An Isoperimetric Inequality for the N-Dimensional Free Membrane Problem

H. F. WEINBERGER

Communicated by C. LOEWNER

1. This paper is concerned with the eigenvalue problem

$$\Delta u + \mu u = 0 \quad \text{in} \quad R, \tag{1.1}$$

$$\partial u / \partial n = 0 \quad \text{on} \quad \dot{R}. \tag{1.2}$$

R is a region in N-dimensional Euclidean space and $\dot{R}$ is its boundary. If the eigenvalues are arranged in non-decreasing order

$$\mu_1 \leqq \mu_2 \leqq \cdots , \tag{1.3}$$

it is easily seen that

$$\mu_1 = 0. \tag{1.4}$$

The corresponding eigenfunction is u = constant. Thus, μ_2 is the first eigenvalue of interest.

For $N = 2$, μ_2 is proportional to the square of the lowest resonant frequency of vibration of a membrane R whose edges are free. It is also proportional to the square of the cutoff frequency of the lowest H-mode of a wave guide of cross section R [1]. For $N = 3$, μ_2 is proportional to the square of the lowest resonant frequency of an acoustical resonator R with perfectly rigid walls.

The following assertion was made by E. T. KORNHAUSER & I. STAKGOLD [1] and proved by G. SZEGÖ [2]: *For all two dimensional simply connected domains R of given area, the circle yields the maximum value of μ_2* . The proof of SZEGÖ leans heavily upon conformal mapping and hence cannot be extended to more than two dimensions.

In this paper we prove the following generalization of the above assertion:

J. Rational Mech. Anal., Vol. 5, No. 4 (1956).

Theorem. *For all N-dimensional domains R of given N-volume, the N-sphere yields the maximum value of μ_2 .*

For $N = 2$ we have an alternative of SZEGÖ's proof. The requirement of simple connectedness is eliminated. For $N = 3$ the above maximum property of the sphere was conjectured by L. I. SCHIFF (in conversation with G. SZEGÖ), and the problem was suggested to the author by Professor SZEGÖ.

It was noted by G. SZEGÖ and the author (in conversation) that SZEGÖ's proof actually yields the stronger result that among simply connected domains R of given area the circle minimizes the quantity $(1/\mu_2) + (1/\mu_3)$. This follows from the invariance of the Dirichlet integral under conformal mapping together with POINCARÉ's formulation of the eigenvalue problem [3]. The proof presented in this paper does not yield this result.

2. Let R be an N-dimensional region with N-volume V. Its boundary R is assumed to be sufficiently smooth for the eigenvalue problem (1.1), (1.2) to be well set. The eigenvalue μ_2 is given by the minimum principle

$$\mu_2 = \min \frac{\int_R |\text{grad } f|^2 \, dV}{\int_R f^2 \, dV}, \tag{2.1}$$

the minimum being taken with respect to all piecewise continuously differentiable functions f on R satisfying

$$\int_R f \, dV = 0. \tag{2.2}$$

Let ρ be the radius of the N-sphere of volume V and let $\bar{\mu}_2$ be the eigenvalue μ_2 for the sphere. This eigenvalue is easily seen to have multiplicity N with the corresponding eigenfunctions

$$g(r)x_i/r, \qquad i = 1, \cdots, N. \tag{2.3}$$

Here r is the distance from the origin, and the x_i are Cartesian coordinates. $g(r)$ satisfies the differential equation

$$g'' + \frac{N-1}{r} g' + \left(\bar{\mu}_2 - \frac{N-1}{r^2}\right) g = 0 \qquad 0 < r < \rho \tag{2.4}$$

and vanishes at $r = 0$. The point $r = \rho$ is the first zero of $g'(r)$. (The equation (2.4) comes from substituting (2.3) in (1.1).)

We define the function

$$G(r) = \begin{cases} g(r) & r \leqq \rho \\ g(\rho) & r > \rho \end{cases} \tag{2.5}$$

and substitute each of the functions

$$f = G(r)x_i/r \tag{2.6}$$

in (2.1). This is permissible providing the auxiliary condition (2.2) is satisfied.

We show that the origin may be chosen in such a way that

$$\int_R [G(r)x_i/r]\, dV = 0. \tag{2.7}$$

Consider the N-vector

$$\int_R [G(r)x_i/r]\, dV \tag{2.8}$$

as a function of the origin of the x_i coordinates. It is clearly a continuous vector field. If the origin is taken on the convex hull of R, the integral (2.8) is a mean value of the inward pointing vector x_i with respect to the positive weight function $G(r)/r$. Hence, on the convex hull of R the vector (2.8) points inward. It follows from BROUWER's fixed point theorem [4] that it must vanish for some point inside the convex hull of R. We fix our origin at this point so that equations (2.7) are satisfied.

Then we have, by direct calculation from (2.1) and (2.6),

$$\mu_2 \leqq \frac{\int_R [G'^2 x_i x_i/r^2 + G^2(1 - x_i x_i/r^2)/r^2]\, dV}{\int_R [G^2 x_i x_i/r^2]\, dV}. \tag{2.9}$$

We multiply each of these inequalities by the denominator on the right and sum the resulting inequalities. Thus we obtain

$$\mu_2 \leqq \frac{\int_R [G'^2(r) + (N-1)G^2(r)/r^2]\, dV}{\int_R G^2(r)\, dV}. \tag{2.10}$$

Let S be the sphere of radius ρ centered at the origin. Let R_1 be the intersection of S and R. Since ρ is the first zero of $g'(r)$, $G(r)$ is non-decreasing for $r > 0$. Thus,

$$\begin{aligned} \int_R G^2(r)\, dV &= \int_{R_1} G^2(r)\, dV + \int_{R-R_1} G^2(r)\, dV \\ &\geqq \int_{R_1} G^2(r)\, dV + G^2(\rho) \int_{R-R_1} dV, \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} \int_S G^2(r)\, dV &= \int_{R_1} G^2(r)\, dV + \int_{S-R_1} G^2(r)\, dV \\ &\leqq \int_{R_1} G^2(r)\, dV + G^2(\rho) \int_{S-R_1} dV. \end{aligned} \tag{2.12}$$

Since the volume of S equals that of R by definition, (2.11) and (2.12) yield(*)

$$\int_R G^2(r)\, dV \geqq \int_S G^2(r)\, dV = \int_S g^2(r)\, dV. \tag{2.13}$$

---

*This method of proof was given in a special case by E. SCHMIDT in the Schwarz Festschrift, Berlin, 1914.

Differentiating the integrand in the numerator of (2.10), we have

$$\frac{d}{dr}\left[G'^2 + (N-1)\frac{G^2}{r^2}\right] = 2G'G'' + 2(N-1)(rGG' - G^2)/r^3. \tag{2.14}$$

For $r > \rho$ this is clearly negative since G is constant there. For $r \leqq \rho$ we use the fact that $G(r) \equiv g(r)$ which satisfies (2.4) to show that

$$\frac{d}{dr}\left[G'^2 + (N-1)\frac{G^2}{r^2}\right] = -2\bar{\mu}_2 GG' - (N-1)(rG' - G)^2/r^3 < 0. \tag{2.15}$$

Thus, the integrand in the numerator is decreasing for $r > 0$ and we prove in the same way as we proved (2.13) that

$$\int_R \left[G'^2 + (N-1)\frac{G^2}{r^2}\right] dV \leqq \int_S \left[g'^2 + (N-1)\frac{g^2}{r^2}\right] dV. \tag{2.16}$$

The equality holds only if R is a sphere (except for a set of measure zero). Integration by parts yields

$$\int_S \left[g'^2 + (N-1)\frac{g^2}{r^2}\right] dV = \bar{\mu}_2 \int_S g^2\, dV. \tag{2.17}$$

Inserting this together with (2.16) and (2.13) in (2.10) yields

$$\mu_2 \leqq \bar{\mu}_2 . \tag{2.18}$$

This inequality is the theorem announced in the introduction. We note that equality holds only when R is a sphere.

This research was supported by the United States Air Force Office of Scientific Research.

Institute for Fluid Dynamics and Applied Mathematics
University of Maryland

Bibliography

[1] E. T. Kornhauser & I. Stakgold: A Variational Theorem for $\nabla^2 u + \lambda u = 0$ and its Application. *Journal of Mathematics and Physics*, **31** (1952), pp. 45–54.

[2] G. Szegö: Inequalities for Certain Eigenvalues of a Membrane of Given Area. *Journal of Rational Mechanics and Analysis*, **3** (1954), pp. 343–356.

[3] H. Poincaré: Sur les équations aux dérivées partielles de la physique mathématique. *American Journal of Mathematics*, **12** (1890), pp. 259–261.

[4] L. E. J. Brouwer: Über Abbildung von Mannigfaltigkeiten. *Mathematische Annalen*, **71** (1912), pp. 97–115.