Research Article

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An iterative approach using Sawi transform for fractional telegraph equation in diversified dimensions

https://doi.org/10.1515/nleng-2022-0285
received January 3, 2023; accepted February 25, 2023

Abstract: In the present study, 1D, 2D, and 3D fractional hyperbolic telegraph equations in Caputo sense have been solved using an iterative method using Sawi transform. These equations serve as a model for signal analysis of electrical impulse transmission and propagation. Along with a table of Sawi transform of some popular functions, some helpful results on Sawi transform are provided. To demonstrate the effectiveness of the suggested method, five examples in 1D, one example in 2D, and one example in 3D are solved using the proposed scheme. Error analysis comparing approximate and exact solutions using graphs and tables has been provided. The proposed scheme is robust, effective, and easy to implement and can be implemented on variety of fractional partial differential equations to obtain precise series approximations.

Keywords: Caputo derivative, fractional hyperbolic telegraph equation, Sawi transform

1 Introduction

Fractional calculus deals with derivatives of fractional orders, and the concept can be dated back to a letter from Leibnitz to L'Hospital discussing the possibility of fractional order derivatives [1]. In the present day, multiple different fractional derivatives exist, for example Caputo derivative, Riemann–Liouville derivative [2], Atangana–Baleanu Caputo derivative [3], *etc*. There exists a wide variety of fractional partial differential equations (FPDEs), one such type of equation being the fractional hyperbolic telegraph equation, which is used as a model in signal analyses of transmission and propagation of electrical impulses and other fields [4]. Many methods have been developed and used to solve FPDEs, such as iterative Laplace transform method [5], homotopy analysis method [6], finite difference method [7], Adomian decomposition method and variational iteration method [8], computational model based on hybrid B-spline collocation method [9], *etc.*

In the 1880s, Oliver Heaviside developed the telegraph equation to describe the time and distance on an electric transmission line with current and voltage [10]. As fractional hyperbolic telegraph equations (FHTEs) are a kind of FPDE, it is difficult to solve them by the usual means. Therefore, multiple techniques have been developed and applied to solve FHTEs, such as Chebyshev Tau method [11], Sinc Legendre collocation method [12], He's variational iteration Method [13], fractional skewed grid Crank–Nicolson scheme [14], meshless method using radial basis function [15], hybrid meshless method by combining GFDM in space domain and Houbolt method in temporal dimension [16], shifted Jacobi collocation scheme [17], finite difference scheme based on extended cubic-B spline method [18], least square homotopy perturbation technique [19], *etc.*

Definition 1. One, two, and three dimensional (1D, 2D, and 3D) FHTEs are given as follows [20]:

$$D_t^{\omega}\delta + \beta\delta(\varepsilon, t) + \alpha\delta_t = \delta_{\varepsilon\varepsilon} + g(\varepsilon, t), \tag{1}$$

$$D_t^{2\omega}\delta + 2\omega D_t^{\omega}\delta + \beta^2\delta = \delta_{\varepsilon\varepsilon} + \delta_{\sigma\sigma} + g(\varepsilon, \sigma, t), \tag{2}$$

$$D_t^{2\omega}\delta + 2\omega D_t^{\omega}\delta + \beta^2\delta = \delta_{\varepsilon\varepsilon} + \delta_{\sigma\sigma} + \delta_{\tau\tau} + g(\varepsilon, \sigma, \tau, t), \tag{3}$$

where β and α are arbitrary constants and δ is an unknown function, ω is the fractional order of the equation, t is the time variable, ε, σ, and τ are the x, y, and z dimensions, respectively. D_t^{ω} is the fractional order derivative of order

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ω in Caputo sense. Alternatively, δ_t^{ω} can be used as the fractional order derivative of order ω in Caputo sense of the function δ. When dealing with space fractional telegraph equations, x would be used to refer to the $x$dimension. $\delta(\varepsilon, \sigma, 0) = g_1(\varepsilon, \sigma)$, $\delta_t(\varepsilon, \sigma, 0) = g_2(\varepsilon, \sigma)$, $\delta(\varepsilon, \sigma, \tau, 0) = g_1(\varepsilon, \sigma, \tau)$ and $\delta_t(\varepsilon, \sigma, \tau, 0) = g_2(\varepsilon, \sigma, \tau)$. For the sake of consistency, t, ω, ε, σ, and τ will have the same meaning as stated above throughout the manuscript.

Different transforms like Natural transform [21], Sumudu transform [22], Mohand transform [23] *etc.*, can be applied on pre-existing techniques to solve FPDEs, for example, Natural transform has been used with Adomian decomposition method, also known as Natural transform decomposition method to solve FHTEs [24], Shehu Transform is used in an analytical approach to solve time-fractional Schrödinger equations [25], differential transform method has been used to solve FPDEs like Bagley–Torvik equation and composite fractional oscillation equation [26], a method developed by combining time discretization and Laplace transform method has been used to numerically solve fractional differential equations *via* quadrature rule [27], Sumudu transform has been combined with homotopy perturbation method to solve non-linear fractional differential equations [28], inverse fractional Shehu transform method has been used to solve fractional differential equations [29] *etc.* In the present study, Sawi transform is used in an iterative approach, which is based on an analytical approach using Shehu transform to solve FHTEs [20], to obtain a series solution to FHTEs in 1D, 2D, and 3D.

Definition 2. Sawi transform of a function $\delta(t)$ is as follows [30]:

$$S[\delta(t)] = \frac{1}{\mu^2}\int_0^{\infty} e^{-\frac{t}{\mu}}\delta(t)\mathrm{d}t = \bar{\delta}(\mu), \quad t \geq 0, \mu > 0, \tag{4}$$

where μ is the transformed variable. μ will be the default transformed variable for the entire manuscript.

Definition 3. Linearity property of Sawi transform is as follows [30]:

$$\begin{aligned} S[a\delta_1(t) + b\delta_2(t)] &= \frac{1}{\mu^2}\int_0^{\infty} e^{-\frac{t}{\mu}}(a\delta_1(t) + b\delta_2(t))\mathrm{d}t \\ &= a\frac{1}{\mu^2}\int_0^{\infty} e^{-\frac{t}{\mu}}\delta_1(t)\mathrm{d}t + b\frac{1}{\mu^2}\int_0^{\infty} \mathrm{e}^{-\frac{t}{\mu}}\delta_2(t)\mathrm{d}t \\ &= aS[\delta_1(t)] + bS[\delta_2(t)]. \end{aligned} \tag{5}$$

Definition 4. Scaling property of Sawi transform is as follows [30]:

$$S[\delta(t)] = \frac{1}{\mu^2}\int_0^{\infty} e^{-\frac{t}{\mu}}\delta(t)\mathrm{d}t = \bar{\delta}(\mu), \quad t \geq 0, \mu > 0$$

Then,

$$S[\delta(at)] = \frac{1}{\mu^2}\int_0^{\infty} e^{-\frac{t}{\mu}}\delta(at)\mathrm{d}t.$$

Let $at = p$, then $\mathrm{d}p = a\mathrm{d}t$

$$\begin{aligned} S[\delta(at)] &= \frac{1}{a\mu^2}\int_0^{\infty} \mathrm{e}^{-\frac{p}{a\mu}}\delta(p)\mathrm{d}p \\ &= \frac{a}{(a\mu)^2}\int_0^{\infty} \mathrm{e}^{-\frac{p}{a\mu}}\delta(p)\mathrm{d}p = a\bar{\delta}(a\mu). \end{aligned} \tag{6}$$

Definition 5. Translation property of Sawi transform is as follows [30]:

$$S[\delta(t)] = \frac{1}{\mu^2}\int_0^{\infty} \mathrm{e}^{-\frac{t}{\mu}}\delta(t)\mathrm{d}t = \bar{\delta}(\mu), \quad t \geq 0, \mu > 0$$

Then,

$$S[e^{kt}\delta(t)] = \frac{1}{(1-k\mu)^2}\bar{\delta}\left(\frac{\mu}{1-k\mu}\right). \tag{7}$$

Definition 6. Caputo derivative of a function $\delta(t)$ is as follows [31]:

$${}_0^C D^{\omega}\delta(t) = \frac{1}{\Gamma(m-\omega)}\int_0^{t} (t-\theta)^{m-\omega-1}\delta^{(m)}(\theta)\mathrm{d}\theta, \tag{8}$$

where $m - 1 < \omega \leq m$, $m \in \mathbb{N}$, $\varepsilon > 0$.

Definition 7. Sawi transform of Caputo derivative of a function $\delta(t)$ is as follows [32]:

$$S[D_t^{\omega}\delta(t)] = \frac{1}{\mu^{\omega}}S[\delta(t)] - \sum_{k=0}^{m-1}\left(\frac{1}{\mu}\right)^{\omega-(k-1)}\delta^{(k)}(0). \tag{9}$$

2 Outline of the study

Outline of the study has been provided below.

- In Section 3, general formula for the 1D FHTEs is developed. For the 2D and 3D FHTEs, Appendixes A and B have been referenced.
- In Section 4, a total of seven examples have been solved to illustrate the efficacy of the proposed method. Examples 1, 4, and 5 involve 1D time FHTEs, Examples 6 and 7 involve 1D space FHTEs, Example 2 involves 2D time FHTE and Example 3 involves 3D time FHTE. A series solution is developed for each example using the proposed method.
- In Section 5, graphs and tables for Examples 1, 2, 4, 5, 6, and 7 have been provided to perform error analysis.
- In Section 6, the conclusion has been provided.

3 Development of the formula

The form of 1D time telegraph equation is as follows [20]:

$$D_t^{\omega}\delta(\varepsilon,t)+L[\delta(\varepsilon,t)]+N[\delta(\varepsilon,t)]=q(\varepsilon,t),$$

where L refers to the linear operator and N refers to the nonlinear operator. $D_t^{\omega}\delta(\varepsilon,t)$ is the Caputo derivative of $\delta(\varepsilon,t)$. Applying Sawi transform on equation

$$S[D_t^{\omega}\delta(\varepsilon,t)]+S[L[\delta(\varepsilon,t)]+N[\delta(\varepsilon,t)]]=S[q(\varepsilon,t)],$$

$$\frac{1}{\mu^{\omega}}S[\delta(\varepsilon,t)]-\sum_{k=0}^{m-1}\left(\frac{1}{\mu}\right)^{\omega-(k-1)}\delta^{(k)}(0)$$
$$=S[q(\varepsilon,t)]-S[L[\delta(\varepsilon,t)]]-S[N[\delta(\varepsilon,t)]],$$

$$\Rightarrow\delta(\varepsilon,t)=S^{-1}\left[\mu^{\omega}\left(\sum_{k=0}^{m-1}\left(\frac{1}{\mu}\right)^{\omega-(k-1)}\delta^{(k)}(0)+S[q(\varepsilon,t)]\right)\right]$$
$$-S^{-1}[\mu^{\omega}(S[L[\delta(\varepsilon,t)]]+S[N[\delta(\varepsilon,t)]])].$$

Now, linear and nonlinear operators can be decomposed in the following manner:

$$L[\delta]=L\left[\sum_{r=0}^{\infty}\delta_r(\varepsilon,t)\right]$$
$$=L[\delta_0(\varepsilon,t)]+\sum_{r=1}^{\infty}\left[L\left[\sum_{i=0}^{r}\delta_i(\varepsilon,t)\right]-L\left[\sum_{i=0}^{r-1}\delta_i(\varepsilon,t)\right]\right].$$

$$N[\delta]=N\left[\sum_{r=0}^{\infty}\delta_r(\varepsilon,t)\right]$$
$$=N[\delta_0(\varepsilon,t)]+\sum_{r=1}^{\infty}\left[N\left[\sum_{i=0}^{r}\delta_i(\varepsilon,t)\right]-N\left[\sum_{i=0}^{r-1}\delta_i(\varepsilon,t)\right]\right],$$

$$\Rightarrow\sum_{k=0}^{\infty}\delta_k(\varepsilon,t)=S^{-1}\left[\mu^{\omega}\left(\sum_{k=0}^{m-1}\left(\frac{1}{\mu}\right)^{\omega-(k-1)}\delta^{(k)}(0)+S[q(\varepsilon,t)]\right)\right]$$
$$-S^{-1}[\mu^{\omega}S(L[\delta_0(\varepsilon,t)]+N[\delta_0(\varepsilon,t)])]$$
$$-S^{-1}\left[\mu^{\omega}S\left(\sum_{r=1}^{\infty}L[\delta_r(\varepsilon,t)]\right.\right.$$
$$\left.\left.+\sum_{r=1}^{\infty}\left[N\left[\sum_{i=0}^{r}\delta_i(\varepsilon,t)\right]-N\left[\sum_{i=0}^{r-1}\delta_i(\varepsilon,t)\right]\right]\right)\right],$$

$$\Rightarrow\delta_0(\varepsilon,t)=S^{-1}\left[\mu^{\omega}\left(\sum_{k=0}^{m-1}\left(\frac{1}{\mu}\right)^{\omega-(k-1)}\delta^{(k)}(0)+S[q(\varepsilon,t)]\right)\right],$$

$$\Rightarrow\delta_1(\varepsilon,t)=-S^{-1}[\mu^{\omega}S(L[\delta_0(\varepsilon,t)]+N[\delta_0(\varepsilon,t)])],$$

$$\Rightarrow\delta_{r+1}(\varepsilon,t)=-S^{-1}\left[\mu^{\omega}S\left(\sum_{r=1}^{\infty}L[\delta_r(\varepsilon,t)]\right.\right.$$
$$\left.\left.+\sum_{r=1}^{\infty}\left[N\left[\sum_{i=0}^{r}\delta_i(\varepsilon,t)\right]-N\left[\sum_{i=0}^{r-1}\delta_i(\varepsilon,t)\right]\right]\right)\right],$$
$$r=1,2,3,4,5,\ldots.$$

The general formulae for 2D and 3D can also be derived in a similar manner. Their proofs have been provided in Appendixes A and B, respectively. It can be observed that the first term δ_0 depends on the initial conditions.

4 Examples and calculations

Each example has a series solution calculated at a specific ω and the exact solution is also provided. Examples 1, 4, and 5 contain 1D time FHTEs, Examples 6 and 7 contain 1D space FHTEs, Example 2 contains 2D time FHTE and Example 3 contains 3D time FHTE.

Example 1. Consider the following 1D time FHTE [20]:

$$D_t^{\omega}\delta=\delta-2\delta_t-\delta_{\varepsilon\varepsilon},\tag{10}$$

where $\delta(\varepsilon,0)=e^{\varepsilon}$ and $\delta_t(\varepsilon,0)=-2e^{\varepsilon}$, $0<\omega\le 2$ are the initial conditions.

$$\delta_0(\varepsilon,t)=\delta(\varepsilon,0)+t\delta_t(\varepsilon,0)=e^{\varepsilon}-t(2e^{\varepsilon})=e^{\varepsilon}(1-2t).$$

Applying Sawi transform on Eq. (10)

$$S[D_t^{\omega}\delta(\varepsilon,t)]=S[\delta-2\delta_t-\delta_{\varepsilon\varepsilon}],$$

$$\Rightarrow \delta = S^{-1}\left[\mu^{\omega}\left(\sum_{k=0}^{m-1}\left(\frac{1}{\mu}\right)^{\omega-(k-1)}\delta^{(k)}(0)\right)\right] + S^{-1}[\mu^{\omega}S[R(\delta)]],$$

$$\Rightarrow \delta_0 = S^{-1}\left[\mu^{\omega}\left(\sum_{k=0}^{m-1}\left(\frac{1}{\mu}\right)^{\omega-(k-1)}\delta^{(k)}(0)\right)\right], \quad (11)$$

$$\Rightarrow \delta_{r+1} = S^{-1}[\mu^{\omega}S[R(\delta_r)]]. \quad (12)$$

Considering $m = 1$ in Eq. (11),

$$\delta_0 = S^{-1}\left[\frac{1}{\mu}\delta(0)\right] = \delta(0) = e^{\varepsilon}(1-2t),$$

$$R[\delta_0] = \delta_0 - 2(\delta_0)_t - (\delta_0)_{\varepsilon\varepsilon} = 4e^{\varepsilon}.$$

Using Eq. (12)

$$\delta_1 = S^{-1}[\mu^{\omega}(S[4e^{\varepsilon}])] \Rightarrow \delta_1 = 4e^{\varepsilon}S^{-1}[\mu^{\omega-1}] \Rightarrow \delta_1(\varepsilon,t) = \frac{4e^{\varepsilon}t^{\omega}}{\Gamma(\omega+1)},$$

$$R[\delta_1] = \delta_1 - 2(\delta_1)_t - (\delta_1)_{\varepsilon\varepsilon} = \frac{-8e^{\varepsilon}\omega t^{\omega-1}}{\Gamma(\omega+1)}.$$

Using Eq. (12)

$$\delta_2 = S^{-1}\left[\mu^{\omega}\left(S\left[\frac{-8e^{\varepsilon}\omega t^{\omega-1}}{\Gamma(\omega+1)}\right]\right)\right] \Rightarrow \delta_2 = \frac{-8e^{\varepsilon}\omega}{\Gamma(\omega+1)}S^{-1}[\Gamma(\omega)\mu^{2\omega-2}] \Rightarrow \delta_2(\varepsilon,t) = \frac{-8e^{\varepsilon}\omega\Gamma(\omega)t^{2\omega-1}}{\Gamma(\omega+1)\Gamma(2\omega)},$$

$$R[\delta_2] = \delta_2 - 2(\delta_2)_t - (\delta_2)_{\varepsilon\varepsilon} = \frac{16e^{\varepsilon}\omega(2\omega-1)\Gamma(\omega)t^{2\omega-2}}{\Gamma(\omega+1)\Gamma(2\omega)}.$$

Using Eq. (12)

$$\delta_3 = S^{-1}\left[\mu^{\omega}\left(S\left[\frac{16e^{\varepsilon}\omega(2\omega-1)\Gamma(\omega)t^{2\omega-2}}{\Gamma(\omega+1)\Gamma(2\omega)}\right]\right)\right] \Rightarrow \delta_2 = \frac{16e^{\varepsilon}\omega(2\omega-1)\Gamma(\omega)}{\Gamma(\omega+1)\Gamma(2\omega)}S^{-1}[\Gamma(2\omega-1)\mu^{3\omega-3}] \Rightarrow \delta_3(\varepsilon,t) = \frac{16e^{\varepsilon}\omega(2\omega-1)\Gamma(\omega)\Gamma(2\omega-1)t^{3\omega-2}}{\Gamma(\omega+1)\Gamma(2\omega)\Gamma(3\omega-1)},$$

$$\delta(\varepsilon,t) = \delta_0(\varepsilon,t) + \delta_1(\varepsilon,t) + \delta_2(\varepsilon,t) + \delta_3(\varepsilon,t) + \ldots$$

$$\Rightarrow \delta(\varepsilon,t) = e^{\varepsilon}(1-2t) + \frac{4e^{\varepsilon}t^{\omega}}{\Gamma(\omega+1)} - \frac{8e^{\varepsilon}\omega\Gamma(\omega)t^{2\omega-1}}{\Gamma(\omega+1)\Gamma(2\omega)} + \frac{16e^{\varepsilon}\omega(2\omega-1)\Gamma(\omega)\Gamma(2\omega-1)t^{3\omega-2}}{\Gamma(\omega+1)\Gamma(2\omega)\Gamma(3\omega-1)} - \ldots$$

Putting $\omega = 2$, the series solution is as follows:

$$\delta(\varepsilon,t) = e^{\varepsilon}\left(1 - \frac{2t}{1!} + \frac{(2t)^2}{2!} - \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} - \ldots\right).$$

Also, the fractional equation becomes,

$$\delta_{tt} = \delta - 2\delta_t - \delta_{\varepsilon\varepsilon}.$$

Given the same initial conditions, the exact solution of this differential equation is $\delta(\varepsilon,t) = e^{\varepsilon-2t}$ [20], and it agrees with the series solution obtained by the iterative scheme.

Example 2. Consider the following 2D time FHTE [20]:

$$D_t^{2\omega}\delta + 3D_t^{\omega}\delta + 2\delta = \delta_{\varepsilon\varepsilon} + \delta_{\sigma\sigma}, \quad (13)$$

where $\delta(\varepsilon,\sigma,0) = e^{\varepsilon+\sigma}$ and $\delta_t(\varepsilon,\sigma,0) = -3e^{\varepsilon+\sigma}$ are the initial conditions.

$$\delta_0(\varepsilon,\sigma,t) = \delta(\varepsilon,\sigma,0) + t\delta_t(\varepsilon,\sigma,0) = e^{\varepsilon+\sigma} - 3t(e^{\varepsilon+\sigma}) = e^{\varepsilon+\sigma}(1-3t).$$

Applying Sawi transform on Eq. (13)

$$S[D_t^{2\omega}\delta + 3D_t^{\omega}\delta + 2\delta] = S[\delta_{\varepsilon\varepsilon} + \delta_{\sigma\sigma}],$$

$$\Rightarrow \delta(\varepsilon,\sigma,t) = S^{-1}\left[\mu^{2\omega}\left(\sum_{k=0}^{m-1}\left(\frac{1}{\mu}\right)^{2\omega-(k-1)}\delta^{(k)}(0)\right)\right] + S^{-1}[\mu^{2\omega}(S[R[\delta] - 3D_t^{\omega}\delta])],$$

$$\Rightarrow \delta_0 = S^{-1}\left[\mu^{2\omega}\left(\sum_{k=0}^{m-1}\left(\frac{1}{\mu}\right)^{2\omega-(k-1)}\delta^{(k)}(0)\right)\right], \quad (14)$$

$$\delta_{r+1} = S^{-1}[\mu^{2\omega}(S[R[\delta] - 3D_t^{\omega}\delta])]. \quad (15)$$

$$r = 0, 1, 2, 3, \ldots$$

Considering $m = 1$ in Eq. (14),

$$\delta_0 = S^{-1}\left[\frac{1}{\mu}\delta(0)\right] = \delta(0) = e^{\varepsilon+\sigma}(1-3t),$$

$$R[\delta_0] = (\delta_0)_{\varepsilon\varepsilon} + (\delta_0)_{\sigma\sigma} - 2\delta_0 = 0.$$

Using Eq. (15)

$$\delta_1 = S^{-1}[\mu^{2\omega}(-3S[D_t^{\omega}\delta_0])] = -3S^{-1}[\mu^{2\omega}(S[D_t^{\omega}\delta_0])],$$

$$S[D_t^{\omega}\delta_0] = \tfrac{1}{\mu^{\omega}}S[\delta_0] - \tfrac{1}{\mu^{\omega+1}}\delta_0(0) \Rightarrow S[D_t^{\omega}\delta_0] = \tfrac{-3e^{\varepsilon+\sigma}}{\mu^{\omega}}$$

$$\Rightarrow \delta_1 = -3S^{-1}\left[\mu^{2\omega}\left(\frac{-3e^{\varepsilon+\sigma}}{\mu^{\omega}}\right)\right] \Rightarrow \delta_1 = \frac{9e^{\varepsilon+\sigma}t^{\omega+1}}{\Gamma(\omega+2)},$$

$$R[\delta_1] = (\delta_1)_{\varepsilon\varepsilon} + (\delta_1)_{\sigma\sigma} - 2\delta_1 = 0.$$

Using Eq. (15)

$$\delta_2 = S^{-1}[\mu^{2\omega}(-3S[D_t^{\omega}\delta_1])] = -3S^{-1}[\mu^{2\omega}(S[D_t^{\omega}\delta_1])],$$

$$S[D_t^{\omega}\delta_1] = \tfrac{1}{\mu^{\omega}}S[\delta_1] - \tfrac{1}{\mu^{\omega+1}}\delta_1(0) \Rightarrow S[D_t^{\omega}\delta_1] = 9e^{\varepsilon+\sigma},$$

$$\Rightarrow \delta_1 = -3S^{-1}[\mu^{2\omega}(9e^{\varepsilon+\sigma})] \Rightarrow \delta_1 = \frac{-27e^{\varepsilon+\sigma}t^{2\omega+1}}{\Gamma(2\omega+2)},$$

$$R[\delta_2] = (\delta_2)_{\varepsilon\varepsilon} + (\delta_2)_{\sigma\sigma} - 2\delta_1 = 0.$$

Using Eq. (15)

$$\delta_3 = S^{-1}[\mu^{2\omega}(-3S[D_t^{\omega}\delta_2])] = -3S^{-1}[\mu^{2\omega}(S[D_t^{\omega}\delta_2])],$$

$$S[D_t^{\omega}\delta_2] = \tfrac{1}{\mu^{\omega}}S[\delta_2] - \tfrac{1}{\mu^{\omega+1}}\delta_2(0) \Rightarrow S[D_t^{\omega}\delta_2] = -27e^{\varepsilon+\sigma}\mu^{\omega},$$

$$\Rightarrow \delta_3 = -3S^{-1}[\mu^{2\omega}(-27e^{\varepsilon+\sigma}\mu^{\omega})] \Rightarrow \delta_3 = \frac{81e^{\varepsilon+\sigma}t^{3\omega+1}}{\Gamma(3\omega+2)},$$

$$\delta(\varepsilon,\sigma,t) = \delta_0(\varepsilon,\sigma,t) + \delta_1(\varepsilon,\sigma,t) + \delta_2(\varepsilon,\sigma,t) + \delta_3(\varepsilon,\sigma,t) + \ldots,$$

$$\Rightarrow \delta(\varepsilon,\sigma,t) = e^{\varepsilon+\sigma}(1-2t) + \frac{9e^{\varepsilon+\sigma}t^{\omega+1}}{\Gamma(\omega+2)} - \frac{27e^{\varepsilon+\sigma}t^{2\omega+1}}{\Gamma(2\omega+2)} + \frac{81e^{\varepsilon+\sigma}t^{3\omega+1}}{\Gamma(3\omega+2)} - \ldots.$$

Putting $\omega = 1$, the series solution is as follows:

$$\delta(\varepsilon,\sigma,t) = e^{\varepsilon+\sigma}\left(1 - \frac{3t}{1!} + \frac{(3t)^2}{2!} - \frac{(3t)^3}{3!} + \frac{(3t)^4}{4!} - \ldots\right).$$

Also, the fractional equation becomes

$$\delta_{tt} + 3\delta_t + 2\delta = \delta_{\varepsilon\varepsilon} + \delta_{\sigma\sigma}.$$

Given the same initial conditions, the exact solution of this differential equation is $\delta(\varepsilon,\sigma,t) = e^{\varepsilon+\sigma-3t}$ [20], and it agrees with the series solution obtained by the iterative scheme.

Example 3. Consider the following 3D time FHTE [20]

$$D_t^{2\omega}\delta + 2D_t^{\omega}\delta + 3\delta = \delta_{\varepsilon\varepsilon} + \delta_{\sigma\sigma} + \delta_{\tau\tau}, \tag{16}$$

where $\delta(\varepsilon,\sigma,\tau,0) = \sinh\varepsilon\sinh\sigma\sinh\tau$ and $\delta_t(\varepsilon,\sigma,\tau,0) = -\sinh\varepsilon\sinh\sigma\sinh\tau$, $0 < \omega \le 1$ are the initial conditions.

$$\begin{aligned}\delta_0(\varepsilon,\sigma,t) &= \delta(\varepsilon,\sigma,\tau,0) + t\delta_t(\varepsilon,\sigma,\tau,0)\\ &= \sinh\varepsilon\sinh\sigma\sinh\tau - t(\sinh\varepsilon\sinh\sigma\sinh\tau)\\ &= \sinh\varepsilon\sinh\sigma\sinh\tau(1-t).\end{aligned}$$

Applying Sawi transform on Eq. (16)

$$S[D_t^{2\omega}\delta + 2D_t^{\omega}\delta + 3\delta] = S[\delta_{\varepsilon\varepsilon} + \delta_{\sigma\sigma} + \delta_{\tau\tau}],$$

$$\Rightarrow \delta(\varepsilon,\sigma,\tau,t) = S^{-1}\left[\mu^{2\omega}\left(\sum_{k=0}^{m-1}\left(\frac{1}{\mu}\right)^{2\omega-(k-1)}\delta^{(k)}(0)\right)\right] + S^{-1}[\mu^{2\omega}(S[R[\delta] - 2D_t^{\omega}\delta])],$$

$$\Rightarrow \delta_0(\varepsilon,\sigma,\tau,t) = S^{-1}\left[\mu^{2\omega}\left(\sum_{k=0}^{m-1}\left(\frac{1}{\mu}\right)^{2\omega-(k-1)}\delta^{(k)}(0)\right)\right], \tag{17}$$

$$\delta_{r+1}(\varepsilon,\sigma,\tau,t) = S^{-1}[\mu^{2\omega}(S[R[\delta] - 2D_t^{\omega}\delta])], \quad r = 0,1,2,3,\ldots \tag{18}$$

Considering $m = 1$ in Eq. (17),

$$\delta_0 = S^{-1}\left[\frac{1}{\mu}\delta(0)\right] = \delta(0) = \sinh\varepsilon\sinh\sigma\sinh\tau(1-t),$$

$$R[\delta_0] = (\delta_0)_{\varepsilon\varepsilon} + (\delta_0)_{\sigma\sigma} + (\delta_0)_{\tau\tau} - 3\delta_0 = 0.$$

Using Eq. (18)

$$\delta_1 = S^{-1}[\mu^{2\omega}(-2S[D_t^{\omega}\delta_0])] = -2S^{-1}[\mu^{2\omega}(S[D_t^{\omega}\delta_0])],$$

$$S[D_t^{\omega}\delta_0] = \tfrac{1}{\mu^{\omega}}S[\delta_0] - \tfrac{1}{\mu^{\omega+1}}\delta_0(0) \Rightarrow S[D_t^{\omega}\delta_0] = \frac{-\sinh\varepsilon\sinh\sigma\sinh\tau}{\mu^{\omega}},$$

$$\Rightarrow \delta_1 = -2S^{-1}\left[\mu^{2\omega}\left(\frac{-\sinh\varepsilon\sinh\sigma\sinh\tau}{\mu^{\omega}}\right)\right] \Rightarrow \delta_1 = \frac{2\sinh\varepsilon\sinh\sigma\sinh\tau t^{\omega+1}}{\Gamma(\omega+2)},$$

$$R[\delta_1] = (\delta_1)_{\varepsilon\varepsilon} + (\delta_1)_{\sigma\sigma} + (\delta_1)_{\tau\tau} - 3\delta_1 = 0.$$

Using Eq. (18)

$$\delta_2 = S^{-1}[\mu^{2\omega}(-2S[D_t^{\omega}\delta_1])] = -2S^{-1}[\mu^{2\omega}(S[D_t^{\omega}\delta_1])],$$

$$S[D_t^{\omega}\delta_1] = \tfrac{1}{\mu^{\omega}}S[\delta_1] - \tfrac{1}{\mu^{\omega+1}}\delta_1(0) \Rightarrow S[D_t^{\omega}\delta_1] = 2\sinh\varepsilon\sinh\sigma\sinh\tau,$$

$$\Rightarrow \delta_1 = -2S^{-1}[\mu^{2\omega}(2\sinh\varepsilon\sinh\sigma\sinh\tau)] \Rightarrow \delta_1 = \frac{-4\sinh\varepsilon\sinh\sigma\sinh\tau t^{2\omega+1}}{\Gamma(2\omega+2)},$$

$$R[\delta_2] = (\delta_2)_{\varepsilon\varepsilon} + (\delta_2)_{\sigma\sigma} + (\delta_2)_{\tau\tau} - 3\delta_2 = 0.$$

Using Eq. (18)

$$\delta_3 = S^{-1}[\mu^{2\omega}(-2S[D_t^{\omega}\delta_2])] = -2S^{-1}[\mu^{2\omega}(S[D_t^{\omega}\delta_2])],$$

$$S[D_t^{\omega}\delta_2] = \tfrac{1}{\mu^{\omega}}S[\delta_2] - \tfrac{1}{\mu^{\omega+1}}\delta_2(0) \Rightarrow S[D_t^{\omega}\delta_2] = -4\sinh\varepsilon\sinh\sigma\sinh\tau\mu^{\omega},$$

$$\Rightarrow \delta_3 = -2S^{-1}[\mu^{2\omega}(-4\sinh\varepsilon\sinh\sigma\sinh\tau\mu^{\omega})] \Rightarrow \delta_3 = \frac{8\sinh\varepsilon\sinh\sigma\sinh\tau t^{3\omega+1}}{\Gamma(3\omega+2)},$$

$$\delta(\varepsilon,\sigma,\tau,t) = \delta_0(\varepsilon,\sigma,\tau,t) + \delta_1(\varepsilon,\sigma,\tau,t) + \delta_2(\varepsilon,\sigma,\tau,t) + \delta_3(\varepsilon,\sigma,\tau,t) + \ldots,$$

$$\Rightarrow \delta(\varepsilon,\sigma,\tau,t) = \sinh\varepsilon\sinh\sigma\sinh\tau(1-t) + \frac{2\sinh\varepsilon\sinh\sigma\sinh\tau t^{\omega+1}}{\Gamma(\omega+2)} - \frac{4\sinh\varepsilon\sinh\sigma\sinh\tau t^{2\omega+1}}{\Gamma(2\omega+2)} + \frac{8\sinh\varepsilon\sinh\sigma\sinh\tau t^{3\omega+1}}{\Gamma(3\omega+2)} - \ldots.$$

Putting $\omega = 1$, the series solution is as follows:

$$\delta(\varepsilon, \sigma, \tau, t) = \sinh\varepsilon\sinh\sigma\sinh\tau\left(1 - \frac{t}{1!} + \frac{2t^2}{2!} - \frac{4t^3}{3!} + \frac{8t^4}{4!} - \ldots\right).$$

Example 4. Consider the following FHTE [31]:

$$D_t^{2\omega}\delta + 2D_t^{\omega}\delta + \delta = \delta_{\varepsilon\varepsilon}, \tag{19}$$

where $\delta(\varepsilon, 0) = \sinh\varepsilon$ and $\delta_t(\varepsilon, 0) = -2\sinh\varepsilon$, $0 < \omega \le 2$ are the initial conditions.

$$\delta_0(\varepsilon, t) = \delta(\varepsilon, 0) + t\delta_t(\varepsilon, 0) = \sinh\varepsilon - t(2\sinh\varepsilon)$$
$$= \sinh\varepsilon(1 - 2t).$$

Applying Sawi transform on Eq. (19)

$$S[D_t^{2\omega}\delta + 2D_t^{\omega}\delta + \delta] = S[\delta_{\varepsilon\varepsilon}],$$

$$S[D_t^{2\omega}\delta] = S[\delta_{\varepsilon\varepsilon} - \delta - 2D_t^{\omega}\delta],$$

$$\Rightarrow \delta(\varepsilon, t) = S^{-1}\left[\mu^{2\omega}\left(\sum_{k=0}^{m-1}\left(\frac{1}{\mu}\right)^{2\omega-(k-1)}\delta^{(k)}(0)\right)\right] + S^{-1}[\mu^{2\omega}(S[R[\delta] - 2D_t^{\omega}\delta])],$$

$$\Rightarrow \delta_0 = S^{-1}\left[\mu^{2\omega}\left(\sum_{k=0}^{m-1}\left(\frac{1}{\mu}\right)^{2\omega-(k-1)}\delta^{(k)}(0)\right)\right], \tag{20}$$

$$\delta_{r+1} = S^{-1}[\mu^{2\omega}(S[R[\delta] - 2D_t^{\omega}\delta])], \tag{21}$$
$$r = 0, 1, 2, 3, \ldots$$

Considering $m = 1$ in Eq. (20),

$$\delta_0 = S^{-1}\left[\frac{1}{\mu}\delta(0)\right] = \delta(0) = \sinh\varepsilon(1 - 2t),$$

$$R[\delta_0] = (\delta_0)_{\varepsilon\varepsilon} - \delta_0 = 0,$$

Using Eq. (21)

$$\delta_1 = S^{-1}[\mu^{2\omega}(-2S[D_t^{\omega}\delta_0])] = -2S^{-1}[\mu^{2\omega}(S[D_t^{\omega}\delta_0])],$$

$$S[D_t^{\omega}\delta_0] = \frac{1}{\mu^{\omega}}S[\delta_0] - \frac{1}{\mu^{\omega+1}}\delta_0(0) \Rightarrow S[D_t^{\omega}\delta_0] = \frac{-2\sinh\varepsilon}{\mu^{\omega}},$$

$$\Rightarrow \delta_1 = -2S^{-1}\left[\mu^{2\omega}\left(\frac{-2\sinh\varepsilon}{\mu^{\omega}}\right)\right] \Rightarrow \delta_1 = \frac{4\sinh\varepsilon t^{\omega+1}}{\Gamma(\omega + 2)},$$

$$R[\delta_1] = (\delta_1)_{\varepsilon\varepsilon} - \delta_1 = 0.$$

Using Eq. (21)

$$\delta_2 = S^{-1}[\mu^{2\omega}(-2S[D_t^{\omega}\delta_1])] = -2S^{-1}[\mu^{2\omega}(S[D_t^{\omega}\delta_1])],$$

$$S[D_t^{\omega}\delta_1] = \frac{1}{\mu^{\omega}}S[\delta_1] - \frac{1}{\mu^{\omega+1}}\delta_1(0) \Rightarrow S[D_t^{\omega}\delta_1] = 4\sinh\varepsilon,$$

$$\Rightarrow \delta_1 = -2S^{-1}[\mu^{2\omega}(4\sinh\varepsilon)] \Rightarrow \delta_1 = \frac{-8\sinh\varepsilon t^{2\omega+1}}{\Gamma(2\omega + 2)},$$

$$R[\delta_2] = (\delta_2)_{\varepsilon\varepsilon} - \delta_2 = 0.$$

Using Eq. (21)

$$\delta_3 = S^{-1}[\mu^{2\omega}(-2S[D_t^{\omega}\delta_2])] = -2S^{-1}[\mu^{2\omega}(S[D_t^{\omega}\delta_2])],$$

$$S[D_t^{\omega}\delta_2] = \frac{1}{\mu^{\omega}}S[\delta_2] - \frac{1}{\mu^{\omega+1}}\delta_2(0)$$
$$\Rightarrow S[D_t^{\omega}\delta_2] = -8\sinh\varepsilon\mu^{\omega},$$

$$\Rightarrow \delta_3 = -2S^{-1}[\mu^{2\omega}(-8\sinh\varepsilon\mu^{\omega})] \Rightarrow \delta_3 = \frac{16\sinh\varepsilon t^{3\omega+1}}{\Gamma(3\omega + 2)},$$

$$\delta(\varepsilon, t) = \delta_0(\varepsilon, t) + \delta_1(\varepsilon, t) + \delta_2(\varepsilon, t) + \delta_3(\varepsilon, t) + \ldots,$$

$$\Rightarrow \delta(\varepsilon, t) = \sinh\varepsilon \quad (1 - 2t) + \frac{4\sinh\varepsilon t^{\omega+1}}{\Gamma(\omega + 2)} - \frac{8\sinh\varepsilon t^{2\omega+1}}{\Gamma(2\omega + 2)} + \frac{16\sinh\varepsilon t^{3\omega+1}}{\Gamma(3\omega + 2)} - \ldots.$$

Putting $\omega = 1$, the series solution is as follows:

$$\delta(\varepsilon, t) = \sinh\varepsilon\left(1 - \frac{2t}{1!} + \frac{(2t)^2}{2!} - \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} - \ldots\right).$$

Also, the fractional equation becomes

$$\delta_{tt} + 2\delta_t + \delta = \delta_{\varepsilon\varepsilon}.$$

Given the same initial conditions, the exact solution of this differential equation is $\delta(\varepsilon, t) = \sinh\varepsilon e^{-2t}$ [31], and it agrees with the series solution obtained by the iterative scheme.

Example 5. Consider the following 1D time FHTE [20],

$$D_t^{\omega}\delta = \delta - \delta_t - \delta_{\varepsilon\varepsilon}, \tag{22}$$

where $\delta(\varepsilon, 0) = e^{\varepsilon}$ and $\delta_t(\varepsilon, 0) = -e^{\varepsilon}$, $0 < \omega \le 2$ are the initial conditions.

$$\delta_0(\varepsilon, t) = \delta(\varepsilon, 0) + t\delta_t(\varepsilon, 0) = e^{\varepsilon} - t(2e^{\varepsilon}) = e^{\varepsilon}(1 - t).$$

Applying Sawi transform on Eq. (22)

$$S[D_t^{\omega}\delta(\varepsilon, t)] = S[\delta - \delta_t - \delta_{\varepsilon\varepsilon}],$$

$$\Rightarrow \delta = S^{-1}\left[\mu^{\omega}\left(\sum_{k=0}^{m-1}\left(\frac{1}{\mu}\right)^{\omega-(k-1)}\delta^{(k)}(0)\right)\right] + S^{-1}[\mu^{\omega}S[R(\delta)]],$$

$$\Rightarrow \delta_0 = S^{-1}\left[\mu^{\omega}\left(\sum_{k=0}^{m-1}\left(\frac{1}{\mu}\right)^{\omega-(k-1)}\delta^{(k)}(0)\right)\right], \tag{23}$$

$$\Rightarrow \delta_{r+1} = S^{-1}[\mu^{\omega}S[R(\delta_r)]]. \tag{24}$$

Considering $m = 1$ in Eq. (23),

$$\delta_0 = S^{-1}\left[\frac{1}{\mu}\delta(0)\right] = \delta(0) = e^{\varepsilon}(1 - t).$$

$$R[\delta_0] = \delta_0 - (\delta_0)_t - (\delta_0)_{\varepsilon\varepsilon} = e^{\varepsilon}.$$

Using Eq. (24)

$$\delta_1 = S^{-1}[\mu^{\omega}(S[e^{\varepsilon}])] \Rightarrow \delta_1 = e^{\varepsilon}S^{-1}[\mu^{\omega-1}] \Rightarrow \delta_1(\varepsilon, t) = \frac{e^{\varepsilon}t^{\omega}}{\Gamma(\omega + 1)},$$

$$R[\delta_1] = \delta_1 - 2(\delta_1)_t - (\delta_1)_{\varepsilon\varepsilon} = \frac{-e^{\varepsilon}\omega t^{\omega-1}}{\Gamma(\omega + 1)}.$$

Using Eq. (24)

$$\delta_2 = S^{-1}\left[\mu^{\omega}\left(S\left[\frac{-e^{\varepsilon}\omega t^{\omega-1}}{\Gamma(\omega + 1)}\right]\right)\right] \Rightarrow \delta_2 = \frac{-e^{\varepsilon}\omega}{\Gamma(\omega + 1)}S^{-1}[\Gamma(\omega)\mu^{2\omega-2}] \Rightarrow \delta_2(\varepsilon, t) = \frac{-e^{\varepsilon}\omega\Gamma(\omega)t^{2\omega-1}}{\Gamma(\omega + 1)\Gamma(2\omega)},$$

$$R[\delta_2] = \delta_2 - 2(\delta_2)_t - (\delta_2)_{\varepsilon\varepsilon} = \frac{e^{\varepsilon}\omega(2\omega - 1)\Gamma(\omega)t^{2\omega-2}}{\Gamma(\omega + 1)\Gamma(2\omega)}.$$

Using Eq. (24)

$$\delta_3 = S^{-1}\left[\mu^{\omega}\left(S\left[\frac{e^{\varepsilon}\omega(2\omega - 1)\Gamma(\omega)t^{2\omega-2}}{\Gamma(\omega + 1)\Gamma(2\omega)}\right]\right)\right] \Rightarrow \delta_2 = \frac{e^{\varepsilon}\omega(2\omega - 1)\Gamma(\omega)}{\Gamma(\omega + 1)\Gamma(2\omega)}S^{-1}[\Gamma(2\omega - 1)\mu^{3\omega-3}] \Rightarrow \delta_3(\varepsilon, t) = \frac{e^{\varepsilon}\omega(2\omega - 1)\Gamma(\omega)\Gamma(2\omega - 1)t^{3\omega-2}}{\Gamma(\omega + 1)\Gamma(2\omega)\Gamma(3\omega - 1)}.$$

$$\delta(\varepsilon, t) = \delta_0(\varepsilon, t) + \delta_1(\varepsilon, t) + \delta_2(\varepsilon, t) + \delta_3(\varepsilon, t) + \dots$$

$$\Rightarrow \delta(\varepsilon, t) = e^{\varepsilon}(1 - t) + \frac{e^{\varepsilon}t^{\omega}}{\Gamma(\omega + 1)} - \frac{e^{\varepsilon}\omega\Gamma(\omega)t^{2\omega-1}}{\Gamma(\omega + 1)\Gamma(2\omega)} + \frac{e^{\varepsilon}\omega(2\omega - 1)\Gamma(\omega)\Gamma(2\omega - 1)t^{3\omega-2}}{\Gamma(\omega + 1)\Gamma(2\omega)\Gamma(3\omega - 1)} - \dots$$

Putting $\omega = 2$, the series solution is as follows:

$$\delta(\varepsilon, t) = e^{\varepsilon}\left(1 - \frac{t}{1!} + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \dots\right)$$

Also, the fractional equation becomes

$$\delta_{tt} = \delta - \delta_t - \delta_{\varepsilon\varepsilon}.$$

Given the same initial conditions, the exact solution of this differential equation is $\delta(\varepsilon, t) = e^{\varepsilon - t}$ [20], and it agrees with the series solution obtained by the iterative scheme.

Example 6. Consider the following 1D space FHTE [34]:

$$\delta_x^{2\omega} = \delta_{tt} + 4\delta_t + 4\delta, \tag{25}$$

where $\delta(0, t) = 1 + e^{-2t}$, $\delta(x, 0) = 1 + e^{2x}$, $\delta_x(0, t) = 2$, $\delta_t(x, 0) = -2$, $t \geq 0$, $0 < \omega < 1$ are the initial conditions.

$$\delta_0(x, t) = \delta(0, t) + x\delta_x(0, t) = 1 + e^{-2t} + x(2) = 1 + 2x + e^{-2t}.$$

Applying Sawi transform on Eq. (25)

$$S[\delta_x^{2\omega}] = S[\delta_{tt} + 4\delta_t + 4\delta],$$

$$\Rightarrow \delta = S^{-1}\left[\mu^{2\omega}\left(\sum_{k=0}^{m-1}\left(\frac{1}{\mu}\right)^{2\omega-(k-1)}\delta^{(k)}(0)\right)\right] + S^{-1}[\mu^{2\omega}S[R(\delta)]],$$

$$\Rightarrow \delta_0 = S^{-1}\left[\mu^{2\omega}\left(\sum_{k=0}^{m-1}\left(\frac{1}{\mu}\right)^{2\omega-(k-1)}\delta^{(k)}(0)\right)\right], \tag{26}$$

$$\Rightarrow \delta_{r+1} = S^{-1}[\mu^{2\omega}S[R(\delta_r)]]. \tag{27}$$

Considering $m = 1$ in Eq. (26)

$$\delta_0 = S^{-1}\left[\frac{1}{\mu}\delta(0)\right] = \delta(0) = 1 + 2x + e^{-2t},$$

$$R[\delta_0] = (\delta_0)_{tt} + 4(\delta_0)_t + 4\delta_0 = 4 + 8x.$$

Using Eq. (27)

$$\delta_1 = S^{-1}[\mu^{2\omega}(S[4 + 8x])] \Rightarrow \delta_1 = S^{-1}[4\mu^{2\omega-1} + 8\mu^{2\omega}] \Rightarrow \delta_1(x, t) = \frac{4x^{2\omega}}{\Gamma(2\omega + 1)} + \frac{8x^{2\omega+1}}{\Gamma(2\omega + 2)},$$

$$R[\delta_1] = (\delta_1)_{tt} + 4(\delta_1)_t + 4\delta_1 = \frac{16x^{2\omega}}{\Gamma(2\omega + 1)} + \frac{32x^{2\omega+1}}{\Gamma(2\omega + 2)}.$$

Using Eq. (27)

$$\delta_2 = S^{-1}\left[\mu^{2\omega}\left(S\left[\frac{16x^{2\omega}}{\Gamma(2\omega + 1)} + \frac{32x^{2\omega+1}}{\Gamma(2\omega + 2)}\right]\right)\right] \Rightarrow \delta_2 = S^{-1}[16\mu^{4\omega-1} + 32\mu^{4\omega}] \Rightarrow \delta_2(x, t) = \frac{16x^{4\omega}}{\Gamma(4\omega + 1)} + \frac{32x^{4\omega+1}}{\Gamma(4\omega + 2)},$$

$$R[\delta_2] = (\delta_2)_{tt} + 4(\delta_2)_t + 4\delta_2 = \frac{64x^{4\omega}}{\Gamma(4\omega + 1)} + \frac{128x^{4\omega+1}}{\Gamma(4\omega + 2)}.$$

Using Eq. (27)

$$\delta_3 = S^{-1}\left[\mu^{2\omega}\left(S\left[\frac{64x^{4\omega}}{\Gamma(4\omega + 1)} + \frac{128x^{4\omega+1}}{\Gamma(4\omega + 2)}\right]\right)\right] \Rightarrow \delta_3 = S^{-1}[64\mu^{6\omega-1} + 128\mu^{6\omega}], \Rightarrow \delta_3(x, t) = \frac{64x^{6\omega}}{\Gamma(6\omega + 1)} + \frac{128x^{6\omega+1}}{\Gamma(6\omega + 2)}.$$

$$\delta(x,t) = \delta_0(x,t) + \delta_1(x,t) + \delta_2(x,t) + \delta_3(x,t) + \ldots$$

$$\Rightarrow \delta(x,t) = (1 + 2x + e^{-2t}) + \left(\frac{4x^{2\omega}}{\Gamma(2\omega+1)} + \frac{8x^{2\omega+1}}{\Gamma(2\omega+2)}\right) + \left(\frac{16x^{4\omega}}{\Gamma(4\omega+1)} + \frac{32x^{4\omega+1}}{\Gamma(4\omega+2)}\right) + \left(\frac{64x^{6\omega}}{\Gamma(6\omega+1)} + \frac{128x^{6\omega+1}}{\Gamma(6\omega+2)}\right) + \ldots$$

Putting $\omega = 1$, the series solution is as follows:

$$\delta(x,t) = e^{-2t} + 1 + \frac{2x}{1!} + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \frac{(2x)^5}{5!} + \frac{(2x)^6}{6!} + \frac{(2x)^7}{7!} + \ldots$$

Also, the fractional equation becomes

$$\delta_{xx} = \delta_{tt} + 4\delta_t + 4\delta.$$

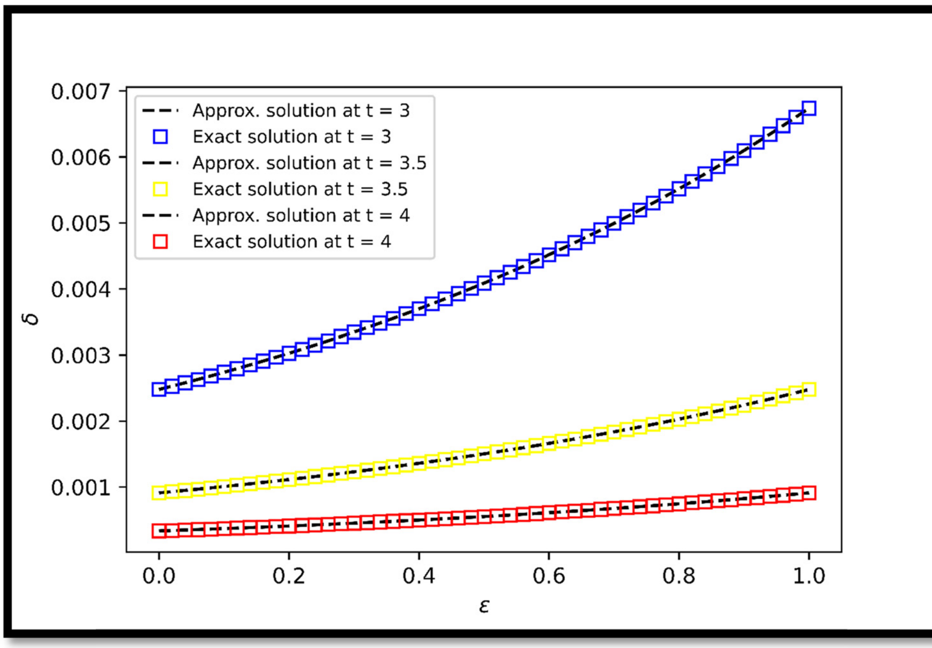

Figure 1: Comparing exact and approximate solutions of Example 1 at t = 3, 3.5, and 4.

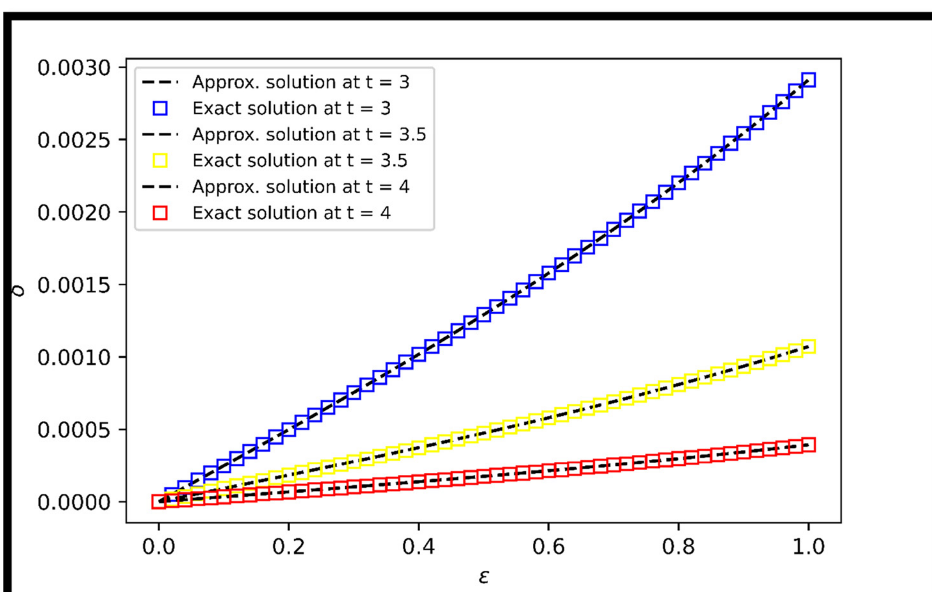

Figure 2: Comparing exact and approximate solutions of Example 4 at t = 3, 3.5, and 4.

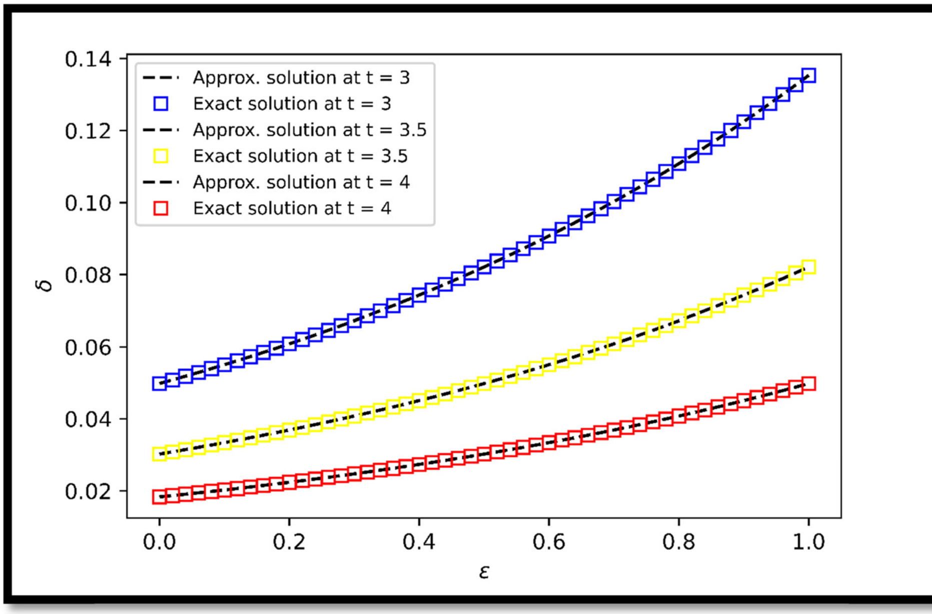

Figure 3: Comparing exact and approximate solutions of Example 5 at $t = 3$, 3.5, and 4.

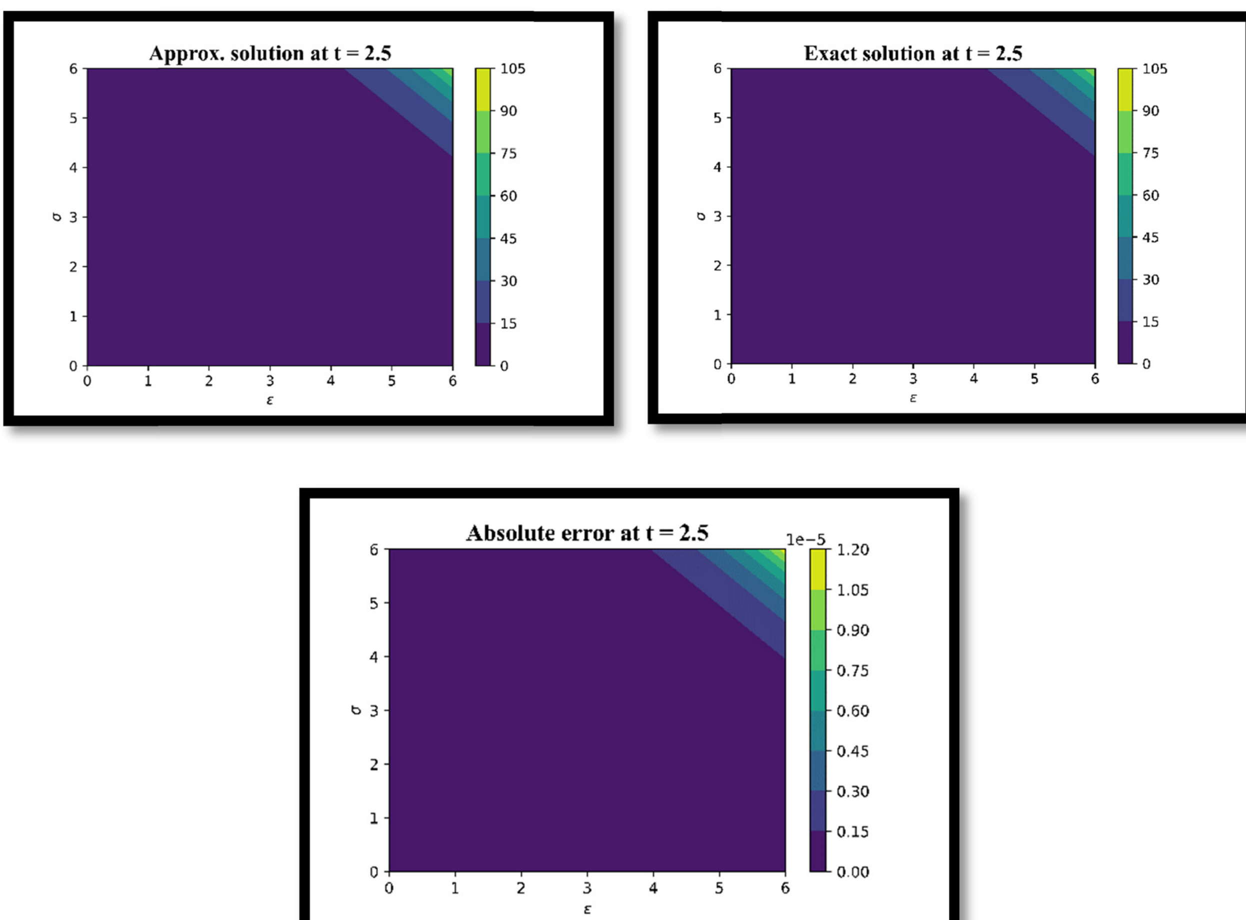

Figure 4: Comparing exact and approximate solutions of Example 2 at $t = 2.5$.

Given the same initial conditions, the exact solution of this differential equation is $\delta(x,t) = e^{-2t} + e^{2x}$ [34], and it agrees with the series solution obtained by the iterative scheme.

Example 7. Consider the following 1D space FHTE [35]:

$$\delta_x^{2\omega} = \delta_{tt} + \delta_t + \delta - t + 1 - x^2, \tag{28}$$

where $\delta(0,t) = t$, $\delta(x,0) = x^2$, $\delta_x(0,t) = 0$, $t \geq 0$, $0 < \omega < 1$ are the initial conditions.

$$\delta_0(x,t) = \delta(0,t) + x\delta_x(0,t) = t + x(0) = t.$$

Applying Sawi transform on Eq. (28)

$$S[\delta_x^{2\omega}] = S[\delta_{tt} + \delta_t + \delta - t + 1 - x^2],$$

$$\Rightarrow \delta = S^{-1}\left[\mu^{2\omega}\left(\sum_{k=0}^{m-1}\left(\frac{1}{\mu}\right)^{2\omega-(k-1)}\delta^{(k)}(0)\right)\right] + S^{-1}[\mu^{2\omega}S[R(\delta) - t + 1 - x^2]],$$

$$\Rightarrow \delta_0 = S^{-1}\left[\mu^{2\omega}\left(\sum_{k=0}^{m-1}\left(\frac{1}{\mu}\right)^{2\omega-(k-1)}\delta^{(k)}(0)\right)\right], \tag{29}$$

$$\Rightarrow \delta_1 = S^{-1}[\mu^{2\omega}S[R(\delta_1) - t + 1 - x^2]], \tag{30}$$

$$\Rightarrow \delta_{r+1} = S^{-1}[\mu^{2\omega}S[R(\delta_r)]]. \tag{31}$$

Considering $m = 1$ in Eq. (29)

$$\delta_0 = S^{-1}\left[\frac{1}{\mu}\delta(0)\right] = \delta(0) = t,$$

$$R[\delta_0] = (\delta_0)_{tt} + (\delta_0)_t + \delta_0 = 1 + t.$$

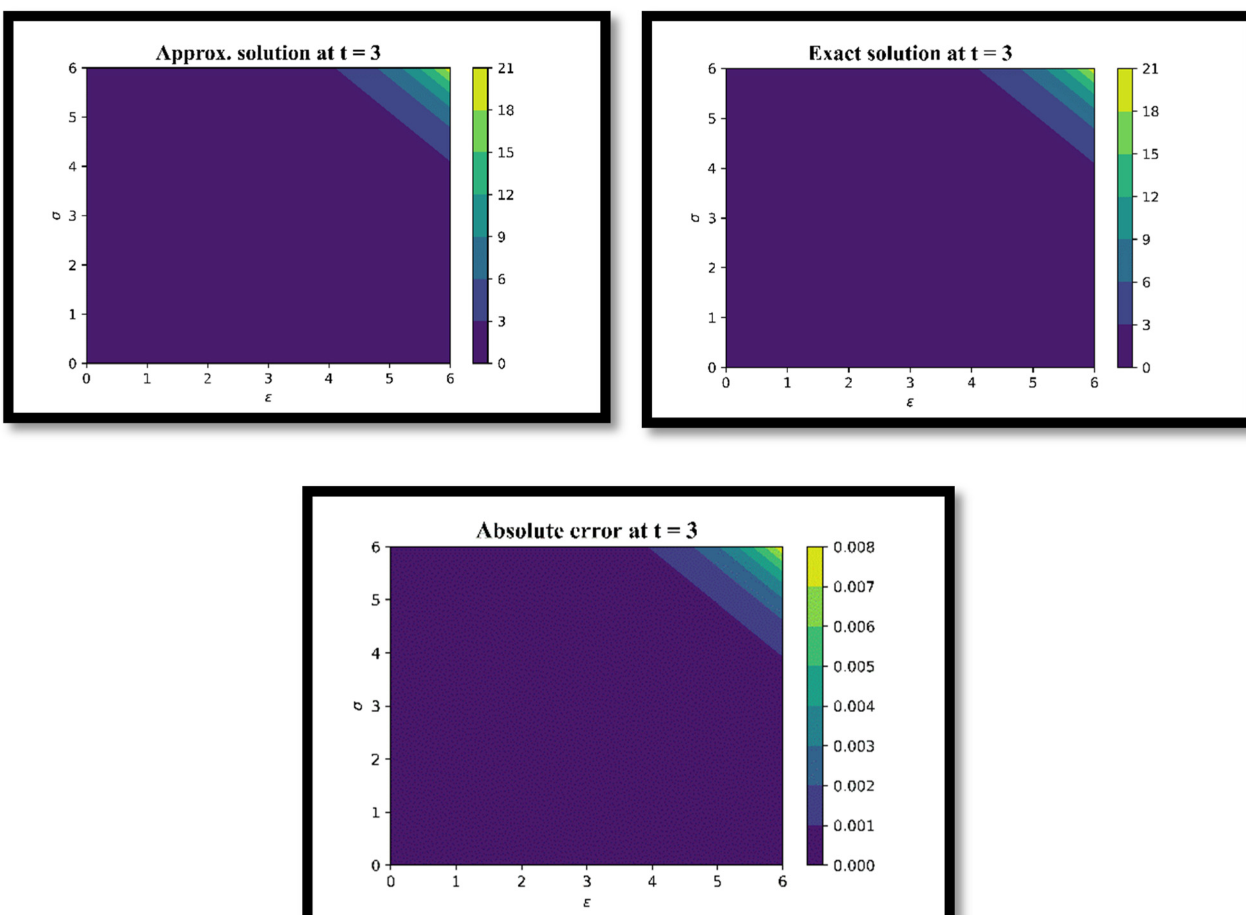

Figure 5: Comparing exact and approximate solutions of Example 2 at $t = 3$.

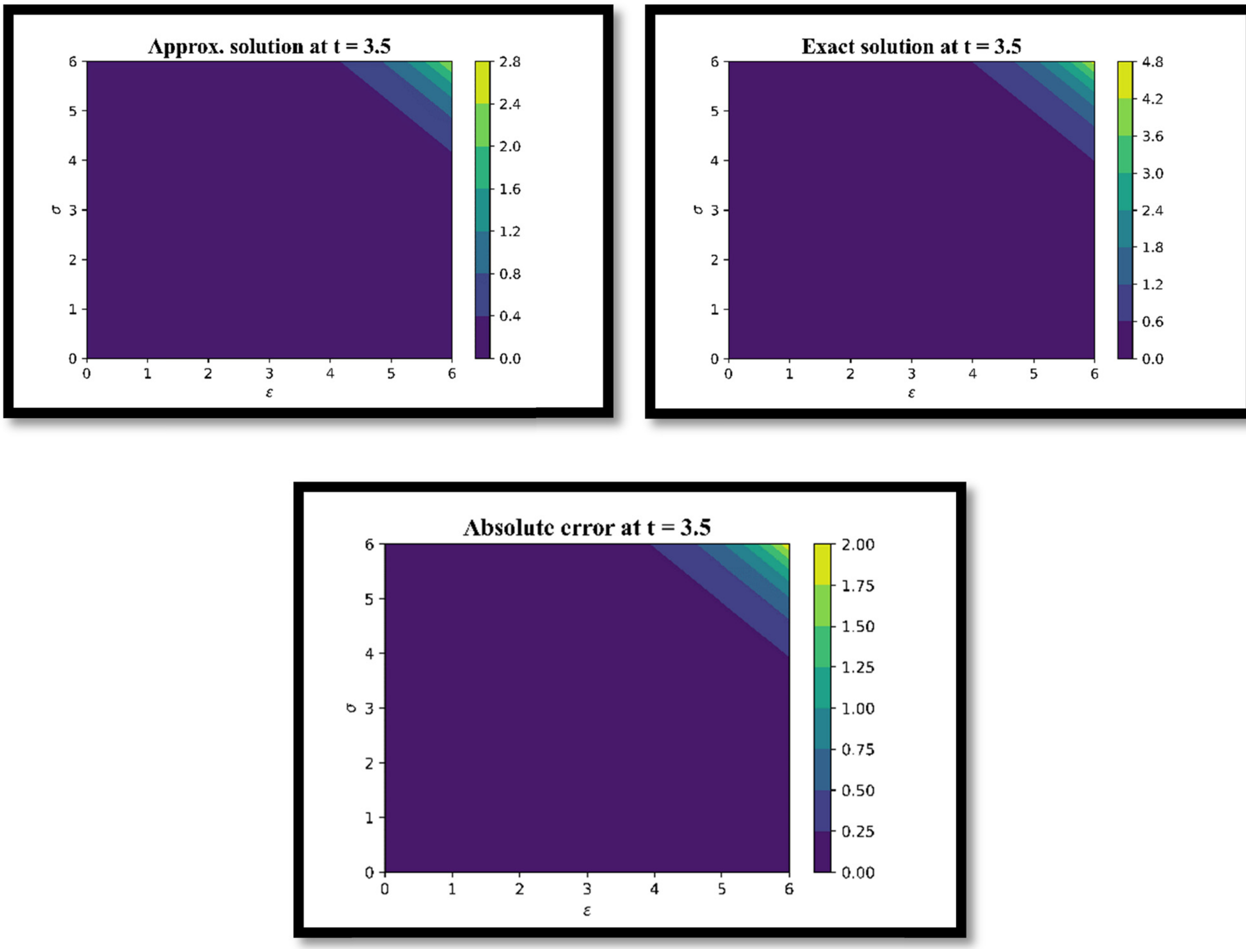

Figure 6: Comparing exact and approximate solutions of Example 2 at $t = 3.5$.

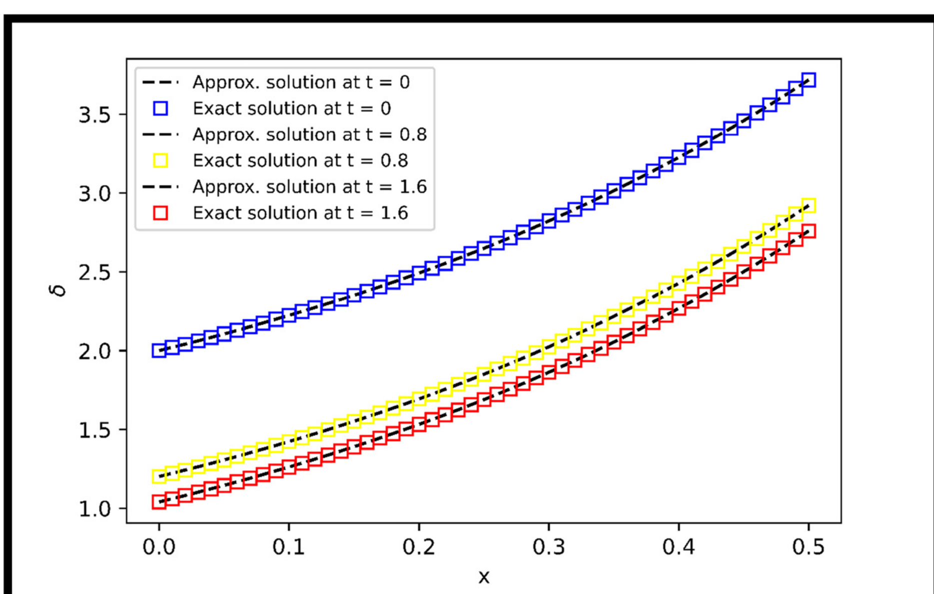

Figure 7: Comparing exact and approximate solutions of Example 6 at $t = 0, 0.8$, and 1.6.

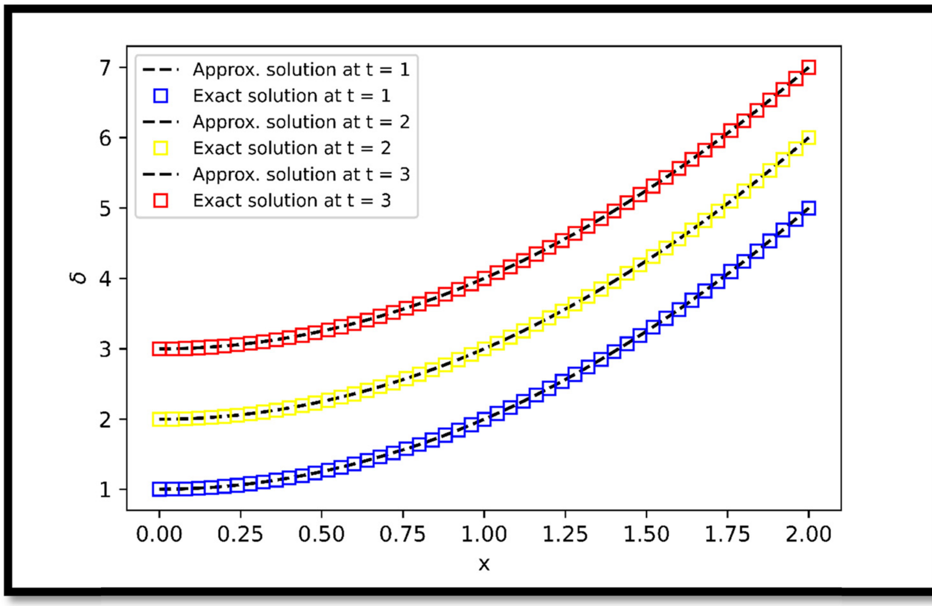

Figure 8: Comparing exact and approximate solutions of Example 7 at $t = 1$, 2, and 3.

Table 1: Sawi transform of common functions [33]

	$f(t)$	$S[f(t)] = q(\mu)$
1	1	$\frac{1}{\mu}$
2	t	1
3	t^n	$\Gamma(n+1)\mu^{n-1}$
4	$\sin at$	$\frac{a}{1+(a\mu)^2}$
5	$\cos at$	$\frac{a}{\mu(1+(a\mu)^2)}$
6	$\sinh at$	$\frac{a}{1-(a\mu)^2}$
7	$\cosh at$	$\frac{a}{\mu(1-(a\mu)^2)}$
8	e^{at}	$\frac{1}{\mu(1-a\mu)}$

Table 2: Inverse Sawi transform of common functions [33]

	$q(\mu)$	$S^{-1}[Q(\mu)] = f(t)$
1	$\frac{1}{\mu}$	1
2	1	t
3	μ^n	$\frac{t^{n+1}}{\Gamma(n+2)}$
4	$\frac{a}{1+(a\mu)^2}$	$\sin at$
5	$\frac{a}{\mu(1+(a\mu)^2)}$	$\cos at$
6	$\frac{a}{1-(a\mu)^2}$	$\sinh at$
7	$\frac{a}{\mu(1-(a\mu)^2)}$	$\cosh at$
8	$\frac{1}{\mu(1-a\mu)}$	e^{at}

Using Eq. (30)

$$\delta_1 = S^{-1}[\mu^{2\omega}(S[1+t-t+1-x^2])] \Rightarrow \delta_1$$
$$= S^{-1}[2\mu^{2\omega-1} - 2\mu^{2\omega+1}] \Rightarrow \delta_1(x,t) = \frac{2x^{2\omega}}{\Gamma(2\omega+1)}$$
$$- \frac{2x^{2\omega+2}}{\Gamma(2\omega+3)}.$$

$$R[\delta_1] = (\delta_1)_{tt} + (\delta_1)_t + \delta_1 = \frac{2x^{2\omega}}{\Gamma(2\omega+1)} - \frac{2x^{2\omega+2}}{\Gamma(2\omega+3)}.$$

Using Eq. (31)

$$\delta_2 = S^{-1}\left[\mu^{2\omega}\left(S\left[\frac{2x^{2\omega}}{\Gamma(2\omega+1)} - \frac{2x^{2\omega+2}}{\Gamma(2\omega+3)}\right]\right)\right] \Rightarrow \delta_2$$
$$= S^{-1}[2\mu^{4\omega-1} - 2\mu^{4\omega+1}] \Rightarrow \delta_2(x,t) = \frac{2x^{4\omega}}{\Gamma(4\omega+1)}$$
$$- \frac{2x^{4\omega+2}}{\Gamma(4\omega+3)},$$

$$R[\delta_2] = (\delta_2)_{tt} + (\delta_2)_t + \delta_2 = \frac{2x^{4\omega}}{\Gamma(4\omega+1)} - \frac{2x^{4\omega+2}}{\Gamma(4\omega+3)}.$$

Using Eq. (31)

$$\delta_3 = S^{-1}\left[\mu^{2\omega}\left(S\left[\frac{2x^{4\omega}}{\Gamma(4\omega+1)} - \frac{2x^{4\omega+2}}{\Gamma(4\omega+3)}\right]\right)\right] \Rightarrow \delta_3$$
$$= S^{-1}[2\mu^{6\omega-1} - 2\mu^{6\omega+1}] \Rightarrow \delta_3(x,t) = \frac{2x^{6\omega}}{\Gamma(6\omega+1)}$$
$$- \frac{2x^{6\omega+2}}{\Gamma(6\omega+3)}.$$

Table 3: Error analysis for Example 1

N	Exact value at t = 1	Approx. value at t = 1	L_∞ err. at t = 1	Exact value at t = 2	Approx. value at t = 2	L_∞ err. at t = 2	Exact value at t = 3	Approx. value at t = 3	L_∞ err. at t = 3
10	0.367879	0.367999	0.000119	0.049787	0.262911	0.213124	0.006738	16.32522	16.31849
20	0.367879	0.367879	1.02×10^{-13}	0.049787	0.049787	1.98×10^{-7}	0.006738	0.007653	0.000915
30	0.29523	0.29523	1.28×10^{-15}	0.033373	0.033373	4.55×10^{-15}	0.006738	0.006738	3.69×10^{-10}
			$\downarrow$ Up to 10^{-15}			$\downarrow$ Up to 10^{-15}			$\downarrow$ Up to 10^{-10}

Table 4: Error analysis for Example 2

N	Exact value at t = 0.4	Approx. value at t = 0.4	L_∞ err. at t = 0.4	Exact value at t = 0.5	Approx. value at t = 0.5	L_∞ err. at t = 0.5	Exact value at t = 0.6	Approx. value at t = 0.6	L_∞ err. at t = 0.6
10	49020.8	49020.83	0.027523	36315.5	36315.82	0.313187	26903.19	26905.46	2.275563
15	49020.8	49020.8	1.34×10^{-7}	36315.5	36315.5	4.69×10^{-6}	26903.19	26903.19	8.54×10^{-5}
20	49020.8	49020.8	6.55×10^{-11}	12088.38	12088.38	2.91×10^{-11}	26903.19	26903.19	7.13×10^{-10}
			$\downarrow$ Up to 10^{-11}			$\downarrow$ Up to 10^{-11}			$\downarrow$ Up to 10^{-10}

$$\delta(x,t) = \delta_0(x,t) + \delta_1(x,t) + \delta_2(x,t) + \delta_3(x,t) + \dots$$

$$\Rightarrow \delta(x,t) = (t) + \left(\frac{2x^{2\omega}}{\Gamma(2\omega+1)} - \frac{2x^{2\omega+2}}{\Gamma(2\omega+3)}\right) + \left(\frac{2x^{4\omega}}{\Gamma(4\omega+1)} - \frac{2x^{4\omega+2}}{\Gamma(4\omega+3)}\right) + \left(\frac{2x^{6\omega}}{\Gamma(6\omega+1)} - \frac{2x^{6\omega+2}}{\Gamma(6\omega+3)}\right) + \dots$$

Putting $\omega = 1$, the series solution is as follows:

$$\delta(x,t) = t + 2\left(\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \dots\right).$$

Also, the fractional equation becomes

$$\delta_{xx} = \delta_{tt} + \delta_t + \delta - t + 1 - x^2.$$

Given the same initial conditions, the exact solution of this differential equation is $\delta(x,t) = t + x^2$ [35], and it agrees with the series solution obtained by the iterative scheme.

Exact and approximate solutions of Examples 1, 4 and 5 are compared for $t = 3, 3.5$ and 4 in Figures 1–3. Contour graphs comparing exact and approximate solutions of Example 2 at $t = 2.5$ and absolute error graph are provided in Figure 4. Contour graphs comparing exact and approximate solutions of Example 2 at $t = 3$ and absolute error graph are provided in Figure 5. Contour graphs comparing exact and approximate solutions of Example 2 at $t = 3.5$ and absolute error graph are provided in Figure 6. Exact and approximate solutions of Example 6 are compared for $t = 0, 0.8,$ and 1.6. Figure 8 provides the exact and approximate solutions of Example 7 at $t = 1, 2,$ and 3. Each figure considers the value of ω used in its respective example for comparing the exact solution for the same ω. Each figure takes $N = 35$ for approximate solutions, where N is the number of terms in approximate solutions, *i.e.* the parameter for convergence. The aim of using multiple values of N is to demonstrate numerical convergence of the approximate solution to the exact solution. The graphs in Figures 1–3 are plotted with 51 equally spaced ε values, the graphs in Figures 7 and 8 are plotted with 51 equally spaced x values and the contour graphs in Figures 4–6 are plotted with 61 equally spaced ε and σ values. Using the graphs, we can conclude that the proposed method provides good compatibility between approximate and exact solutions of FPDEs over wide variety of time scales (Tables 1 and 2).

For consistency, the number of equally spaced values for L_∞ error in the tables is the same as their respective graphs of the examples. For the same reason, the range at

Table 5: Error analysis for Example 4

N	Exact value at $t = 1$	Approx. value at $t = 1$	L_∞ err. at $t = 1$	Exact value at $t = 2$	Approx. value at $t = 2$	L_∞ err. at $t = 2$	Exact value at $t = 3$	Approx. value at $t = 3$	L_∞ err. at $t = 3$
10	0.159046	0.159098	5.16×10^{-5}	0.021525	0.113665	0.09214	0.002913	7.057923	7.05501
20	0.159046	0.159046	4.41×10^{-14}	0.021525	0.021525	8.55×10^{-8}	0.002913	0.003309	0.000396
30	0.146795	0.146795	8.05×10^{-16}	0.017259	0.017259	1.6×10^{-15}	0.002913	0.002913	1.6×10^{-10}
			↓ Up to 10^{-16}			↓ Up to 10^{-15}			↓ Up to 10^{-10}

Table 6: Error analysis for Example 5

N	Exact value at $t = 1$	Approx. value at $t = 1$	L_∞ err. at $t = 1$	Exact value at $t = 2$	Approx. value at $t = 2$	L_∞ err. at $t = 2$	Exact value at $t = 3$	Approx. value at $t = 3$	L_∞ err. at $t = 3$
10	1	1	6.28×10^{-8}	0.367879	0.367999	0.000119	0.135335	0.144955	0.00962
20	0.398519	0.398519	2.22×10^{-16}	0.367879	0.367879	1.02×10^{-13}	0.135335	0.135335	4.89×10^{-10}
30	0.398519	0.398519	2.22×10^{-16}	0.29523	0.29523	1.28×10^{-15}	0.132655	0.132655	4.44×10^{-15}
			↓ Up to 10^{-16}			↓ Up to 10^{-15}			↓ Up to 10^{-15}

Table 7: Error analysis for Example 6

N	Exact value at $t = 1$	Approx. value at $t = 1$	L_∞ err. at $t = 1$	Exact value at $t = 2$	Approx. value at $t = 2$	L_∞ err. at $t = 2$	Exact value at $t = 3$	Approx. value at $t = 3$	L_∞ err. at $t = 3$
4	2.853617	2.853589	2.79×10^{-5}	2.736597	2.73657	2.79×10^{-5}	2.720761	2.720733	2.79×10^{-5}
7	2.853617	2.853617	1.23×10^{-11}	2.736597	2.736597	1.23×10^{-11}	2.720761	2.720761	1.23×10^{-11}
10	1.957454	1.957454	4.44×10^{-16}	2.527606	2.527606	4.44×10^{-16}	2.098414	2.098414	4.44×10^{-16}
			↓ Up to 10^{-16}			↓ Up to 10^{-16}			↓ Up to 10^{-16}

Table 8: Error Analysis for Example 7

N	Exact value at $t = 1$	Approx. value at $t = 1$	L_∞ err. at $t = 1$	Exact value at $t = 2$	Approx value at $t = 2$	L_∞ err. at $t = 2$	Exact value at $t = 3$	Approx. value at $t = 3$	L_∞ err. at $t = 3$
4	5	4.987302	0.012698	6	5.987302	0.012698	7	6.987302	0.012698
7	5	5	3.76×10^{-7}	6	6	3.76×10^{-7}	7	7	3.76×10^{-7}
10	5	5	8.62×10^{-13}	6	6	8.62×10^{-13}	7	7	8.62×10^{-13}
			↓ Up to 10^{-13}			↓ Up to 10^{-13}			↓ Up to 10^{-13}

which L_∞ error is considered is also the same as the respective graphs of the examples. The columns containing the exact and approximate values at any given t are taken at the point where the error is maximum, *i.e.* the L_∞ error. In Table 3, the exact, approximate, and L_∞ error values for Example 1 have been provided at $t = 1, 2$, and 3, and as N increases to 30, the error reduces up to 10^{-15}. For Example 2, Table 4 provides the exact, approximate, and L_∞ error values at $t = 1, 2$, and 3, and as N increases to 20, the error reduces up to 10^{-11}. In Table 5, the exact, approximate, and L_∞ error values for Example 4 have been provided at $t = 1, 2$, and 3, and as N increases to 30, the error reduces up to 10^{-16}. For Example 5, Table 6 provides the exact, approximate, and L_∞ error values at $t = 1, 2$, and 3, and as N increases to 30, the error reduces up to 10^{-16}. In Table 7, the exact, approximate, and L_∞ error values for Example 6 have been provided at $t = 1, 2$, and 3, and as N increases to 10, the error reduces up to 10^{-16}. For Example 7, Table 8 provides the exact, approximate, and L_∞ error values at $t = 1, 2$, and 3, and as N increases to 10, the error reduces up to 10^{-13}. From these tables, we can thus conclude that as the parameter of convergence increases, the series solutions approximate the exact solutions with increasing accuracy, thus illustrating the efficacy of the proposed method.

5 Conclusion

In the present study, an iterative scheme was proposed involving the Sawi transform for solving FHTEs. A general formula for the proposed method was developed for 1D, 2D, and 3D FHTEs. Seven Examples in total was solved using the proposed method, Examples 1, 4, and 5 of which were for 1D time FHTEs, Examples 6 and 7 for 1D space FHTEs, Example 2 for *2D* time FHTE, and Example 3 for 3D time FHTE. Graphs for Examples 1, 2, 4, 5, 6, and 7 were created and the approximate and exact solutions were compared using them. It was found that the approximate and exact solutions were showing good compatibility across wide variety of time scales. Tables for the same examples as the graphs were created and contained the exact, approximate, and L_∞ error values. For each table, it was observed that as the parameter of convergence increased, the L_∞ error reduced, thus illustrating that the series generated by the proposed scheme approximates the exact solution with acceptable level of accuracy. Thus, it can be concluded that the proposed method is a simple and efficient method for obtaining solutions to 1D, 2D, and 3D time and space FHTEs.

Funding information: Not available.

Author contributions: Conceptualization: M.K.; programming: S.K.; drafting: S.K.; calculation: M.K. and S.K.

Conflict of interest: Not applicable.

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Appendix A

The form of 2D time telegraph equation is as follows [20]:

$$D_t^{\omega}\delta(\varepsilon,\sigma,t)+L[\delta(\varepsilon,\sigma,t)]+N[\delta(\varepsilon,\sigma,t)]=q(\varepsilon,\sigma,t),$$

where L refers to the linear operator and N refers to the nonlinear operator. $D_t^{\omega}\delta(\varepsilon,t)$ is the Caputo derivative of $\delta(\varepsilon,t)$. Applying Sawi transform on equation

$$S[D_t^{\omega}\delta(\varepsilon,t)]+S[L[\delta(\varepsilon,t)]+N[\delta(\varepsilon,t)]]=S[q(\varepsilon,t)],$$

$$\frac{1}{\mu^{\omega}}S[\delta(\varepsilon,t)]-\sum_{k=0}^{m-1}\left(\frac{1}{\mu}\right)^{\omega-(k-1)}\delta^{(k)}(0)=S[q(\varepsilon,t)]-S[L[\delta(\varepsilon,t)]]-S[N[\delta(\varepsilon,t)]]$$

$$\Rightarrow\delta(\varepsilon,t)=S^{-1}\left[\mu^{\omega}\left(\sum_{k=0}^{m-1}\left(\frac{1}{\mu}\right)^{\omega-(k-1)}\delta^{(k)}(0)+S[q(\varepsilon,t)]\right)\right]-S^{-1}[\mu^{\omega}(S[L[\delta(\varepsilon,\sigma,t)]]+S[N[\delta(\varepsilon,\sigma,t)]])].$$

Now,

$$L[\delta]=L\left[\sum_{r=0}^{\infty}\delta_r(\varepsilon,\sigma,t)\right]=L[\delta_0(\varepsilon,\sigma,t)]+\sum_{r=1}^{\infty}\left[L\left[\sum_{i=0}^{r}\delta_i(\varepsilon,\sigma,t)\right]-L\left[\sum_{i=0}^{r-1}\delta_i(\varepsilon,\sigma,t)\right]\right],$$

$$N[\delta]=N\left[\sum_{r=0}^{\infty}\delta_r(\varepsilon,\sigma,t)\right]=N[\delta_0(\varepsilon,\sigma,t)]+\sum_{r=1}^{\infty}\left[N\left[\sum_{i=0}^{r}\delta_i(\varepsilon,\sigma,t)\right]-N\left[\sum_{i=0}^{r-1}\delta_i(\varepsilon,\sigma,t)\right]\right],$$

$$\Rightarrow\sum_{k=0}^{\infty}\delta_k(\varepsilon,\sigma,t)=S^{-1}\left[\mu^{\omega}\left(\sum_{k=0}^{m-1}\left(\frac{1}{\mu}\right)^{\omega-(k-1)}\delta^{(k)}(0)+S[q(\varepsilon,\sigma,t)]\right)\right]-S^{-1}[\mu^{\omega}S(L[\delta_0(\varepsilon,\sigma,t)]+N[\delta_0(\varepsilon,\sigma,t)])]-S^{-1}\left[\mu^{\omega}S\left(\sum_{r=1}^{\infty}L[\delta_r(\varepsilon,\sigma,t)]+\sum_{r=1}^{\infty}\left[N\left[\sum_{i=0}^{r}\delta_i(\varepsilon,\sigma,t)\right]-N\left[\sum_{i=0}^{r-1}\delta_i(\varepsilon,\sigma,t)\right]\right]\right)\right],$$

$$\Rightarrow\delta_0(\varepsilon,\sigma,t)=S^{-1}\left[\mu^{\omega}\left(\sum_{k=0}^{m-1}\left(\frac{1}{\mu}\right)^{\omega-(k-1)}\delta^{(k)}(0)+S[q(\varepsilon,\sigma,t)]\right)\right],$$

$$\Rightarrow\delta_1(\varepsilon,\sigma,t)=-S^{-1}[\mu^{\omega}S(L[\delta_0(\varepsilon,\sigma,t)]+N[\delta_0(\varepsilon,\sigma,t)])]$$

$$\Rightarrow\delta_{r+1}(\varepsilon,\sigma,t)=-S^{-1}\left[\mu^{\omega}S\left(\sum_{r=1}^{\infty}L[\delta_r(\varepsilon,\sigma,t)]+\sum_{r=1}^{\infty}\left[N\left[\sum_{i=0}^{r}\delta_i(\varepsilon,\sigma,t)\right]-N\left[\sum_{i=0}^{r-1}\delta_i(\varepsilon,\sigma,t)\right]\right]\right)\right],$$

$$r=1,2,3,4,5,\ldots$$

Appendix B

The form of 3D time telegraph equation is as follows [20]:

$$D_t^{\omega}\delta(\varepsilon,\sigma,\tau,t)+L[\delta(\varepsilon,\sigma,\tau,t)]+N[\delta(\varepsilon,\sigma,\tau,t)]=q(\varepsilon,\sigma,\tau,t),$$

where L refers to the linear operator and N refers to the nonlinear operator. $D_t^{\omega}\delta(\varepsilon,t)$ is the Caputo derivative of $\delta(\varepsilon,t)$. Applying Sawi transform on equation

$$S[D_t^{\omega}\delta(\varepsilon,t)]+S[L[\delta(\varepsilon,t)]+N[\delta(\varepsilon,t)]]=S[q(\varepsilon,t)],$$

$$\frac{1}{\mu^{\omega}}S[\delta(\varepsilon,t)]-\sum_{k=0}^{m-1}\left(\frac{1}{\mu}\right)^{\omega-(k-1)}\delta^{(k)}(0)=S[q(\varepsilon,t)]-S[L[\delta(\varepsilon,t)]]-S[N[\delta(\varepsilon,t)]],$$

$$\Rightarrow\delta(\varepsilon,t)=S^{-1}\left[\mu^{\omega}\left(\sum_{k=0}^{m-1}\left(\frac{1}{\mu}\right)^{\omega-(k-1)}\delta^{(k)}(0)+S[q(\varepsilon,t)]\right)\right]-S^{-1}[\mu^{\omega}(S[L[\delta(\varepsilon,\sigma,\tau,t)]]+S[N[\delta(\varepsilon,\sigma,\tau,t)]])].$$

Now,

$$L[\delta]=L\left[\sum_{r=0}^{\infty}\delta_r(\varepsilon,\sigma,\tau,t)\right]=L[\delta_0(\varepsilon,\sigma,\tau,t)]+\sum_{r=1}^{\infty}\left[L\left[\sum_{i=0}^{r}\delta_i(\varepsilon,\sigma,\tau,t)\right]-L\left[\sum_{i=0}^{r-1}\delta_i(\varepsilon,\sigma,\tau,t)\right]\right],$$

$$N[\delta]=N\left[\sum_{r=0}^{\infty}\delta_r(\varepsilon,\sigma,\tau,t)\right]=N[\delta_0(\varepsilon,\sigma,\tau,t)]+\sum_{r=1}^{\infty}\left[N\left[\sum_{i=0}^{r}\delta_i(\varepsilon,\sigma,\tau,t)\right]-N\left[\sum_{i=0}^{r-1}\delta_i(\varepsilon,\sigma,\tau,t)\right]\right],$$

$$\Rightarrow \sum_{k=0}^{\infty} \delta_k(\varepsilon, \sigma, \tau, t) = S^{-1}\left[\mu^{\omega}\left(\sum_{k=0}^{m-1}\left(\frac{1}{\mu}\right)^{\omega-(k-1)} \delta^{(k)}(0) + S[q(\varepsilon, \sigma, \tau, t)]\right)\right] - S^{-1}[\mu^{\omega} S(L[\delta_0(\varepsilon, \sigma, \tau, t)] + N[\delta_0(\varepsilon, \sigma, \tau, t)])] - S^{-1}\left[\mu^{\omega} S\left(\sum_{r=1}^{\infty} L[\delta_r(\varepsilon, \sigma, \tau, t)] + \sum_{r=1}^{\infty}\left[N\left[\sum_{i=0}^{r}\delta_i(\varepsilon, \sigma, \tau, t)\right] - N\left[\sum_{i=0}^{r-1}\delta_i(\varepsilon, \sigma, \tau, t)\right]\right]\right)\right],$$

$$\Rightarrow \delta_0(\varepsilon, \sigma, \tau, t) = S^{-1}\left[\mu^{\omega}\left(\sum_{k=0}^{m-1}\left(\frac{1}{\mu}\right)^{\omega-(k-1)} \delta^{(k)}(0) + S[q(\varepsilon, \sigma, \tau, t)]\right)\right],$$

$$\Rightarrow \delta_1(\varepsilon, \sigma, \tau, t) = -S^{-1}[\mu^{\omega} S(L[\delta_0(\varepsilon, \sigma, \tau, t)] + N[\delta_0(\varepsilon, \sigma, \tau, t)])],$$

$$\Rightarrow \delta_{r+1}(\varepsilon, \sigma, \tau, t) = -S^{-1}\left[\mu^{\omega} S\left(\sum_{r=1}^{\infty} L[\delta_r(\varepsilon, \sigma, \tau, t)] + \sum_{r=1}^{\infty}\left[N\left[\sum_{i=0}^{r}\delta_i(\varepsilon, \sigma, \tau, t)\right] - N\left[\sum_{i=0}^{r-1}\delta_i(\varepsilon, \sigma, \tau, t)\right]\right]\right)\right],$$

$r = 1, 2, 3, 4, 5, \ldots$