^2 \\ &= (\langle x^2 \cosh\alpha x\rangle/\langle\cosh\alpha x\rangle) - \varphi^2 \\ &\geqq \langle x^2\rangle - \varphi^2 ,\end{aligned} \tag{2.25}$$

since x^2 is a monotone function of $(\cosh\alpha x)$. If we write

$$\psi(\alpha) = \hat{x}\tanh(\alpha\hat{x}) , \tag{2.26}$$

then Eq. (2.2) gives

$$(\partial\psi/\partial\alpha) = (\hat{x})^2 - \psi^2 = \langle x^2\rangle - \psi^2 , \tag{2.27}$$

and therefore

$$(\partial\varphi/\partial\alpha) + \varphi^2 \geqq (\partial\psi/\partial\alpha) + \psi^2 . \tag{2.28}$$

Suppose if possible that $\varphi < \psi$ for some positive α_1. Then Eq. (2.28) gives

$$(\partial/\partial\alpha)(\varphi - \psi) > 0 , \qquad \alpha = \alpha_1 . \tag{2.29}$$

But

$$\varphi = \psi = 0 , \qquad \alpha = 0 . \tag{2.30}$$

Therefore the differentiable function $(\varphi - \psi)$ must have a negative minimum for some α_2 in the range $0 < \alpha_2 < \alpha_1$. But then Eq. (2.28) would be violated at $\alpha = \alpha_2$. Hence $\varphi \geqq \psi$ for all positive α, and the lemma is proved.

III. Digression on the Classical Heisenberg Model

Consider a classical Heisenberg version of the hierarchical model M_N. The Hamiltonian is still given by Eq. (1.8), (1.9), but each μ_j is now a unit vector free to vary over the surface of a sphere in 3-dimensional Euclidean space. The product $(S_{p,2r-1} S_{p,2r})$ in Eq. (1.8) means the scalar product of the two vectors. The long-range-order parameter m^2 of the infinite model M is defined by Eqs. (1.10)–(1.12) as before, with $(S_{p,r})^2$ meaning the square of the length of $S_{p,r}$.

The proof of Theorem 3 can be repeated almost word-for-word for the classical Heisenberg case. The scalar product $(s \cdot t)$ appears instead of st in Eqs. (2.4), (2.5), (2.8). Instead of Lemma 1 we use

Lemma 2. *Under the same hypotheses as in Lemma 1,*

$$[\langle\exp(\alpha x)\rangle/\langle x^{-1}\exp(\alpha x)\rangle] \geqq \hat{x}\coth\alpha\hat{x} . \tag{3.1}$$

Lemma 2 has the same physical meaning for the classical Heisenberg ferromagnet as Lemma 1 has for the Ising ferromagnet. Remarkably, the proof of Lemma 1 also applies without change to Lemma 2. When the average over the directions of the vectors s and t is taken in Eqs. (2.5) and (2.8), the resulting ratio is of the form of the left side of Eq. (3.1). Lemma 2 then gives instead of Eq. (2.11)

$$f_N(N) \geqq \tfrac{1}{2} f_N(N-1) + 2^{1-2N} \hat{x} \left[\coth(\alpha \hat{x}) - (\alpha \hat{x})^{-1}\right]. \tag{3.2}$$

From this it is simple to deduce

$$f_N(N) \geqq f_{N-1}(N-1) - (\beta b_N)^{-1}. \tag{3.3}$$

So we have proved

Theorem 5. *The classical Heisenberg hierarchical model has a phase-transition if*

$$B = \sum_{p=1}^{\infty} b_p^{-1} < \infty. \tag{3.4}$$

It has long-range order so long as

$$\beta > B. \tag{3.5}$$

This theorem can probably be proved in essentially the same way for the quantum Heisenberg model. It is considerably stronger than Theorem 7 of Dyson [5]. It also seems likely that for sequences b_p which are positive and increasing with p the condition (3.4) is necessary for a phase transition in Heisenberg hierarchical models. If so, we see here a qualitative difference in behavior between Ising and Heisenberg ferromagnets. In the Ising case, there exists a borderline model $b_p = J \log p$ which is the "weakest" ferromagnet for which a transition occurs, and this borderline model shows a Thouless effect. In the Heisenberg case, there exists no borderline model since there is no "most slowly converging" series (3.4). Thus we do not expect to find a Thouless effect in any one-dimensional Heisenberg hierarchical ferromagnet. The argument of Thouless [8] would also not lead one to expect a Thouless effect in a linear Heisenberg ferromagnet.

IV. Proof of Theorems 2 and 4

The basic idea of the proofs of Theorems 2 and 4 is the same as that which we used for Theorem 3. We seek to establish an inequality giving an upper bound for the mean-square sum of 2^{p+1} spins in terms of the mean-square sum of 2^p spins. According to Eq. (1.2) this means that we require an upper bound for $f_N(p+1)$ in terms of $f_N(p)$. In the case of

Theorem 3 it was sufficient to deal only with $f_N(N)$ and $f_{N-1}(N-1)$, since the monotonicity properties of $f_N(p)$ imply

$$m^2 \geqq \overline{\lim_{N\to\infty}} f_N(N).$$

However, an upper bound for m^2 cannot be obtained from the $f_N(N)$ alone. For Theorems 2 and 4 we are compelled to keep N and p as independent parameters in order to pass to the limit according to Eqs. (1.10), (1.11). We have to work with a block of 2^{p+1} spins deep inside a model M_N with $N > p$, and the details of the analysis thereby become much more complicated.

The tool which replaces Lemma 1 in the proof of Theorems 2 and 4 is the inequality (4.17) with (4.18) below. Eq. (4.17) has a physical interpretation similar to that of Lemma 1. The essential difference is that ξ_{Np} given by Eq. (4.18) is no longer a mean-square spin but involves the mean-fourth-powers of the total spins according to Eq. (4.20).

We begin the proofs of Theorems 2 and 4 together. Once Eq. (4.17) is reached, Theorem 2 can be disposed of easily. After proving Theorem 2, we shall first discuss the physical motivation for the additional steps which are required to prove Theorem 4, and then continue with the details.

Consider a finite hierarchical Ising model M_N with Hamiltonian (1.8). Choose an integer $p < N$, and let s, t be sums of neighbouring blocks of 2^p spins,

$$s = S_{p,2r-1}, \qquad t = S_{p,2r}, \tag{4.1}$$

$$s + t = S_{p+1,r}. \tag{4.2}$$

We write then

$$H_N = H_s + H_t + H_u - B_p st - (s+t)\,u. \tag{4.3}$$

Here H_s, H_t are the parts of the Hamiltonian referring to spins internal to s and t, and H_u is the part referring to spins external to $(s+t)$. Also

$$B_p = 2^{-1-2p}\, b_{p+1}, \tag{4.4}$$

and

$$u = \sum_j u_j \mu_j \tag{4.5}$$

is a sum over the spins external to $(s+t)$ with positive coefficients u_j. Further,

$$|u| \leqq \Sigma u_j \leqq U_p = \sum_{q=p+2}^{\infty} 2^{-q}\, b_q. \tag{4.6}$$

The partition-function of M_N can be written

$$Z_N = \sum_s \sum_t \sum_u Y(s)\ Y(t)\ W(u) \exp\left[\beta B_p st + \beta u\,(s+t)\right], \tag{4.7}$$

where $Y(s)$ and $W(u)$ are positive and even functions of s and u. Summing over the possible combinations of signs of s, t, u gives

$$Z_N = \sum_s \sum_t \sum_u Z(s,t,u),$$

$$\begin{aligned} Z(s,t,u) = {} & Y(s)\ Y(t)\ W(u)\left[\cosh\beta B_p st \cosh\beta us \cosh\beta ut\right. \\ & \left. + \sinh\beta B_p st \sinh\beta us \sinh\beta ut\right]. \end{aligned} \tag{4.8}$$

We write as in Dyson [5], using Eq. (1.12),

$$c_N(p+1) = 2 f_N(p+1) - f_N(p) = 2^{-2p} \langle st \rangle_N . \tag{4.9}$$

Here, by virtue of Eq. (4.7) and (4.8),

$$\begin{aligned} \langle st \rangle_N &= (Z_N)^{-1} \sum_s \sum_t \sum_u Y(s)\ Y(t)\ W(u)\, st \exp\left[\beta B_p st + \beta u\,(s+t)\right] \\ &= (Z_N)^{-1} \sum_s \sum_t \sum_u Z(s,t,u)\, st\ T(s,t,u), \end{aligned} \tag{4.10}$$

with

$$T(s,t,u) = \frac{\tanh\beta B_p st + \tanh\beta us \tanh\beta ut}{1 + \tanh\beta B_p st \tanh\beta us \tanh\beta ut}. \tag{4.11}$$

Both Z and stT are even functions of s, t, u, and so we may take s, t, u all positive in estimating (4.10).

Since $\log \tanh x$ is a concave function of $\log x$, we have

$$\tanh x \tanh y \leqq \tanh^2 (xy)^{\frac{1}{2}} < \tanh (xy)^{\frac{1}{2}} . \tag{4.12}$$

Therefore for positive s, t,

$$T(s,t,u) \leqq \tanh\left[\beta B_p st + \beta U_p (st)^{\frac{1}{2}}\right]. \tag{4.13}$$

Then Eq. (4.10) implies

$$\langle st \rangle_N \leqq \langle x \tanh(\beta B_p x + \beta U_p x^{\frac{1}{2}}) \rangle , \tag{4.14}$$

where $x = |st|$ is a random variable with the probability distribution

$$p(x) = (Z_N)^{-1} \sum\sum\sum_{|st| = x} Z(s,t,u). \tag{4.15}$$

Now we use the fact that

$$\tanh(\beta B_p x + \beta U_p x^{\frac{1}{2}}) \tag{4.16}$$

is a concave function of x for positive x. Therefore

$$\langle x \tanh(\beta B_p x + \beta U_p x^{\frac{1}{2}})\rangle \leqq \langle x\rangle \tanh(\beta B_p \xi_{Np} + \beta U_p \xi_{Np}^{\frac{1}{2}}), \quad (4.17)$$

with

$$\xi_{Np} = \langle x^2\rangle / \langle x\rangle . \quad (4.18)$$

The averages in Eq. (4.17) and (4.18) are taken with the probability-distribution (4.15). Thus

$$\langle x\rangle = \langle |st|\rangle_N \leqq \tfrac{1}{2}\langle s^2 + t^2\rangle_N = 2^{2p} f_N(p), \quad (4.19)$$

$$\langle x^2\rangle = \langle s^2 t^2\rangle_N , \quad (4.20)$$

and

$$\xi_{Np} \leqq \mathrm{Max}(|st|) = 2^{2p} . \quad (4.21)$$

Putting together Eqs. (4.9), (4.14), (4.17) and (4.19), we deduce

$$\begin{aligned} f_N(p+1) &\leqq \tfrac{1}{2} f_N(p)\,[1 + \tanh(\beta B_p \xi_{Np} + \beta U_p \xi_{Np}^{\frac{1}{2}})] \\ &= f_N(p)\,[1 + \exp(-2\beta(B_p \xi_{Np} + U_p \xi_{Np}^{\frac{1}{2}}))]^{-1} . \end{aligned} \quad (4.22)$$

Since $f_N(0) = 1$, this implies

$$f_N(p) \leqq \prod_{q=0}^{p-1} [1 + \exp(-2\beta(B_q \xi_{Nq} + U_q \xi_{Nq}^{\frac{1}{2}}))]^{-1} . \quad (4.23)$$

We can now rapidly proceed to prove Theorem 2. Suppose that Eq. (1.15) holds. Then for any fixed β and all $p \geqq p_0$, with p_0 depending on β, we have from Eq. (4.4) and (4.6),

$$2^{2p} B_p < (4\beta)^{-1} \log p , \quad (4.24)$$

$$2^p U_p < (4\beta)^{-1} \log p . \quad (4.25)$$

Then Eq. (4.23) with (4.21) implies

$$f_N(p) \leqq \prod_{q=p_0}^{p-1} [1 + q^{-1}]^{-1} = [p_0/p] . \quad (4.26)$$

From Eq. (1.10), (1.11) we have $m^2 = 0$, and Theorem 2 is proved.

The proof of Theorem 4 is also based on Eq. (4.23), but the argument is much more delicate. To explain the strategy of the proof, we first suppose that instead of Eq. (4.18) we had

$$\xi_{Np} = \langle x\rangle . \quad \text{(false!)} \quad (4.27)$$

For sufficiently large N and p we shall have roughly

$$f_N(p) \sim m^2 . \quad (4.28)$$

If we could use Eq. (4.27), then by Eqs. (1.19), (4.4), (4.6) and (4.19), we would find

$$B_q \xi_{Nq} + U_q \xi_{Nq}^{\frac{1}{2}} \sim \tfrac{1}{2} J(m^2 + m) \log q, \tag{4.29}$$

for large N and q. Hence Eq. (4.23) would become, in the limit $N, p \to \infty$,

$$m^2 \leqq \prod_{q=p_0}^{\infty} [1 + q^{-\beta J(m^2+m)}]^{-1}. \tag{4.30}$$

Eq. (4.30) is exactly what is required to produce a Thouless effect. At any temperature for which $m^2 > 0$, the product on the right must converge away from zero, and therefore

$$(m^2 + m)\beta J > 1. \tag{4.31}$$

This means that Eqs. (1.5), (1.6) must hold with

$$m_c > (kT_c/2J), \tag{4.32}$$

and so Theorem 4 would be proved.

Why cannot this simple argument be right? The trouble is that it should apply equally well when β is allowed to vary as N and p tend to infinity. Since $f_N(p)$ is a continuous function of β for any finite N and p, we can certainly choose a sequence β_{Np}, depending on N and p, such that

$$\operatorname*{Lim}_{p\to\infty} \operatorname*{Lim}_{N\to\infty} \beta_{Np} = \beta_c, \tag{4.33}$$

$$\operatorname*{Lim}_{p\to\infty} \operatorname*{Lim}_{N\to\infty} f_N(p) = \mu^2 \tag{4.34}$$

where μ is any number in the range $0 < \mu < m_c$. The argument would then give the result

$$\mu^2 \leqq \prod_{q=p_0}^{\infty} [1 + q^{-\beta_c J(\mu^2+\mu)}]^{-1}, \tag{4.35}$$

which is certainly false when μ is small but positive.

We see from this discussion that the true ξ_{Np} given by Eq. (4.18) must be considerably larger than $\langle x \rangle$ in the immediate neighbourhood of the phase-transition. Just at the transition the fluctuations of long-range order are very great, and so $\langle x^2 \rangle$ can be much larger than $\langle x \rangle^2$. Our strategy in proving Theorem 4 must be designed to make these fluctuations harmless. The key to the proof is the fact that the large fluctuations occur only over a range of temperature which tends to zero as $N, p \to \infty$. The strategy is therefore to integrate Eq. (4.23) over a finite range of temperatures, holding the range of integration fixed while passing to the limit $N, p \to \infty$. The technical complications of the proof come mainly from the need to transform Eq. (4.23) into an inequality which can be integrated explicitly with respect to β.

The proof of Theorem 4 is now resumed. To linearize Eq. (4.23), we use the fact that $\log[1+\exp(-x)]$ is a convex function of x. Thus Eq. (4.23) implies

$$\log f_N(p) \leqq -p\log(1+\exp(-y)) < -p[1+\exp(y)]^{-1}, \quad (4.36)$$

with

$$y = 2\beta p^{-1}\sum_{q=0}^{p-1}(B_q\xi_{Nq} + U_q\xi_{Nq}^{\frac{1}{2}}). \quad (4.37)$$

If we write

$$w = p^{-1}\sum_{q=0}^{p-1} B_q\xi_{Nq}, \quad (4.38)$$

$$z = p^{-1}\sum_{q=0}^{p-1}(U_q)^2(B_q)^{-1}, \quad (4.39)$$

then Eq. (4.37) with Cauchy's inequality gives

$$y \leqq 2\beta(w + (wz)^{\frac{1}{2}}). \quad (4.40)$$

Eqs. (4.36) and (4.40) imply respectively

$$y > \log[-1 - (p/\log f_N(p))], \quad (4.41)$$

and

$$w \geqq z[y/(y+2\beta z)]^2. \quad (4.42)$$

Eq. (4.38) and (4.42) give the desired linear inequality involving the ξ_{Nq}.

We now return to Eq. (4.18), (4.20). From the form of the partition function (4.7) it follows that

$$\langle s^2t^2\rangle_N = [\langle st\rangle_N]^2 + D\langle st\rangle_N, \quad (4.43)$$

where

$$D = [\partial/\partial(\beta B_p)] \quad (4.44)$$

means a differentiation with respect to βB_p as it appears in Eq. (4.7), leaving all other interaction constants fixed. Now

$$(\partial/\partial\beta)\langle st\rangle_N = B_p D\langle st\rangle_N + R, \quad (4.45)$$

where R is a sum of derivatives of $\langle st\rangle_N$ with respect to the other interaction constants. According to the inequalities of Griffiths [8], R is non-negative. Eqs. (4.18), (4.19), (4.20), (4.43) and (4.45) then give

$$\xi_{Np} \leqq \langle st\rangle_N + B_p^{-1}(\partial/\partial\beta)\log\langle st\rangle_N. \quad (4.46)$$

Eq. (4.38) with (4.9) then implies

$$w \leqq p^{-1} \sum_{q=0}^{p-1} \{2^{2q} B_q c_N(q+1) + (\partial/\partial\beta) \log c_N(q+1)\} . \qquad (4.47)$$

We integrate Eqs. (4.42) and (4.47) with respect to β over the interval

$$\beta_1 < \beta < \beta_2 = \beta_1 + \alpha . \qquad (4.48)$$

The correlation-functions $f_N(p)$ and $c_N(q+1)$ are monotone non-decreasing functions of β. The integration therefore gives

$$p^{-1} \sum_{q=0}^{p-1} \{2^{2q} B_q \alpha [c_N(q+1)]_{\beta_2} + [\log c_N(q+1)]_{\beta_2} - [\log c_N(q+1)]_{\beta_1}\}$$

$$\geqq \int_{\beta_1}^{\beta_2} w \, d\beta \geqq \alpha z [y_1/(y_1 + 2\beta_2 z)]^2 , \qquad (4.49)$$

with

$$y_1 = \log\{-1 - p/[\log f_N(p)]_{\beta_1}\} . \qquad (4.50)$$

Since Eq. (1.19) is assumed to hold, Theorem 3 ensures that there is a phase-transition at some critical β_c with

$$\beta_c J \leqq 8 \qquad (4.51)$$

and

$$m^2 > 0 \quad \text{for} \quad \beta > \beta_c . \qquad (4.52)$$

Let β_1 and β_2 be chosen arbitrarily subject to

$$\beta_c < \beta_1 < \beta_2 . \qquad (4.53)$$

Holding β_1 and β_2 fixed, we let first $N \to \infty$, then $p \to \infty$ in Eqs. (4.49), (4.50). In this limit

$$[f_N(p)]_{\beta_j} \to m_j^2 > 0 , \qquad (4.54)$$

$$[c_N(p)]_{\beta_j} \to m_j^2 > 0 , \qquad (4.55)$$

$$m_j^2 = [m^2]_{\beta_j} , \quad j = 1, 2, \qquad (4.56)$$

and the convergence of the logarithms in Eqs. (4.49), (4.50) is bounded so long as Eq. (4.53) holds. From Eqs. (4.4), (4.6), (4.39) with (1.19) we have

$$(2^{2p} B_p/(\log p)) \to \tfrac{1}{2} J , \qquad (4.57)$$

$$(2^p U_p/(\log p)) \to \tfrac{1}{2} J , \qquad (4.58)$$

$$(z/(\log p)) \to \tfrac{1}{2} J , \qquad (4.59)$$

as $p \to \infty$. In the limit, only the terms proportional to $(\alpha \log p)$ survive in Eqs. (4.49), (4.50). We thus obtain

$$m_2^2 \geqq [1 + \beta_2 J]^{-2}. \tag{4.60}$$

Although all reference to β_1 has now disappeared, it was essential in proving Eq. (4.60) that the quantities $\log c_N(p)$ and $\log f_N(p)$ remain bounded as $p \to \infty$ when $\beta = \beta_1$.

Since β_2 can take any value greater than β_c, while m_2 is a non-decreasing function of β_2, Eq. (4.60) implies

$$m \geqq [1 + \beta_c J]^{-1} \quad \text{for} \quad \beta > \beta_c. \tag{4.61}$$

We have thus proved Eq. (1.5) with m_c satisfying (1.20), and this completes the proof of Theorem 4.

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