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An isoperimetric inequality in the plane with a log-convex density

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Abstract Given a positive lower semi-continuous density f on $\mathbb{R}^2$ the weighted volume $V_f := f\mathscr{L}^2$ is defined on the $\mathscr{L}^2$-measurable sets in $\mathbb{R}^2$. The f-weighted perimeter of a set of finite perimeter E in $\mathbb{R}^2$ is written $P_f(E)$. We study minimisers for the weighted isoperimetric problem

$$I_f(v) := \inf\Big\{P_f(E) : E \text{ is a set of finite perimeter in } \mathbb{R}^2 \text{ and } V_f(E) = v\Big\}$$

for $v > 0$. Suppose f takes the form $f : \mathbb{R}^2 \to (0, +\infty); x \mapsto e^{h(|x|)}$ where $h : [0, +\infty) \to \mathbb{R}$ is a non-decreasing convex function. Let $v > 0$ and B a centred ball in $\mathbb{R}^2$ with $V_f(B) = v$. We show that B is a minimiser for the above variational problem and obtain a uniqueness result.

Keywords Isoperimetric problem · Log-convex density · Generalised mean curvature

Mathematics Subject Classification 49Q20

1 Introduction

Let f be a positive lower semi-continuous density on $\mathbb{R}^2$. The weighted volume $V_f := f\mathscr{L}^2$ is defined on the $\mathscr{L}^2$-measurable sets in $\mathbb{R}^2$. Let E be a set of finite perimeter in $\mathbb{R}^2$. The weighted perimeter of E is defined by

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$$P_f(E) := \int_{\mathbb{R}^2} f\, d|D\chi_E| \in [0, +\infty]. \tag{1.1}$$

We study minimisers for the weighted isoperimetric problem

$$I_f(v) := \inf\left\{P_f(E) : E \text{ is a set of finite perimeter in } \mathbb{R}^2 \text{ and } V_f(E) = v\right\} \tag{1.2}$$

for $v > 0$. To be more specific we suppose that f takes the form

$$f : \mathbb{R}^2 \to (0, +\infty); x \mapsto e^{h(|x|)} \tag{1.3}$$

where $h : [0, +\infty) \to \mathbb{R}$ is a non-decreasing convex function. Our first main result is the following. It contains the classical isoperimetric inequality (cf. [9,12]) as a special case; namely, when h is constant on $[0, +\infty)$.

Theorem 1.1 *Let f be as in (1.3) where $h : [0, +\infty) \to \mathbb{R}$ is a non-decreasing convex function. Let $v > 0$ and B a centred ball in $\mathbb{R}^2$ with $V_f(B) = v$. Then B is a minimiser for (1.2).*

For $x \geq 0$ and $v \geq 0$ define the directional derivative of h in direction v by

$$h'_+(x, v) := \lim_{t\downarrow 0} \frac{h(x + tv) - h(x)}{t} \in \mathbb{R}$$

and define $h'_-(x, v)$ similarly for $x > 0$ and $v \leq 0$. We introduce the notation

$$\rho_+ := h'_+(\cdot, +1),\ \rho_- := -h'_+(\cdot, -1) \text{ and } \rho := (1/2)(\rho_+ + \rho_-)$$

on $(0, +\infty)$. The function h is locally of bounded variation and is differentiable a.e. with $h' = \rho$ a.e. on $(0, +\infty)$. Our second main result is a uniqueness theorem.

Theorem 1.2 *Let f be as in (1.3) where $h : [0, +\infty) \to \mathbb{R}$ is a non-decreasing convex function. Suppose that $R := \inf\{\rho > 0\} \in [0, +\infty)$ and set $v_0 := V(B(0, R))$. Let $v > 0$ and E a minimiser for (1.2). The following hold:*

(i) if $v \leq v_0$ then E is a.e. equivalent to a ball B in $\overline{B}(0, R)$ with $V(B) = V(E)$;
(ii) if $v > v_0$ then E is a.e. equivalent to a centred ball B with $V(B) = V(E)$.

Theorem 1.1 is a generalisation of Conjecture 3.12 in [24] (due to K. Brakke) in the sense that less regularity is required of the density f: in the latter, h is supposed to be smooth on $(0, +\infty)$ as well as convex and non-decreasing. This conjecture springs in part from the observation that the weighted perimeter of a local volume-preserving perturbation of a centred ball is non-decreasing ([24] Theorem 3.10). In addition, the conjecture holds for log-convex Gaussian densities of the form $h : [0, +\infty) \to \mathbb{R}; t \mapsto e^{ct^2}$ with $c > 0$ ([3,24] Theorem 5.2). In subsequent work partial forms of the conjecture were proved in the literature. In [19] it is shown to hold for large v provided that h is uniformly convex in the sense that $h'' \geq 1$ on $(0, +\infty)$ (see [19] Corollary

6.8). A complemen tary result is contained in [11] Theorem 1.1 which establishes the conjecture for small v on condition that h'' is locally uniformly bounded away from zero on $[0, +\infty)$. The above-mentioned conjecture is proved in large part in [7] (see Theorem 1.1) in dimension $n \geq 2$ (see also [4]). There it is assumed that the function h is of class C^3 on $(0, +\infty)$ and is convex and even (meaning that h is the restriction of an even function on $\mathbb{R}$ to $[0, +\infty)$). A uniqueness result is also obtained ([7] Theorem 1.2). We obtain these results under weaker hypotheses in the 2-dimensional case and our proofs proceed along different lines.

We give a brief outline of the article. In Sect. 2 we discuss some preliminary material. In Sect. 3 we show that (1.2) admits an open minimiser E with C^1 boundary M (Theorem 3.8). The argument draws upon the regularity theory for almost minimal sets (cf. [27]) and includes an adaptation of [21] Proposition 3.1. In Sect. 4 it is shown that the boundary M is of class $C^{1,1}$ (and has weakly bounded curvature). This result is contained in [21] Corollary 3.7 (see also [8]) but we include a proof for completeness. This Section also includes the result that E may be supposed to possess spherical cap symmetry (Theorem 4.5). Section 5 contains further results on spherical cap symmetric sets useful in the sequel. The main result of Sect. 6 is Theorem 6.5 which shows that the generalised (mean) curvature is conserved along M in a weak sense. In Sect. 7 it is shown that there exist convex minimisers of (1.2). Sections 8 and 9 comprise an analytic interlude and are devoted to the study of solutions of the first-order differential equation that appears in Theorem 6.6 subject to Dirichlet boundary conditions. Section 9 for example contains a comparison theorem for solutions to a Ricatti equation (Theorem 9.15 and Corollary 9.16). These are new as far as the author is aware. Section 10 concludes the proof of our main theorems.

2 Some preliminaries

Geometric measure theory. We use $|\cdot|$ to signify the Lebesgue measure on $\mathbb{R}^2$ (or occasionally $\mathscr{L}^2$). Let E be a $\mathscr{L}^2$-measurable set in $\mathbb{R}^2$. The set of points in E with density $t \in [0, 1]$ is given by

$$E^t := \left\{ x \in \mathbb{R}^2 : \lim_{\rho \downarrow 0} \frac{|E \cap B(x, \rho)|}{|B(x, \rho)|} = t \right\}.$$

As usual $B(x, \rho)$ denotes the open ball in $\mathbb{R}^2$ with centre $x \in \mathbb{R}^2$ and radius $\rho > 0$. The set E^1 is the measure-theoretic interior of E while E^0 is the measure-theoretic exterior of E. The essential boundary of E is the set $\partial^\star E := \mathbb{R}^2 \backslash (E^0 \cup E^1)$.

Recall that an integrable function u on $\mathbb{R}^2$ is said to have bounded variation if the distributional derivative of u is representable by a finite Radon measure Du (cf. [1] Definition 3.1 for example) with total variation $|Du|$; in this case, we write $u \in BV(\mathbb{R}^2)$. The set E has finite perimeter if χ_E belongs to $BV_{loc}(\mathbb{R}^2)$. The reduced boundary $\mathscr{F}E$ of E is defined by

$$\mathscr{F}E := \Big\{ x \in \operatorname{supp}|D\chi_E| : \nu^E(x) := \lim_{\rho \downarrow 0} \frac{D\chi_E(B(x,\rho))}{|D\chi_E|(B(x,\rho))} \text{ exists in } \mathbb{R}^2 \text{ and } |\nu^E(x)| = 1 \Big\}$$

(cf. [1] Definition 3.54) and is a Borel set (cf. [1] Theorem 2.22 for example). We use $\mathscr{H}^k$ ($k \in [0, +\infty)$) to stand for k-dimensional Hausdorff measure. If E is a set of finite perimeter in $\mathbb{R}^2$ then

$$\mathscr{F}E \subset E^{1/2} \subset \partial^* E \text{ and } \mathscr{H}^1(\partial^* E \backslash \mathscr{F}E) = 0 \tag{2.1}$$

by [1] Theorem 3.61.

Let f be a positive locally Lipschitz density on $\mathbb{R}^2$. Let E be a set of finite perimeter and U a bounded open set in $\mathbb{R}^2$. The weighted perimeter of E relative to U is defined by

$$P_f(E, U) := \sup\left\{ \int_U \operatorname{div}(fX)\, dx : X \in C_c^\infty(U, \mathbb{R}^2), \|X\|_\infty \leq 1 \right\}.$$

By the Gauss–Green formula ([1] Theorem 3.36 for example) and a convolution argument,

$$\begin{aligned} P_f(E, U) &= \sup\left\{ \int_{\mathbb{R}^2} f\langle \nu^E, X\rangle\, d|D\chi_E| : X \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2), \right. \\ &\qquad \left. \operatorname{supp}[X] \subset U, \|X\|_\infty \leq 1 \right\} \\ &= \sup\left\{ \int_{\mathbb{R}^2} f\langle \nu^E, X\rangle\, d|D\chi_E| : X \in C_c(\mathbb{R}^2, \mathbb{R}^2), \right. \\ &\qquad \left. \operatorname{supp}[X] \subset U, \|X\|_\infty \leq 1 \right\} \\ &= \int_U f\, d|D\chi_E| \end{aligned} \tag{2.2}$$

where we have also used [1] Propositions 1.47 and 1.23.

Lemma 2.1 *Let φ be a C^1 diffeomeorphism of $\mathbb{R}^2$ which coincides with the identity map on the complement of a compact set and $E \subset \mathbb{R}^2$ with $\chi_E \in \mathrm{BV}(\mathbb{R}^2)$. Then*

(i) $\chi_{\varphi(E)} \in \mathrm{BV}(\mathbb{R}^2)$;
(ii) $\partial^\star \varphi(E) = \varphi(\partial^\star E)$;
(iii) $\mathscr{H}^1(\mathscr{F}\varphi(E) \Delta \varphi(\mathscr{F}E)) = 0$.

Proof Part *(i)* follows from [1] Theorem 3.16 as φ is a proper Lipschitz function. Given $x \in E^0$ we claim that $y := \varphi(x) \in \varphi(E)^0$. Let M stand for the Lipschitz constant of φ and L stand for the Lipschitz constant of φ^{-1}. Note that $B(y, r) \subset \varphi(B(x, Lr))$ for each $r > 0$. As φ is a bijection and using [1] Proposition 2.49,

$$\begin{aligned} |\varphi(E) \cap B(y, r)| &\leq |\varphi(E) \cap \varphi(B(x, Lr)| \\ &= |\varphi(E \cap B(x, Lr))| \leq M^2 |E \cap B(x, Lr)|. \end{aligned}$$

This means that

$$\frac{|\varphi(E) \cap B(y,r)|}{|B(y,r)|} \leq (LM)^2 \frac{|E \cap B(x, Lr)|}{|B(x, Lr)|}$$

for $r > 0$ and this proves the claim. This entails that $\varphi(E^0) \subset [\varphi(E)]^0$. The reverse inclusion can be seen using the fact that φ is a bijection. In summary $\varphi(E^0) = [\varphi(E)]^0$. The corresponding identity for E^1 can be seen in a similar way. These identities entail *(ii)*. From (2.1) and *(ii)* we may write $\mathscr{F}\varphi(E) \cup N_1 = \varphi(\mathscr{F}E) \cup \varphi(N_2)$ for $\mathscr{H}^1$-null sets N_1, N_2 in $\mathbb{R}^2$. Item *(iii)* follows. □

Curves with weakly bounded curvature. Suppose the open set E in $\mathbb{R}^2$ has C^1 boundary M. Denote by $n : M \to \mathbb{S}^1$ the inner unit normal vector field. Given $p \in M$ we choose a tangent vector $t(p) \in \mathbb{S}^1$ in such a way that the pair $\{t(p), n(p)\}$ forms a positively oriented basis for $\mathbb{R}^2$. There exists a local parametrisation $\gamma_1 : I \to M$ where $I = (-\delta, \delta)$ for some $\delta > 0$ of class C^1 with $\gamma_1(0) = p$. We always assume that γ_1 is parametrised by arc-length and that $\dot{\gamma}_1(0) = t(p)$ where the dot signifies differentiation with respect to arc-length. Let X be a vector field defined in some neighbourhood of p in M. Then

$$(D_t X)(p) := \frac{d}{ds}\Big|_{s=0} (X \circ \gamma_1)(s) \tag{2.3}$$

if this limit exists and the divergence $\operatorname{div}^M X$ of X along M at p is defined by

$$\operatorname{div}^M X := \langle D_t X, t \rangle \tag{2.4}$$

evaluated at p. Suppose that X is a vector field in $C^1(U, \mathbb{R}^2)$ where U is an open neighbourhood of p in $\mathbb{R}^2$. Then

$$\operatorname{div} X = \operatorname{div}^M X + \langle D_n X, n \rangle \tag{2.5}$$

at p. If $p \in M \backslash \{0\}$ let $\sigma(p)$ stand for the angle measured anti-clockwise from the position vector p to the tangent vector $t(p)$; $\sigma(p)$ is uniquely determined up to integer multiples of 2π.

Let E be an open set in $\mathbb{R}^2$ with $C^{1,1}$ boundary M. Let $x \in M$ and $\gamma_1 : I \to M$ a local parametrisation of M in a neighbourhood of x. There exists a constant $c > 0$ such that

$$|\dot{\gamma}_1(s_2) - \dot{\gamma}_1(s_1)| \leq c|s_2 - s_1|$$

for $s_1, s_2 \in I$; a constraint on average curvature (cf. [10,18]). That is, $\dot{\gamma}_1$ is Lipschitz on I. So $\dot{\gamma}_1$ is absolutely continuous and differentiable a.e. on I with

$$\dot{\gamma}_1(s_2) - \dot{\gamma}_1(s_1) = \int_{s_1}^{s_2} \ddot{\gamma}_1 \, ds \tag{2.6}$$

for any $s_1, s_2 \in I$ with $s_1 < s_2$. Moreover, $|\ddot{\gamma}_1| \leq c$ a.e. on I (cf. [1] Corollary 2.23). As $\langle \dot{\gamma}_1, \dot{\gamma}_1 \rangle = 1$ on I we see that $\langle \dot{\gamma}_1, \ddot{\gamma}_1 \rangle = 0$ a.e. on I. The (geodesic) curvature k_1 is then defined a.e. on I via the relation

$$\ddot{\gamma}_1 = k_1 n_1 \tag{2.7}$$

as in [18]. The curvature k of M is defined $\mathscr{H}^1$-a.e. on M by

$$k(x) := k_1(s) \tag{2.8}$$

whenever $x = \gamma_1(s)$ for some $s \in I$ and $k_1(s)$ exists. We sometimes write $H(\cdot, E) = k$.

Let E be an open set in $\mathbb{R}^2$ with C^1 boundary M. Let $x \in M$ and $\gamma_1 : I \to M$ a local parametrisation of M in a neighbourhood of x. In case $\gamma_1 \neq 0$ let θ_1 stand for the angle measured anti-clockwise from e_1 to the position vector γ_1 and σ_1 stand for the angle measured anti-clockwise from the position vector γ_1 to the tangent vector $t_1 = \dot{\gamma}_1$. Put $r_1 := |\gamma_1|$ on I. Then $r_1, \theta_1 \in C^1(I)$ and

$$\dot{r}_1 = \cos\sigma_1; \tag{2.9}$$

$$r_1\dot{\theta}_1 = \sin\sigma_1; \tag{2.10}$$

on I provided that $\gamma_1 \neq 0$. Now suppose that M is of class $C^{1,1}$. Let α_1 stand for the angle measured anti-clockwise from the fixed vector e_1 to the tangent vector t_1 (uniquely determined up to integer multiples of 2π). Then $t_1 = (\cos\alpha_1, \sin\alpha_1)$ on I so α_1 is absolutely continuous on I. In particular, α_1 is differentiable a.e. on I with $\dot{\alpha}_1 = k_1$ a.e. on I. This means that $\alpha_1 \in C^{0,1}(I)$. In virtue of the identities $r_1 \cos\sigma_1 = \langle \gamma_1, t_1 \rangle$ and $r_1 \sin\sigma_1 = -\langle \gamma_1, n_1 \rangle$ we see that σ_1 is absolutely continuous on I and $\sigma_1 \in C^{0,1}(I)$. By choosing an appropriate branch we may assume that

$$\alpha_1 = \theta_1 + \sigma_1 \tag{2.11}$$

on I. We may choose σ in such a way that $\sigma \circ \gamma_1 = \sigma_1$ on I.
Flows. Recall that a diffeomorphism $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ is said to be proper if $\varphi^{-1}(K)$ is compact whenever $K \subset \mathbb{R}^2$ is compact. Given $X \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$ there exists a 1-parameter group of proper C^∞ diffeomorphisms $\varphi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ as in [20] Lemma 2.99 that satisfy

$$\begin{aligned} \partial_t\varphi(t, x) &= X(\varphi(t, x)) \text{ for each } (t, x) \in \mathbb{R} \times \mathbb{R}^2; \\ \varphi(0, x) &= x \text{ for each } x \in \mathbb{R}^2. \end{aligned} \tag{2.12}$$

We often use φ_t to refer to the diffeomorphism $\varphi(t, \cdot) : \mathbb{R}^2 \to \mathbb{R}^2$.

Lemma 2.2 *Let $X \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$ and φ be the corresponding flow as above. Then*

(i) there exists $R \in C^\infty(\mathbb{R} \times \mathbb{R}^2, \mathbb{R}^2)$ and $K > 0$ such that

$$\varphi(t, x) = \begin{cases} x + tX(x) + R(t, x) & \textit{for } x \in \text{supp}[X]; \\ x & \textit{for } x \notin \text{supp}[X]; \end{cases}$$

where $|R(t,x)| \leq Kt^2$ *for* $(t,x) \in \mathbb{R} \times \mathbb{R}^2$*;*

(ii) there exists $R^{(1)} \in C^\infty(\mathbb{R} \times \mathbb{R}^2, M_2(\mathbb{R}))$ *and* $K_1 > 0$ *such that*

$$d\varphi(t,x) = \begin{cases} I + t dX(x) + R^{(1)}(t,x) & \textit{for } x \in \text{supp}[X]; \\ I & \textit{for } x \notin \text{supp}[X]; \end{cases}$$

where $|R^{(1)}(t,x)| \leq K_1 t^2$ *for* $(t,x) \in \mathbb{R} \times \mathbb{R}^2$*;*

(iii) there exists $R^{(2)} \in C^\infty(\mathbb{R} \times \mathbb{R}^2, \mathbb{R})$ *and* $K_2 > 0$ *such that*

$$J_2 d\varphi(t,x) = \begin{cases} 1 + t \operatorname{div} X(x) + R^{(2)}(t,x) & \textit{for } x \in \text{supp}[X]; \\ 1 & \textit{for } x \notin \text{supp}[X]; \end{cases}$$

where $|R^{(2)}(t,x)| \leq K_2 t^2$ *for* $(t,x) \in \mathbb{R} \times \mathbb{R}^2$.

Let $x \in \mathbb{R}^2$, v *a unit vector in* $\mathbb{R}^2$ *and* M *the line though* x *perpendicular to* v*. Then*

(iv) there exists $R^{(3)} \in C^\infty(\mathbb{R} \times \mathbb{R}^2, \mathbb{R})$ *and* $K_3 > 0$ *such that*

$$J_1 d^M \varphi(t,x) = \begin{cases} 1 + t(\operatorname{div}^M X)(x) + R^{(3)}(t,x) & \textit{for } x \in \text{supp}[X]; \\ 1 & \textit{for } x \notin \text{supp}[X]; \end{cases}$$

where $|R^{(3)}(t,x)| \leq K_3 t^2$ *for* $(t,x) \in \mathbb{R} \times \mathbb{R}^2$.

Proof (i) First notice that $\varphi \in C^\infty(\mathbb{R} \times \mathbb{R}^2)$ by [16] Theorem 3.3 and Exercise 3.4. The statement for $x \notin \text{supp}[X]$ follows by uniqueness (cf. [16] Theorem 3.1); the assertion for $x \in \text{supp}[X]$ follows from Taylor's theorem. *(ii)* follows likewise: note, for example, that

$$[\partial_{tt} d\varphi]_{\alpha\beta}|_{t=0} = X^\alpha_{,\beta\delta} X^\delta + X^\alpha_{,\gamma} X^\gamma_{,\beta}$$

where the subscript $_,$ signifies partial differentiation. *(iii)* follows from *(ii)* and the definition of the 2-dimensional Jacobian (cf. [1] Definition 2.68). *(iv)* Using [1] Definition 2.68 together with the Cauchy–Binet formula [1] Proposition 2.69, $J_1 d^M \varphi(t,x) = |d\varphi(t,x)v|$ for $t \in \mathbb{R}$ and the result follows from *(ii)*. □

Let I be an open interval in $\mathbb{R}$ containing 0. Let $Z : I \times \mathbb{R}^2 \to \mathbb{R}^2$; $(t,x) \mapsto Z(t,x)$ be a continuous time-dependent vector field on $\mathbb{R}^2$ with the properties

(Z.1) $Z(t,\cdot) \in C^1_c(\mathbb{R}^2, \mathbb{R}^2)$ for each $t \in I$;
(Z.2) $\text{supp}[Z(t,\cdot)] \subset K$ for each $t \in I$ for some compact set $K \subset \mathbb{R}^2$.

By [16] Theorems I.1.1, I.2.1, I.3.1, I.3.3 there exists a unique flow $\varphi : I \times \mathbb{R}^2 \to \mathbb{R}^2$ such that

(F.1) $\varphi : I \times \mathbb{R}^2 \to \mathbb{R}^2$ is of class C^1;
(F.2) $\varphi(0,x) = x$ for each $x \in \mathbb{R}^2$;
(F.3) $\partial_t \varphi(t,x) = Z(t, \varphi(x,t))$ for each $(t,x) \in I \times \mathbb{R}^2$;
(F.4) $\varphi_t := \varphi(t,\cdot) : \mathbb{R}^2 \to \mathbb{R}^2$ is a proper diffeomorphism for each $t \in I$.

Lemma 2.3 *Let Z be a time-dependent vector field with the properties* $(Z.1)$–$(Z.2)$ *and φ be the corresponding flow. Then*

(i) for $(t, x) \in I \times \mathbb{R}^2$,

$$d\varphi(t, x) = \begin{cases} I + t dZ_0(x) + t R(t, x) & \text{for } x \in K; \\ I & \text{for } x \notin K; \end{cases}$$

where $\sup_K |R(t, \cdot)| \to 0$ *as* $t \to 0$.

Let $x \in \mathbb{R}^2$, v a unit vector in $\mathbb{R}^2$ and M the line though x perpendicular to v. Then

(ii) for $(t, x) \in I \times \mathbb{R}^2$,

$$J_1 d^M \varphi(t, x) = \begin{cases} 1 + t(\mathrm{div}^M Z_0)(x) + t R^{(1)}(t, x) & \text{for } x \in K; \\ 1 & \text{for } x \notin K. \end{cases}$$

where $\sup_K |R^{(1)}(t, \cdot)| \to 0$ *as* $t \to 0$.

Proof *(i)* We first remark that the flow $\varphi : I \times \mathbb{R}^2 \to \mathbb{R}^2$ associated to Z is continuously differentiable in t, x in virtue of (Z.1) by [16] Theorem I.3.3. Put $y(t, x) := d\varphi(t, x)$ for $(t, x) \in I \times \mathbb{R}^2$. By [16] Theorem I.3.3,

$$\dot{y}(t, x) = dZ(t, \varphi(t, x)) y(t, x)$$

for each $(t, x) \in I \times \mathbb{R}^2$ and $y(0, x) = I$ for each $x \in \mathbb{R}^2$ where I stands for the 2×2-identity matrix. For $x \in K$ and $t \in I$,

$$\begin{aligned} d\varphi(t, x) &= I + d\varphi(t, x) - d\varphi(0, x) \\ &= I + t\dot{y}(0, x) + t\left\{ \frac{d\varphi(t, x) - d\varphi(0, x)}{t} - \dot{y}(0, x) \right\} \\ &= I + t dZ(0, x) + t\left\{ \frac{y(t, x) - y(0, x)}{t} - \dot{y}(0, x) \right\} \\ &= I + t dZ_0(x) + t\left\{ \frac{y(t, x) - y(0, x)}{t} - \dot{y}(0, x) \right\}. \end{aligned}$$

Applying the mean-value theorem component-wise and using uniform continuity of the matrix $\dot{y}$ in its arguments we see that

$$\frac{y(t, \cdot) - y(0, \cdot)}{t} - \dot{y}(0, \cdot) \to 0$$

uniformly on K as $t \to 0$. This leads to *(i)*. Part *(ii)* follows as in Lemma 2.2. □

Let E be a set of finite perimeter in $\mathbb{R}^2$ with $V_f(E) < +\infty$. The first variation of weighted volume resp. perimeter along $X \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$ is defined by

$$\delta V_f(X) := \frac{d}{dt}\Big|_{t=0} V_f(\varphi_t(E)), \tag{2.13}$$

$$\delta P_f^+(X) := \lim_{t \downarrow 0} \frac{P_f(\varphi_t(E)) - P_f(E)}{t}, \tag{2.14}$$

whenever the limit exists. By Lemma 2.1 the f-perimeter in (2.14) is well-defined. *Convex functions.* Suppose that $h : [0, +\infty) \to \mathbb{R}$ is a convex function. For $x \geq 0$ and $v \geq 0$ define

$$h'_+(x, v) := \lim_{t \downarrow 0} \frac{h(x + tv) - h(x)}{t} \in \mathbb{R}$$

and define $h'_-(x, v)$ similarly for $x > 0$ and $v \leq 0$. For future use we introduce the notation

$$\rho_+ := h'(\cdot, +1), \, \rho_- := -h'(\cdot, -1) \text{ and } \rho := (1/2)(\rho_+ + \rho_-)$$

on $(0, +\infty)$. It holds that h is differentiable a.e. and $h' = \rho$ a.e. on $(0, +\infty)$. Define $[\rho] := \rho_+ - \rho_-$. Then $[\rho] \geq 0$ and vanishes a.e. on $(0, +\infty)$.

Lemma 2.4 *Suppose that the function f takes the form (1.3) where $h : [0, +\infty) \to \mathbb{R}$ is a convex function. Then*

(i) the directional derivative $f'_+(x, v)$ exists in $\mathbb{R}$ for each $x \in \mathbb{R}^2$ and $v \in \mathbb{R}^2$;
(ii) for $v \in \mathbb{R}^2$,

$$f'_+(x, v) = \begin{cases} f(x) h'_+(|x|, \operatorname{sgn}\langle x, v\rangle) \frac{|\langle x, v\rangle|}{|x|} & \text{for } x \in \mathbb{R}^2 \setminus \{0\}; \\ f(0) h'_+(0, +1)|v| & \text{for } x = 0; \end{cases}$$

(iii) if M is a C^1 hypersurface in $\mathbb{R}^2$ such that $\cos\sigma \neq 0$ on M then f is differentiable $\mathscr{H}^1$-a.e. on M and

$$(\nabla f)(x) = f(x) \rho(|x|) \frac{\langle x, \cdot \rangle}{|x|}$$

for $\mathscr{H}^1$-a.e. $x \in M$.

Proof The assertion in *(i)* follows from the monotonicity of chords property while *(ii)* is straightforward. *(iii)* Let $x \in M$ and $\gamma_1 : I \to M$ be a C^1-parametrisation of M near x as above. Now $r_1 \in C^1(I)$ and $\dot{r_1}(0) = \cos\sigma(x) \neq 0$ so we may assume that $r_1 : I \to r_1(I) \subset (0, +\infty)$ is a C^1 diffeomorphism. The differentiability set $D(h)$ of h has full Lebesgue measure in $[0, +\infty)$. It follows by [1] Proposition 2.49 that $r_1^{-1}(D(h))$ has full measure in I. This entails that f is differentiable $\mathscr{H}^1$-a.e. on $\gamma_1(I) \subset M$. □

3 Existence and C^1 regularity

We start with an existence theorem.

Theorem 3.1 *Assume that f is a positive radial lower-semicontinuous non-decreasing density on $\mathbb{R}^2$ which diverges to infinity. Then for each $v > 0$,*

(i) (1.2) admits a minimiser;
(ii) any minimiser of (1.2) is essentially bounded.

Proof See [22] Theorems 3.3 and 5.9. □

But the bulk of this section will be devoted to a discussion of C^1 regularity.

Proposition 3.2 *Let f be a positive locally Lipschitz density on $\mathbb{R}^2$. Let $E \subset \mathbb{R}^2$ be a bounded set with finite perimeter. Let $X \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$. Then*

$$\delta V_f(X) = \int_E \operatorname{div}(fX)\,dx = -\int_{\mathscr{F}E} f\,\langle \nu^E, X\rangle\,d\mathscr{H}^1.$$

Proof Let $t \in \mathbb{R}$. By the area formula ([1] Theorem 2.71 and (2.74)),

$$V_f(\varphi_t(E)) = \int_{\varphi_t(E)} f\,dx = \int_E (f \circ \varphi_t)\,J_2 d(\varphi_t)_x\,dx \tag{3.1}$$

and

$$\begin{aligned} V_f(\varphi_t(E)) - V_f(E) &= \int_E (f \circ \varphi_t) J_2 d\varphi_t - f\,dx \\ &= \int_E (f \circ \varphi_t)(J_2 d\varphi_t - 1)\,dx + \int_E f \circ \varphi_t - f\,dx. \end{aligned}$$

The density f is locally Lipschitz and in particular differentiable a.e. on $\mathbb{R}^2$ (see [1] 2.3 for example). By the dominated convergence theorem and Lemma 2.2,

$$\begin{aligned} \delta V_f(X) &= \int_E \left\{ f \operatorname{div}(X) + \langle \nabla f, X\rangle \right\} dx = \int_E \operatorname{div}(fX)\,dx \\ &= -\int_{\mathscr{F}E} f\,\langle \nu^E, X\rangle\,d\mathscr{H}^1 \end{aligned}$$

by the generalised Gauss–Green formula [1] Theorem 3.36. □

Proposition 3.3 *Let f be a positive locally Lipschitz density on $\mathbb{R}^2$. Let $E \subset \mathbb{R}^2$ be a bounded set with finite perimeter. Let $X \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$. Then there exist constants $C > 0$ and $\delta > 0$ such that*

$$|P_f(\varphi_t(E)) - P_f(E)| \le C|t|$$

for $|t| < \delta$.

Proof Let $t \in \mathbb{R}$. By Lemma 2.1 and [1] Theorem 3.59,

$$P_f(\varphi_t(E)) = \int_{\mathbb{R}^2} f\, d|D\chi_{\varphi_t(E)}| = \int_{\mathscr{F}\varphi_t(E)} f\, d\mathscr{H}^1 = \int_{\varphi_t(\mathscr{F}E)} f\, d\mathscr{H}^1.$$

As $\mathscr{F}E$ is countably 1-rectifiable ([1] Theorem 3.59) we may use the generalised area formula [1] Theorem 2.91 to write

$$P_f(\varphi_t(E)) = \int_{\mathscr{F}E} (f \circ \varphi_t) J_1 d^{\mathscr{F}E}(\varphi_t)_x\, d\mathscr{H}^1.$$

For each $x \in \mathscr{F}E$ and any $t \in \mathbb{R}$,

$$|(f \circ \varphi_t)(x) - f(x)| \leq K|\varphi(t, x) - x| \leq K\|X\|_\infty |t|$$

where K is the Lipschitz constant of f on supp$[X]$. The result follows upon writing

$$P_f(\varphi_t(E)) - P_f(E) = \int_{\mathscr{F}E} (f \circ \varphi_t)(J_1 d^{\mathscr{F}E}(\varphi_t)_x - 1) + [f \circ \varphi_t - f]\, d\mathscr{H}^1 \tag{3.2}$$

and using Lemma 2.2. □

Lemma 3.4 *Let f be a positive locally Lipschitz density on $\mathbb{R}^2$. Let $E \subset \mathbb{R}^2$ be a bounded set with finite perimeter and $p \in \mathscr{F}E$. For any $r > 0$ there exists $X \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$ with* supp$[X] \subset B(p, r)$ *such that $\delta V_f(X) = 1$.*

Proof By (2.2) and [1] Theorem 3.59 and (3.57) in particular,

$$P_f(E, B(p, r)) = \int_{B(p,r)\cap\mathscr{F}E} f\, d\mathscr{H}^1 > 0$$

for any $r > 0$. By the variational characterisation of the f-perimeter relative to $B(p, r)$ we can find $Y \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$ with supp$[Y] \subset B(p, r)$ such that

$$0 < \int_{E\cap B(p,r)} \mathrm{div}(fY)\, dx = -\int_{\mathscr{F}E\cap B(p,r)} f\langle \nu^E, Y\rangle\, d\mathscr{H}^1 =: c$$

where we make use of the generalised Gauss–Green formula (cf. [1] Theorem 3.36). Put $X := (1/c)Y$. Then $X \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$ with supp$[X] \subset B(p, r)$ and $\delta V_f(X) = 1$ according to Proposition 3.2. □

Proposition 3.5 *Let f be a positive lower semi-continuous density on $\mathbb{R}^2$. Let U be a bounded open set in $\mathbb{R}^2$ with Lipschitz boundary. Let E, F_1, F_2 be bounded sets in $\mathbb{R}^2$ with finite perimeter. Assume that $E\Delta F_1 \subset\subset U$ and $E\Delta F_2 \subset\subset \mathbb{R}^2\setminus\overline{U}$. Define*

$$F := \Big[F_1 \cap U\Big] \cup \Big[F_2\setminus U\Big].$$

Then F is a set of finite perimeter in $\mathbb{R}^2$ and

$$P_f(E) + P_f(F) = P_f(F_1) + P_f(F_2).$$

Proof The function $\chi_E|_U \in \mathrm{BV}(U)$ and $D(\chi_E|_U) = (D\chi_E)|_U$. We write χ_E^U for the boundary trace of $\chi_E|_U$ (see [1] Theorem 3.87); then $\chi_E^U \in L^1(\partial U, \mathscr{H}^1 \llcorner \partial U)$ (cf. [1] Theorem 3.88). We use similar notation elsewhere. By [1] Corollary 3.89,

$$\begin{aligned}
D\chi_E &= D\chi_E \llcorner U + (\chi_E^U - \chi_E^{\mathbb{R}^2\setminus\overline{U}})\nu^U \mathscr{H}^1 \llcorner \partial U + D\chi_E \llcorner (\mathbb{R}^2\setminus\overline{U});\\
D\chi_F &= D\chi_{F_1} \llcorner U + (\chi_{F_1}^U - \chi_{F_2}^{\mathbb{R}^2\setminus\overline{U}})\nu^U \mathscr{H}^1 \llcorner \partial U + D\chi_{F_2} \llcorner (\mathbb{R}^2\setminus\overline{U});\\
D\chi_{F_1} &= D\chi_{F_1} \llcorner U + (\chi_{F_1}^U - \chi_E^{\mathbb{R}^2\setminus\overline{U}})\nu^U \mathscr{H}^1 \llcorner \partial U + D\chi_E \llcorner (\mathbb{R}^2\setminus\overline{U});\\
D\chi_{F_2} &= D\chi_E \llcorner U + (\chi_E^U - \chi_{F_2}^{\mathbb{R}^2\setminus\overline{U}})\nu^U \mathscr{H}^1 \llcorner \partial U + D\chi_{F_2} \llcorner (\mathbb{R}^2\setminus\overline{U}).
\end{aligned}$$

From the definition of the total variation measure ([1] Definition 1.4),

$$\begin{aligned}
|D\chi_E| &= |D\chi_E| \llcorner U + |\chi_E^U - \chi_E^{\mathbb{R}^2\setminus\overline{U}}| \mathscr{H}^1 \llcorner \partial U + |D\chi_E| \llcorner (\mathbb{R}^2\setminus\overline{U});\\
|D\chi_F| &= |D\chi_{F_1}| \llcorner U + |\chi_E^U - \chi_E^{\mathbb{R}^2\setminus\overline{U}}| \mathscr{H}^1 \llcorner \partial U + |D\chi_{F_2}| \llcorner (\mathbb{R}^2\setminus\overline{U});\\
|D\chi_{F_1}| &= |D\chi_{F_1}| \llcorner U + |\chi_E^U - \chi_E^{\mathbb{R}^2\setminus\overline{U}}| \mathscr{H}^1 \llcorner \partial U + |D\chi_E| \llcorner (\mathbb{R}^2\setminus\overline{U});\\
|D\chi_{F_2}| &= |D\chi_E| \llcorner U + |\chi_E^U - \chi_E^{\mathbb{R}^2\setminus\overline{U}}| \mathscr{H}^1 \llcorner \partial U + |D\chi_{F_2}| \llcorner (\mathbb{R}^2\setminus\overline{U});
\end{aligned}$$

where we also use the fact that $\chi_{F_1}^U = \chi_E^U$ as $E \Delta F_1 \subset\subset U$ and similarly for F_2. The result now follows. □

Proposition 3.6 *Assume that f is a positive locally Lipschitz density on $\mathbb{R}^2$. Let $v > 0$ and suppose that the set E is a bounded minimiser of (1.2). Let U be a bounded open set in $\mathbb{R}^2$. There exist constants $C > 0$ and $\delta > 0$ with the following property. For any $x \in U$ and $0 < r < \delta$,*

$$P_f(E) - P_f(F) \leq C\big|V_f(E) - V_f(F)\big| \tag{3.3}$$

where F is any set with finite perimeter in $\mathbb{R}^2$ such that $E \Delta F \subset\subset B(x, r)$.

Proof The proof follows that of [21] Proposition 3.1. We assume to the contrary that

$$(\forall\, C > 0)(\forall\, \delta > 0)(\exists\, x \in U)(\exists\, r \in (0, \delta))(\exists\, F \subset \mathbb{R}^2)$$

$$\Big[F \Delta E \subset\subset B(x, r) \wedge \Delta P_f > C|\Delta V_f|\Big] \tag{3.4}$$

in the language of quantifiers where we have taken some liberties with notation.

Choose $p_1, p_2 \in \mathscr{F}E$ with $p_1 \neq p_2$. Choose $r_0 > 0$ such that the open balls $B(p_1, r_0)$ and $B(p_2, r_0)$ are disjoint. Choose vector fields $X_j \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$ with $\text{supp}[X_j] \subset B(p_j, r_0)$ such that

$$\delta V_f(X_j) = 1 \text{ and } |P_f(\varphi_t^{(j)}(E)) - P_f(E)| \le a_j|t| \text{ for } |t| < \delta_j \text{ and } j = 1, 2 \tag{3.5}$$

as in Lemma 3.4 and Proposition 3.3. Put $a := \max\{a_1, a_2\}$. By (3.5),

$$V_f(\varphi_t^{(j)}(E)) - V_f(E) = t + o(t) \text{ as } t \to 0 \text{ for } j = 1, 2.$$

So there exist $\varepsilon > 0$ and $1 > \eta > 0$ such that

$$\begin{aligned} &t - \eta|t| < V_f(\varphi_t^{(j)}(E)) - V_f(E) < t + \eta|t|;\\ &|P_f(\varphi_t^{(j)}(E)) - P_f(E)| < (a+1)|t|; \end{aligned} \tag{3.6}$$

for $|t| < \varepsilon$ and $j = 1, 2$. In particular,

$$\begin{aligned} &|V_f(\varphi_t^{(j)}(E)) - V_f(E)| > (1-\eta)|t|;\\ &|P_f(\varphi_t^{(j)}(E)) - P_f(E)| < \frac{1+a}{1-\eta}|V_f(\varphi_t^{(j)}(E)) - V_f(E)| \text{ for } |t| < \varepsilon; \end{aligned} \tag{3.7}$$

for $|t| < \varepsilon$ and $j = 1, 2$.

In (3.4) choose $C = (1+a)/(1-\eta)$ and $\delta > 0$ such that

(a) $0 < 2\delta < \text{dist}(B(p_1, r_0), B(p_2, r_0))$,
(b) $\sup\{V_f(B(x, \delta)) : x \in U\} < (1-\eta)\,\varepsilon$.

Choose x, r and F_1 as in (3.4). In light of (a) we may assume that $B(x,r) \cap B(p_1, r_0) = \emptyset$. By (b),

$$|V_f(F_1) - V_f(E)| \le V_f(B(x,r)) \le V_f(B(x,\delta)) < (1-\eta)\,\varepsilon. \tag{3.8}$$

From (3.6) and (3.8) we can find $t \in (-\varepsilon, \varepsilon)$ such that with $F_2 := \varphi_t^{(1)}(E)$,

$$V_f(F_2) - V_f(E) = -\Big\{V_f(F_1) - V_f(E)\Big\} \tag{3.9}$$

by the intermediate value theorem. From (3.4),

$$P_f(F_1) < P_f(E) - C|V_f(F_1) - V_f(E)| \tag{3.10}$$

while from (3.7),

$$P_f(F_2) < P_f(E) + C|V_f(F_2) - V_f(E)|. \tag{3.11}$$

Let F be the set

$$F := \Big[F_1 \backslash B(p_1, r_0))\Big] \cup \Big[B(p_1, r_0) \cap F_2\Big].$$

Note that $E \Delta F_2 \subset\subset B(p_1, r_0)$. By Proposition 3.5, F is a bounded set of finite perimeter in $\mathbb{R}^2$ and

$$P_f(E) + P_f(F) = P_f(F_1) + P_f(F_2).$$

We then infer from (3.10), (3.11) and (3.9) that

$$\begin{aligned} P_f(F) &= P_f(F_1) + P_f(F_2) - P_f(E) \\ &< P_f(E) - C|V_f(F_1) - V_f(E)| + P_f(E) \\ &\quad + C|V_f(F_2) - V_f(E)| - P_f(E) \ = P_f(E). \end{aligned}$$

On the other hand, $V_f(F) = V_f(F_1) + V_f(F_2) - V_f(E) = V_f(E)$ by (3.9). We therefore obtain a contradiction to the f-isoperimetric property of E. □

Let E be a set of finite perimeter in $\mathbb{R}^2$ and U a bounded open set in $\mathbb{R}^2$. The minimality excess is the function ψ defined by

$$\psi(E, U) := P(E, U) - \nu(E, U) \tag{3.12}$$

where

$$\nu(E, U) := \inf\{P(F, U) : F \text{ is a set of finite perimeter with } F \Delta E \subset\subset U\}$$

as in [27] (1.9). We recall that the boundary of E is said to be almost minimal in $\mathbb{R}^2$ if for each bounded open set U in $\mathbb{R}^2$ there exists $T > 0$ and a positive constant K such that for every $x \in U$ and $r \in (0, T)$,

$$\psi(E, B(x, r)) \leq K r^2. \tag{3.13}$$

This definition corresponds to [27] Definition 1.5.

Theorem 3.7 *Assume that f is a positive locally Lipschitz density on $\mathbb{R}^2$. Let $v > 0$ and assume that E is a bounded minimiser of (1.2). Then the boundary of E is almost minimal in $\mathbb{R}^2$.*

Proof Let U be a bounded open set in $\mathbb{R}^2$ and $C > 0$ and $\delta > 0$ as in Proposition 3.6. The open δ-neighbourhood of U is denoted $I_\delta(U)$. Let $x \in U$ and $r \in (0, \delta)$. Put $V := I_{2\delta}(U)$. For the sake of brevity write $m := \inf_{B(x,r)} f$ and $M := \sup_{B(x,r)} f$. Let F be a set of finite perimeter in $\mathbb{R}^2$ such that $F \Delta E \subset\subset B(x, r)$. By Proposition 3.6,

$$
\begin{aligned}
&P(E, B(x,r)) - P(F, B(x,r)) \\
&\quad \le \frac{1}{m} P_f(E, B(x,r)) - \frac{1}{M} P_f(F, B(x,r)) \\
&\quad = \frac{1}{m}\Big(P_f(E, B(x,r)) - P_f(F, B(x,r))\Big) + \Big(\frac{1}{m} - \frac{1}{M}\Big) P_f(F, B(x,r)) \\
&\quad \le \frac{1}{m}\Big(P_f(E, B(x,r)) - P_f(F, B(x,r))\Big) + \frac{M-m}{m^2} P_f(F, B(x,r)) \\
&\quad \le \frac{C}{\inf_V f} |V_f(E) - V_f(F)| + (2Lr)\frac{\sup_V f}{(\inf_V f)^2} P(F, B(x,r)) \\
&\quad \le C\pi r^2 \frac{\sup_V f}{\inf_V f} + (2Lr)\frac{\sup_V f}{(\inf_V f)^2} P(F, B(x,r))
\end{aligned}
$$

where L stands for the Lipschitz constant of the restriction of f to V. We then derive that

$$
\psi(E, B(x,r)) \le C\pi r^2 \frac{\sup_V f}{\inf_V f} + (2Lr)\frac{\sup_V f}{(\inf_V f)^2} \nu(E, B(x,r)).
$$

By [13] (5.14), $\nu(E, B(x,r)) \le \pi r$. The inequality in (3.13) now follows. □

Theorem 3.8 *Assume that f is a positive locally Lipschitz density on $\mathbb{R}^2$. Let $v > 0$ and suppose that E is a bounded minimiser of (1.2). Then there exists a set $\widetilde{E} \subset \mathbb{R}^2$ such that*

(i) $\widetilde{E}$ is a bounded minimiser of (1.2);
(ii) $\widetilde{E}$ is equivalent to E;
(iii) $\widetilde{E}$ is open and $\partial \widetilde{E}$ is a C^1 hypersurface in $\mathbb{R}^2$.

Proof By [13] Proposition 3.1 there exists a Borel set F equivalent to E with the property that

$$
\partial F = \{x \in \mathbb{R}^2 : 0 < |F \cap B(x,\rho)| < \pi\rho^2 \text{ for each } \rho > 0\}.
$$

By Theorem 3.7 and [27] Theorem 1.9, ∂F is a C^1 hypersurface in $\mathbb{R}^2$ (taking note of differences in notation). The set

$$
\widetilde{E} := \{x \in \mathbb{R}^2 : |F \cap B(x,\rho)| = \pi\rho^2 \text{ for some } \rho > 0\}
$$

satisfies *(i)–(iii)*. □

4 Weakly bounded curvature and spherical cap symmetry

Theorem 4.1 *Assume that f is a positive locally Lipschitz density on $\mathbb{R}^2$. Let $v > 0$ and suppose that E is a bounded minimiser of (1.2). Then there exists a set $\widetilde{E} \subset \mathbb{R}^2$ such that*

(i) $\widetilde{E}$ is a bounded minimiser of (1.2);
(ii) $\widetilde{E}$ is equivalent to E;

(iii) $\widetilde{E}$ *is open and* $\partial\widetilde{E}$ *is a* $C^{1,1}$ *hypersurface in* $\mathbb{R}^2$.

Proof We may assume that E has the properties listed in Theorem 3.8. Put $M := \partial E$. Let $x \in M$ and U a bounded open set containing x. Choose $C > 0$ and $\delta > 0$ as in Proposition 3.6. Let $0 < r < \delta$ and $X \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$ with $\text{supp}[X] \subset B(x, r)$. Then

$$P_f(E) - P_f(\varphi_t(E)) \leq C|V_f(E) - V_f(\varphi_t(E))|$$

for each $t \in \mathbb{R}$. From the identity (3.2),

$$\begin{aligned}
-\int_M (f \circ \varphi_t)(J_1 d^M(\varphi_t)_x - 1)\, d\mathscr{H}^1 &\leq C|V_f(E) - V_f(\varphi_t(E))| \\
&\quad + \int_M [f \circ \varphi_t - f]\, d\mathscr{H}^1 \\
&\leq C|V_f(E) - V_f(\varphi_t(E))| + \sqrt{2}K\|X\|_\infty \mathscr{H}^1(M \cap \text{supp}[X])t
\end{aligned}$$

where K stands for the Lipschitz constant of f restricted to U. On dividing by t and taking the limit $t \to 0$ we obtain

$$-\int_M f \,\text{div}^M X \, d\mathscr{H}^1 \leq C\Big|\int_M f\langle n, X\rangle \, d\mathscr{H}^1\Big| + \sqrt{2}K\|X\|_\infty \mathscr{H}^1(M \cap \text{supp}[X])$$

upon using Lemma 2.2 and Proposition 3.2. Replacing X by $-X$ we derive that

$$\Big|\int_M f \,\text{div}^M X \, d\mathscr{H}^1\Big| \leq C_1\|X\|_\infty \mathscr{H}^1(M \cap \text{supp}[X])$$

where $C_1 = C\|f\|_{L^\infty(U)} + \sqrt{2}K$. Let $\gamma_1 : I \to M$ be a local C^1 parametrisation of M near x. Suppose that $Y \in C_c^1(I, \mathbb{R}^2)$ with $\text{supp}[Y] \subset I$ and that $\gamma_1(I) \subset M \cap B(x, r)$. Note that there exists $X \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$ with $\text{supp}[X] \subset B(x, r)$ such that $X \circ \gamma_1 = Y$ on I. The above estimate entails that

$$\Big|\int_I (f \circ \gamma_1)\langle \dot{Y}, t\rangle \, ds\Big| \leq C_1 \Big|\text{supp}[Y]\Big| \|Y\|_\infty.$$

This means that the function $(f \circ \gamma_1)t$ belongs to $\text{BV}(I)$ and this implies in turn that $t \in \text{BV}(I)$. For $s_1, s_2 \in I$ with $s_1 < s_2$,

$$\begin{aligned}
|t(s_2) - t(s_1)| = |Dt((s_1, s_2))| &\leq |Dt|((s_1, s_2)) \\
&= \sup\Big\{\int_{(s_1,s_2)} \langle t, \dot{Y}\rangle \, ds : Y \in C_c^1((s_1, s_2)) \text{ and } \|Y\|_\infty \leq 1\Big\} \\
&\leq c \sup\Big\{\int_{(s_1,s_2)} (f \circ \gamma_1)\langle t, \dot{Y}\rangle \, ds : Y \in C_c^1((s_1, s_2)) \text{ and } \|Y\|_\infty \leq 1\Big\} \\
&\leq cC_1|s_2 - s_1|
\end{aligned}$$

where $1/c = \inf_{\overline{U}} f > 0$. It follows that M is of class $C^{1,1}$. □

We turn to the topic of spherical cap symmetrisation. Denote by $\mathbb{S}^1_\tau$ the centred circle in $\mathbb{R}^2$ with radius $\tau > 0$. We sometimes write $\mathbb{S}^1$ for $\mathbb{S}^1_1$. Given $x \in \mathbb{R}^2$, $v \in \mathbb{S}^1$ and $\alpha \in (0, \pi]$ the open cone with vertex x, axis v and opening angle 2α is the set

$$C(x, v, \alpha) := \Big\{ y \in \mathbb{R}^2 : \langle y - x, v \rangle > |y - x| \cos \alpha \Big\}.$$

Let E be an $\mathscr{L}^2$-measurable set in $\mathbb{R}^2$ and $\tau > 0$. The τ-section E_τ of E is the set $E_\tau := E \cap \mathbb{S}^1_\tau$. Put

$$L(\tau) = L_E(\tau) := \mathscr{H}^1(E_\tau) \text{ for } \tau > 0 \tag{4.1}$$

and $p(E) := \{\tau > 0 : L(\tau) > 0\}$. The function L is $\mathscr{L}^1$-measurable by [1] Theorem 2.93. Given $\tau > 0$ and $0 < \alpha \leq \pi$ the spherical cap $C(\tau, \alpha)$ is the set

$$C(\tau, \alpha) := \begin{cases} \mathbb{S}^1_\tau \cap C(0, e_1, \alpha) & \text{if } 0 < \alpha < \pi; \\ \mathbb{S}^1_\tau & \text{if } \alpha = \pi; \end{cases}$$

and has $\mathscr{H}^1$-measure $s(\tau, \alpha) := 2\alpha\tau$. The spherical cap symmetral E^{sc} of the set E is defined by

$$E^{sc} := \bigcup_{\tau \in p(E)} C(\tau, \alpha) \tag{4.2}$$

where $\alpha \in (0, \pi]$ is determined by $s(\tau, \alpha) = L(\tau)$. Observe that E^{sc} is a $\mathscr{L}^2$-measurable set in $\mathbb{R}^2$ and $V_f(E^{sc}) = V_f(E)$. Note also that if B is a centred open ball then $B^{sc} = B \backslash \{0\}$. We say that E is spherical cap symmetric if $\mathscr{H}^1((E \Delta E^{sc})_\tau) = 0$ for each $\tau > 0$. This definition is broad but suits our purposes.

The result below is stated in [22] Theorem 6.2 and a sketch proof given. A proof along the lines of [2] Theorem 1.1 can be found in [23]. First, let B be a Borel set in $(0, +\infty)$; then the annulus $A(B)$ over B is the set $A(B) := \{x \in \mathbb{R}^2 : |x| \in B\}$.

Theorem 4.2 *Let E be a set of finite perimeter in $\mathbb{R}^2$. Then E^{sc} is a set of finite perimeter and*

$$P(E^{sc}, A(B)) \leq P(E, A(B)) \tag{4.3}$$

for any Borel set $B \subset (0, \infty)$ and the same inequality holds with E^{sc} replaced by any set F that is $\mathscr{L}^2$-equivalent to E^{sc}.

Corollary 4.3 *Let f be a positive lower semi-continuous radial function on $\mathbb{R}^2$. Let E be a set of finite perimeter in $\mathbb{R}^2$. Then $P_f(E^{sc}) \leq P_f(E)$.*

Proof Assume that $P_f(E) < +\infty$. We remark that f is Borel measurable as f is lower semi-continuous. Let (f_h) be a sequence of simple Borel measurable radial functions on $\mathbb{R}^2$ such that $0 \leq f_h \leq f$ and $f_h \uparrow f$ on $\mathbb{R}^2$ as $h \to \infty$. By Theorem 4.2,

$$P_{f_h}(E^{sc}) = \int_{\mathbb{R}^2} f_h \, d|D\chi_{E^{sc}}| \leq \int_{\mathbb{R}^2} f_h \, d|D\chi_E| = P_{f_h}(E)$$

for each h. Taking the limit $h \to \infty$ the monotone convergence theorem gives $P_f(E^{sc}) \leq P_f(E)$. □

Lemma 4.4 *Let E be an $\mathscr{L}^2$-measurable set in $\mathbb{R}^2$ such that $E \backslash \{0\} = E^{sc}$. Then there exists an $\mathscr{L}^2$-measurable set F equivalent to E such that*

(i) $\partial F = \{x \in \mathbb{R}^2 : 0 < |F \cap B(x, \rho)| < |B(x, \rho)|$ for any $\rho > 0\}$;
(ii) F is spherical cap symmetric.

Proof Put

$$E_1 := \{x \in \mathbb{R}^2 : |E \cap B(x, \rho)| = |B(x, \rho)| \text{ for some } \rho > 0\};$$
$$E_0 := \{x \in \mathbb{R}^2 : |E \cap B(x, \rho)| = 0 \text{ for some } \rho > 0\}.$$

We claim that E_1 is spherical cap symmetric. For take $x \in E_1$ with $\tau = |x| > 0$ and $|\theta(x)| \in (0, \pi]$. Now $|E \cap B(x, \rho)| = |B(x, \rho)|$ for some $\rho > 0$. Let $y \in \mathbb{R}^2$ with $|y| = \tau$ and $|\theta(y)| < |\theta(x)|$. Choose a rotation $O \in \mathrm{SO}(2)$ such that $OB(x, \rho) = B(y, \rho)$. As $E \backslash \{0\} = E^{sc}$, $|E \cap B(y, \rho)| = |O(E \cap B(x, \rho))| = |E \cap B(x, \rho)| = |B(x, \rho)| = |B(y, \rho)|$. The claim follows. It follows in a similar way that $\mathbb{R}^2 \backslash E_0$ is spherical cap symmetric. It can then be seen that the set $F := (E_1 \cup E) \backslash E_0$ inherits this property. As in [13] Proposition 3.1 the set F is equivalent to E and enjoys the property in *(i)*. □

Theorem 4.5 *Let f be as in (1.3) where $h : [0, +\infty) \to \mathbb{R}$ is a non-decreasing convex function. Given $v > 0$ let E be a bounded minimiser of (1.2). Then there exists an $\mathscr{L}^2$-measurable set $\widetilde{E}$ with the properties*

(i) $\widetilde{E}$ is a minimiser of (1.2);
(ii) $L_{\widetilde{E}} = L$ a.e. on $(0, +\infty)$;
(iii) $\widetilde{E}$ is open, bounded and has $C^{1,1}$ boundary;
(iv) $\widetilde{E} \backslash \{0\} = \widetilde{E}^{sc}$.

Proof Let E be a bounded minimiser for (1.2). Then $E_1 := E^{sc}$ is a bounded minimiser of (1.2) by Corollary 4.3 and $L_E = L_{E_1}$ on $(0, +\infty)$. Now put $E_2 := F$ with F as in Lemma 4.4. Then $L_{E_2} = L$ a.e. on $(0, +\infty)$ as E_2 is equivalent to E_1, E_2 is a bounded minimiser of (1.2) and E_2 is spherical cap symmetric. Moreover, $\partial E_2 = \{x \in \mathbb{R}^2 : 0 < |E_2 \cap B(x, \rho)| < |B(x, \rho)|$ for any $\rho > 0\}$. As in the proof of Theorem 3.8, ∂E_2 is a C^1 hypersurface in $\mathbb{R}^2$. Put

$$\widetilde{E} := \{x \in \mathbb{R}^2 : |E_2 \cap B(x, \rho)| = |B(x, \rho)| \text{ for some } \rho > 0\}.$$

Then $\widetilde{E}$ is equivalent to E_2 so that *(ii)* holds, and is a bounded minimiser of (1.2); $\widetilde{E}$ is open and $\partial \widetilde{E} = \partial E_2$ is C^1. In fact, $\partial \widetilde{E}$ is of class $C^{1,1}$ by Theorem 4.1. As E_2

is spherical cap symmetric the same is true of $\widetilde{E}$. But $\widetilde{E}$ is open which entails that $\widetilde{E}\backslash\{0\} = \widetilde{E}^{sc}$. $\square$

5 More on spherical cap symmetry

Let

$$H := \{x = (x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$$

stand for the open upper half-plane in $\mathbb{R}^2$ and

$$S : \mathbb{R}^2 \to \mathbb{R}^2; x = (x_1, x_2) \mapsto (x_1, -x_2)$$

for reflection in the x_1-axis. Let $O \in \mathrm{SO}(2)$ represent rotation anti-clockwise through $\pi/2$.

Lemma 5.1 *Let E be an open set in $\mathbb{R}^2$ with C^1 boundary M and assume that $E\backslash\{0\} = E^{sc}$. Let $x \in M\backslash\{0\}$. Then*

(i) $Sx \in M\backslash\{0\}$*;*
(ii) $n(Sx) = Sn(x)$*;*
(iii) $\cos\sigma(Sx) = -\cos\sigma(x)$.

Proof *(i)* The closure $\overline{E}$ of E is spherical cap symmetric. The spherical cap symmetral $\overline{E}$ is invariant under S from the representation (4.2). *(ii)* is a consequence of this last observation. *(iii)* Note that $t(Sx) = O^\star n(Sx) = O^\star Sn(x)$. Then

$$\begin{aligned}\cos\sigma(Sx) &= \langle Sx, t(Sx)\rangle = \langle Sx, O^\star Sn(x)\rangle = \langle x, SO^\star Sn(x)\rangle \\ &= \langle x, On(x)\rangle = -\langle x, O^\star n(x)\rangle = \cos\sigma(x)\end{aligned}$$

as $SO^\star S = O$ and $O = -O^\star$. $\square$

We introduce the projection $\pi : \mathbb{R}^2 \to [0, +\infty); x \mapsto |x|$.

Lemma 5.2 *Let E be an open set in $\mathbb{R}^2$ with boundary M and assume that $E\backslash\{0\} = E^{sc}$.*

(i) *Suppose $0 \neq x \in \mathbb{R}^2\backslash\overline{E}$ and $\theta(x) \in (0, \pi]$. Then there exists an open interval I in $(0, +\infty)$ containing τ and $\alpha \in (0, \theta(x))$ such that $A(I)\backslash\overline{S}(\alpha) \subset \mathbb{R}^2\backslash\overline{E}$.*
(ii) *Suppose $0 \neq x \in E$ and $\theta(x) \in [0, \pi)$. Then there exists an open interval I in $(0, +\infty)$ containing τ and $\alpha \in (\theta(x), \pi)$ such that $A(I) \cap S(\alpha) \subset E$.*
(iii) *For each $0 < \tau \in \pi(M)$, M_τ is the union of two closed spherical arcs in $\mathbb{S}^1_\tau$ symmetric about the x_1-axis.*

Proof *(i)* We can find $\alpha \in (0, \theta(x))$ such that $\mathbb{S}^1_\tau\backslash S(\alpha) \subset \mathbb{R}^2\backslash\overline{E}$ as can be seen from definition (4.2). This latter set is compact so $\mathrm{dist}(\mathbb{S}^1_\tau\backslash S(\alpha), \overline{E}) > 0$. This means that the ε-neighbourhood of $\mathbb{S}^1_\tau\backslash S(\alpha)$ is contained in $\mathbb{R}^2\backslash\overline{E}$ for $\varepsilon > 0$ small. The claim

follows. *(ii)* Again from (4.2) we can find $\alpha \in (\theta(x), \pi)$ such that $\overline{\mathbb{S}^1_\tau \cap S(\alpha)} \subset E$ and the assertion follows as before.
(iii) Suppose x_1, x_2 are distinct points in M_τ with $0 \leq \theta(x_1) < \theta(x_2) \leq \pi$. Suppose y lies in the interior of the spherical arc joining x_1 and x_2. If $y \in \mathbb{R}^2 \backslash \overline{E}$ then $x_2 \in \mathbb{R}^2 \backslash \overline{E}$ by *(i)* and hence $x_2 \notin M$. If $y \in E$ we obtain the contradiction that $x_1 \in E$ by *(ii)*. Therefore $y \in M$. We infer that the closed spherical arc joining x_1 and x_2 lies in M_τ. The claim follows noting that M_τ is closed. □

Lemma 5.3 *Let E be an open set in $\mathbb{R}^2$ with C^1 boundary M. Let $x \in M$. Then*

$$\liminf_{\overline{E} \ni y \to x} \left\langle \frac{y - x}{|y - x|}, n(x) \right\rangle \geq 0.$$

Proof Assume for a contradiction that

$$\liminf_{\overline{E} \ni y \to x} \left\langle \frac{y - x}{|y - x|}, n(x) \right\rangle \in [-1, 0).$$

There exists $\eta \in (0, 1)$ and a sequence (y_h) in E such that $y_h \to x$ as $h \to \infty$ and

$$\left\langle \frac{y_h - x}{|y_h - x|}, n(x) \right\rangle < -\eta \tag{5.1}$$

for each $h \in \mathbb{N}$. Choose $\alpha \in (0, \pi/2)$ such that $\cos\alpha = \eta$. As M is C^1 there exists $r > 0$ such that

$$B(x, r) \cap C(x, -n(x), \alpha) \cap E = \emptyset.$$

By choosing h sufficiently large we can find $y_h \in B(x, r)$ with the additional property that $y_h \in C(x, -n(x), \alpha)$ by (5.1). We are thus led to a contradiction. □

Lemma 5.4 *Let E be an open set in $\mathbb{R}^2$ with C^1 boundary M and assume that $E \backslash \{0\} = E^{sc}$. For each $0 < \tau \in \pi(M)$,*

(i) $|\cos\sigma|$ *is constant on* M_τ*;*
(ii) $\cos\sigma = 0$ *on* $M_\tau \cap \{x_2 = 0\}$*;*
(iii) $\langle Ox, n(x) \rangle \leq 0$ *for* $x \in M_\tau \cap H$
(iv) $\cos\sigma \leq 0$ *on* $M_\tau \cap H$*;*

and if $\cos\sigma \not\equiv 0$ *on* M_τ *then*

(v) $\tau \in p(E)$*;*
(vi) M_τ *consists of two disjoint singletons in* $\mathbb{S}^1_\tau$ *symmetric about the* x_1*-axis;*
(vii) $L(\tau) \in (0, 2\pi\tau)$*;*
(viii) $M_\tau = \{(\tau\cos(L(\tau)/2\tau), \pm\tau\sin(L(\tau)/2\tau)\}$.

Proof (i) By Lemma 5.2, M_τ is the union of two closed spherical arcs in $\mathbb{S}^1_\tau$ symmetric about the x_1-axis. In case $M_\tau \cap \overline{H}$ consists of a singleton the assertion follows from Lemma 5.1. Now suppose that $M_\tau \cap \overline{H}$ consists of a spherical arc in $\mathbb{S}^1_\tau$ with non-empty interior. It can be seen that $\cos\sigma$ vanishes on the interior of this arc as $0 = r_1' = \cos\sigma_1$ in a local parametrisation by (2.9). By continuity $\cos\sigma = 0$ on M_τ. *(ii)* follows from Lemma 5.1. *(iii)* Let $x \in M_\tau \cap H$ so $\theta(x) \in (0,\pi)$. Then $S(\theta(x)) \cap \mathbb{S}^1_\tau \subset \overline{E}$ as $\overline{E}$ is spherical cap symmetric. Then

$$0 \leq \lim_{S(\theta(x))\cap\mathbb{S}^1_\tau \ni y \to x} \left\langle \frac{y-x}{|y-x|}, n(x) \right\rangle = -\langle Ox, n(x)\rangle$$

by Lemma 5.3. *(iv)* The adjoint transformation $O^\star$ represents rotation clockwise through $\pi/2$. Let $x \in M_\tau \cap H$. By *(iii)*,

$$0 \geq \langle Ox, n(x)\rangle = \langle x, O^\star n(x)\rangle = \langle x, t(x)\rangle = \tau \cos\sigma(x)$$

and this leads to the result. *(v)* As $\cos\sigma \not\equiv 0$ on M_τ we can find $x \in M_\tau \cap H$. We claim that $\mathbb{S}^1_\tau \cap S(\theta(x)) \subset E$. For suppose that $y \in \mathbb{S}^1_\tau \cap S(\theta(x))$ but $y \notin E$. We may suppose that $0 \leq \theta(y) < \theta(x) < \pi$. If $y \in \mathbb{R}^2\backslash\overline{E}$ then $x \in \mathbb{R}^2\backslash\overline{E}$ by Lemma 5.2. On the other hand, if $y \in M$ then the spherical arc in H joining y to x is contained in M again by Lemma 5.2. This arc also has non-empty interior in $\mathbb{S}^1_\tau$. Now $\cos\sigma = 0$ on its interior so $\cos(\sigma(x)) = 0$ by *(i)* contradicting the hypothesis. A similar argument deals with *(vi)* and this together with *(v)* in turn entails *(vii)* and *(viii)*. □

Lemma 5.5 *Let E be an open set in $\mathbb{R}^2$ with C^1 boundary M and assume that $E\backslash\{0\} = E^{sc}$. Suppose that $0 \in M$. Then*

(i) $(\sin\sigma)(0+) = 0$;
(ii) $(\cos\sigma)(0+) = -1$.

Proof *(i)* Let γ_1 be a C^1 parametrisation of M in a neighbourhood of 0 with $\gamma_1(0) = 0$ as above. Then $n(0) = n_1(0) = e_1$ and hence $t(0) = t_1(0) = -e_2$. By Taylor's Theorem $\gamma_1(s) = \gamma_1(0) + t_1(0)s + o(s) = -e_2 s + o(s)$ for $s \in I$. This means that $r_1(s) = |\gamma_1(s)| = s + o(s)$ and

$$\cos\theta_1 = \frac{\langle e_1, \gamma_1\rangle}{r_1} = \frac{\langle e_1, \gamma_1\rangle}{s}\frac{s}{r_1} \to 0$$

as $s \to 0$ which entails that $(\cos\theta_1)(0-) = 0$. Now t_1 is continuous on I so $t_1 = -e_2 + o(1)$ and $\cos\alpha_1 = \langle e_1, t_1\rangle = o(1)$. We infer that $(\cos\alpha_1)(0-) = 0$. By (2.11), $\cos\alpha_1 = \cos\sigma_1\cos\theta_1 - \sin\sigma_1\sin\theta_1$ on I and hence $(\sin\sigma_1)(0-) = 0$. We deduce that $(\sin\sigma)(0+) = 0$. Item *(ii)* follows from *(i)* and Lemma 5.4. □

The set

$$\Omega := \pi\Big[(M\backslash\{0\}) \cap \{\cos\sigma \neq 0\}\Big] \tag{5.2}$$

plays an important rôle in the proof of Theorem 1.1.

Lemma 5.6 *Let E be an open set in $\mathbb{R}^2$ with C^1 boundary M and assume that $E\backslash\{0\} = E^{sc}$. Then Ω is an open set in $(0,+\infty)$.*

Proof Suppose $0 < \tau \in \Omega$. Choose $x \in M_\tau \cap \{\cos\sigma \neq 0\}$. Let $\gamma_1 : I \to M$ be a local C^1 parametrisation of M in a neighbourhood of x such that $\gamma_1(0) = x$ as before. By shrinking I if necessary we may assume that $r_1 \neq 0$ and $\cos\sigma_1 \neq 0$ on I. Then the set $\{r_1(s) : s \in I\} \subset \Omega$ is connected and so an interval in $\mathbb{R}$ (see for example [25] Theorems 6.A and 6.B). By (2.9), $r_1'(0) = \cos\sigma_1(0) = \cos\sigma(p) \neq 0$. This means that the set $\{r_1(s) : s \in I\}$ contains an open interval about τ. □

6 Generalised (mean) curvature

Given a set E of finite perimeter in $\mathbb{R}^2$ the first variation $\delta V_f(Z)$ resp. $\delta P_f^+(Z)$ of weighted volume and perimeter along a time-dependent vector field Z are defined as in (2.13) and (2.14).

Proposition 6.1 *Let f be as in (1.3) where $h : [0, +\infty) \to \mathbb{R}$ is a non-decreasing convex function. Let E be a bounded open set in $\mathbb{R}^2$ with C^1 boundary M. Let Z be a time-dependent vector field. Then*

$$\delta P_f^+(Z) = \int_M f_+'(\cdot, Z_0) + f \operatorname{div}^M Z_0 \, d\mathscr{H}^1$$

where $Z_0 := Z(0, \cdot) \in C_c^1(\mathbb{R}^2, \mathbb{R}^2)$.

Proof The identity (3.2) holds for each $t \in I$ with M in place of $\mathscr{F}E$. The assertion follows on appealing to Lemma 2.3 and Lemma 2.4 with the help of the dominated convergence theorem. □

Given $X, Y \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$ let ψ resp. χ stand for the 1-parameter group of C^∞ diffeomorphisms of $\mathbb{R}^2$ associated to the vector fields X resp. Y as in (2.12). Let I be an open interval in $\mathbb{R}$ containing the point 0. Suppose that the function $\sigma : I \to \mathbb{R}$ is C^1. Define a flow via

$$\varphi : I \times \mathbb{R}^2 \to \mathbb{R}^2; (t, x) \mapsto \chi(\sigma(t), \psi(t, x)).$$

Lemma 6.2 *The time-dependent vector field Z associated with the flow φ is given by*

$$Z(t, x) = \sigma'(t) Y(\chi(\sigma(t), \psi(t, x))) + d\chi(\sigma(t), \psi(t, x)) X(\psi(t, x)) \tag{6.1}$$

for $(t, x) \in I \times \mathbb{R}^2$ and satisfies (Z.1) and (Z.2).

Proof For $t \in I$ and $x \in \mathbb{R}^2$ we compute using (2.12),

$$\partial_t \varphi(t, x) = (\partial_t \chi)(\sigma(t), \psi(t, x)) \sigma'(t) + d\chi(\sigma(t), \psi(t, x)) \partial_t \psi(t, x)$$

and this gives (6.1). Put $K_1 := \operatorname{supp}[X]$, $K_2 := \operatorname{supp}[Y]$ and $K := K_1 \cup K_2$. Then $(Z.2)$ holds with this choice of K. □

Let E be a bounded open set in $\mathbb{R}^2$ with C^1 boundary M. Define $\Lambda := (M \backslash \{0\}) \cap \{\cos\sigma = 0\}$ and

$$\Lambda_1 := \{x \in M : \mathscr{H}^1(\Lambda \cap B(x,\rho)) = \mathscr{H}^1(M \cap B(x,\rho)) \text{ for some } \rho > 0\}. \quad (6.2)$$

For future reference put $\Lambda_1^{\pm} := \Lambda_1 \cap \{x \in M : \pm\langle x, n\rangle > 0\}$.

Lemma 6.3 *Let f be as in (1.3) where $h : [0, +\infty) \to \mathbb{R}$ is a non-decreasing convex function. Let E be a bounded open set in $\mathbb{R}^2$ with $C^{1,1}$ boundary M and suppose that $E\backslash\{0\} = E^{sc}$. Then*

(i) Λ_1 is a countable disjoint union of well-separated open circular arcs centred at 0;
(ii) $\mathscr{H}^1(\overline{\Lambda_1}\backslash\Lambda_1) = 0$;
(iii) f is differentiable $\mathscr{H}^1$-a.e. on $M\backslash\overline{\Lambda_1}$.

The term well-separated in *(i)* means the following: if Γ is an open circular arc in Λ_1 with $\Gamma \cap (\Lambda_1\backslash\Gamma) = \emptyset$ then $d(\Gamma, \Lambda_1\backslash\Gamma) > 0$.

Proof (i) Let $x \in \Lambda_1$ and $\gamma_1 : I \to M$ a $C^{1,1}$ parametrisation of M near x. By shrinking I if necessary we may assume that $\gamma_1(I) \subset M \cap B(x,\rho)$ with ρ as in (6.2). So $\cos\sigma = 0$ $\mathscr{H}^1$-a.e. on $\gamma_1(I)$ and hence $\cos\sigma_1 = 0$ a.e. on I. This means that $\cos\sigma_1 = 0$ on I as $\sigma_1 \in C^{0,1}(I)$ and that r_1 is constant on I by (2.9). Using (2.10) it can be seen that $\gamma_1(I)$ is an open circular arc centred at 0. By compactness of M it follows that Λ_1 is a countable disjoint union of open circular arcs centred on 0. The well-separated property flows from the fact that M is C^1. *(ii)* follows as a consequence of this property. *(iii)* Let $x \in M\backslash\overline{\Lambda_1}$ and $\gamma_1 : I \to M$ a $C^{1,1}$ parametrisation of M near x with properties as before. We assume that x lies in the upper half-plane H. By shrinking I if necessary we may assume that $\gamma_1(I) \subset (M\backslash\overline{\Lambda_1}) \cap H$. Let $s_1, s_2, s_3 \in I$ with $s_1 < s_2 < s_3$. Then $y := \gamma_1(s_2) \in M\backslash\overline{\Lambda_1}$. So $\mathscr{H}^1(M \cap \{\cos\sigma \neq 0\} \cap B(y,\rho)) > 0$ for each $\rho > 0$. This means that for small $\eta > 0$ the set $\gamma_1((s_2-\eta, s_2+\eta)) \cap \{\cos\sigma \neq 0\}$ has positive $\mathscr{H}^1$-measure. Consequently, $r_1(s_3) - r_1(s_1) = \int_{s_1}^{s_3} \cos\sigma_1\, ds < 0$ bearing in mind Lemma 5.4. This shows that r_1 is strictly decreasing on I. So h is differentiable a.e. on $r_1(I) \subset (0, +\infty)$ in virtue of the fact that h is convex and hence locally Lipschitz. This entails *(iii)*. □

Proposition 6.4 *Let f be as in (1.3) where $h : [0, +\infty) \to \mathbb{R}$ is a non-decreasing convex function. Given $v > 0$ let E be a minimiser of (1.2). Assume that E is a bounded open set in $\mathbb{R}^2$ with C^1 boundary M and suppose that $E\backslash\{0\} = E^{sc}$. Suppose that $M\backslash\overline{\Lambda_1} \neq \emptyset$. Then there exists $\lambda \in \mathbb{R}$ such that for any $X \in C_c^1(\mathbb{R}^2, \mathbb{R}^2)$,*

$$0 \leq \int_M \left\{ f'_+(\cdot, X) + f \operatorname{div}^M X - \lambda f \langle n, X\rangle \right\} d\mathscr{H}^1.$$

Proof Let $X \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$. Let $x \in M$ and $r > 0$ such that $M \cap B(x,r) \subset M\backslash\overline{\Lambda_1}$. Choose $Y \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$ with $\operatorname{supp}[Y] \subset B(x,r)$ as in Lemma 3.4. Let ψ resp. χ stand for the 1-parameter group of C^∞ diffeomorphisms of $\mathbb{R}^2$ associated to the vector fields X resp. Y as in (2.12). For each $(s,t)\in\mathbb{R}^2$ the set $\chi_s(\psi_t(E))$ is an open set in $\mathbb{R}^2$ with C^1 boundary and $\partial(\chi_s\circ\psi_t)(E)=(\chi_s\circ\psi_t)(M)$ by Lemma 2.1. Define

$$\begin{aligned}V(s,t) &:= V_f(\chi_t(\psi_s(E)))-V_f(E),\\ P(s,t) &:= P_f(\chi_t(\psi_s(E))),\end{aligned}$$

for $(s,t)\in\mathbb{R}^2$. We write $F=(\chi_t\circ\psi_s)(E)$. Arguing as in Proposition 3.2,

$$\begin{aligned}\partial_t V(s,t) &= \lim_{h\to 0}(1/h)\{V_f(\chi_h(F))-V_f(F)\}=\int_F \operatorname{div}(fY)\,dx\\ &= \int_E (\operatorname{div}(fY)\circ\chi_t\circ\psi_s)\; J_2 d(\chi_t\circ\psi_s)_x\,dx\end{aligned}$$

with an application of the area formula (cf. [1] Theorem 2.71). This last varies continuously in (s,t). The same holds for partial differentiation with respect to s. Indeed, put $\eta:=\chi_t\circ\psi_s$. Then noting that $J_2d(\eta\circ\psi_h)=(J_2d\eta)\circ\psi_h J_2d\psi_h$ and using the dominated convergence theorem,

$$\begin{aligned}\partial_s V(s,t) &= \lim_{h\to 0}(1/h)\Big\{V_f(\eta(\psi_h(E)))-V_f(\eta(E))\Big\}\\ &= \lim_{h\to 0}(1/h)\Big\{\int_E (f\circ\eta\circ\psi_h)J_2d(\eta\circ\psi_h)_x\,dx-\int_E (f\circ\eta)J_2d\eta_x\,dx\Big\}\\ &= \lim_{h\to 0}(1/h)\Big\{\int_E [(f\circ\eta\circ\psi_h)-(f\circ\eta)]J_2d(\eta\circ\psi_h)_x\,dx\\ &\quad+\int_E (f\circ\eta)[(J_2d\eta\circ\psi_h-J_2d\eta]J_2d\psi_h\,dx\\ &\quad+\int_E (f\circ\eta)J_2d\eta[J_2d\psi_h-1]\,dx\Big\}\\ &= \int_E\langle\nabla(f\circ\eta),X\rangle J_2d\eta_x\,dx+\int_E (f\circ\eta)\langle\nabla J_2d\eta,X\rangle\,dx\\ &\quad+\int_E (f\circ\eta)J_2d\eta\operatorname{div}X\,dx\end{aligned}$$

where the explanation for the last term can be found in the proof of Proposition 3.2. In this regard we note that $d(d\chi_t)$ (for example) is continuous on $I\times\mathbb{R}^2$ (cf. [1] Theorem 3.3 and Exercise 3.2) and in particular $\nabla J_2d\chi_t$ is continuous on $I\times\mathbb{R}^2$. The expression above also varies continuously in (s,t) as can be seen with the help of the dominated convergence theorem. This means that $V(\cdot,\cdot)$ is continuously differentiable on $\mathbb{R}^2$. Note that

$$\partial_t V(0,0)=\int_E \operatorname{div}(fY)\,dx=1$$

by choice of Y. By the implicit function theorem there exists $\eta > 0$ and a C^1 function $\sigma : (-\eta, \eta) \to \mathbb{R}$ such that $\sigma(0) = 0$ and $V(s, \sigma(s)) = 0$ for $s \in (-\eta, \eta)$; moreover,

$$\begin{aligned}\sigma'(0) = -\partial_s V(0,0) &= -\int_E \left\{ \langle \nabla f, X \rangle + f \operatorname{div} X \right\} dx \\ &= -\int_E \operatorname{div}(fX)\, dx = \int_M f \langle n, X \rangle \, d\mathscr{H}^1\end{aligned}$$

by the Gauss–Green formula (cf. [1] Theorem 3.36).

The mapping

$$\varphi : (-\eta, \eta) \times \mathbb{R}^2 \to \mathbb{R}^2; t \mapsto \chi(\sigma(t), \psi(t, x))$$

satisfies conditions (F.1)–(F.4) above with $I = (-\eta, \eta)$ where the associated time-dependent vector field Z is given as in (6.1) and satisfies (Z.1) and (Z.2); moreover, $Z_0 = Z(0, \cdot) = \sigma'(0)Y + X$. Note that $Z_0 = X$ on $M \backslash B(x, r)$.

The mapping $I \to \mathbb{R}; t \mapsto P_f(\varphi_t(E))$ is right-differentiable at $t = 0$ as can be seen from Proposition 6.1 and has non-negative right-derivative there. By Proposition 6.1 and Lemma 6.3,

$$\begin{aligned}0 \le \delta P_f^+(Z) &= \int_M f'_+(\cdot, Z_0) + f \operatorname{div}^M Z_0 \, d\mathscr{H}^1 \\ &= \int_{M \backslash \overline{\Lambda_1}} f'_+(\cdot, Z_0) + f \operatorname{div}^M Z_0 \, d\mathscr{H}^1 \\ &\quad + \int_{\overline{\Lambda_1}} f'_+(\cdot, X) + f \operatorname{div}^M X \, d\mathscr{H}^1 \\ &= \int_{M \backslash \overline{\Lambda_1}} \sigma'(0) \langle \nabla f, Y \rangle + \langle \nabla f, X \rangle \\ &\quad + \sigma'(0) f \operatorname{div}^M Y + f \operatorname{div}^M X \, d\mathscr{H}^1 \\ &\quad + \int_{\overline{\Lambda_1}} f'_+(\cdot, X) + f \operatorname{div}^M X \, d\mathscr{H}^1 \\ &= \int_M f'_+(\cdot, X) + f \operatorname{div}^M X \, d\mathscr{H}^1 \\ &\quad + \sigma'(0) \int_M f'_+(\cdot, Y) + f \operatorname{div}^M Y \, d\mathscr{H}^1. \end{aligned} \tag{6.3}$$

The identity then follows upon inserting the expression for $\sigma'(0)$ above with $\lambda = -\int_M f'_+(\cdot, Y) + f \operatorname{div}^M Y \, d\mathscr{H}^1$. The claim follows for $X \in C_c^1(\mathbb{R}^2, \mathbb{R}^2)$ by a density argument. □

Theorem 6.5 *Let f be as in (1.3) where $h : [0, +\infty) \to \mathbb{R}$ is a non-decreasing convex function. Given $v > 0$ let E be a minimiser of (1.2). Assume that E is a bounded open set in $\mathbb{R}^2$ with $C^{1,1}$ boundary M and suppose that $E \backslash \{0\} = E^{sc}$. Suppose that $M \backslash \overline{\Lambda_1} \neq \emptyset$. Then there exists $\lambda \in \mathbb{R}$ such that*

(i) $k + \rho \sin \sigma + \lambda = 0$ $\mathscr{H}^1$*-a.e. on* $M \backslash \overline{\Lambda_1}$*;*
(ii) $\rho_- - \lambda \leq k \leq \rho_+ - \lambda$ *on* Λ_1^+*;*
(iii) $-\rho_+ - \lambda \leq k \leq -\rho_- - \lambda$ *on* Λ_1^-.

The expression $k + \rho \sin \sigma$ is called the generalised (mean) curvature of M.

Proof (i) Let $x \in M$ and $r > 0$ such that $M \cap B(x, r) \subset M \backslash \overline{\Lambda_1}$. Choose $X \in C_c^1(\mathbb{R}^2, \mathbb{R}^2)$ with supp$[X] \subset B(x, r)$. We know from Lemma 6.3 that f is differentiable $\mathscr{H}^1$-a.e. on supp$[X]$. Let λ be as in Proposition 6.4. Replacing X by $-X$ we deduce from Proposition 6.4 that

$$0 = \int_M \Big\{ \langle \nabla f, X \rangle + f \operatorname{div}^M X - \lambda f \langle n, X \rangle \Big\} \, d\mathscr{H}^1.$$

The divergence theorem on manifolds (cf. [1] Theorem 7.34) holds also for $C^{1,1}$ manifolds. So

$$\begin{aligned} \int_M \langle \nabla f, X \rangle + f \operatorname{div}^M X \, d\mathscr{H}^1 &= \int_M \partial_n f \, \langle n, \; X \rangle + \langle \nabla^M f, \; X \rangle + f \operatorname{div}^M X \, d\mathscr{H}^1 \\ &= \int_M \partial_n f \, \langle n, \; X \rangle + \operatorname{div}^M (f X) \, d\mathscr{H}^1 \\ &= \int_M \partial_n f \, \langle n, \; X \rangle - H f \langle n, \; X \rangle \, d\mathscr{H}^1 \\ &= \int_M f u \, \{ \partial_n \log f - H \} \, d\mathscr{H}^1 \end{aligned}$$

where $u = \langle n, X \rangle$. Combining this with the equality above we see that

$$\int_M u f \, \{ \partial_n \log f - H - \lambda \} \, d\mathscr{H}^1 = 0$$

for all $X \in C_c^1(\mathbb{R}^2, \mathbb{R}^2)$. This leads to the result.
(ii) Let $x \in M$ and $r > 0$ such that $M \cap B(x, r) \subset \Lambda_1^+$. Let $\phi \in C^1(\mathbb{S}_r^1)$ with support in $\mathbb{S}_r^1 \cap B(x, r)$. We can construct $X \in C_c^1(\mathbb{R}^2, \mathbb{R}^2)$ with the property that $X = \phi n$ on $M \cap B(x, r)$. By Lemma 2.4,

$$f_+'(\cdot, X) = f h_+'(|x|, \operatorname{sgn}\langle x, X \rangle) |\langle n, X \rangle| = f h_+'(|x|, \operatorname{sgn} \phi \langle x, n \rangle) |\phi|$$

on Λ_1. Let us assume that $\phi \geq 0$. As $\langle \cdot, n \rangle > 0$ on Λ_1^+ we have that $f_+'(\cdot, X) = f \phi h_+'(|x|, +1) = f \phi \rho_+$ so by Proposition 6.4,

$$\begin{aligned} 0 &\leq \int_M \Big\{ f_+'(\cdot, X) + f \operatorname{div}^M X - \lambda f \langle n, X \rangle \Big\} d\mathscr{H}^1 \\ &= \int_M f \phi \Big\{ \rho_+ - k - \lambda \Big\} \, d\mathscr{H}^1. \end{aligned}$$

We conclude that $\rho_+ - k - \lambda \geq 0$ on $M \cap B(x, r)$. Now assume that $\phi \leq 0$. Then $f'_+(\cdot, X) = -f\phi h'_+(|x|, -1) = f\phi\rho_-$ so

$$0 \leq \int_M f\phi\Big\{\rho_- - k - \lambda\Big\}\, d\mathscr{H}^1$$

and hence $\rho_- - k - \lambda \leq 0$ on $M \cap B(x, r)$. This shows *(ii)*.
(iii) The argument is similar. Assume in the first instance that $\phi \geq 0$. Then $f'_+(\cdot, X) = f\phi h'_+(|x|, -1) = -f\phi\rho_-$ so

$$0 \leq \int_M f\phi\Big\{-\rho_- - k - \lambda\Big\}\, d\mathscr{H}^1.$$

We conclude that $-\rho_- - k - \lambda \geq 0$ on $M \cap B(x, r)$. Next suppose that $\phi \leq 0$. Then $f'_+(\cdot, X) = -f\phi h'_+(|x|, +1) = -f\phi\rho_+$ so

$$0 \leq \int_M f\phi\Big\{-\rho_+ - k - \lambda\Big\}\, d\mathscr{H}^1$$

and $-\rho_+ - k - \lambda \leq 0$ on $M \cap B(x, r)$. □

Let E be an open set in $\mathbb{R}^2$ with C^1 boundary M and assume that $E\backslash\{0\} = E^{sc}$ and that Ω is as in (5.2). Bearing in mind Lemma 5.4 we may define

$$\theta_2 : \Omega \to (0, \pi);\ \tau \mapsto L(\tau)/2\tau; \tag{6.4}$$

$$\gamma : \Omega \to M;\ \tau \mapsto (\tau\cos\theta_2(\tau), \tau\sin\theta_2(\tau)). \tag{6.5}$$

The function

$$u : \Omega \to [-1, 1];\ \tau \mapsto \sin(\sigma(\gamma(\tau))). \tag{6.6}$$

plays a key role.

Theorem 6.6 *Let f be as in (1.3) where $h : [0, +\infty) \to \mathbb{R}$ is a non-decreasing convex function. Given $v > 0$ let E be a bounded minimiser of (1.2). Assume that E is open with $C^{1,1}$ boundary M and that $E\backslash\{0\} = E^{sc}$. Suppose that $M\backslash\overline{\Lambda_1} \neq \emptyset$ and let λ be as in Theorem 6.5. Then $u \in C^{0,1}(\Omega)$ and*

$$u' + (1/\tau + \rho)u + \lambda = 0$$

a.e. on Ω.

Proof Let $\tau \in \Omega$ and x a point in the open upper half-plane such that $x \in M_\tau$. There exists a $C^{1,1}$ parametrisation $\gamma_1 : I \to M$ of M in a neighbourhood of x with $\gamma_1(0) = x$ as above. Put $u_1 := \sin\sigma_1$ on I. By shrinking the open interval I if necessary we may assume that $r_1 : I \to r_1(I)$ is a diffeomorphism and that $r_1(I) \subset\subset \Omega$. Note that $\gamma = \gamma_1 \circ r_1^{-1}$ and $u = u_1 \circ r_1^{-1}$ on $r_1(I)$. It follows that $u \in C^{0,1}(\Omega)$. By (2.9),

$$u' = \frac{\dot{u}_1}{\dot{r}_1} \circ r_1^{-1} = \dot{\sigma}_1 \circ r_1^{-1}$$

a.e. on $r_1(I)$. As $\dot{\alpha}_1 = k_1$ a.e. on I and using the identity (2.10) we see that $\dot{\sigma}_1 = \dot{\alpha}_1 - \dot{\theta}_1 = k_1 - (1/r_1)\sin\sigma_1$ a.e on I. Thus,

$$u' = k - (1/\tau)\sin(\sigma \circ \gamma) = k - (1/\tau)u$$

a.e. on $r_1(I)$. By Theorem 6.5 there exists $\lambda \in \mathbb{R}$ such that $k + \rho \sin\sigma + \lambda = 0$ $\mathscr{H}^1$-a.e. on M. So

$$u' = -\rho(\tau)u - \lambda - (1/\tau)u = -(1/\tau + \rho(\tau))u - \lambda$$

a.e. on $r_1(I)$. The result follows. □

Lemma 6.7 *Suppose that E is a bounded open set in $\mathbb{R}^2$ with C^1 boundary M and that $E\backslash\{0\} = E^{sc}$. Then*

(i) $\theta_2 \in C^1(\Omega)$;
(ii) $\theta_2' = -\frac{1}{\tau}\frac{u}{\sqrt{1-u^2}}$ *on* Ω.

Proof Let $\tau \in \Omega$ and x a point in the open upper half-plane such that $x \in M_\tau$. There exists a C^1 parametrisation $\gamma_1 : I \to M$ of M in a neighbourhood of x with $\gamma_1(0) = x$ as above. By shrinking the open interval I if necessary we may assume that $r_1 : I \to r_1(I)$ is a diffeomorphism and that $r_1(I) \subset\subset \Omega$. It then holds that

$$\theta_2 = \theta_1 \circ r_1^{-1} \text{ and } \sigma \circ \gamma = \sigma_1 \circ r_1^{-1}$$

on $r_1(I)$ by choosing an appropriate branch of θ_1. It follows that $\theta_2 \in C^1(\Omega)$. By the chain-rule, (2.10) and (2.9),

$$\begin{aligned}\theta_2' &= \frac{\dot{\theta}_1}{\dot{r}_1} \circ r_1^{-1} = \left(\frac{1}{r_1}\tan\sigma_1\right) \circ r_1^{-1} \\ &= (1/\tau)\tan(\sigma \circ \gamma)\end{aligned}$$

on $r_1(I)$. By Lemma 5.4, $\cos(\sigma \circ \gamma) = -\sqrt{1-u^2}$ on Ω. This entails *(ii)*. □

7 Convexity

Lemma 7.1 *Let E be a bounded open set in $\mathbb{R}^2$ with $C^{1,1}$ boundary M and assume that $E\backslash\{0\} = E^{sc}$. Put $d := \sup\{|x| : x \in M\} > 0$ and $b := (d, 0)$. Let $\gamma_1 : I \to M$ be a $C^{1,1}$ parametrisation of M near b with $\gamma_1(0) = b$. Then*

$$\lim_{\delta \downarrow 0} \left\{ \operatorname{ess\,sup}_{[-\delta,\delta]} k_1 \right\} \geq 1/d.$$

Proof For $s \in I$,

$$\gamma_1(s) = de_1 + se_2 + \int_0^s \left\{\dot{\gamma}_1(u) - \dot{\gamma}_1(0)\right\} du$$

and

$$\dot{\gamma}_1(u) - \dot{\gamma}_1(0) = \int_0^u k_1 n_1 \, dv$$

by (2.6). By the Fubini–Tonelli Theorem,

$$\gamma_1(s) = de_1 + se_2 + \int_0^s (s-u)k_1(u)n_1(u)\, du = de_1 + se_2 + R(s)$$

for $s \in I$. Assume for a contradiction that

$$\lim_{\delta \downarrow 0} \left\{\text{ess sup}_{[-\delta,\delta]} k_1\right\} < l < 1/d$$

for some $l \in \mathbb{R}$. Then we can find $\delta > 0$ such that $k_1 < l$ a.e. on $[-\delta, \delta]$. So

$$\langle R(s), e_1\rangle = \int_0^s (s-u)k_1(u)\langle n_1(u), e_1\rangle \, du > -(1/2)s^2 l(1+o(1))$$

as $s \downarrow 0$ and

$$\begin{aligned} r_1(s)^2 - d^2 &= 2d\langle R(s), e_1\rangle + s^2 + o(s^2) \\ &> -dls^2(1+o(1)) + s^2 + o(s^2) \end{aligned}$$

as $s \downarrow 0$. Alternatively,

$$\frac{r_1(s)^2 - d^2}{s^2} > 1 - dl + o(1).$$

As $1 - dl > 0$ we can find $s \in I$ with $r_1(s) > d$, contradicting the definition of d. □

Lemma 7.2 *Let f be as in (1.3) where $h : [0, +\infty) \to \mathbb{R}$ is a non-decreasing convex function. Given $v > 0$ let E be a bounded minimiser of (1.2). Assume that E is open with $C^{1,1}$ boundary M and that $E \setminus \{0\} = E^{sc}$. Suppose that $M \setminus \overline{\Lambda_1} \neq \emptyset$. Then $\lambda \leq -1/d - \rho_-(d) < 0$ with λ as in Theorem 6.5.*

Proof We write M as the disjoint union $M = (M \setminus \overline{\Lambda_1}) \cup \overline{\Lambda_1}$. Let b be as above. Suppose that $b \in \overline{\Lambda_1}$. Then $b \in \Lambda_1$; in fact, $b \in \Lambda_1^-$. By Theorem 6.5, $\lambda \leq -\rho_- - k$ at b. By Lemma 7.1, $\lambda \leq -1/d - \rho_-(d)$ upon considering an appropriate sequence in M converging to b. Now suppose that b lies in the open set $M \setminus \overline{\Lambda_1}$ in M. Let $\gamma_1 : I \to M$ be a $C^{1,1}$ parametrisation of M near b with $\gamma_1(I) \subset M \setminus \overline{\Lambda_1}$. By Theorem 6.5, $k_1 + \rho(r_1)\sin\sigma_1 + \lambda = 0$ a.e. on I. Now $\sin\sigma_1(s) \to 1$ as $s \to 0$. In light of Lemma 7.1, $1/d + \rho(d-) + \lambda \leq 0$ and $\lambda \leq -1/d - \rho_-(d)$. □

Theorem 7.3 *Let f be as in (1.3) where $h : [0, +\infty) \to \mathbb{R}$ is a non-decreasing convex function. Given $v > 0$ let E be a bounded minimiser of (1.2). Assume that E is open with $C^{1,1}$ boundary M and that $E \setminus \{0\} = E^{sc}$. Suppose that $M \setminus \overline{\Lambda_1} \neq \emptyset$. Then E is convex.*

Proof The proof runs along similar lines as [22] Theorem 6.5. By Theorem 6.5, $k + \rho \sin \sigma + \lambda = 0$ $\mathscr{H}^1$-a.e. on $M \setminus \overline{\Lambda_1}$. By Lemma 7.2,

$$0 \leq k + \rho_-(d) + \lambda \leq k - 1/d$$

and $k \geq 1/d$ $\mathscr{H}^1$-a.e. on $M \setminus \overline{\Lambda_1}$. On Λ_1^+, $k \geq \rho_- - \lambda \geq \rho_- + \rho_-(d) + 1/d > 0$; on the other hand, $k < 0$ on Λ_1^+. So in fact $\Lambda_1^+ = \emptyset$. If $b \in \Lambda_1^-$ then $k = 1/d$. On $\Lambda_1^- \cap B(0, d)$, $k \geq -\rho_+ - \lambda \geq -\rho_+ + \rho_-(d) + 1/d \geq 1/d$. Therefore $k \geq 1/d > 0$ $\mathscr{H}^1$-a.e. on M. The set E is then convex by a modification of [26] Theorem 1.8 and Proposition 1.4. It is sufficient that the function f (here α_1) in the proof of the former theorem is non-decreasing. □

8 A reverse Hermite–Hadamard inequality

Let $0 \leq a < b < +\infty$ and $\rho \geq 0$ be a non-decreasing bounded function on $[a, b]$. Let h be a primitive of ρ on $[a, b]$ so that $h \in C^{0,1}([a, b])$ and introduce the functions

$$\mathtt{f} : [a, b] \to \mathbb{R}; x \mapsto e^{h(x)}; \tag{8.1}$$

$$g : [a, b] \to \mathbb{R}; x \mapsto x\mathtt{f}(x). \tag{8.2}$$

Then

$$g' = (1/x + \rho)g = \mathtt{f} + g\rho \tag{8.3}$$

a.e. on (a, b). Define

$$m = m(\rho, a, b) := \frac{g(b) - g(a)}{\int_a^b g\, dt}. \tag{8.4}$$

If ρ takes the constant value $\mathbb{R} \ni \lambda \geq 0$ on $[a, b]$ we use the notation $m(\lambda, a, b)$ and we write $m_0 = m(0, a, b)$. A computation gives

$$m_0 = m(0, a, b) = A(a, b)^{-1} \tag{8.5}$$

where $A(a, b) := (a + b)/2$ stands for the arithmetic mean of a and b.

Lemma 8.1 *Let $0 \leq a < b < +\infty$ and $\rho \geq 0$ be a non-decreasing bounded function on $[a, b]$. Then $m_0 \leq m$.*

Proof Note that g is convex on $[a, b]$ as can be seen from (8.3). By the Hermite-Hadamard inequality (cf. [15,17]),

$$\frac{1}{b-a}\int_a^b g\,dt \le \frac{g(a)+g(b)}{2}. \tag{8.6}$$

The inequality $(b-a)(g(a)+g(b)) \le (a+b)(g(b)-g(a))$ entails

$$\int_a^b g\,dt \le \frac{a+b}{2}(g(b)-g(a))$$

and the result follows on rearrangement. □

Lemma 8.2 *Let* $0 \le a < b < +\infty$ *and* $\lambda > 0$. *Then* $m(\lambda, a, b) < \lambda + A(a, b)^{-1}$.

Proof First suppose that $\lambda = 1$ and take $h : [a, b] \to \mathbb{R}; t \mapsto t$. In this case,

$$\int_a^b g\,dt = \int_a^b te^t\,dt = (b-1)e^b - (a-1)e^a$$

and

$$m(1, a, b) = \frac{be^b - ae^a}{(b-1)e^b - (a-1)e^a}.$$

The inequality in the statement is equivalent to

$$(a+b)(be^b - ae^a) < ((b-1)e^b - (a-1)e^a)(2+a+b)$$

which in turn is equivalent to the statement $\tanh[(b-a)/2] < (b-a)/2$ which holds for any $b > a$.

For $\lambda > 0$ take $h : [a, b] \to \mathbb{R}; t \mapsto \lambda t$. Substitution gives

$$\int_a^b g\,dt = (1/\lambda)^2[(\lambda b - 1)e^{\lambda b} - (\lambda a - 1)e^{\lambda a}] \text{ and}$$
$$g(b) - g(a) = (1/\lambda)[\lambda be^{\lambda b} - \lambda ae^{\lambda a}]$$

so from above

$$m(\lambda, a, b) = \lambda m(1, \lambda a, \lambda b) < \lambda\left\{1 + A(\lambda a, \lambda b)^{-1}\right\} = \lambda + A(a, b)^{-1}.$$

□

Theorem 8.3 *Let* $0 \le a < b < +\infty$ *and* $\rho \ge 0$ *be a non-decreasing bounded function on* $[a, b]$. *Then*

(i) $m(\rho, a, b) \le \rho(b-) + A(a, b)^{-1}$;

(ii) equality holds if and only if $\rho \equiv 0$ *on* $[a, b)$.

Proof (i) Define $h := \int_a^{\cdot} \rho \, d\tau$ on $[a, b]$ so that $h' = \rho$ a.e. on (a, b). Define $h_1 : [a, b] \to \mathbb{R}; t \mapsto h(b) - \rho(b-)(b - t)$. Then $h_1(b) = h(b), h_1' = \rho(b-) \geq \rho = h'$ a.e. on (a, b) and hence $h \geq h_1$ on $[a, b]$. We derive

$$\int_a^b g \, dt = \int_a^b te^{h(t)} \, dt \geq \int_a^b te^{h_1(t)} \, dt = \int_a^b g_1 \, dt$$

and

$$\begin{aligned} g(b) - g(a) = be^{h(b)} - ae^{h(a)} &= be^{h_1(b)} - ae^{h(a)} \\ &\leq be^{h_1(b)} - ae^{h_1(a)} = g_1(b) - g_1(a) \end{aligned}$$

with obvious notation. This entails that $m(\rho, a, b) \leq m(\rho(b-), a, b)$ and the result follows with the help of Lemma 8.2.
(ii) Suppose that $\rho \not\equiv 0$ on $[a, b)$. If ρ is constant on $[a, b]$ the assertion follows from Lemma 8.2. Assume then that ρ is not constant on $[a, b)$. Then $h \not\equiv h_1$ on $[a, b]$ in the above notation and $\int_a^b te^{h(t)} \, dt > \int_a^b te^{h_1(t)} \, dt$ which entails strict inequality in *(i)*. □

With the above notation define

$$\hat{m} = \hat{m}(\rho, a, b) := \frac{g(a) + g(b)}{\int_a^b g \, dt}. \tag{8.7}$$

A computation gives

$$\hat{m}_0 := \hat{m}(0, a, b) = \frac{2}{b - a}. \tag{8.8}$$

Lemma 8.4 *Let* $0 \leq a < b < +\infty$ *and* $\rho \geq 0$ *be a non-decreasing bounded function on* $[a, b]$. *Then* $\hat{m} \geq \hat{m}_0$.

Proof This follows by the Hermite-Hadamard inequality (8.6). □

We prove a reverse Hermite-Hadamard inequality.

Theorem 8.5 *Let* $0 \leq a < b < +\infty$ *and* $\rho \geq 0$ *be a non-decreasing bounded function on* $[a, b]$. *Then*

(i) $(b - a)\hat{m}(\rho, a, b) \leq 2 + a\rho(a+) + b\rho(b-)$;
(ii) equality holds if and only if $\rho \equiv 0$ *on* $[a, b)$.

This last inequality can be written in the form

$$\frac{g(a) + g(b)}{2 + a\rho(a+) + b\rho(b-)} \leq \frac{1}{b - a} \int_a^b g \, dt;$$

comparing with (8.6) justifies naming this a reverse Hermite-Hadamard inequality.

Proof (i) We assume in the first instance that $\rho \in C^1((a,b))$. We prove the above result in the form

$$\int_a^b g\,dt \geq (b-a)\frac{g(a)+g(b)}{2+a\rho(a)+b\rho(b)}. \tag{8.9}$$

Put

$$w := \frac{(t-a)(g(a)+g)}{2+a\rho(a)+t\rho}$$

for $t \in [a,b]$ so that

$$\int_a^b w'\,dt = (b-a)\frac{g(a)+g(b)}{2+a\rho(a)+b\rho(b)}.$$

Then using (8.3),

$$\begin{aligned} w' &= \frac{(g(a)+g+(t-a)g')(2+a\rho(a)+t\rho)-(t-a)(g(a)+g)(\rho+t\rho')}{(2+a\rho(a)+t\rho)^2} \\ &= \frac{(g(a)-ag'+(2+t\rho)g)(2+a\rho(a)+t\rho)-(t-a)(g(a)+g)(\rho+t\rho')}{(2+a\rho(a)+t\rho)^2} \\ &= \frac{(2+t\rho)(2+a\rho(a)+t\rho)}{(2+a\rho(a)+t\rho)^2}g \\ &\quad + \frac{(g(a)-ag')(2+a\rho(a)+t\rho)-(t-a)(g(a)+g)(\rho+t\rho')}{(2+a\rho(a)+t\rho)^2} \\ &\leq g - \frac{2g(a)}{(2+a\rho(a)+b\rho(b))^2}(t-a)\rho \\ &\leq g \end{aligned} \tag{8.10}$$

on (a,b) as

$$g(a)-ag' = a(\mathtt{f}(a)-(1/t+\rho)g) = a(\mathtt{f}(a)-\mathtt{f}-\rho g) \leq 0.$$

An integration over $[a,b]$ gives the result.

Let us now assume that $\rho \geq 0$ is a non-decreasing bounded function on $[a,b]$. Extend ρ to $\mathbb{R}$ via

$$\widetilde{\rho}(t) := \begin{cases} \rho(a+) & \text{for } t \in (-\infty, a]; \\ \rho(t) & \text{for } t \in (a,b]; \\ \rho(b-) & \text{for } t \in (b,+\infty); \end{cases}$$

for $t \in \mathbb{R}$. Let $(\psi_\varepsilon)_{\varepsilon>0}$ be a family of mollifiers (see e.g. [1] 2.1) and set $\widetilde{\rho}_\varepsilon := \widetilde{\rho} \star \psi_\varepsilon$ on $\mathbb{R}$ for each $\varepsilon > 0$. Then $\widetilde{\rho}_\varepsilon \in C^\infty(\mathbb{R})$ and is non-decreasing on $\mathbb{R}$ for each $\varepsilon > 0$. Put $\rho_\varepsilon := \widetilde{\rho}_\varepsilon\,|_{[a,b]}$ for each $\varepsilon > 0$. Then $(\rho_\varepsilon)_{\varepsilon>0}$ converges to ρ in $L^1((a,b))$ by [1]

2.1 for example. Note that $h_\varepsilon := \int_a^\cdot \rho_\varepsilon\,dt \to h$ pointwise on $[a,b]$ as $\varepsilon \downarrow 0$ and that (h_ε) is uniformly bounded on $[a,b]$. Moreover, $\rho_\varepsilon(a) \to \rho(a+)$ and $\rho_\varepsilon(b) \to \rho(b-)$ as $\varepsilon \downarrow 0$. By the above result,

$$(b-a)\hat{m}(\rho_\varepsilon, a, b) \le 2 + a\rho_\varepsilon(a) + b\rho_\varepsilon(b)$$

for each $\varepsilon > 0$. The inequality follows on taking the limit $\varepsilon \downarrow 0$ with the help of the dominated convergence theorem.

(ii) We now consider the equality case. We claim that

$$(b-a)\frac{g(a)+g(b)}{2+a\rho(a+)+b\rho(b-)} \le \int_a^b g\,dt - \frac{2g(a)}{(2+a\rho(a+)+b\rho(b-))^2}\int_a^b (t-a)\rho\,dt; \tag{8.11}$$

this entails the equality condition in *(ii)*. First suppose that $\rho \in C^1((a,b))$. In this case the inequality in (8.10) implies (8.11) upon integration. Now suppose that $\rho \ge 0$ is a non-decreasing bounded function on $[a,b]$. Then (8.11) holds with ρ_ε in place of ρ for each $\varepsilon > 0$. The inequality for ρ follows by the dominated convergence theorem. □

9 Comparison theorems for first-order differential equations

Let $\mathscr{L}$ stand for the collection of Lebesgue measurable sets in $[0,+\infty)$. Define a measure μ on $([0,+\infty),\mathscr{L})$ by $\mu(dx) := (1/x)\,dx$. Let $0 \le a < b < +\infty$. Suppose that $u : [a,b] \to \mathbb{R}$ is an $\mathscr{L}^1$-measurable function with the property that

$$\mu(\{u > t\}) < +\infty \text{ for each } t > 0. \tag{9.1}$$

The distribution function $\mu_u : (0,+\infty) \to [0,+\infty)$ of u with respect to μ is given by

$$\mu_u(t) := \mu(\{u > t\}) \text{ for } t > 0.$$

Note that μ_u is right-continuous and non-increasing on $(0,\infty)$ and $\mu_u(t) \to 0$ as $t \to \infty$.

Let u be a Lipschitz function on $[a,b]$. Define

$$Z_1 := \{u \text{ differentiable and } u' = 0\},\ Z_2 := \{u \text{ not differentiable}\} \text{ and } Z := Z_1 \cup Z_2.$$

By [1] Lemma 2.96, $Z \cap \{u = t\} = \emptyset$ for $\mathscr{L}^1$-a.e. $t \in \mathbb{R}$ and hence $N := u(Z) \subset \mathbb{R}$ is $\mathscr{L}^1$-negligible. We make use of the coarea formula ([1] Theorem 2.93 and (2.74)),

$$\int_{[a,b]} \phi|u'|\,dx = \int_{-\infty}^{\infty}\int_{\{u=t\}} \phi\,d\mathscr{H}^0\,dt \tag{9.2}$$

for any $\mathscr{L}^1$-measurable function $\phi : [a, b] \to [0, \infty]$.

Lemma 9.1 *Let* $0 \le a < b < +\infty$ *and* u *a Lipschitz function on* $[a, b]$. *Then*

(i) $\mu_u \in \mathrm{BV}_{\mathrm{loc}}((0, +\infty))$;
(ii) $D\mu_u = -u_\sharp \mu$;
(iii) $D\mu_u^a = D\mu_u \llcorner ((0, +\infty)\backslash N)$;
(iv) $D\mu_u^s = D\mu_u \llcorner N$;
(v) $A := \Big\{t \in (0, +\infty) : \mathscr{L}^1(Z \cap \{u = t\}) > 0\Big\}$ *is the set of atoms of* $D\mu_u$ *and* $D\mu_u^j = D\mu_u \llcorner A$;
(vi) μ_u *is differentiable* $\mathscr{L}^1$*-a.e. on* $(0, +\infty)$ *with derivative given by*

$$\mu_u'(t) = -\int_{\{u=t\}\backslash Z} \frac{1}{|u'|} \frac{d\mathscr{H}^0}{\tau}$$

for $\mathscr{L}^1$*-a.e.* $t \in (0, +\infty)$;
(vii) $\mathrm{Ran}(u) \cap [0, +\infty) = \mathrm{supp}(D\mu_u)$.

The notation above $D\mu_u^a$, $D\mu_u^s$, $D\mu_u^j$ stands for the absolutely continuous resp. singular resp. jump part of the measure $D\mu_u$ (see [1] 3.2 for example).

Proof For any $\varphi \in C_c^\infty((0, +\infty))$ with $\mathrm{supp}[\varphi] \subset (\tau, +\infty)$ for some $\tau > 0$,

$$\begin{aligned}\int_0^\infty \mu_u \varphi' \, dt &= \int_{[a,b]} \varphi \circ u \, d\mu \\ &= \int_{[a,b]} \chi_{\{u>\tau\}} \varphi \circ u \, d\mu \end{aligned} \tag{9.3}$$

by Fubini's theorem; so $\mu_u \in \mathrm{BV}_{\mathrm{loc}}((0, +\infty))$ and $D\mu_u$ is the push-forward of μ under u, $D\mu_u = -u_\sharp \mu$ (cf. [1] 1.70). By (9.2),

$$\begin{aligned} D\mu_u \llcorner ((0, +\infty)\backslash N)(A) &= -\mu(\{u \in A\}\backslash Z) \\ &= -\int_A \int_{\{u=t\}\backslash Z} \frac{1}{|u'|} \frac{d\mathcal{H}^0}{\tau} \, dt \end{aligned}$$

for any $\mathscr{L}^1$-measurable set A in $(0, +\infty)$. In light of the above, we may identify $D\mu_u^a = D\mu_u \llcorner ((0, +\infty)\backslash N)$ and $D\mu_u^s = D\mu_u \llcorner N$. The set of atoms of $D\mu_u$ is defined by $A := \{t \in (0, +\infty) : D\mu_u(\{t\}) \neq 0\}$. For $t > 0$,

$$\begin{aligned} D\mu_u(\{t\}) = D\mu_u^s(\{t\}) &= (D\mu_u \llcorner N)((\{t\}) \\ &= -u_\sharp \mu(N \cap \{t\}) = -\mu(Z \cap \{u = t\}) \end{aligned}$$

and this entails *(v)*. The monotone function μ_u is a good representative within its equivalence class and is differentiable $\mathscr{L}^1$-a.e. on $(0, +\infty)$ with derivative given by the density of $D\mu_u$ with respect to $\mathscr{L}^1$ by [1] Theorem 3.28. Item *(vi)* follows from (9.2) and *(iii)*. Item *(vii)* follows from *(ii)*. □

Let $0 < a < b < +\infty$ and $\rho \geq 0$ be a non-decreasing bounded function on $[a, b]$. Let $\eta \in \{\pm 1\}^2$. We study solutions to the first-order linear ordinary differential equation

$$u' + (1/x + \rho)u + \lambda = 0 \text{ a.e. on } (a, b) \text{ with } u(a) = \eta_1 \text{ and } u(b) = \eta_2 \tag{9.4}$$

where $u \in C^{0,1}([a, b])$ and $\lambda \in \mathbb{R}$. In case $\rho \equiv 0$ on $[a, b]$ we use the notation u_0.

Lemma 9.2 *Let $0 < a < b < +\infty$ and $\rho \geq 0$ be a non-decreasing bounded function on $[a, b]$. Let $\eta \in \{\pm 1\}^2$. Then*

(i) there exists a solution (u, λ) of (9.4) with $u \in C^{0,1}([a, b])$ and $\lambda = \lambda_\eta \in \mathbb{R}$;
(ii) the pair (u, λ) in (i) is unique;
(iii) λ_η is given by

$$-\lambda_{(1,1)} = \lambda_{(-1,-1)} = m; \ \lambda_{(1,-1)} = -\lambda_{(-1,1)} = \hat{m};$$

(iv) if $\eta = (1, 1)$ or $\eta = (-1, -1)$ then u is uniformly bounded away from zero on $[a, b]$.

Proof *(i)* For $\eta = (1, 1)$ define $u : [a, b] \to \mathbb{R}$ by

$$u(t) := \frac{m \int_a^t g\, ds + g(a)}{g(t)} \text{ for } t \in [a, b] \tag{9.5}$$

with m as in (8.4). Then $u \in C^{0,1}([a, b])$ and satisfies (9.4) with $\lambda = -m$. For $\eta = (1, -1)$ set $u = (-\hat{m} \int_a^\cdot g\, ds + g(a))/g$ with $\lambda = \hat{m}$. The cases $\eta = (-1, -1)$ and $\eta = (-1, 1)$ can be dealt with using linearity. *(ii)* We consider the case $\eta = (1, 1)$. Suppose that (u_1, λ_1) resp. (u_2, λ_2) solve (9.4). By linearity $u := u_1 - u_2$ solves

$$u' + (1/x + \rho)u + \lambda = 0 \text{ a.e. on } (a, b) \text{ with } u(a) = u(b) = 0$$

where $\lambda = \lambda_1 - \lambda_2$. An integration gives that $u = (-\lambda \int_a^\cdot g\, ds + c)/g$ for some constant $c \in \mathbb{R}$ and the boundary conditions entail that $\lambda = c = 0$. The other cases are similar. *(iii)* follows as in *(i)*. *(iv)* If $\eta = (1, 1)$ then $u > 0$ on $[a, b]$ from (9.5) as $m > 0$. □

The boundary condition $\eta_1 \eta_2 = -1$.

Lemma 9.3 *Let $0 < a < b < +\infty$ and $\rho \geq 0$ be a non-decreasing bounded function on $[a, b]$. Let (u, λ) solve (9.4) with $\eta = (1, -1)$. Then*

(i) there exists a unique $c \in (a, b)$ with $u(c) = 0$;
(ii) $u' < 0$ a.e. on $[a, c]$ and u is strictly decreasing on $[a, c]$;
(iii) $D\mu_u^s = 0$.

Proof *(i)* We first observe that $u' \leq -\hat{m} < 0$ a.e. on $\{u \geq 0\}$ in view of (9.4). Suppose $u(c_1) = u(c_2) = 0$ for some $c_1, c_2 \in (a, b)$ with $c_1 < c_2$. We may assume that $u \geq 0$ on $[c_1, c_2]$. This contradicts the above observation. Item *(ii)* is plain. For any $\mathscr{L}^1$-measurable set B in $(0, +\infty)$, $D\mu_u^s(B) = \mu(\{u \in B\} \cap Z) = 0$ using Lemma 9.1 and *(ii)*. □

Lemma 9.4 *Let* $0 < a < b < +\infty$ *and* $\rho \geq 0$ *be a non-decreasing bounded function on* $[a, b]$. *Let* (u, λ) *solve* (9.4) *with* $\eta = (1, -1)$. *Assume that*

- *(a)* u *is differentiable at both* a *and* b *and that* (9.4) *holds there;*
- *(b)* $u'(a) < 0$ *and* $u'(b) < 0$*;*
- *(c)* ρ *is differentiable at* a *and* b.

Put $v := -u$. *Then*

- *(i)* $\int_{\{v=1\}\setminus Z_v} \frac{1}{|v'|} \frac{d\mathscr{H}^0}{\tau} \geq \int_{\{u=1\}\setminus Z_u} \frac{1}{|u'|} \frac{d\mathscr{H}^0}{\tau}$;
- *(ii)* *equality holds if and only if* $\rho \equiv 0$ *on* $[a, b)$.

Proof First, $\{u = 1\} = \{a\}$ by Lemma 9.3. Further $0 < -au'(a) = 1 + a[\hat{m} + \rho(a)]$ from (9.4). On the other hand $\{v = 1\} \supset \{b\}$ and $0 < bv'(b) = -1 + b[\hat{m} - \rho(b)]$. Thus

$$\int_{\{v=1\}\setminus Z_v} \frac{1}{|v'|} \frac{d\mathscr{H}^0}{\tau} - \int_{\{u=1\}\setminus Z_u} \frac{1}{|u'|} \frac{d\mathscr{H}^0}{\tau} \geq \frac{1}{-1 + b[\hat{m} - \rho(b)]} - \frac{1}{1 + a[\hat{m} + \rho(a)]}.$$

By Theorem 8.5, $0 \leq 2 + (a - b)\hat{m} + a\rho(a) + b\rho(b)$, noting that $\rho(a) = \rho(a+)$ in virtue of (c) and similarly at b. A rearrangement leads to the inequality. The equality assertion follows from Theorem 8.5. □

Theorem 9.5 *Let* $0 < a < b < +\infty$ *and* $\rho \geq 0$ *be a non-decreasing bounded function on* $[a, b]$. *Suppose that* (u, λ) *solves* (9.4) *with* $\eta = (1, -1)$ *and set* $v := -u$. *Assume that* $u > -1$ *on* $[a, b)$. *Then*

- *(i)* $-\mu_v' \geq -\mu_u'$ *for* $\mathscr{L}^1$*-a.e.* $t \in (0, 1)$*;*
- *(ii)* *if* $\rho \not\equiv 0$ *on* $[a, b)$ *then there exists* $t_0 \in (0, 1)$ *such that* $-\mu_v' > -\mu_u'$ *for* $\mathscr{L}^1$*-a.e.* $t \in (t_0, 1)$*;*
- *(iii)* *for* $t \in [-1, 1]$,

$$\mu_{u_0}(t) = \log\left\{\frac{-(b-a)t + \sqrt{(b-a)^2 t^2 + 4ab}}{2a}\right\}$$

and $\mu_{v_0} = \mu_{u_0}$ *on* $[-1, 1]$*;*

in obvious notation.

Proof *(i)* The set

$$Y_u := Z_{2,u} \cup \left(\{u' + (1/x + \rho)u + \lambda \neq 0\}\setminus Z_{2,u}\right) \cup \{\rho \text{ not differentiable}\} \subset [a, b]$$

(in obvious notation) is a null set in $[a, b]$ and likewise for Y_v. By [1] Lemma 2.95 and Lemma 2.96, $\{u = t\} \cap (Y_u \cup Z_{1,u}) = \emptyset$ for a.e. $t \in (0, 1)$ and likewise for the function v. Let $t \in (0, 1)$ and assume that $\{u = t\} \cap (Y_u \cup Z_{1,u}) = \emptyset$ and $\{v = t\} \cap (Y_v \cup Z_{1,v}) = \emptyset$. Put $c := \max\{u \geq t\}$. Then $c \in (a, b)$, $\{u > t\} = [a, c)$ by Lemma 9.3 and u is differentiable at c with $u'(c) < 0$. Put $d := \max\{v \leq t\} = \max\{u \geq -t\}$. As u is continuous on $[a, b]$ it holds that $a < c < d < b$. Moreover, $u'(d) < 0$ as $v(d) = t$ and $d \notin Z_v$. Put $\widetilde{u} := u/t$ and $\widetilde{v} := v/t$ on $[c, d]$. Then

$$\begin{aligned}
&\widetilde{u}' + (1/\tau + \rho)\widetilde{u} + \hat{m}/t = 0 \text{ a.e. on } (c, d) \text{ and } \widetilde{u}(c) = -\widetilde{u}(d) = 1;\\
&\widetilde{v}' + (1/\tau + \rho)\widetilde{v} - \hat{m}/t = 0 \text{ a.e. on } (c, d) \text{ and } -\widetilde{v}(c) = \widetilde{v}(d) = 1.
\end{aligned}$$

By Lemma 9.4,

$$\begin{aligned}
\int_{\{v=t\}\setminus Z_v} \frac{1}{|v'|} \frac{d\mathscr{H}^0}{\tau} &\geq \int_{[c,d]\cap\{v=t\}\setminus Z_v} \frac{1}{|v'|} \frac{d\mathscr{H}^0}{\tau}\\
&= (1/t) \int_{[c,d]\cap\{\widetilde{v}=1\}\setminus Z_v} \frac{1}{|\widetilde{v}'|} \frac{d\mathscr{H}^0}{\tau}\\
&\geq (1/t) \int_{[c,d]\cap\{\widetilde{u}=1\}\setminus Z_u} \frac{1}{|\widetilde{u}'|} \frac{d\mathscr{H}^0}{\tau}\\
&= \int_{\{u=t\}\setminus Z_u} \frac{1}{|u'|} \frac{d\mathscr{H}^0}{\tau}.
\end{aligned}$$

By Lemma 9.1,

$$-\mu_u'(t) = \int_{\{u=t\}\setminus Z_u} \frac{1}{|u'|} \frac{d\mathscr{H}^0}{\tau}$$

for $\mathscr{L}^1$-a.e. $t \in (0, 1)$ and a similar formula holds for v. The assertion in *(i)* follows.
(ii) Assume that $\rho \not\equiv 0$ on $[a, b)$. Put $\alpha := \inf\{\rho > 0\} \in [a, b)$. Note that $\max\{v \leq t\} \to b$ as $t \uparrow 1$ as $v < 1$ on $[a, b)$ by assumption. Choose $t_0 \in (0, 1)$ such that $\max\{v \leq t_0\} > \alpha$. Then for $t > t_0$,

$$a < \max\{u \geq t\} < \max\{u \geq -t_0\} = \max\{v \leq t_0\} < \max\{v \leq t\} < d;$$

that is, the interval $[c, d]$ with c, d as described above intersects $(\alpha, b]$. So for $\mathscr{L}^1$-a.e. $t \in (t_0, 1)$,

$$\int_{\{v=t\}\setminus Z_v} \frac{1}{|v'|} \frac{d\mathscr{H}^0}{\tau} > \int_{\{u=t\}\setminus Z_u} \frac{1}{|u'|} \frac{d\mathscr{H}^0}{\tau}.$$

by the equality condition in Lemma 9.4. The conclusion follows from the representation of μ_u resp. μ_v in Lemma 9.1.

(iii) A direct computation gives

$$u_0(\tau) = \frac{1}{b-a}\left\{ -\tau + \frac{ab}{\tau} \right\}$$

for $\tau \in [a, b]$; u_0 is strictly decreasing on its domain. This leads to the formula in *(iii)*. A similar computation gives

$$\mu_{v_0}(t) = \log\left\{ \frac{2b}{(b-a)t + \sqrt{(b-a)^2 t^2 + 4ab}} \right\}$$

for $t \in [-1, 1]$. Rationalising the denominator results in the stated equality. □

Corollary 9.6 *Let* $0 < a < b < +\infty$ *and* $\rho \geq 0$ *be a non-decreasing bounded function on* $[a, b]$. *Suppose that* (u, λ) *solves (9.4) with* $\eta = (1, -1)$ *and set* $v := -u$. *Assume that* $u > -1$ *on* $[a, b)$. *Then*

(i) $\mu_u(t) \leq \mu_v(t)$ *for each* $t \in (0, 1)$;
(ii) if $\rho \not\equiv 0$ *on* $[a, b)$ *then* $\mu_u(t) < \mu_v(t)$ *for each* $t \in (0, 1)$.

Proof (i) By [1] Theorem 3.28 and Lemma 9.3,

$$\begin{aligned}\mu_u(t) &= \mu_u(t) - \mu_u(1) = -D\mu_u((t, 1]) \\ &= -D\mu_u^a((t, 1]) - D\mu_u^s((t, 1]) \\ &= -\int_{(t,1]} \mu_u' \, ds\end{aligned}$$

for each $t \in (0, 1)$ as $\mu_u(1) = 0$. On the other hand,

$$\begin{aligned}\mu_v(t) &= \mu_v(1) + (\mu_v(t) - \mu_v(1)) = \mu_v(1) - D\mu_v((t, 1]) \\ &= \mu_v(1) - \int_{(t,1]} \mu_v' \, ds - D\mu_v^s((t, 1])\end{aligned}$$

for each $t \in (0, 1)$. The claim follows from Theorem 9.5 noting that $D\mu_v^s((t, 1]) \leq 0$ as can be seen from Lemma 9.1. Item *(ii)* follows from Theorem 9.5 *(ii)*. □

Corollary 9.7 *Let* $0 < a < b < +\infty$ *and* $\rho \geq 0$ *be a non-decreasing bounded function on* $[a, b]$. *Suppose that* (u, λ) *solves (9.4) with* $\eta = (1, -1)$. *Assume that* $u > -1$ *on* $[a, b)$. *Let* $\varphi \in C^1((-1, 1))$ *be an odd strictly increasing function with* $\varphi \in L^1((-1, 1))$. *Then*

(i) $\int_{\{u>0\}} \varphi(u) \, d\mu < +\infty$;
(ii) $\int_a^b \varphi(u) \, d\mu \leq 0$;
(iii) equality holds in (ii) if and only if $\rho \equiv 0$ *on* $[a, b)$.

In particular,

(iv) $\int_a^b \frac{u}{\sqrt{1-u^2}} \, d\mu \leq 0$ *with equality if and only if* $\rho \equiv 0$ *on* $[a, b)$.

Proof (i) Put $I := \{1 > u > 0\}$. The function $u : I \to (0, 1)$ is $C^{0,1}$ and $u' \leq -\hat{m}$ a.e. on I by Lemma 9.3. It has $C^{0,1}$ inverse $v : (0, 1) \to I$, $v' = 1/(u' \circ v)$ and $|v'| \leq 1/\hat{m}$ a.e. on (0, 1). By a change of variables,

$$\int_{\{u>0\}} \varphi(u)\, d\mu = \int_0^1 \varphi(v'/v)\, dt$$

from which the claim is apparent. *(ii)* The integral is well-defined because $\varphi(u)^+ = \varphi(u)\chi_{\{u>0\}} \in L^1((a, b), \mu)$ by *(i)*. By Lemma 9.3 the set $\{u = 0\}$ consists of a singleton and has μ-measure zero. So

$$\begin{aligned}\int_a^b \varphi(u)\, d\mu &= \int_{\{u>0\}} \varphi(u)\, d\mu + \int_{\{u<0\}} \varphi(u)\, d\mu \\ &= \int_{\{u>0\}} \varphi(u)\, d\mu - \int_{\{v>0\}} \varphi(v)\, d\mu\end{aligned}$$

where $v := -u$ as φ is an odd function. We remark that in a similar way to (9.3),

$$\begin{aligned}\int_0^1 \varphi' \mu_u\, dt &= \int_{\{u>0\}} \Big\{\varphi(u) - \varphi(0)\Big\} \\ d\mu &= \int_{\{u>0\}} \varphi(u)\, d\mu\end{aligned}$$

using oddness of φ and an analogous formula holds with v in place of u. Thus we may write

$$\begin{aligned}\int_a^b \varphi(u)\, d\mu &= \int_0^1 \varphi' \mu_u\, dt - \int_0^1 \varphi' \mu_v\, dt \\ &= \int_0^1 \varphi' \Big\{\mu_u - \mu_v\Big\}\, dt \leq 0\end{aligned}$$

by Corollary 9.6 as $\varphi' > 0$ on (0, 1). *(iii)* Suppose that $\rho \not\equiv 0$ on $[a, b)$. Then strict inequality holds in the above by Corollary 9.6. If $\rho \equiv 0$ on $[a, b)$ the equality follows from Theorem 9.5. *(iv)* follows from *(ii)* and *(iii)* with the particular choice $\varphi : (-1, 1) \to \mathbb{R}; t \mapsto t/\sqrt{1 - t^2}$. □

The boundary condition $\eta_1\eta_2 = 1$. Let $0 < a < b < +\infty$ and $\rho \geq 0$ be a non-decreasing bounded function on $[a, b]$. We study solutions of the auxilliary Riccati equation

$$w' + \lambda w^2 = (1/x + \rho)w \text{ a.e. on } (a, b) \text{ with } w(a) = w(b) = 1; \tag{9.6}$$

with $w \in C^{0,1}([a, b])$ and $\lambda \in \mathbb{R}$. If $\rho \equiv 0$ on $[a, b]$ then we write w_0 instead of w. Suppose (u, λ) solves (9.4) with $\eta = (1, 1)$. Then $u > 0$ on $[a, b]$ by Lemma 9.2 and we may set $w := 1/u$. Then $(w, -\lambda)$ satisfies (9.6).

Lemma 9.8 *Let* $0 < a < b < +\infty$ *and* $\rho \geq 0$ *be a non-decreasing bounded function on* $[a, b]$. *Then*

(i) *there exists a solution* (w, λ) *of* (9.6) *with* $w \in C^{0,1}([a, b])$ *and* $\lambda \in \mathbb{R}$;
(ii) *the pair* (w, λ) *in (i) is unique;*
(iii) $\lambda = m$.

Proof *(i)* Define $w : [a, b] \to \mathbb{R}$ by

$$w(t) := \frac{g(t)}{m \int_a^t g\, ds + g(a)} \text{ for } t \in [a, b].$$

Then $w \in C^{0,1}([a, b])$ and (w, m) satisfies (9.6). *(ii)* We claim that $w > 0$ on $[a, b]$ for any solution (w, λ) of (9.6). For otherwise, $c := \min\{w = 0\} \in (a, b)$. Then $u := 1/w$ on $[a, c)$ satisfies

$$u' + \left(\frac{1}{\tau} + \rho\right) u - \lambda = 0 \text{ a.e. on } (a, c) \text{ and } u(a) = 1, u(c-) = +\infty.$$

Integrating, we obtain

$$gu - g(a) - \lambda \int_a^{\cdot} g\, dt = 0 \text{ on } [a, c)$$

and this entails the contradiction that $u(c-) < +\infty$. We may now use the uniqueness statement in Lemma 9.2. *(iii)* follows from *(ii)* and the particular solution given in *(i)*. □

We introduce the mapping

$$\omega : (0, \infty) \times (0, \infty) \to \mathbb{R}; (t, x) \mapsto -(2/t) \coth(x/2).$$

For $\xi > 0$,

$$|\omega(t, x) - \omega(t, y)| \leq \operatorname{cosech}^2[\xi/2](1/t)|x - y| \tag{9.7}$$

for $(t, x), (t, y) \in (0, \infty) \times (\xi, \infty)$ and ω is locally Lipschitzian in x on $(0, \infty) \times (0, \infty)$ in the sense of [16] I.3. Let $0 < a < b < +\infty$ and set $\lambda := A/G > 1$. Here, $A = A(a, b)$ stands for the arithmetic mean of a, b as introduced in the previous Section while $G = G(a, b) := \sqrt{|ab|}$ stands for their geometric mean. We refer to the inital value problem

$$z' = \omega(t, z) \text{ on } (0, \lambda) \text{ and } z(1) = \mu((a, b)). \tag{9.8}$$

Define

$$z_0 : (0, \lambda) \to \mathbb{R}; t \mapsto 2 \log \left\{ \frac{\lambda + \sqrt{\lambda^2 - t^2}}{t} \right\}.$$

Lemma 9.9 *Let* $0 < a < b < +\infty$. *Then*

(i) $w_0(\tau) = \frac{2A\tau}{G^2+\tau^2}$ *for* $\tau \in [a, b]$;
(ii) $\|w_0\|_\infty = \lambda$;
(iii) $\mu_{w_0} = z_0$ *on* $[1, \lambda)$;
(iv) z_0 *satisfies (9.8) and this solution is unique;*
(v) $\int_{\{w_0=1\}} \frac{1}{|w_0'|} \frac{d\mathscr{H}^0}{\tau} = 2\coth(\mu((a, b))/2)$;
(vi) $\int_a^b \frac{1}{\sqrt{w_0^2-1}} \frac{dx}{x} = \pi$.

Proof *(i)* follows as in the proof of Lemma 9.8 with $g(t) = t$ while *(ii)* follows by calculus. *(iii)* follows by solving the quadratic equation $t\tau^2 - 2A\tau + G^2 t = 0$ for τ with $t \in (0, \lambda)$. Uniqueness in *(iv)* follows from [16] Theorem 3.1 as ω is locally Lipschitzian with respect to x in $(0, \infty) \times (0, \infty)$. For *(v)* note that $|aw_0'(a)| = 1-a/A$ and $|bw_0'(b)| = b/A - 1$ and

$$2\coth(\mu((a, b))/2) = 2(a + b)/(b - a).$$

(vi) We may write

$$\int_a^b \frac{1}{\sqrt{w_0^2 - 1}} \frac{d\tau}{\tau} = \int_a^b \frac{ab + \tau^2}{\sqrt{(a + b)^2\tau^2 - (ab + \tau^2)^2}} \frac{d\tau}{\tau} = \int_a^b \frac{ab + \tau^2}{\sqrt{(\tau^2 - a^2)(b^2 - \tau^2)}} \frac{d\tau}{\tau}.$$

The substitution $s = \tau^2$ followed by the Euler substitution (cf. [14] 2.251)

$$\sqrt{(s - a^2)(b^2 - s)} = t(s - a^2)$$

gives

$$\int_a^b \frac{1}{\sqrt{w_0^2 - 1}} \frac{d\tau}{\tau} = \int_0^\infty \frac{1}{1 + t^2} + \frac{ab}{b^2 + a^2t^2}\, dt = \pi.$$

□

Lemma 9.10 *Let* $0 < a < b < +\infty$. *Then*

(i) for $y > a$ *the function* $x \mapsto \frac{by-ax}{(y-a)(b-x)}$ *is strictly increasing on* $(-\infty, b]$;
(ii) the function $y \mapsto \frac{(b-a)y}{(y-a)(b-y)}$ *is strictly increasing on* $[G, b]$;
(iii) for $x < b$ *the function* $y \mapsto \frac{by-ax}{(y-a)(b-x)}$ *is strictly decreasing on* $[a, +\infty)$

Proof The proof is an exercise in calculus. □

Lemma 9.11 *Let* $0 < a < b < +\infty$ *and* $\rho \geq 0$ *be a non-decreasing bounded function on* $[a, b]$. *Let* (w, λ) *solve (9.6). Assume*

(i) w *is differentiable at both* a *and* b *and that* (9.6) *holds there;*
(ii) $w'(a) > 0$ *and* $w'(b) < 0$*;*
(iii) $w > 1$ *on* (a, b)*;*
(iv) ρ *is differentiable at* a *and* b*.*

Then

$$\int_{\{w=1\}\setminus Z_w} \frac{1}{|w'|}\frac{d\mathscr{H}^0}{\tau} \geq 2\coth(\mu((a,b))/2)$$

with equality if and only if $\rho \equiv 0$ *on* $[a, b)$.

Proof At the end-points $x = a, b$ the condition *(i)* entails that $w' + m - \rho = 1/x = w_0' + m_0$ so that

$$w' - w_0' = m_0 - m + \rho \text{ at } x = a, b. \tag{9.9}$$

We consider the four cases

(a) $w'(a) \geq w_0'(a)$ and $w'(b) \geq w_0'(b)$;
(b) $w'(a) \geq w_0'(a)$ and $w'(b) \leq w_0'(b)$;
(c) $w'(a) \leq w_0'(a)$ and $w'(b) \geq w_0'(b)$;
(d) $w'(a) \leq w_0'(a)$ and $w'(b) \leq w_0'(b)$;

in turn.

(a) Condition (a) together with (9.9) means that $m_0 - m + \rho(a) \geq 0$; that is, $m - \rho(a) \leq m_0$. By *(i)* and *(ii)*, $bm - b\rho(b) - 1 = -bw'(b) > 0$; or $m - \rho(b) > 1/b$. Therefore,

$$0 < 1/b < m - \rho(b) \leq m - \rho(a) \leq 1/A$$

by (8.5). Put $x := 1/(m - \rho(b))$ and $y := 1/(m - \rho(a))$. Then

$$a < A \leq y \leq x < b.$$

We write

$$aw'(a) = -(m - \rho(a))a + 1 = -(1/y)a + 1 > 0;$$
$$bw'(b) = -(m - \rho(b))b + 1 = -(1/x)b + 1 < 0.$$

Making use of assumption *(iii)*,

$$\int_{\{w=1\}\setminus Z_w} \frac{1}{|w'|}\frac{d\mathscr{H}^0}{x} = \frac{1}{-(1/y)a + 1} - \frac{1}{-(1/x)b + 1}$$
$$= \frac{by - ax}{(y - a)(b - x)}.$$

By Lemma 9.10 *(i)* then *(ii)*,

$$\int_{\{w=1\}} \frac{1}{|w'|}\frac{d\mathscr{H}^0}{x} \geq \frac{(b-a)y}{(y-a)(b-y)} \geq \frac{(b-a)A}{(A-a)(b-A)}$$
$$= 2\frac{a+b}{b-a} = 2\coth(\mu((a,b))/2).$$

If equality holds then $\rho(a) = \rho(b)$ and ρ is constant on $[a,b]$. By Theorem 8.3 we conclude that $\rho \equiv 0$ on $[a,b)$.

(b) Condition (b) together with (9.9) entails that $0 \leq m_0 - m + \rho(a)$ and $0 \leq -m_0 + m - \rho(b)$ whence $0 \leq \rho(a) - \rho(b)$ upon adding; so ρ is constant on the interval $[a,b]$ by monotonicity. Define x and y as above. Then $x = y$ and $y \geq A$. The result now follows in a similar way to case (a).

(c) In this case,

$$\frac{1}{aw'(a)} - \frac{1}{bw'(b)} \geq \frac{1}{aw_0'(a)} - \frac{1}{bw_0'(b)}$$
$$= 2\coth(\mu((a,b))/2)$$

by Lemma 9.9. If equality holds then $w'(b) = w_0'(b)$ so that $m_0 - m + \rho(b) = 0$ and ρ vanishes on $[a,b]$ by Theorem 8.3.

(d) Condition (d) together with (9.9) means that $m_0 - m + \rho(b) \leq 0$; that is, $m \geq \rho(b) + m_0$. On the other hand, by Theorem 8.3, $m \leq \rho(b) + m_0$. In consequence, $m = \rho(b) + m_0$. It then follows that $\rho \equiv 0$ on $[a,b]$ by Theorem 8.3. Now use Lemma 9.9.

□

Lemma 9.12 *Let* $\phi : (0,+\infty) \to (0,+\infty)$ *be a convex non-increasing function with* $\inf_{(0,+\infty)} \phi > 0$*. Let* Λ *be an at most countably infinite index set and* $(x_h)_{h\in\Lambda}$ *a sequence of points in* $(0,+\infty)$ *with* $\sum_{h\in\Lambda} x_h < +\infty$*. Then*

$$\sum_{h\in\Lambda}\phi(x_h) \geq \phi\left(\sum_{h\in\Lambda} x_h\right)$$

and the left-hand side takes the value $+\infty$ *in case* Λ *is countably infinite and is otherwise finite.*

Proof Suppose $0 < x_1 < x_2 < +\infty$. By convexity $\phi(x_1) + \phi(x_2) \geq 2\phi(\frac{x_1+x_2}{2}) \geq \phi(x_1 + x_2)$ as ϕ is non-increasing. The result for finite Λ follows by induction. □

Theorem 9.13 *Let* $0 < a < b < +\infty$ *and* $\rho \geq 0$ *be a non-decreasing bounded function on* $[a,b]$*. Let* (w,λ) *solve (9.6). Assume that* $w > 1$ *on* (a,b)*. Then*

(i) for $\mathscr{L}^1$*-a.e.* $t \in (1, \|w\|_\infty)$*,*

$$-\mu_w' \geq (2/t)\coth((1/2)\mu_w); \tag{9.10}$$

(ii) if $\rho \not\equiv 0$ on $[a, b)$ then there exists $t_0 \in (1, \|w\|_\infty)$ such that strict inequality holds in (9.10) for $\mathscr{L}^1$-a.e. $t \in (1, t_0)$.

Proof (i) The set

$$Y_w := Z_{2,w} \cup \Big(\{w' + mw^2 \neq (1/x + \rho)w\} \backslash Z_{2,w}\Big)$$
$$\cup\{\rho \text{ not differentiable}\} \subset [a, b]$$

is a null set in $[a, b]$. By [1] Lemma 2.95 and Lemma 2.96, $\{w = t\} \cap (Y_w \cap Z_{1,w}) = \emptyset$ for a.e. $t > 1$. Let $t \in (1, \|w\|_\infty)$ and assume that $\{w = t\} \cap (Y_w \cap Z_{1,w}) = \emptyset$. We write $\{w > t\} = \bigcup_{h \in \Lambda} I_h$ where Λ is an at most countably infinite index set and $(I_h)_{h \in \Lambda}$ are disjoint non-empty well-separated open intervals in (a, b). The term well-separated means that for each $h \in \Lambda$, $\inf_{k \in \Lambda \backslash \{h\}} d(I_h, I_k) > 0$. This follows from the fact that $w' \neq 0$ on ∂I_h for each $h \in \Lambda$. Put $\widetilde{w} := w/t$ on $\overline{\{w > t\}}$ so

$$\widetilde{w}' + (mt)\widetilde{w}^2 = (1/x + \rho)\widetilde{w} \text{ a.e. on } \{w > t\} \text{ and } \widetilde{w} = 1 \text{ on } \{w = t\}.$$

We use the fact that the mapping $\phi : (0, +\infty) \to (0, +\infty); t \mapsto \coth t$ satisfies the hypotheses of Lemma 9.12. By Lemmas 9.11 and 9.12,

$$\begin{aligned}
(0, +\infty] \ni \int_{\{w=t\}\backslash Z_w} \frac{1}{|w'|} \frac{d\mathscr{H}^0}{x} &= (1/t) \int_{\{\widetilde{w}=1\}} \frac{1}{|\widetilde{w}'|} \frac{d\mathscr{H}^0}{\tau} \\
&= (1/t) \sum_{h \in \Lambda} \int_{\partial I_h} \frac{1}{|\widetilde{w}'|} \frac{d\mathscr{H}^0}{\tau} \\
&\geq (2/t) \sum_{h \in \Lambda} \coth((1/2)\mu(I_h)) \\
&\geq (2/t) \coth\left((1/2) \sum_{h \in \Lambda} \mu(I_h)\right) \\
&= (2/t) \coth((1/2)\mu(\{w > t\})) \\
&= (2/t) \coth((1/2)\mu_w(t)).
\end{aligned}$$

The statement now follows from Lemma 9.1.
(ii) Suppose that $\rho \not\equiv 0$ on $[a, b)$. Put $\alpha := \min\{\rho > 0\} \in [a, b)$. Now that $\{w > t\} \uparrow (a, b)$ as $t \downarrow 1$ as $w > 1$ on (a, b). Choose $t_0 \in (1, \|w\|_\infty)$ such that $\{w > t_0\} \cap (\alpha, b) \neq \emptyset$. Then for each $t \in (1, t_0)$ there exists $h \in \Lambda$ such that $\rho \not\equiv 0$ on I_h. The statement then follows by Lemma 9.11. □

Lemma 9.14 *Let $\emptyset \neq S \subset \mathbb{R}$ be bounded and suppose S has the property that for each $s \in S$ there exists $\delta > 0$ such that $[s, s + \delta) \subset S$. Then S is $\mathscr{L}^1$-measurable and $|S| > 0$.*

Proof For each $s \in S$ put $t_s := \inf\{t > s : t \notin S\}$. Then $s < t_s < +\infty$, $[s, t_s) \subset S$ and $t_s \notin S$. Define

$$\mathscr{C} := \Big\{[s, t] : s \in S \text{ and } t \in (s, t_s)\Big\}.$$

Then $\mathscr{C}$ is a Vitali cover of S (see [6] Chapter 16 for example). By Vitali's Covering Theorem (cf. [6] Theorem 16.27) there exists an at most countably infinite subset $\Lambda \subset \mathscr{C}$ consisting of pairwise disjoint intervals such that

$$\left| S \backslash \bigcup_{I \in \Lambda} I \right| = 0.$$

Note that $I \subset S$ for each $I \in \Lambda$. Consequently, $S = \bigcup_{I \in \Lambda} I \cup N$ where N is an $\mathscr{L}^1$-null set and hence S is $\mathscr{L}^1$-measurable. The positivity assertion is clear. □

Theorem 9.15 *Let* $0 < a < b < +\infty$ *and* $\rho \geq 0$ *be a non-decreasing bounded function on* $[a, b]$. *Let* (w, λ) *solve (9.6). Assume that* $w > 1$ *on* (a, b). *Put* $T := \min\{\|w_0\|_\infty, \|w\|_\infty\} > 1$. *Then*

(i) $\mu_w(t) \leq \mu_{w_0}(t)$ *for each* $t \in [1, T)$;
(ii) $\|w\|_\infty \leq \|w_0\|_\infty$;
(iii) if $\rho \not\equiv 0$ *on* $[a, b)$ *then there exists* $t_0 \in (1, \|w\|_\infty)$ *such that* $\mu_w(t) < \mu_{w_0}(t)$ *for each* $t \in (1, t_0)$.

Proof (i) We adapt the proof of [16] Theorem I.6.1. The assumption entails that $\mu_w(1) = \mu_{w_0}(1) = \mu((a, b))$. Suppose for a contradiction that $\mu_w(t) > \mu_{w_0}(t)$ for some $t \in (1, T)$.

For $\varepsilon > 0$ consider the initial value problem

$$z' = \omega(t, z) + \varepsilon \text{ and } z(1) = \mu((a, b)) + \varepsilon \tag{9.11}$$

on $(0, T)$. Choose $\upsilon \in (0, 1)$ and $\tau \in (t, T)$. By [16] Lemma I.3.1 there exists $\varepsilon_0 > 0$ such that for each $0 \leq \varepsilon < \varepsilon_0$ (9.11) has a continuously differentiable solution z_ε defined on $[\upsilon, \tau]$ and this solution is unique by [16] Theorem I.3.1. Moreover, the sequence $(z_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ converges uniformly to z_0 on $[\upsilon, \tau]$.

Given $0 < \varepsilon < \eta < \varepsilon_0$ it holds that $z_0 \leq z_\varepsilon \leq z_\eta$ on $[1, \tau]$ by [16] Theorem I.6.1. Note for example that $z_0' \leq \omega(\cdot, z_0) + \varepsilon$ on $(1, \tau)$. In fact, $(z_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ decreases strictly to z_0 on $(1, \tau)$. For if, say, $z_0(s) = z_\varepsilon(s)$ for some $s \in (1, \tau)$ then $z_\varepsilon'(s) = \omega(s, z_\varepsilon(s)) + \varepsilon > \omega(s, z_0(s)) = z_0'(s)$ by (9.11); while on the other hand $z_\varepsilon'(s) \leq z_0'(s)$ by considering the left-derivative at s and using the fact that $z_\varepsilon \geq z_0$ on $[1, \tau]$. This contradicts the strict inequality.

Choose $\varepsilon_1 \in (0, \varepsilon_0)$ such that $z_\varepsilon(t) < \mu_w(t)$ for each $0 < \varepsilon < \varepsilon_1$. Now μ_w is right-continuous and strictly decreasing as $\mu_w(t) - \mu_w(s) = -\mu(\{s < w \leq t\}) < 0$ for $1 \leq s < t < \|w\|_\infty$ by continuity of w. So the set $\{z_\varepsilon < \mu_w\} \cap (1, t)$ is open and non-empty in $(0, +\infty)$ for each $\varepsilon \in (0, \varepsilon_1)$. Thus there exists a unique $s_\varepsilon \in [1, t)$ such that

$$\mu_w > z_\varepsilon \text{ on } (s_\varepsilon, t] \text{ and } \mu_w(s_\varepsilon) = z_\varepsilon(s_\varepsilon)$$

for each $\varepsilon \in (0, \varepsilon_1)$. As $z_\varepsilon(1) > \mu((a, b))$ it holds that each $s_\varepsilon > 1$. Note that $1 < s_\varepsilon < s_\eta$ whenever $0 < \varepsilon < \eta$ as $(z_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ decreases strictly to z_0 as $\varepsilon \downarrow 0$.

Define

$$S := \left\{ s_\varepsilon : 0 < \varepsilon < \varepsilon_1 \right\} \subset (1, t).$$

We claim that for each $s \in S$ there exists $\delta > 0$ such that $[s, s + \delta) \subset S$. This entails that S is $\mathscr{L}^1$-measurable with positive $\mathscr{L}^1$-measure by Lemma 9.14.

Suppose $s = s_\varepsilon \in S$ for some $\varepsilon \in (0, \varepsilon_1)$ and put $z := z_\varepsilon(s) = \mu_w(s)$. Put $k := \operatorname{cosech}^2(z_0(t)/2)$. For $0 \le \zeta < \eta < \varepsilon_1$ define

$$\Omega_{\zeta,\eta} := \left\{ (u, y) \in \mathbb{R}^2 : u \in (0, t) \text{ and } z_\zeta(u) < y < z_\eta(u) \right\}$$

and note that this is an open set in $\mathbb{R}^2$. We remark that for each $(u, y) \in \Omega_{\zeta,\eta}$ there exists a unique $\nu \in (\zeta, \eta)$ such that $y = z_\nu(u)$. Given $r > 0$ with $s + r < t$ set

$$Q = Q_r := \left\{ (u, y) \in \mathbb{R}^2 : s \le u < s + r \text{ and } |y - z| < \|z_\varepsilon - z\|_{C([s,s+r])} \right\}.$$

Choose $r \in (0, t - s)$ and $\varepsilon_2 \in (\varepsilon, \varepsilon_1)$ such that

(a) $Q_r \subset \Omega_{0,\varepsilon_1}$;
(b) $\|z_\varepsilon - z\|_{C([s,s+r])} < s\varepsilon/(2k)$;
(c) $\sup_{\eta\in(\varepsilon,\varepsilon_2)} \|z_\eta - z\|_{C([s,s+r])} \le \|z_\varepsilon - z\|_{C([s,s+r])}$;
(d) $z_\eta < \mu_w$ on $[s + r, t]$ for each $\eta \in (\varepsilon, \varepsilon_2)$.

We can find $\delta \in (0, r)$ such that $z_\varepsilon < \mu_w < z_{\varepsilon_2}$ on $(s, s + \delta)$ as $z_{\varepsilon_2}(s) > z$; in other words, the graph of μ_w restricted to $(s, s + \delta)$ is contained in $\Omega_{\varepsilon,\varepsilon_2}$.

Let $u \in (s, s + \delta)$. Then $\mu_w(u) = z_\eta(u)$ for some $\eta \in (\varepsilon, \varepsilon_2)$ as above. We claim that $u = s_\eta$ so that $u \in S$. This implies in turn that $[s, s + \delta) \subset S$. Suppose for a contradiction that $z_\eta \not< \mu_w$ on $(u, t]$. Then there exists $v \in (u, t]$ such that $\mu_w(v) = z_\eta(v)$. In view of condition (d), $v \in (u, s + r)$. By [1] Theorem 3.28 and Theorem 9.13,

$$\begin{aligned}\mu_w(v) - \mu_w(u) &= D\mu_w((u, v]) = D\mu_w^a((u, v]) + D\mu_w^s((u, v]) \\ &\le D\mu_w^a((u, v]) = \int_u^v \mu_w' \, d\tau \le \int_u^v \omega(\cdot, \mu_w) \, d\tau.\end{aligned}$$

On the other hand,

$$z_\eta(v) - z_\eta(u) = \int_u^v z_\eta' \, d\tau = \int_u^v \omega(\cdot, z_\eta) \, d\tau + \eta(v - u).$$

We derive that

$$\varepsilon(v-u) \le \eta(v-u) \le \int_u^v \Big\{\omega(\cdot,\mu_w) - \omega(\cdot, z_\eta)\Big\}\, d\tau$$
$$\le k \int_u^v |\mu_w - z_\eta|\, d\mu$$

using the estimate (9.7). Thus

$$\begin{aligned}\varepsilon &\le k\frac{1}{v-u}\int_u^v |\mu_w - z_\eta|\, d\mu \\ &\le (k/s)\|\mu_w - z_\eta\|_{C([u,v])} \\ &\le (k/s)\Big\{\|\mu_w - z\|_{C([s,s+r])} + \|z_\eta - z\|_{C([s,s+r])}\Big\} \\ &\le (2k/s)\|z_\varepsilon - z\|_{C([s,s+r])} < \varepsilon\end{aligned}$$

by (b) and (c) giving rise to the desired contradiction.

By Theorem 9.13, $\mu_w' \le \omega(\cdot,\mu_w)$ for $\mathscr{L}^1$-a.e. $t \in S$. Choose $s \in S$ such that μ_w is differentiable at s and the latter inequality holds at s. Let $\varepsilon \in (0,\varepsilon_1)$ such that $s = s_\varepsilon$. For any $u \in (s,t)$,

$$\mu_w(u) - \mu_w(s) > z_\varepsilon(u) - z_\varepsilon(s).$$

We deduce that $\mu_w'(s) \ge z_\varepsilon'(s)$. But then

$$\mu_w'(s) \ge z_\varepsilon'(s) = \omega(s, z_\epsilon(s)) + \varepsilon > \omega(s,\mu_w(s)).$$

This strict inequality holds on a set of full measure in S. This contradicts Theorem 9.13.

(ii) Use the fact that $\|w\|_\infty = \sup\{t > 0 : \mu_w(t) > 0\}$.
(iii) Assume that $\rho \not\equiv 0$ on $[a,b)$. Let $t_0 \in (1,\|w\|_\infty)$ be as in Lemma 9.13. Then for $t \in (1,t_0)$,

$$\begin{aligned}\mu_w(t) - \mu_w(1) = D\mu_w((1,t]) &= D\mu_w^a((1,t]) \\ &\quad + D\mu_w^s((1,t]) \le D\mu_w^a((1,t]) \\ &= \int_{(1,t]} \mu_w'\, ds < \int_{(1,t]} \omega(s,\mu_w)\, ds \\ &\le \int_{(1,t]} \omega(s,\mu_{w_0})\, ds = \mu_{w_0}(t) - \mu_{w_0}(1)\end{aligned}$$

by Theorem 9.13, Lemma 9.9 and the inequality in *(i)*.

□

Corollary 9.16 *Let $0 < a < b < +\infty$ and $\rho \geq 0$ be a non-decreasing bounded function on $[a, b]$. Suppose that (w, λ) solves (9.6). Assume that $w > 1$ on (a, b). Let $0 \leq \varphi \in C^1((1, +\infty))$ be strictly decreasing with $\int_a^b \varphi(w_0)\, d\mu < +\infty$. Then*

(i) $\int_a^b \varphi(w)\, d\mu \geq \int_a^b \varphi(w_0)\, d\mu$;
(ii) equality holds in (i) if and only if $\rho \equiv 0$ on $[a, b)$.

In particular,

(iii) $\int_a^b \frac{1}{\sqrt{w^2-1}}\, d\mu \geq \pi$ *with equality if and only if $\rho \equiv 0$ on $[a, b)$.*

Proof (i) Let $\varphi \geq 0$ be a decreasing function on $(1, +\infty)$ which is piecewise C^1. Suppose that $\varphi(1+) < +\infty$. By Tonelli's Theorem,

$$\begin{aligned}\int_{[1,+\infty)} \varphi' \mu_w\, ds &= \int_{[1,+\infty)} \varphi' \Big\{ \int_{(a,b)} \chi_{\{w>s\}}\, d\mu \Big\}\, ds \\ &= \int_{(a,b)} \Big\{ \int_{[1,+\infty)} \varphi' \chi_{\{w>s\}}\, ds \Big\}\, d\mu \\ &= \int_{(a,b)} \Big\{ \varphi(w) - \varphi(1) \Big\}\, d\mu \\ &= \int_{(a,b)} \varphi(w)\, d\mu - \varphi(1)\mu((a,b))\end{aligned}$$

and a similar identity holds for μ_{w_0}. By Theorem 9.15, $\int_a^b \varphi(w)\, d\mu \geq \int_a^b \varphi(w_0)\, d\mu$. Now suppose that $0 \leq \varphi \in C^1((1, +\infty))$ is strictly decreasing with $\int_a^b \varphi(w_0)\, d\mu < +\infty$. The inequality holds for the truncated function $\varphi \wedge n$ for each $n \in \mathbb{N}$. An application of the monotone convergence theorem establishes the result for φ.
(ii) Suppose that equality holds in *(i)*. For $c \in (1, +\infty)$ put $\varphi_1 := \varphi \vee \varphi(c) - \varphi(c)$ and $\varphi_2 := \varphi \wedge \varphi(c)$. By *(i)* we deduce

$$\int_a^b \varphi_2(w)\, d\mu = \int_a^b \varphi_2(w_0)\, d\mu;$$

and hence by the above that

$$\int_{[c,+\infty)} \varphi' \Big\{ \mu_w - \mu_{w_0} \Big\}\, ds = 0.$$

This means that $\mu_w = \mu_{w_0}$ on $(c, +\infty)$ and hence on $(1, +\infty)$. By Theorem 9.15 we conclude that $\rho \equiv 0$ on $[a, b)$. *(iii)* flows from *(i)* and *(ii)* noting that the function $\varphi : (1, +\infty) \to \mathbb{R}; t \mapsto 1/\sqrt{t^2 - 1}$ satisfies the integral condition by Lemma 9.9. □

The case $a = 0$. Let $0 < b < +\infty$ and $\rho \geq 0$ be a non-decreasing bounded function on $[0, b]$. We study solutions to the first-order linear ordinary differential equation

$$u' + (1/x + \rho)u + \lambda = 0 \text{ a.e. on } (0, b) \text{ with } u(0) = 0 \text{ and } u(b) = 1 \qquad (9.12)$$

where $u \in C^{0,1}([0, b])$ and $\lambda \in \mathbb{R}$. If $\rho \equiv 0$ on $[0, b]$ then we write u_0 instead of u.

Lemma 9.17 *Let* $0 < b < +\infty$ *and* $\rho \geq 0$ *be a non-decreasing bounded function on* $[0, b]$. *Then*

(i) *there exists a solution* (u, λ) *of* (9.12) *with* $u \in C^{0,1}([0, b])$ *and* $\lambda \in \mathbb{R}$;
(ii) λ *is given by* $\lambda = -g(b)/G(b)$ *where* $G := \int_0^{\cdot} g\, ds$;
(iii) *the pair* (u, λ) *in (i) is unique;*
(iv) $u > 0$ *on* $(0, b]$.

Proof *(i)* The function $u : [a, b] \to \mathbb{R}$ given by

$$u = \frac{g(b)}{G(b)} \frac{G}{g} \tag{9.13}$$

on $[0, b]$ solves (9.12) with λ as in *(ii)*. *(iii)* Suppose that (u_1, λ_1) resp. (u_2, λ_2) solve (9.12). By linearity $u := u_1 - u_2$ solves

$$u' + (1/x + \rho)u + \lambda = 0 \text{ a.e. on } (0, b) \text{ with } u(0) = u(b) = 0$$

where $\lambda = \lambda_1 - \lambda_2$. An integration gives that $u = (-\lambda G + c)/g$ for some constant $c \in \mathbb{R}$ and the boundary conditions entail that $\lambda = c = 0$. *(iv)* follows from the formula (9.13) and unicity. □

Lemma 9.18 *Suppose* $-\infty < a < b < +\infty$ *and that* $\phi : [a, b] \to \mathbb{R}$ *is convex. Suppose that there exists* $\xi \in (a, b)$ *such that*

$$\phi(\xi) = \frac{b - \xi}{b - a}\phi(a) + \frac{\xi - a}{b - a}\phi(b).$$

Then

$$\phi(c) = \frac{b - c}{b - a}\phi(a) + \frac{c - a}{b - a}\phi(b)$$

for each $c \in [a, b]$.

Proof Let $c \in (\xi, b)$. By monotonicity of chords,

$$\frac{\phi(\xi) - \phi(a)}{\xi - a} \leq \frac{\phi(c) - \phi(\xi)}{c - \xi}$$

so

$$\begin{aligned}\phi(c) &\geq \frac{c - a}{\xi - a}\phi(\xi) - \frac{c - \xi}{\xi - a}\phi(a) \\ &= \frac{c - a}{\xi - a}\left\{\frac{b - \xi}{b - a}\phi(a) + \frac{\xi - a}{b - a}\phi(b)\right\} - \frac{c - \xi}{\xi - a}\phi(a)\end{aligned}$$

$$= \frac{b-c}{b-a}\phi(a) + \frac{c-a}{b-a}\phi(b)$$

and equality follows. The case $c \in (a, \xi)$ is similar. □

Lemma 9.19 *Let $0 < b < +\infty$ and $\rho \geq 0$ be a non-decreasing bounded function on $[0, b]$. Let (u, λ) satisfy (9.12). Then*

(i) $u \geq u_0$ on $[0, b]$;
(ii) if $\rho \not\equiv 0$ on $[0, b)$ then $u > u_0$ on $(0, b)$.

Proof *(i)* The mapping $G : [0, b] \to [0, G(b)]$ is a bijection with inverse G^{-1}. Define $\eta : [0, G(b)] \to \mathbb{R}$ via $\eta := (tg) \circ G^{-1}$. Then

$$\eta' = \frac{(tg)'}{g} \circ G^{-1} = (2 + t\rho) \circ G^{-1}$$

a.e. on $(0, G(b))$ so η' is non-decreasing there. This means that η is convex on $[0, G(b)]$. In particular, $\eta(s) \leq [\eta(G(b))/G(b)]s$ for each $s \in [0, G(b)]$. For $t \in [0, b]$ put $s := G(t)$ to obtain $tg(t) \leq (bg(b)/G(b))G(t)$. A rearrangement gives $u \geq u_0$ on $[0, b]$ noting that $u_0 : [0, b] \to \mathbb{R}; t \mapsto t/b$. *(ii)* Assume $\rho \not\equiv 0$ on $[0, b)$. Suppose that $u(c) = u_0(c)$ for some $c \in (0, b)$. Then $\eta(G(c)) = [\eta(G(b))/G(b)]G(c)$. By Lemma 9.18, $\eta' = 0$ on $(0, G(b))$. This implies that $\rho \equiv 0$ on $[0, b)$. □

Lemma 9.20 *Let $0 < b < +\infty$. Then $\int_0^b \frac{u_0}{\sqrt{1-u_0^2}}\,d\mu = \pi/2$.*

Proof The integral is elementary as $u_0(t) = t/b$ for $t \in [0, b]$. □

10 Proof of main results

Lemma 10.1 *Let $x \in H$ and v be a unit vector in $\mathbb{R}^2$ such that the pair $\{x, v\}$ forms a positively oriented orthogonal basis for $\mathbb{R}^2$. Put $b := (\tau, 0)$ where $|x| = \tau$ and $\gamma := \theta(x) \in (0, \pi)$. Let $\alpha \in (0, \pi/2)$ such that*

$$\frac{\langle v, x-b \rangle}{|x-b|} = \cos\alpha.$$

Then

(i) $C(x, v, \alpha) \cap H \cap \overline{C}(0, e_1, \gamma) = \emptyset$;
(ii) for any $y \in C(x, v, \alpha) \cap H \setminus \overline{B}(0, \tau)$ the line segment $[b, y]$ intersects $\mathbb{S}^1_\tau$ outside the closed cone $\overline{C}(0, e_1, \gamma)$.

We point out that $C(0, e_1, \gamma)$ is the open cone with vertex 0 and axis e_1 which contains the point x on its boundary. We note that $\cos\alpha \in (0, 1)$ because

$$\begin{aligned}\langle v, x-b\rangle &= -\langle v, b\rangle = -\langle (1/\tau)Ox, b\rangle \\ &= -\langle Op, e_1\rangle = \langle x, O^{\star}e_1\rangle = \langle x, e_2\rangle > 0 \end{aligned} \tag{10.1}$$

and if $|x-b| = \langle v, x-b\rangle$ then $b = x-\lambda v$ for some $\lambda \in \mathbb{R}$ and hence $x_1 = \langle e_1, x\rangle = \tau$ and $x_2 = 0$.

Proof (i) For $\omega \in \mathbb{S}^1$ define the open half-space

$$H_\omega := \{y \in \mathbb{R}^2 : \langle y, \omega\rangle > 0\}.$$

We claim that $C(x, v, \alpha) \subset H_v$. For given $y \in C(x, v, \alpha)$,

$$\langle y, v\rangle = \langle y-x, v\rangle > |y-x|\cos\alpha > 0.$$

On the other hand, it holds that $\overline{C}(0, e_1, \gamma) \cap H \subset \overline{H}_{-v}$. This establishes *(i)*.
(ii) By some trigonometry $\gamma = 2\alpha$. Suppose that ω is a unit vector in $C(b, -e_1, \pi/2 - \alpha)$. Then $\lambda := \langle \omega, e_1\rangle < \cos\alpha$ since upon rewriting the membership condition for $C(b, -e_1, \pi/2-\alpha)$ we obtain the quadratic inequality

$$\lambda^2 - 2\cos^2\alpha\lambda + \cos\gamma > 0.$$

For ω a unit vector in $\overline{C}(0, e_1, \gamma)$ the opposite inequality $\langle \omega, e_1\rangle \geq \cos\alpha$ holds. This shows that

$$C(b, -e_1, \pi/2-\alpha) \cap \overline{C}(0, e_1, \gamma) \cap \mathbb{S}^1_\tau = \emptyset.$$

The set $C(x, v, \alpha) \cap H$ is contained in the open convex cone $C(b, -e_1, \pi/2-\alpha)$. Suppose $y \in C(x, v, \alpha) \cap H \backslash \overline{B}(0, \tau)$. Then the line segment $[b, y]$ is contained in $C(b, -e_1, \pi/2-\alpha) \cup \{b\}$. Now the set $C(b, -e_1, \pi/2-\alpha) \cap \mathbb{S}^1_\tau$ disconnects $C(b, -e_1, \pi/2-\alpha) \cup \{b\}$. This entails that $(b, y] \cap C(b, -e_1, \pi/2-\alpha) \cap \mathbb{S}^1_\tau \neq \emptyset$. The foregoing paragraph entails that $(b, y] \cap \overline{C}(0, e_1, \gamma) \cap \mathbb{S}^1_\tau = \emptyset$. This establishes the result. □

Lemma 10.2 *Let E be an open set in $\mathbb{R}^2$ such that $M := \partial E$ is a $C^{1,1}$ hypersurface in $\mathbb{R}^2$. Assume that $E\backslash\{0\} = E^{sc}$. Suppose*

(i) $x \in (M\backslash\{0\}) \cap H$;
(ii) $\sin(\sigma(x)) = -1$.

Then E is not convex.

Proof Let $\gamma_1 : I \to M$ be a $C^{1,1}$ parametrisation of M in a neighbourhood of x with $\gamma_1(0) = x$ as above. As $\sin(\sigma(x)) = -1$, $n(x)$ and hence $n_1(0)$ point in the direction of x. Put $v := -t_1(0) = -t(x)$. We may write

$$\gamma_1(s) = \gamma_1(0) + st_1(0) + R_1(s) = x - sv + R_1(s)$$

for $s \in I$ where $R_1(s) = s\int_0^1 \dot{\gamma}_1(ts) - \dot{\gamma}_1(0)\,dt$ and we can find a finite positive constant K such that $|R_1(s)| \leq Ks^2$ on a symmetric open interval I_0 about 0 with $I_0 \subset\subset I$. Then

$$\begin{aligned}\frac{\langle \gamma_1(s) - x, v\rangle}{|\gamma_1(s) - x|} &= \frac{\langle -sv + R_1, v\rangle}{|-sv + R_1|}\\ &= \frac{1 - \langle (R_1/s), v\rangle}{|v - R_1/s|} \to 1\end{aligned}$$

as $s \uparrow 0$. Let α be as in Lemma 10.1 with x and v as just mentioned. The above estimate entails that $\gamma_1(s) \in C(x, v, \alpha)$ for small $s < 0$. By (2.9) and Lemma 5.4 the function r_1 is non-increasing on I. In particular, $r_1(s) \geq r_1(0) = |x| =: \tau$ for $I \ni s < 0$ and $\gamma_1(s) \notin B(0, \tau)$.

Choose $\delta_1 > 0$ such that $\gamma_1(s) \in C(x, v, \alpha) \cap H$ for each $s \in [-\delta_1, 0)$. Put $\beta := \inf\{s \in [-\delta_1, 0] : r_1(s) = \tau\}$. Suppose first that $\beta \in [-\delta_1, 0)$. Then E is not convex (see Lemma 5.2). Now suppose that $\beta = 0$. Let γ be as in Lemma 10.1. Then the open circular arc $\mathbb{S}^1_\tau \setminus \overline{C}(0, e_1, \gamma)$ does not intersect $\overline{E}$: for otherwise, M intersects $\mathbb{S}^1_\tau \setminus \overline{C}(0, e_1, \gamma)$ and $\beta < 0$ bearing in mind Lemma 5.2. Choose $s \in [-\delta_1, 0)$. Then the points b and $\gamma_1(s)$ lie in $\overline{E}$. But by Lemma 10.1 the line segment $[b, \gamma_1(s)]$ intersects $\mathbb{S}^1_\tau$ in $\mathbb{S}^1_\tau \setminus \overline{C}(0, e_1, \gamma)$. Let $c \in [b, \gamma_1(s)] \cap \mathbb{S}^1_\tau$. Then $c \notin \overline{E}$. This shows that $\overline{E}$ is not convex. But if E is convex then $\overline{E}$ is convex. Therefore E is not convex. □

Theorem 10.3 *Let f be as in (1.3) where $h : [0, +\infty) \to \mathbb{R}$ is a non-decreasing convex function. Given $v > 0$ let E be a bounded minimiser of (1.2). Assume that E is open, $M := \partial E$ is a $C^{1,1}$ hypersurface in $\mathbb{R}^2$ and $E\setminus\{0\} = E^{sc}$. Put*

$$R := \inf\{\rho > 0\} \in [0, +\infty). \tag{10.2}$$

Then $\Omega \cap (R, +\infty) = \emptyset$ with Ω as in (5.2).

Proof Suppose that $\Omega \cap (R, +\infty) \neq \emptyset$. As Ω is open in $(0, +\infty)$ by Lemma 5.6 we may write Ω as a countable union of disjoint open intervals in $(0, +\infty)$. By a suitable choice of one of these intervals we may assume that $\Omega = (a, b)$ for some $0 \leq a < b < +\infty$ and that $\Omega \cap (R, +\infty) \neq \emptyset$. Let us assume for the time being that $a > 0$. Note that $[a, b] \subset \pi(M)$ and $\cos\sigma$ vanishes on $M_a \cup M_b$.

Let $u : \Omega \to [-1, 1]$ be as in (6.6). Then u has a continuous extension to $[a, b]$ and $u = \pm 1$ at $\tau = a, b$. This may be seen as follows. For $\tau \in (a, b)$ the set $M_\tau \cap \overline{H}$ consists of a singleton by Lemma 5.4. The limit $x := \lim_{\tau\downarrow a} M_\tau \cap \overline{H} \in \mathbb{S}^1_a \cap \overline{H}$ exists as M is C^1. There exists a $C^{1,1}$ parametrisation $\gamma_1 : I \to M$ with $\gamma_1(0) = x$ as above. By (2.9) and Lemma 5.4, r_1 is decreasing on I. So $r_1 > a$ on $I \cap \{s < 0\}$ for otherwise the C^1 property fails at x. It follows that $\gamma_1 = \gamma \circ r_1$ and $\sigma_1 = \sigma \circ \gamma \circ r_1$ on $I \cap \{s < 0\}$. Thus $\sin(\sigma \circ \gamma) \circ r_1 = \sin\sigma_1$ on $I \cap \{s < 0\}$. Now the function $\sin\sigma_1$ is continuous on I. So $u \to \sin\sigma_1(0) \in \{\pm 1\}$ as $\tau \downarrow a$. Put $\eta_1 := u(a)$ and $\eta_2 := u(b)$.

Let us consider the case $\eta = (\eta_1, \eta_2) = (1, 1)$. According to Theorem 6.5 the generalised (mean) curvature is constant $\mathscr{H}^1$-a.e. on M with value $-\lambda$, say. Note that $u < 1$ on (a, b) for otherwise $\cos(\sigma \circ \gamma)$ vanishes at some point in (a, b) bearing in mind Lemma 5.4. By Theorem 6.6 the pair (u, λ) satisfies (9.4) with $\eta = (1, 1)$. By Lemma 9.2, $u > 0$ on $[a, b]$. Put $w := 1/u$. Then $(w, -\lambda)$ satisfies (9.6) and $w > 1$ on (a, b). By Lemma 6.7,

$$\begin{aligned}\theta_2(b) - \theta_2(a) &= \int_a^b \theta_2' \, d\tau = -\int_a^b \frac{u}{\sqrt{1-u^2}} \frac{d\tau}{\tau} \\ &= -\int_a^b \frac{1}{\sqrt{w^2-1}} \frac{d\tau}{\tau}.\end{aligned}$$

By Corollary 9.16, $|\theta_2(b) - \theta_2(a)| > \pi$. But this contradicts the definition of θ_2 in (6.4) as θ_2 takes values in $(0, \pi)$ on (a, b). If $\eta = (-1, -1)$ then $\lambda > 0$ by Lemma 9.2; this contradicts Lemma 7.2.

Now let us consider the case $\eta = (-1, 1)$. Using the same formula as above, $\theta_2(b) - \theta_2(a) < 0$ by Corollary 9.7. This means that $\theta_2(a) \in (0, \pi]$. As before the limit $x := \lim_{\tau \downarrow a} M_\tau \cap \overline{H} \in \mathbb{S}^1_a \cap \overline{H}$ exists as M is C^1. Using a local parametrisation it can be seen that $\theta_2(a) = \theta(x)$ and $\sin(\sigma(x)) = -1$. If $\theta_2(a) \in (0, \pi)$ then E is not convex by Lemma 10.2. This contradicts Theorem 7.3. Note that we may assume that $\theta_2(a) \in (0, \pi)$. For otherwise, $\langle \gamma, e_2 \rangle < 0$ for $\tau > a$ near a, contradicting the definition of γ (6.5). If $\eta = (1, -1)$ then $\lambda > 0$ by Lemma 9.2 and this contradicts Lemma 7.2 as before.

Suppose finally that $a = 0$. By Lemma 5.5, $u(0) = 0$ and $u(b) = \pm 1$. Suppose $u(b) = 1$. Again employing the formula above, $\theta_2(b) - \theta_2(0) < -\pi/2$ by Lemma 9.19, the fact that the function $\phi : (0, 1) \to \mathbb{R}; t \mapsto t/\sqrt{1-t^2}$ is strictly increasing and Lemma 9.20. This means that $\theta_2(0) > \pi/2$. This contradicts the C^1 property at $0 \in M$. If $u(b) = -1$ then then $\lambda > 0$ by Lemma 9.2 giving a contradiction. □

Lemma 10.4 *Let f be as in (1.3) where $h : [0, +\infty) \to \mathbb{R}$ is a non-decreasing convex function. Let $v > 0$.*

- *(i) Let E be a bounded minimiser of (1.2). Assume that E is open, $M := \partial E$ is a $C^{1,1}$ hypersurface in $\mathbb{R}^2$ and $E \setminus \{0\} = E^{sc}$. Then for any $r > 0$ with $r \geq R$, $M \setminus \overline{B}(0, r)$ consists of a finite union of disjoint centred circles.*
- *(ii) There exists a minimiser E of (1.2) such that ∂E consists of a finite union of disjoint centred circles.*

Proof (i) First observe that

$$\emptyset \neq \pi(M) = \Big[\pi(M) \cap [0, r]\Big] \cup \Big[\pi(M) \cap (r, +\infty)\Big] \setminus \Omega$$

by Lemma 10.3. We assume that the latter member is non-empty. By definition of Ω, $\cos\sigma = 0$ on $M \cap A((r, +\infty))$. Let $\tau \in \pi(M) \cap (r, +\infty)$. We claim that $M_\tau = \mathbb{S}^1_\tau$. Suppose for a contradiction that $M_\tau \neq \mathbb{S}^1_\tau$. By Lemma 5.2, M_τ is the union of two closed spherical arcs in $\mathbb{S}^1_\tau$. Let x be a point on the boundary of one of these spherical arcs relative to $\mathbb{S}^1_\tau$. There exists a $C^{1,1}$ parametrisation $\gamma_1 : I \to M$ of M in a neighbourhood of x with $\gamma_1(0) = x$ as before. By shrinking I if necessary we may assume that $\gamma_1(I) \subset A((r, +\infty))$ as $\tau > r$. By (2.9), $\dot{r}_1 = 0$ on I as $\cos\sigma_1 = 0$ on I

because $\cos\sigma = 0$ on $M \cap A((r, +\infty))$; that is, r_1 is constant on I. This means that $\gamma_1(I) \subset \mathbb{S}^1_\tau$. As the function $\sin\sigma_1$ is continuous on I it takes the value ± 1 there. By (2.10), $r_1\dot{\theta}_1 = \sin\sigma_1 = \pm 1$ on I. This means that θ_1 is either strictly decreasing or strictly increasing on I. This entails that the point x is not a boundary point of M_τ in $\mathbb{S}^1_\tau$ and this proves the claim.

It follows from these considerations that $M \setminus \overline{B}(0, r)$ consists of a finite union of disjoint centred circles. Note that $f \geq e^{h(0)} =: c > 0$ on $\mathbb{R}^2$. As a result, $+\infty > P_f(E) \geq cP(E)$ and in particular the relative perimeter $P(E, \mathbb{R}^2 \setminus \overline{B}(0, r)) < +\infty$. This explains why $M \setminus \overline{B}(0, r)$ comprises only finitely many circles.
(ii) Let E be a bounded minimiser of (1.2) such that E is open, $M := \partial E$ is a $C^{1,1}$ hypersurface in $\mathbb{R}^2$ and $E \setminus \{0\} = E^{sc}$ as in Theorem 4.5. Assume that $R > 0$. By *(i)*, $M \setminus \overline{B}(0, R)$ consists of a finite union of disjoint centred circles. We claim that only one of the possibilities

$$M_R = \emptyset,\ M_R = \mathbb{S}^1_R,\ M_R = \{Re_1\} \text{ or } M_R = \{-Re_1\} \tag{10.3}$$

holds. To prove this suppose that $M_R \neq \emptyset$ and $M_R \neq \mathbb{S}^1_R$. Bearing in mind Lemma 5.2 we may choose $x \in M_R$ such that x lies on the boundary of M_R relative to $\mathbb{S}^1_R$. Assume that $x \in H$. Let $\gamma_1 : I \to M$ be a local parametrisation of M with $\gamma_1(0) = x$ with the usual conventions. We first notice that $\cos(\sigma(x)) = 0$ for otherwise we obtain a contradiction to Theorem 10.3. As r_1 is decreasing on I and x is a relative boundary point it holds that $r_1 < R$ on $I^+ := I \cap \{s > 0\}$. As $M \setminus \overline{\Lambda_1}$ is open in M we may suppose that $\gamma_1(I^+) \subset M \setminus \overline{\Lambda_1}$. According to Theorem 6.5 the curvature k of $\gamma_1(I^+) \cap B(0, R)$ is a.e. constant as ρ vanishes on $(0, R)$. Hence $\gamma_1(I^+) \cap B(0, R)$ consists of a line or circular arc. The fact that $\cos(\sigma(x)) = 0$ means that $\gamma_1(I^+) \cap B(0, R)$ cannot be a line. So $\gamma_1(I^+) \cap B(0, R)$ is an open arc of a circle C containing x in its closure with centre on the line-segment $[0, x]$ and radius $r \in (0, R)$. By considering a local parametrisation, it can be seen that $C \cap B(0, R) \subset M$. But this contradicts the fact that $E \setminus \{0\} = E^{sc}$. In summary, $M_R \subset \{\pm Re_1\}$. Finally note that if $M_R = \{\pm Re_1\}$ then $M_R = \mathbb{S}^1_R$ by Lemma 5.2. This establishes (10.3).

Suppose that $M_R = \emptyset$. As both sets M and $\mathbb{S}^1_R$ are compact, $d(M, \mathbb{S}^1_R) > 0$. Assume first that $\mathbb{S}^1_R \subset E$. Put $F := B(0, R) \setminus E$ and suppose $F \neq \emptyset$. Then F is a set of finite perimeter, $F \subset\subset B(0, R)$ and $P(F) = P(E, B(0, R))$. Let B be a centred ball with $|B| = |F|$. By the classical isoperimetric inequality, $P(B) \leq P(F)$. Define $E_1 := (\mathbb{R}^2 \setminus B) \cap (B(0, R) \cup E)$. Then $V_f(E_1) = V_f(E)$ and $P_f(E_1) \leq P_f(E)$. That is, E_1 is a minimiser of (1.2) such that ∂E_1 consists of a finite union of disjoint centred circles. Now suppose that $\mathbb{S}^1_R \subset \mathbb{R}^2 \setminus \overline{E}$. In like fashion we may redefine E via $E_1 := B \cup (E \setminus \overline{B}(0, R))$ with B a centred ball in $B(0, R)$. The remaining cases in (10.3) can be dealt with in a similar way. The upshot of this argument is that there exists a m inimiser of (1.2) whose boundary M consists of a finite union of disjoint centred circles in case $R > 0$.

Now suppose that $R = 0$. By *(i)*, $M \setminus \overline{B}(0, r)$ consists of a finite union of disjoint centred circles for any $r \in (0, 1)$. If these accumulate at 0 then M fails to be C^1 at the origin. The assertion follows. □

Lemma 10.5 *Suppose that the function* $J : [0, +\infty) \to [0, +\infty)$ *is continuous non-decreasing and* $J(0) = 0$. *Let* $N \in \mathbb{N} \cup \{+\infty\}$ *and* $\{t_h : h = 0, \dots, 2N+1\}$ *a sequence of points in* $[0, +\infty)$ *with*

$$t_0 > t_1 > \cdots > t_{2h} > t_{2h+1} > \cdots \geq 0.$$

Then

$$+\infty \geq \sum_{h=0}^{2N+1} J(t_h) \geq J\left(\sum_{h=0}^{2N+1} (-1)^h t_h\right).$$

Proof We suppose that $N = +\infty$. The series $\sum_{h=0}^{\infty}(-1)^h t_h$ converges by the alternating series test. For each $n \in \mathbb{N}$,

$$\sum_{h=0}^{2n+1} (-1)^h t_h \leq t_0$$

and the same inequality holds for the infinite sum. As in Step 2 in [5] Theorem 2.1,

$$+\infty \geq \sum_{h=0}^{\infty} J(t_h) \geq J(t_0) \geq J\left(\sum_{h=0}^{\infty}(-1)^h t_h\right)$$

as J is non-decreasing. □

Proof of Theorem 1.1 There exists a minimiser E of (1.2) with the property that ∂E consists of a finite union of disjoint centred circles according to Lemma 10.4. As such we may write

$$E = \bigcup_{h=0}^{N} A((a_{2h+1}, a_{2h}))$$

where $N \in \mathbb{N}$ and $+\infty > a_0 > a_1 > \cdots > a_{2N} > a_{2N+1} > 0$. Define

$$\begin{aligned} \mathtt{f} &: [0, +\infty) \to \mathbb{R}; t \mapsto e^{h(t)}; \\ g &: [0, +\infty) \to \mathbb{R}; t \mapsto t\mathtt{f}(t); \\ G &: [0, +\infty) \to \mathbb{R}; t \mapsto \int_0^t g \, d\tau. \end{aligned}$$

Then $G : [0, +\infty) \to [0, +\infty)$ is a bijection with inverse G^{-1}. Define the strictly increasing function

$$J : [0, +\infty) \to \mathbb{R}; t \mapsto g \circ G^{-1}.$$

Put $t_h := G(a_h)$ for $h = 0, \dots, 2N+1$. Then $+\infty > t_0 > t_1 > \cdots > t_{2N} > t_{2N+1} >> 0$. Put $B := B(0, r)$ where $r := G^{-1}(v/2\pi)$ so that $V_f(B) = v$. Note that

$$
\begin{aligned}
v = V_f(E) &= 2\pi \sum_{h=0}^{N} \Big\{ G(a_{2h}) - G(a_{2h+1}) \Big\} \\
&= 2\pi \sum_{h=0}^{2N+1} (-1)^h t_h.
\end{aligned}
$$

By Lemma 10.5,

$$
\begin{aligned}
P_f(E) = 2\pi \sum_{h=0}^{2N+1} g(a_h) &= 2\pi \sum_{h=0}^{2N+1} J(t_h) \\
&\geq 2\pi J\left(\sum_{h=0}^{2N+1} (-1)^h t_h \right) \\
&= 2\pi J(v/2\pi) = P_f(B).
\end{aligned}
$$

□

Proof of Theorem 1.2 Let $v > 0$ and E be a minimiser for (1.2). Then E is essentially bounded by Theorem 3.1. By Theorem 4.5 there exists an $\mathscr{L}^2$-measurable set $\widetilde{E}$ with the properties

(a) $\widetilde{E}$ is a minimiser of (1.2);
(b) $L_{\widetilde{E}} = L_E$ a.e. on $(0, +\infty)$;
(c) $\widetilde{E}$ is open, bounded and has $C^{1,1}$ boundary;
(d) $\widetilde{E} \backslash \{0\} = \widetilde{E}^{sc}$.

(i) Suppose that $0 < v \leq v_0$ so that $R > 0$. Choose $r \in (0, R]$ such that $V(B(0, r)) = V(E) = v$. Suppose that $\widetilde{E} \backslash \overline{B}(0, R) \neq \emptyset$. By Lemma 10.4 there exists $t > R$ such that $\mathbb{S}_t^1 \subset M$. As g is strictly increasing, $g(t) > g(r)$. So $P_f(E) = P_f(\widetilde{E}) \geq \pi g(t) > \pi g(r) = P_f(B(0, r))$. This contradicts the fact that E is a minimiser for (1.2). So $\widetilde{E} \subset \overline{B}(0, R)$ and $L_{\widetilde{E}} = 0$ on $(R, +\infty)$. By property (b), $|E \backslash \overline{B}(0, R)| = 0$. By the uniqueness property in the classical isoperimetric theorem (see for example [12] Theorem 4.11) the set E is equivalent to a ball B in $\overline{B}(0, R)$.
(ii) With $r > 0$ as before, $V(B(0, r)) = V(E) = v > v_0 = V(B(0, R))$ so $r > R$. If $\widetilde{E} \backslash \overline{B}(0, r) \neq \emptyset$ we derive a contradiction in the same way as above. Consequently, $\widetilde{E} = B := B(0, r)$. Thus, $L_E = L_B$ a.e. on $(0, +\infty)$; in particular, $|E \backslash B| = 0$. This entails that E is equivalent to B. □

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