AN L_2 ANALYTIC FOURIER-FEYNMAN TRANSFORM

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INTRODUCTION

The concept of an *analytic Fourier-Feynman transform* was introduced in 1972 by M. D. Brue [2], and it was defined essentially as in (0.2) below. It was based on the analytic Wiener and Feynman integrals [3], for which we now give simplified definitions sufficiently general for this paper.

Definition. Let C[a, b] be the space of real continuous functions $x(\cdot)$ on [a, b] for which $x(a) = 0$. Let F be a functional such that the Wiener integral

$$J(\lambda) = \int_{C[a,b]} F(\lambda^{-1/2} x)\, dx$$

exists for almost all real $\lambda > 0$. If there exists a function $J^*(\lambda)$ analytic in the half-plane $\Re\lambda > 0$ such that $J^*(\lambda) = J(\lambda)$ for almost all real $\lambda > 0$, then we define this "essential analytic extension" of J to be the *analytic Wiener integral of* F *over* C[a, b] *with parameter* λ, and for $\Re\lambda > 0$ we write

$$\int_{C[a,b]}^{\text{anw}_\lambda} F(x)\, dx = J^*(\lambda).$$

Definition. Let q be a real parameter ($q \neq 0$), and let F be a functional whose analytic Wiener integral exists for $\Re\lambda > 0$. Then, if the following limit exists, we call it the *analytic Feynman integral of* F *over* C[a, b] *with parameter* q, and we write

$$(0.1)\qquad \int_{C[a,b]}^{\text{anf}_q} F(x)\, dx = \lim_{\substack{\lambda \to -iq \\ \Re\lambda > 0}} \int_{C[a,b]}^{\text{anw}_\lambda} F(x)\, dx.$$

On the basis of these definitions, we can define Brue's transform as follows:

Definition. If $q \neq 0$ and if for each $y \in C[a, b]$ the analytic Feynman integral

$$(0.2)\qquad T_q^* F \equiv \int_{C[a,b]}^{\text{anf}_q} F(x + y)\, dx$$

exists, then $T_q^* F$ is called the *analytic Fourier-Feynman transform of* F.

Actually, Brue used a slightly more general definition of the analytic Feynman integral, but restricted the definition of his transform to the case $q = -1$, using the case $q = 1$ as the inverse transform.

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He applied this definition to various classes of functionals defined on $C[a, b]$ and also to a class of analytic functionals defined on the space of complex functions whose real and imaginary parts are both in $C[a, b]$. In this complex case, his results are beautifully symmetric, as the space is transformed into itself by T_1^*, and $T_1^* T_{-1}^* F = F$. However, his results for real functionals are not symmetric, and they require either strong or complicated hypotheses. His final theorem deals with functionals of the form

$$\exp\left\{ \int_a^b \theta(t, x(t))\, dt \right\},$$

which seems to be of interest to physicists, but of all his theorems it has the strongest hypotheses. The $L_1(-\infty, \infty)$ Fourier transform is one of the basic concepts upon which Brue's paper is based.

In the present paper we work with simple hypotheses, and by using the L_2 Fourier transform and L_2-theory generally, we obtain symmetric results for functionals defined almost everywhere on $C[a, b]$. Our method requires the concept of the scale-invariant limit in the mean in the $L_2(C[a, b])$-sense, which is defined below.

Throughout the paper, the term *Wiener measurable* will mean measurable with respect to the uncompleted Wiener measure or "strict Wiener measure", as in [7].

Terminology. We shall say that two functionals $F(x)$ and $G(x)$ are *equal s-almost everywhere* if for each $\rho > 0$ the equation $F(\rho x) = G(\rho x)$ holds for almost all $x \in C[a, b]$, in other words, if $F(x) = G(x)$ except for a scale-invariant null set. We denote this equivalence relation between functionals by

$$F \approx G.$$

Definition. Let $\{H_n\}$ and H be measurable functionals such that, for each $\rho > 0$,

$$\lim_{n \to \infty} \int_{C[a,b]} |H_n(\rho y) - H(\rho y)|^2\, dy = 0.$$

Then we write

$$\underset{n \to \infty}{\text{l. i. m.}}\, (w_s)\, H_n \approx H,$$

and we call H *the scale invariant limit in the mean of* H_n over $C[a, b]$. A similar definition is understood when n is replaced by a continuously varying parameter.

We use a limit of this type in defining an L_2 analytic Feynman integral and an L_2 analytic Fourier-Feynman transform, as follows:

Definition. Let q be real, $q \neq 0$. If

$$G(y) \equiv \underset{\substack{\lambda \to -iq \\ \Re \lambda > 0}}{\text{l. i. m.}}\, (w_s) \int_{C[a,b]}^{\text{anw}_\lambda} F(x, y)\, dx$$

exists for a functional F measurable on $C[a, b] \times C[a, b]$, we write

(0.3) $$G(y) = \int_{C[a,b]}^{m\ \mathrm{anf}_q} F(x, y)\, dx,$$

and we call G *the* (*scale invariant*) L_2 *analytic Feynman integral of* F *over* C[a, b] *with parameter* q. The letter m appearing before the symbol anf above the integral sign is intended to distinguish this Feynman integral, which depends on a limit in the mean, from the earlier Feynman integral defined in equation (0.1).

We note that the existence of (0.3) presupposes that for each $\rho > 0$ the analytic Wiener integral $\int_{C[a,b]}^{\mathrm{anw}_\lambda} F(x, \rho y)\, dx$ exists for $\Re \lambda > 0$.

Definition. Let q be real, $q \neq 0$. We define the L_2 analytic Fourier Feynman transform of F by the formula

$$(T_q F)(y) \equiv \int_{C[a,b]}^{m\ \mathrm{anf}_q} F(x + y)\, dx,$$

whenever the integral on the right hand side exists over C[a, b]. (We note that $T_q F$ is defined only s-almost-everywhere.)

We remark that if F is measurable and $T_q F$ exists, then $T_q F$ is measurable. We also call $T_q F$ the *mean Feynman transform* of F.

We next define three classes of spaces of functionals to which T_q applies.

Definition. Let $\mathscr{A}_n$ be the space of functionals F that can be expressed in the form

$$F(x) = f[x(t_1), \cdots, x(t_n)]$$

s-almost-everywhere on C[a, b], where $a < t_1 < t_2 < \cdots < t_n \leq b$, and where $f \in L_2(\mathbb{R}^n)$ and f is Borel measurable.

It will be shown that if $F \in \mathscr{A}_n$, then $T_q F$ exists, $T_q F \in \mathscr{A}_n$, and

(0.4) $$T_{-q} T_q F \approx F.$$

Notation. Let $\Delta_n \equiv \{(t_1, \cdots, t_n) \mid a < t_1 < t_2 < \cdots < t_n \leq b\}$.

Definition. Let $\mathscr{K}_n$ be the space of functions f defined and Borel measurable on $\Delta_n \times \mathbb{R}^n$ such that $f(t_1, \cdots, t_n; \cdots) \in L_2(\mathbb{R}^n)$ and

$$N_n(f) \equiv \sup_{(t_1,\cdots,t_n)\in\Delta_n} \|f(t_1, \cdots, t_n; \cdots)\| < +\infty.$$

In particular, we interpret $\mathscr{K}_0$ to be the set of complex constants, and if $f_0 \in \mathscr{K}_0$, then $N_0(f_0) = |f_0|$, and we note that $\mathscr{K}_1$ is the set of Borel measurable functions f on $[a, b] \times (-\infty, \infty)$ such that $\sup_{a<t\leq b} \|f(t, \cdot)\| < +\infty$.

Remark. It can be shown that if $f \in \mathscr{K}_n$, then for almost every $x \in C[a, b]$ and all $\rho > 0$,

$$\int \overset{\cdots}{\underset{\Delta_n}{}} \int f(t_1, \cdots, t_n; \rho x(t_1), \cdots, \rho x(t_n))\, dt_1 \cdots dt_n < +\infty.$$

The proof is similar to the proof of Lemma 2 of [6].

Definition. Let $\mathscr{S}_n$ be the space of functionals F such that, for some function $f \in \mathscr{K}_n$,

$$F(x) \equiv \int \overset{\cdots}{\underset{\Delta_n}{}} \int f(t_1, \cdots, t_n; x(t_1), \cdots, x(t_n))\, dt_1 \cdots dt_n \tag{0.5}$$

for s-almost-all x. We shall call this function f *a defining kernel for* F. In particular, we interpret $\mathscr{S}_0$ to be the space of constant complex-valued functionals. If $F \in \mathscr{S}_0$, we take $f \equiv F$ to be the kernel of F and write $N_0(f) = |f|$.

As before, it will be shown that T_q is defined on $\mathscr{S}_n$, maps $\mathscr{S}_n$ onto itself, and satisfies (0.4) on $\mathscr{S}_n$.

Finally, we build a larger space by using sums of functionals chosen from each of the spaces $\mathscr{S}_n$.

Definition. Let $\mathscr{S}$ be the space of functionals F such that there exists a sequence $\{F_n\}$ with $F_n \in \mathscr{S}_n$ having corresponding defining kernels $f_n \in \mathscr{K}_n$ such that

$$F \approx \sum_{n=0}^{\infty} F_n$$

and

$$[N_n(f_n)]^{1/n} = o(n^{3/4}) \qquad \text{as } n \to +\infty. \tag{0.6}$$

We shall call $\{F_n\}$ *a defining sequence for* F, and $\{f_n\}$ *a corresponding kernel sequence.*

Again, it will be shown that if $F \in \mathscr{S}$, then $T_q F$ exists, $T_q F \in \mathscr{S}$, and

$$T_{-q} T_q F \approx F.$$

We shall also show that $\mathscr{S}$ contains some interesting functionals; indeed if Φ is an entire function of order less than four and $\theta \in \mathscr{K}_1$, then the functional

$$F(x) \equiv \Phi\left[\int_a^b \theta(t, x(t))\, dt\right] \tag{0.7}$$

belongs to the space $\mathscr{S}$, and thus $T_q F \in \mathscr{S}$ and $T_{-q} T_q F \approx F$.

Since we shall use Plancherel's theorem and related theorems, it will be convenient to introduce a short notation for the type of limiting integral that occurs in their context.

Notation. We denote

$$\underset{\substack{A \to \infty \\ (v_1, \cdots, v_n)}}{\text{l. i. m.}} \int_{-A}^{A} \overset{(n)}{\cdots} \int_{-A}^{A} \Phi(u_1, \cdots, u_n; v_1, \cdots, v_n)\, du_1 \cdots du_n$$

by

$$\overset{(v)}{\int_{-\infty}^{\infty}} \overset{(n)}{\cdots} \int_{-\infty}^{\infty} \Phi(u_1, \cdots, u_n; v_1, \cdots, v_n)\, du_1 \cdots du_n .$$

We shall refer to such an integral as an "L_2-limiting integral".

In the proofs of Lemma 1 and Theorem 2, we shall refer to a Lemma 1 of [8] that is the n-dimensional extension of Lemma 1 of [5]. For the convenience of the reader we state it here.

LEMMA H. *Let* $\Re \lambda \geq 0$ $(\lambda \neq 0)$, *and let* $f \in L_2(\mathbb{R}^n)$. *Let*

$$g(v_1, \cdots, v_n)$$

$$= \left(\frac{\lambda}{2\pi}\right)^{n/2} \int_{-\infty}^{\infty} \overset{(n)}{\cdots} \int_{-\infty}^{\infty} f(u_1, \cdots, u_n) \exp\left(-\frac{\lambda}{2} \sum_{j=1}^{n} (u_j - v_j)^2\right) du_1 \cdots du_n .$$

Then $g \in L_2(\mathbb{R}^n)$, *and*

$$\|g\| \leq \|f\| .$$

If $\Re \lambda = 0$, *the integral is to be interpreted as an* L_2-*limiting integral; moreover, in this case*

$$\|g\| = \|f\| .$$

1. THE EQUIVALENCE RELATION ≈

In this section, we explain why the Wiener limit was defined so as to be scale-invariant. We shall show that the transformation T_q preserves equivalence classes based on the relation $\approx$. We have not defined $F \approx G$ to mean merely $F(x) = G(x)$ almost everywhere on $C[a, b]$ because such an equivalence relation is not preserved under the transformation T_q. Indeed, we shall exhibit two functionals such that $F(x) = G(x)$ a.e. but $(T_q F)(y) \neq (T_q G)(y)$ on a set of positive Wiener measure.

THEOREM 1. *If* q *is real* $(q \neq 0)$, *if* F_1 *and* F_2 *are measurable and* $F_1 \approx F_2$, *and if* $T_q F_1$ *exists, then* $T_q F_2$ *exists and*

$$T_q F_1 \approx T_q F_2 .$$

Proof. Let ρ and σ be any two positive numbers, and let $F = F_1 - F_2$. Then $F(\sqrt{\rho^2 + \sigma^2}\, u) = 0$ for almost all $u \in C[a, b]$, and thus

$$\int_{C[a,b]} |F(\sqrt{\rho^2 + \sigma^2}\, u)|\, du = 0 .$$

By Bearman's Lemma (see [1] or Lemma 2 of [7]),

$$\int_{C[a,b]\times C[a,b]} |F(\rho x + \sigma y)|\, d(x \times y) = 0;$$

therefore $F(\rho x + \sigma y) = 0$ for almost all $(x, y) \in C[a, b] \times C[a, b]$. Thus for each

$\rho > 0$ and each $\sigma > 0$, $\int_{C[a,b]} F(\rho x + \sigma y)dx = 0$ for almost all $y \in C[a, b]$. Since F is measurable, we see by Fubini's theorem that for each σ and for almost every $y \in C[a, b]$,

$$\int_{C[a,b]} F(\rho x + \sigma y)\, dx = 0$$

for almost every ρ. Thus, for each σ and almost every y,

$$\int_{C[a,b]} [F_1(\rho x + \sigma y) - F_2(\rho x + \sigma y)]\, dx = 0$$

for almost every $\rho > 0$, and therefore

$$\int_{C[a,b]}^{\mathrm{anw}_\lambda} [F_1(x + \sigma y) - F_2(x + \sigma y)]\, dx = 0.$$

But since $T_q F_1$ exists, there exists a function $J(\lambda)$, analytic in $\Re\lambda > 0$, such that $\int_{C[a,b]} F_1(\lambda^{-1/2} x + \sigma y)\, dx = J(\lambda)$ for almost all real positive λ. Consequently, the same is true for F_2, and therefore

$$\int_{C[a,b]}^{\mathrm{anw}_\lambda} F_2(x + \sigma y)\, dx$$

exists for all positive σ and almost every y, for $\Re\lambda > 0$. Moreover,

$$\int_{C[a,b]}^{\mathrm{anw}_\lambda} F_1(x + \sigma y)\, dx = \int_{C[a,b]}^{\mathrm{anw}_\lambda} F_2(x + \sigma y)\, dx,$$

and $T_q F_2$ exists and

$$T_q F_1 \approx T_q F_2.$$

Counterexample. In [4] it was shown that there exists a Wiener measurable set $C_1 \subset C[a, b]$ such that $m(C_1) = 1$ and $m(\rho C_1) = 0$ if $\rho > 0$ and $\rho \neq 1$.

Let $F(x) \equiv 1$ and

$$G(x) = \begin{cases} 1 & \text{on } C_1, \\ 0 & \text{elsewhere.} \end{cases}$$

Then $F(x) = G(x)$ almost everywhere, and if $q \neq 0$, $(T_q F)(y) \equiv 1$. Moreover, by Bearman's Lemma, for each $\lambda > 0$,

$$\int_{C[a,b]} \left[\int_{C[a,b]} G(\lambda^{-1/2} x + y)\, dx \right] dy = \int_{C[a,b]} G(\sqrt{\lambda^{-1} + 1}\, z)\, dz.$$

But $G(\sqrt{\lambda^{-1}+1}\,z) = 0$ if $\sqrt{\lambda^{-1}+1}\,z \notin C_1$, and hence $G(\sqrt{\lambda^{-1}+1}\,z) = 0$ for almost all z. Hence, since $G(\sqrt{\lambda^{-1}+1}\,z) = 0$,

$$\int_{C[a,b]} G[\lambda^{-1/2}x + y]\,dx = 0$$

for almost all y. Hence

$$(T_q G)(y) = 0$$

for almost all y, and

$$(T_q F)(y) \neq (T_q G)(y) \quad \text{a.e.}$$

even though $F = G$ a.e.

This example shows the importance of calling sets in Wiener space equivalent only if they differ merely on scale-invariant null sets.

2. THE TRANSFORMATION T_q APPLIED TO FUNCTIONALS $F \in \mathscr{A}_n$

In order to show that $T_q F$ exists and that $T_{-q} T_q$ is the identity transformation when $F \in \mathscr{A}_n$, we shall first prove two lemmas.

Notation. Let $\gamma \equiv \gamma(t) \equiv [(2\pi)^n (t_1 - a) \cdots (t_n - t_{n-1})]^{-1/2}$.

LEMMA 1: *Let*

$$F(x) \equiv f(x(t_1), \cdots, x(t_n)) \in \mathscr{A}_n .$$

Then, for each $y \in C[a, b]$ *and each* λ *with* $\Re\lambda > 0$, *the Wiener integral in the relation*

$$\int_{C[a,b]}^{\text{anw}_\lambda} F(x+y)\,dx = h(y(t_1), y(t_2), \cdots, y(t_n), \lambda) \tag{2.1}$$

exists; here

$$h(w_1, \cdots, w_n, \lambda) = \lambda^{n/2}\gamma \int_{-\infty}^{\infty} \overset{(n)}{\cdots} \int_{-\infty}^{\infty} f(v_1, \cdots, v_n) \exp\left(-\frac{\lambda}{2} \sum_{j=1}^{n} \frac{[(v_j - v_{j-1}) - (w_j - w_{j-1})]^2}{t_j - t_{j-1}} \right) dv_1 \cdots dv_n \tag{2.2}$$

for all real values $w_1, \cdots, w_n$ (*notation:* $v_0 = w_0 = 0$). *Moreover, for fixed* $w_1, \cdots, w_n$ *the function* $h(w_1, \cdots, w_n, \lambda)$ *is analytic in* λ *for* $\Re\lambda > 0$; *also* $h(\cdots, \lambda) \in L_2(\mathbb{R}^n)$,

$$\| h(\cdots, \lambda) \| \leq \| f \| , \tag{2.3}$$

and for each $\rho > 0$,

$$\int_{C[a,b]} |h(\rho y(t_1), \cdots, \rho y(t_n), \lambda)|^2\,dy \leq \rho^{-n}\gamma\, \|f\|^2 \quad \textit{for } \Re\lambda > 0 . \tag{2.4}$$

(To make nonintegral powers of complex λ ($\Re\lambda \geq 0$) well-defined, we shall assume $|\arg\lambda| \leq \pi/2$, and we choose the branch $\lambda^z = \exp[z(\log|\lambda| + i\arg\lambda)]$.)

Proof. If λ is real and positive and $y \in C[a, b]$, then

$$\int_{C[a,b]} F(\lambda^{-1/2}x + y)\,dx = \int_{C[a,b]} f[\lambda^{-1/2}x(t_1) + y(t_1), \cdots, \lambda^{-1/2}x(t_n) + y(t_n)]\,dx$$

$$= \gamma \int_{-\infty}^{\infty} \overset{(n)}{\cdots} \int_{-\infty}^{\infty} f[\lambda^{-1/2}u_1 + y(t_1), \cdots, \lambda^{-1/2}u_n + y(t_n)]$$

$$\exp\left(-\frac{1}{2}\sum_{j=1}^{n}\frac{(u_j - u_{j-1})^2}{(t_j - t_{j-1})}\right)du_1 \cdots du_n \tag{2.5}$$

$$= \lambda^{n/2}\gamma \int_{-\infty}^{\infty} \overset{(n)}{\cdots} \int_{-\infty}^{\infty} f(v_1, \cdots, v_n)$$

$$\exp\left(-\frac{\lambda}{2}\sum_{j=1}^{n}\frac{[(v_j - v_{j-1}) - (y(t_j) - y(t_{j-1}))]^2}{(t_j - t_{j-1})}\right)dv_1 \cdots dv_n$$

$$= h(y(t_1), \cdots, y(t_n), \lambda)\,.$$

It is easy to see that for each real $(w_1, \cdots, w_n)$, the function $h(w_1, \cdots, w_n, \lambda)$ is analytic in λ for $\Re\lambda > 0$ and thus the Wiener integral in (2.5) has an analytic extension to $\Re\lambda > 0$, and consequently the left member of (2.1) exists and equation (2.1) is true.

We now transform (2.2) by setting

$$v_j' = \frac{v_j - v_{j-1}}{\sqrt{t_j - t_{j-1}}} \quad \text{and} \quad w_j' = \frac{w_j - w_{j-1}}{\sqrt{t_j - t_{j-1}}} \quad \text{for } j = 1, \cdots, n, \tag{2.6}$$

so that

$$\sum_{k=1}^{j} \sqrt{t_k - t_{k-1}}\, v_k' = v_j \quad \text{and} \quad \sum_{k=1}^{j} \sqrt{t_k - t_{k-1}}\, w_k' = w_j;$$

thus we obtain the formula

$$h\left(\sqrt{t_1 - t_0}\,w_1', \cdots, \sum_{k=1}^{n}\sqrt{t_k - t_{k-1}}\,w_k', \lambda\right)$$

$$= \left(\frac{\lambda}{2\pi}\right)^{n/2} \int_{-\infty}^{\infty} \overset{(n)}{\cdots} \int_{-\infty}^{\infty} f\left(\sqrt{t_1 - t_0}\,v_1', \cdots, \sum_{k=1}^{n}\sqrt{t_k - t_{k-1}}\,v_k'\right) \tag{2.7}$$

$$\exp\left(-\frac{\lambda}{2}\sum_{j=1}^{n}(v_j' - w_j')^2\right)dv_1' \cdots dv_n'.$$

If we define

$$h^*(w_1', \cdots, w_n', \lambda) \equiv h\left(\sqrt{t_1 - t_0}\, w_1', \cdots, \sum_{k=1}^{n} \sqrt{t_k - t_{k-1}}\, w_k', \lambda\right) = h(w_1, \cdots, w_n, \lambda)$$

and

$$f^*(v_1', \cdots, v_n') \equiv f\left(\sqrt{t_1 - t_0}\, v_1', \cdots, \sum_{k=1}^{n} \sqrt{t_k - t_{k-1}}\, v_k'\right) = f(v_1, \cdots, v_n),$$

equation (2.7) becomes

$$h^*(w_1', \cdots, w_n', \lambda) = \left(\frac{\lambda}{2\pi}\right)^{n/2} \int_{-\infty}^{\infty} \overset{(n)}{\cdots} \int_{-\infty}^{\infty} f^*(v_1', \cdots, v_n') \exp\left(-\frac{\lambda}{2} \sum_{j=1}^{n} (v_j' - w_j')^2\right) dv_1' \cdots dv_n'.$$

By Lemma H, $\|h^*(\cdots, \lambda)\| \leq \|f^*\|$, that is,

$$\int_{-\infty}^{\infty} \overset{(n)}{\cdots} \int_{-\infty}^{\infty} \left| h\left(\sqrt{t_1 - t_0}\, w_1', \cdots, \sum_{k=1}^{n} \sqrt{t_k - t_{k-1}}\, w_n'\right)\right|^2 dw_1' \cdots dw_n'$$

$$\leq \int_{-\infty}^{\infty} \overset{(n)}{\cdots} \int_{-\infty}^{\infty} \left| f\left(\sqrt{t_1 - t_0}\, v_1', \cdots, \sum_{k=1}^{n} \sqrt{t_k - t_{k-1}}\, v_k'\right)\right|^2 dv_1' \cdots dv_n';$$

applying the inverse transformation of (2.6), we obtain the inequality

$$\int_{-\infty}^{\infty} \overset{(n)}{\cdots} \int_{-\infty}^{\infty} |h(w_1, \cdots, w_n, \lambda)|^2\, dw_1 \cdots dw_n$$

$$\leq \int_{-\infty}^{\infty} \overset{(n)}{\cdots} \int_{-\infty}^{\infty} |f(v_1, \cdots, v_n)|^2\, dv_1 \cdots dv_n,$$

and inequality (2.3) is established:

$$\|h(\cdot, \cdots, \cdot, \lambda)\| \leq \|f\|.$$

To establish (2.4), we note that

$$\int_{C[a,b]} |h(\rho y(t_1), \cdots, \rho y(t_n), \lambda)|^2\, dy$$

$$= \gamma \int_{-\infty}^{\infty} \overset{(n)}{\cdots} \int_{-\infty}^{\infty} |h(\rho u_1, \cdots, \rho u_n, \lambda)|^2 \exp\left(-\sum_{j=1}^{n} \frac{(u_j - u_{j-1})^2}{2(t_j - t_{j-1})}\right) du_1 \cdots du_n$$

$$\leq \gamma \int_{-\infty}^{\infty} \overset{(n)}{\cdots} \int_{-\infty}^{\infty} |h(\rho u_1, \cdots, \rho u_n, \lambda)|^2 du_1 \cdots du_n \leq \gamma \|f\|^2 \rho^{-n} ;$$

thus (2.4) is established and the lemma is proved.

Remark. For each function $g \in L_2(\mathbb{R}^n)$, we see by the Schwarz inequality that

$$\left| \int_{-\infty}^{\infty} \overset{(n)}{\cdots} \int_{-\infty}^{\infty} g(v_1, \cdots, v_n) \exp\left(- \sum_{j=1}^{n} \Lambda_j[(v_j - v_{j-1}) - u_j]^2 \right) dv_1 \cdots dv_n \right|$$

$$\leq \|g\| \left(\frac{\pi}{2}\right)^{n/4} [\Lambda_1 \cdots \Lambda_n]^{-1/4}$$

for all positive $\Lambda_1, \cdots, \Lambda_n$ and all real u_j.

LEMMA 2. *Suppose that* $f \in L_2(\mathbb{R}^n)$, *that* q *is real* $(q \neq 0)$, *and that* $a = t_0 < t_1 < \cdots < t_n = b$. *Let* g *be given by*

$$\begin{aligned} g(v_1, \cdots, v_n) = (-iq)^{n/2} \gamma \; &{}^{(v)}\!\int_{-\infty}^{\infty} \overset{(n)}{\cdots} \int_{-\infty}^{\infty} f(u_1, \cdots, u_n) \\ &\exp\left(\frac{iq}{2} \sum_{j=1}^{n} \frac{[(u_j - u_{j-1}) - (v_j - v_{j-1})]^2}{t_j - t_{j-1}} \right) du_1 \cdots du_n, \end{aligned} \tag{2.8}$$

where $u_0 = v_0 = 0$. *Then* $g \in L_2(\mathbb{R}^n)$ *and*

$$\begin{aligned} f(u_1, \cdots, u_n) = (iq)^{n/2} \gamma \; &{}^{(u)}\!\int_{-\infty}^{\infty} \overset{(n)}{\cdots} \int_{-\infty}^{\infty} g(v_1, \cdots, v_n) \\ &\exp\left(- \frac{iq}{2} \sum_{j=1}^{n} \frac{[(v_j - v_{j-1}) - (u_j - u_{j-1})]^2}{t_j - t_{j-1}} \right) dv_1 \cdots dv_n \end{aligned} \tag{2.9}$$

for almost all $u_1, \cdots, u_n$. *Moreover,*

$$\|g\| = \|f\| . \tag{2.10}$$

Conversely, if $g \in L_2(\mathbb{R}^n)$ *and* (2.9) *defines* f, *then* $f \in L_2(\mathbb{R}^n)$, (2.8) *holds a. e., and* (2.10) *is valid.*

Proof. Let $p = |q|$, so that $q = \pm p$. Since $f \in L_2(\mathbb{R}^n)$, it follows that $f^* \in L_2(\mathbb{R}^n)$, where

$$\begin{aligned} &f^*(z_1, \cdots, z_n) \\ &= \exp\left(\pm \frac{i}{2} \sum_{j=1}^{n} z_j^2 \right) f\left(\left(\frac{t_1 - t_0}{p}\right)^{1/2} z_1, \cdots, \sum_{j=1}^{n} \left(\frac{t_j - t_{j-1}}{p}\right)^{1/2} z_j \right). \end{aligned} \tag{2.11}$$

Let us write

$$\tilde{g} = \mathscr{F} f^*, \tag{2.12}$$

where $\mathscr{F}$ denotes the n-dimensional Fourier transform. We shall show that

$$\tilde{g} = (\pm i)^{n/2} g^*, \tag{2.13}$$

where

$$g^*(w_1, \cdots, w_n)$$

$$= \exp\left(\mp\frac{i}{2}\sum_{j=1}^{n} w_j^2\right) g\left(\mp\left(\frac{t_1 - t_0}{p}\right)^{1/2} w_1, \cdots, \mp\sum_{j=1}^{n}\left(\frac{t_j - t_{j-1}}{p}\right)^{1/2} w_j\right). \tag{2.14}$$

By the definition of the L_2 Fourier transform, we see that

$$\tilde{g}(w_1, \cdots, w_n)$$

$$= \underset{A\to\infty}{\text{l. i. m.}}\ (2\pi)^{-n/2} \int \overset{(n)}{\underset{D_A}{\cdots}} \int f^*(z_1, \cdots, z_n) \exp\left(i\sum_{j=1}^{n} w_j z_j\right) dz_1 \cdots dz_n, \tag{2.15}$$

where D_A is a monotone family of bounded regions such that $\bigcup_A D_A = \mathbb{R}^n$. Since we wish to make the transformation

$$z_j = (u_j - u_{j-1})(p/(t_j - t_{j-1}))^{1/2}, \tag{2.16}$$

we choose D_A to be the image of $D'_A \equiv [-A, A]^n$ under this transformation. Making the transformation above and also the transformation

$$w_j = \mp(v_j - v_{j-1})(p/(t_j - t_{j-1}))^{1/2}, \tag{2.17}$$

we deduce from (2.15) that

$$\exp\left(\frac{\pm i}{2}\sum_{j=1}^{n} w_j^2\right)\tilde{g}(w_1, \cdots, w_n)$$

$$= \underset{A\to\infty}{\text{l. i. m.}}\ (2\pi)^{-n/2} \int \overset{(n)}{\underset{D_A}{\cdots}} \int f\left(\left(\frac{t_1 - t_0}{p}\right)^{1/2} z_1, \cdots, \sum_{j=1}^{n}\left(\frac{t_j - t_{j-1}}{p}\right)^{1/2} z_j\right)$$

$$\exp\left(\frac{\pm i}{2}\sum_{j=1}^{n}(z_j \pm w_j)^2\right) dz_1 \cdots dz_n$$

$$= \underset{A\to\infty}{\text{l. i. m.}} \int \overset{(n)}{\underset{D'_A}{\cdots}} \int f(u_1, \cdots, u_n)$$

$$\exp\left(\frac{iq}{2}\sum_{j=1}^{n}\frac{[(u_j - u_{j-1}) - (v_j - v_{j-1})]^2}{t_j - t_{j-1}}\right)\cdot\left(\frac{p}{2\pi}\right)^{n/2}\left(\prod_{j=1}^{n}(t_j - t_{j-1})\right)^{-1/2} du_1 \cdots du_n.$$

Thus g defined by (2.8) exists, and $g \in L_2$. Therefore

$$\exp\left(\pm\frac{i}{2}\sum_{j=1}^{n} w_j^2\right)\tilde{g}(w_1, \cdots, w_n) = (\pm i)^{n/2} g(v_1, \cdots, v_n)$$

$$= (\pm i)^{n/2} g\left(\mp\left(\frac{t_1 - t_0}{p}\right)^{1/2} w_1, \cdots, \mp\sum_{j=1}^{n}\left(\frac{t_j - t_{j-1}}{p}\right)^{1/2} w_j\right)$$

$$= (\pm i)^{n/2} \exp\left(\pm\frac{i}{2}\sum_{j=1}^{n} w_j^2\right)\cdot g^*(w_1, \cdots, w_n) .$$

Thus

$$\tilde{g} = (\pm i)^{n/2} g^* ,$$

and (2.13) is established; from (2.12) it follows that

$$g^* = (\pm i)^{-n/2} \mathscr{F} f^* . \tag{2.18}$$

By the Plancherel theorem

$$\|g^*\| = \|f^*\| ,$$

and

$$f^*(z_1, \cdots, z_n)$$

$$= (\pm i)^{n/2} \underset{A\to\infty}{\text{l. i. m.}} (2\pi)^{-n/2} \int \overset{(n)}{\underset{D_A}{\cdots}} \int g^*(w_1, \cdots, w_n) \exp\left(-i\sum_{j=1}^{n} w_j z_j\right) dw_1 \cdots dw_n . \tag{2.19}$$

Equations (2.11), (2.14), and (2.19) imply that

$$\exp\left(\pm\frac{i}{2}\sum_{j=1}^{n} z_j^2\right) f\left(\left(\frac{t_1 - t_0}{p}\right)^{1/2} z_1, \cdots, \sum_{j=1}^{n}\left(\frac{t_j - t_{j-1}}{p}\right)^{1/2} z_j\right)$$

$$= (\pm i)^{n/2} \underset{A\to\infty}{\text{l. i. m.}} (2\pi)^{-n/2} \int \underset{D_A}{\cdots} \int g\left(\mp\left(\frac{t_1 - t_0}{p}\right)^{1/2} w_1, \cdots, \mp\sum_{j=1}^{n}\left(\frac{t_j - t_{j-1}}{p}\right)^{1/2} w_j\right)$$

$$\exp\left(\mp\frac{i}{2}\sum_{j=1}^{n} w_j^2\right)\exp\left(-i\sum_{j=1}^{n} w_j z_j\right) dw_1 \cdots dw_n .$$

Making the substitutions (2.16) and (2.17), we obtain the relation

$$f(u_1, \cdots, u_n) = (iq)^{n/2}\gamma \operatorname*{l.i.m.}_{A\to\infty} \int \overset{(n)}{\underset{D'_A}{\cdots}} \int g(v_1, \cdots, v_n)$$

$$\exp\left(-\frac{iq}{2}\sum_{j=1}^{n} \frac{[(u_j - u_{j-1}) - (v_j - v_{j-1})]^2}{t_j - t_{j-1}}\right) dv_1 \cdots dv_n ,$$

and (2.9) is established. Moreover, from (2.9), (2.11), and (2.14) we obtain (2.10), and thus the direct statements of the theorem are proved. The converse follows if we interchange f and g and replace q by -q.

Remark. The hypotheses of Lemma 2 imply (2.18) with f^* and g^* defined by (2.11) and (2.14). Conversely, if $f^* \in L_2$ and (2.18) holds, then (2.8) holds with g and f defined by (2.11) and (2.14).

THEOREM 2. *Let* q *be real* ($q \neq 0$), *and let*

$$F(x) = f(x(t_1), \cdots, x(t_n)) \in \mathscr{A}_n . \tag{2.20}$$

Then the mean Feynman transform of F,

$$G \equiv T_q F , \tag{2.21}$$

exists s-*almost-everywhere on* C[a, b], *and*

$$G(y) \approx g(y(t_1), \cdots, y(t_n)) \in \mathscr{A}_n . \tag{2.22}$$

For real $w_1, \cdots, w_n$, *the function* g *is given by the formula*

$$g(w_1, \cdots, w_n) = (-iq)^{n/2}\gamma \; {}^{(w)}\!\int_{-\infty}^{\infty} \overset{(n)}{\cdots} \int_{-\infty}^{\infty} f(v_1, \cdots, v_n) \exp\left(\frac{iq}{2}\sum_{j=1}^{n} \frac{[(v_j - v_{j-1}) - (w_j - w_{j-1})]^2}{t_j - t_{j-1}}\right) dv_1 \cdots dv_n \tag{2.23}$$

($v_0 = w_0 = 0$). *Moreover,*

$$\|g\| = \|f\| . \tag{2.24}$$

Proof. By Lemma 2, g defined by (2.23) exists and is of class L_2. By Lemma 1, for $\Re\lambda > 0$ and $y \in C[a, b]$, the analytic Wiener integral below exists, and

$$\int_{C[a,b]}^{\text{anw}_\lambda} F(x + y)\, dx = h(y(t_1), y(t_2), \cdots, y(t_n), \lambda) , \tag{2.25}$$

where h is given by (2.2) (of Lemma 1).

Thus, to establish (2.21), we shall show that

$$\int_{C[a,b]}^{m\ \mathrm{anf}_q} F(x+y)\,dx = g(y(t_1), \cdots, y(t_n)),$$

that is,

$$\underset{\substack{\lambda \to -iq \\ \Re\lambda > 0}}{\mathrm{l.\,i.\,m.}}\ (w_s)\, h(y(t_1), y(t_2), \cdots, y(t_n), \lambda) = g(y(t_1), \cdots, y(t_n)).$$

In other words, we must show that for each $\rho > 0$,

$$\lim_{\substack{\lambda \to -iq \\ \Re\lambda > 0}} J(\lambda) = 0, \tag{2.26}$$

where

$$J(\lambda) = \int_{C[a,b]} |g(\rho y(t_1), \cdots, \rho y(t_n)) - h(\rho y(t_1), \cdots, \rho y(t_n), \lambda)|^2\, dy. \tag{2.27}$$

To simplify our next computation, we introduce the following notation for the Gaussian density function $p(u, t)$.

Notation. Let $p(u, t) = (2\pi t)^{-1/2} \exp\{-u^2/2t\}$ and

$$P(\vec{v}, \vec{t}, \lambda) = \prod_{j=1}^{n} \{p(v_j - v_{j-1}, (t_j - t_{j-1})/\lambda)\},$$

where $\vec{v} = (v_1, \cdots, v_n)$, $\vec{t} = (t_1, \cdots, t_n)$, $t_0 = a$, and $v_0 = 0$.

Expressing g and h by means of P, we have the equations

$$g(w_1, \cdots, w_n) = {}^{(w)}\int_{-\infty}^{\infty} \overset{(n)}{\cdots} \int_{-\infty}^{\infty} f(v_1, \cdots, v_n) P(\vec{v} - \vec{w}, \vec{t}, -iq)\, dv_1 \cdots dv_n \tag{2.28}$$

and

$$h(w_1, \cdots, w_n, \lambda) = \int_{-\infty}^{\infty} \overset{(n)}{\cdots} \int_{-\infty}^{\infty} f(v_1, \cdots, v_n) P(\vec{v} - \vec{w}, \vec{t}, \lambda)\, dv_1 \cdots dv_n. \tag{2.29}$$

Evaluating the right member of (2.27) by Wiener's formula, we see that

$$\begin{aligned} J(\lambda) &= \int_{-\infty}^{\infty} \overset{(n)}{\cdots} \int_{-\infty}^{\infty} |g(\rho w_1, \cdots, \rho w_n) - h(\rho w_1, \cdots, \rho w_n, \lambda)|^2 P(\vec{w}, \vec{t}, 1)\, dw_1 \cdots dw_n \\ &= \rho^{-n} \int_{-\infty}^{\infty} \overset{(n)}{\cdots} \int_{-\infty}^{\infty} |g(z_1, \cdots, z_n) - h(z_1, \cdots, z_n, \lambda)|^2 \\ &\qquad P(\rho^{-1}\vec{z}, \vec{t}, 1)\, dz_1 \cdots dz_n. \end{aligned} \tag{2.30}$$

Substituting (2.28) and (2.29) in (2.30), we obtain the formula

$$\rho^n J(\lambda) = \int_{-\infty}^{\infty} \overset{(n)}{\cdots} \int_{-\infty}^{\infty} \left| {}^{(w)}\int_{-\infty}^{\infty} \overset{(n)}{\cdots} \int_{-\infty}^{\infty} f(v_1, \cdots, v_n) P(\vec{v} - \vec{z}, \vec{t}, -iq)\, dv_1 \cdots dv_n \right.$$
$$\left. - \int_{-\infty}^{\infty} \overset{(n)}{\cdots} \int_{-\infty}^{\infty} f(v_1, \cdots, v_n) P(\vec{v} - \vec{z}, \vec{t}, \lambda)\, dv_1 \cdots dv_n \right|^2 P(\rho^{-1}\vec{z}, \vec{t}, 1)\, dz_1 \cdots dz_n .$$

Let

$$f_A(\vec{v}) = \begin{cases} 0 & \text{if } |v_j| \leq A \text{ for } j = 1, \cdots, n , \\ f(\vec{v}) & \text{if } |v_j| > A \text{ for some } j \ (j = 1, \cdots, n) . \end{cases}$$

Then, by the triangle inequality

$$[\rho^n J(\lambda)]^{1/2} = \left\| \left[{}^{(w)}\int_{-\infty}^{\infty} \overset{(n)}{\cdots} \int_{-\infty}^{\infty} f(v_1, \cdots, v_n) P(\vec{v} - (\cdot), \vec{t}, -iq)\, dv_1 \cdots dv_n \right.\right.$$
$$\left.\left. - \int_{-\infty}^{\infty} \overset{(n)}{\cdots} \int_{-\infty}^{\infty} f(v_1, \cdots, v_n) P(\vec{v} - (\cdot), \vec{t}, \lambda)\, dv_1 \cdots dv_n \right] P(\rho^{-1}(\cdot), \vec{t}, 1)^{1/2} \right\|$$
$$\leqq \left\| \int_{-A}^{A} \overset{(n)}{\cdots} \int_{-A}^{A} f(v_1, \cdots, v_n) [P(\vec{v} - (\cdot), \vec{t}, -iq) \right.$$
$$\left. - P(\vec{v} - (\cdot), \vec{t}, \lambda)]\, dv_1 \cdots dv_n P(\rho^{-1}(\cdot), \vec{t}, 1)^{1/2} \right\|$$
$$+ \gamma^{1/2} \left\| {}^{(w)}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_A(v_1, \cdots, v_n) P(\vec{v} - (\cdot), \vec{t}, -iq)\, dv_1 \cdots dv_n \right\|$$
$$+ \gamma^{1/2} \left\| \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_A(v_1, \cdots, v_n) P(\vec{v} - (\cdot), \vec{t}, \lambda)\, dv_1 \cdots dv_n \right\| \equiv I_1 + I_2 + I_3 .$$

By Lemma 2 and by (2.3) of Lemma 1 applied to f_A instead of f, we see that

$$I_2 \leq \gamma^{1/2} \|f_A\| \quad \text{and} \quad I_3 \leq \gamma^{1/2} \|f_A\| .$$

To estimate I_1, we note that $\|f_A\| \to 0$ as $A \to +\infty$. Corresponding to any positive number ε, we can choose a number A large enough so that

$$\|f_A\| < \varepsilon / 4\gamma^{1/2}.$$

Then, for $|\lambda + iq| < 1$ and $\Re \lambda > 0$,

$$[\rho^n J(\lambda)]^{1/2} < I_1 + \varepsilon/2$$

$$= \left\{ \int_{-\infty}^{\infty} \overset{(n)}{\cdots} \int_{-\infty}^{\infty} \left| \int_{-A}^{A} \overset{(n)}{\cdots} \int_{-A}^{A} f(v_1, \cdots, v_n)[P(\vec{v} - \vec{w}, \vec{t}, -iq) - P(\vec{v} - \vec{w}, \vec{t}, \lambda)] dv_1 \cdots dv_n \right|^2 P(\rho^{-1}\vec{w}, \vec{t}, 1)\, dw_1 \cdots dw_n \right\}^{1/2} + \varepsilon/2 .$$

By the continuity of $P(\vec{v}, \vec{t}, \lambda)$, the difference of the two values P has the limit zero as $\lambda \to -iq$ $(\Re\lambda > 0)$, for each $\vec{v}$ and $\vec{w}$. Moreover, the integrand of the interior integral is dominated by the function

$$2\gamma(|q| + 1)^{n/2} \, |f(v_1, \cdots, v_n)| ,$$

which is integrable on $[-A, A]^n$. Consequently, the interior integral approaches zero as $\lambda \to -iq$ $(\Re\lambda > 0)$, for each fixed $\vec{w}$. Moreover, the integrand of the outer integral is dominated by the quantity

$$2\gamma(|q| + 1)^{n/2} \int_{-A}^{A} \overset{(n)}{\cdots} \int_{-A}^{A} |f(v_1, \cdots, v_n)| dv_1 \cdots dv_n \, P(\rho^{-1}\vec{w}, \vec{t}, 1) ,$$

which is integrable with respect to $w_1, \cdots, w_n$ over $\mathbb{R}^n$. By virtue of dominated convergence, $I_1 \to 0$ as $\lambda \to -iq$ $(\Re\lambda > 0)$,

$$[\rho^n J(\lambda)]^{1/2} < \varepsilon ,$$

and (2.26) is established. Thus, by (2.25) and (2.27), the theorem is proved.

THEOREM 3. *Let* q *be real* ($q \neq 0$), *and let* $F \in \mathscr{A}_n$. *Then*

$$T_{-q} T_q F \approx F . \tag{2.31}$$

Proof. Since $F \in \mathscr{A}_n$, we can express F by (2.20), with $f \in L_2(\mathbb{R}^n)$. By Theorem 2, T_q is given by (2.21), with (2.22) and (2.23). Since $G \in \mathscr{A}_n$, the functions G and g satisfy the hypotheses placed on F and f in Theorem 2. While G is not defined uniquely by the equation $G = T_q F$, we see by the definition of $T_q F$ that any two representations of it, say G_1 and G_2, satisfy the condition $G_1 \approx G_2$. Since $G_1 \in \mathscr{A}_n$ and $G_2 \in \mathscr{A}_n$, both $T_{-q} G_1$ and $T_{-q} G_2$ exist. Moreover, by Theorem 1, $T_{-q} G_1 \approx T_{-q} G_2$, and $T_{-q} T_q F$ is uniquely defined up to the equivalence relation $\approx$.

By Theorem 2, $T_{-q} G$ is given by the identity

$$T_{-q} G \equiv H, \tag{2.32}$$

where $H(z) \approx h(z(t_1), \cdots, z(t_n))$ and $h \in L_2(\mathbb{R}^n)$ and

$$h(u_1, \cdots, u_n) = (iq)^{n/2} \gamma \, {}^{(u)}\!\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(v_1, \cdots, v_n) \exp\left(-\frac{iq}{2} \sum_{j=1}^{n} \frac{[(v_j - v_{j-1}) - (u_j - u_{j-1})]^2}{t_j - t_{j-1}} \right) dv_1 \cdots dv_n \tag{2.33}$$

for real values $u_1, \cdots, u_n$, with $u_0 = v_0 = 0$.

From (2.9) of Lemma 2 and (2.33), it follows that $h(u_1, \cdots, u_n) = f(u_1, \cdots, u_n)$ a.e. in $\mathbb{R}^n$, and hence for each $\rho > 0$,

$$h(\rho u_1, \cdots, \rho u_n) = f(\rho u_1, \cdots, \rho u_n) \text{ a.e. in } \mathbb{R}^n.$$

Thus $H \approx F$, and (2.32) implies that (2.31) holds and Theorem 3 is proved.

3. THE TRANSFORMATION T_q APPLIED TO FUNCTIONALS $F \in \mathcal{S}_n$

The following lemma concerning analytic Wiener integrals will enable us to establish the existence of $T_q F$ for $F \in \mathcal{S}_n$.

LEMMA 3. *Let*

$$F(x) = \int \cdots \int_{\Delta_n} f(t_1, \cdots, t_n; x(t_1), \cdots, x(t_n)) \, dt_1 \cdots dt_n, \tag{3.1}$$

where $F \in \mathcal{S}_n$ *and* f *is the defining kernel of* F. *Then, for each* $y \in C[a, b]$ *and for* $\Re \lambda > 0$, *the analytic Wiener integral*

$$\int_{C[a,b]}^{\text{anw}_\lambda} F(x + y) \, dx = \int \cdots \int_{\Delta_n} h(t_1, \cdots, t_n; y(t_1), \cdots, y(t_n); \lambda) \, dt_1 \cdots dt_n \tag{3.2}$$

exists; here

$$h(t_1, \cdots, t_n; u_1, \cdots, u_n; \lambda) = \lambda^{n/2} \gamma \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(t_1, \cdots, t_n; v_1, \cdots, v_n) \exp\left(-\frac{\lambda}{2} \sum_{j=1}^{n} \frac{[(u_j - u_{j-1}) - (v_j - v_{j-1})]^2}{t_j - t_{j-1}} \right) dv_1 \cdots dv_n \tag{3.3}$$

for all real $u_1, \cdots, u_n$ *with* $u_0 = v_0 = 0$. *Moreover, for* $\Re \lambda > 0$ *and* $(t_1, \cdots, t_n) \in \Delta_n$, $h(t_1, \cdots, t_n; \cdots; \lambda) \in L_2(\mathbb{R}^n)$ *and*

$$\|h(t_1, \cdots, t_n; \cdots; \lambda)\| \leq \|f(t_1, \cdots, t_n; \cdots)\| \leq N_n(f),$$

$$N_n(h(\cdots; \cdots; \lambda)) \leq N_n(f).$$

Finally, (3.2) *can be written in the form*

$$\int_{C[a,b]}^{\text{anw}_\lambda} F(x+y)\,dx$$

$$= \int_{C[a,b]}^{\text{anw}_\lambda} \int \underset{\Delta_n}{\cdots} \int f(t_1, \cdots, t_n; x(t_1)+y(t_1), \cdots, x(t_n)+y(t_n))\,dt_1 \cdots dt_n\,dx$$

$$= \int \underset{\Delta_n}{\cdots} \int \left(\int_{C[a,b]}^{\text{anw}_\lambda} f(t_1, \cdots, t_n; x(t_1)+y(t_1), \cdots, x(t_n)+y(t_n))\,dx \right) dt_1 \cdots dt_n ,$$

and

$$\left| \int_{C[a,b]}^{\text{anw}_\lambda} F(x+y)\,dx \right| \leq \left(\frac{|\lambda|^2}{4\pi\,\Re\,\lambda} \right)^{n/4} \frac{\left[\Gamma\left(\frac{3}{4}\right) \right]^n}{\Gamma\left(\frac{3n}{4}+1\right)} (b-a)^{3n/4} N_n(f) .$$

Proof. Let

(3.4) $$\Phi(t_1, \cdots, t_n; x) \equiv f(t_1, \cdots, t_n; x(t_1), \cdots, x(t_n)) .$$

Then, for each $(t_1, \cdots, t_n) \in \Delta_n$, the function $\Phi(t_1, \cdots, t_n; \cdot\,)$ satisfies the hypotheses of Lemma 1. Thus, for $\Re\,\lambda > 0$,

(3.5) $$\int_{C[a,b]}^{\text{anw}_\lambda} \Phi(t_1, \cdots, t_n; x+y)\,dx = h(t_1, \cdots, t_n; y(t_1), \cdots, y(t_n); \lambda) ,$$

where

$$h(t_1, \cdots, t_n; u_1, \cdots, u_n; \lambda) = \lambda^{n/2} \gamma \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(t_1, \cdots, t_n; v_1, \cdots, v_n)$$

$$\exp\left(-\frac{\lambda}{2} \sum_{j=1}^{n} \frac{[(v_j - v_{j-1}) - (u_j - u_{j-1})]^2}{(t_j - t_{j-1})} \right) dv_1 \cdots dv_n$$

for real $u_1, \cdots, u_n$ $(u_0 = v_0 = 0)$.

Moreover, for each $(t_1, \cdots, t_n)$ in Δ_n and each $(u_1, \cdots, u_n)$ in $\mathbb{R}^n$, the function $h(t_1, \cdots, t_n; u_1, \cdots, u_n; \lambda)$ is analytic in λ for $\Re\,\lambda > 0$. Furthermore,

$$h(t_1, \cdots, t_n; \cdots; \lambda) \in L_2(\mathbb{R}^n) \quad \text{and} \quad \|h(t_1, \cdots, t_n; \cdots; \lambda)\| \leq \|f(t_1, \cdots, t_n; \cdots)\| .$$

Now, for positive λ, it follows from the n-dimensional generalization of Lemma 1 of [6] that the integrand of the left member below is measurable, and by virtue of Fubini's theorem and Schwarz's inequality

$$\int_{C[a,b]} \int \underset{\Delta_n}{\cdots} \int |f(t_1, \cdots, t_n; \lambda^{-1/2}x(t_1) + y(t_1), \cdots, \lambda^{-1/2}x(t_n) + y(t_n))| dt_1 \cdots dt_n dx$$

$$= \int \underset{\Delta_n}{\cdots} \int \int_{C[a,b]} |f| \, dx \, dt_1 \cdots dt_n$$

$$= \int \underset{\Delta_n}{\cdots} \int \gamma \int_{-\infty}^{\infty} \overset{(n)}{\cdots} \int_{-\infty}^{\infty} |f(t_1, \cdots, t_n; \lambda^{-1/2}u_1 + y(t_1), \cdots, \lambda^{-1/2}u_n + y(t_n))| \cdot$$

$$\exp\left(-\frac{1}{2} \sum_{j=1}^{n} \frac{(u_j - u_{j-1})^2}{(t_j - t_{j-1})} \right) du_1 \cdots du_n \, dt_1 \cdots dt_n$$

$$= \int \underset{\Delta_n}{\cdots} \int \lambda^{n/2} \gamma \int_{-\infty}^{\infty} \overset{(n)}{\cdots} \int_{-\infty}^{\infty} |f(t_1, \cdots, t_n; v_1, \cdots, v_n)| \cdot$$

$$\exp\left(-\frac{\lambda}{2} \sum_{j=1}^{n} \frac{[(v_j - v_{j-1}) - (y(t_j) - y(t_{j-1}))]^2}{t_j - t_{j-1}} \right) dv_1 \cdots dv_n \, dt_1 \cdots dt_n$$

$$\leq \int \underset{\Delta_n}{\cdots} \int \lambda^{n/2} \gamma N_n(f) \cdot$$

$$\left[\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left(-\lambda \sum_{j=1}^{n} \frac{[(v_j - v_{j-1}) - (y(t_j) - y(t_{j-1}))]^2}{t_j - t_{j-1}} \right) dv_1 \cdots dv_n \right]^{1/2} dt_1 \cdots dt_n$$

$$= \int \underset{\Delta_n}{\cdots} \int \left(\frac{\lambda}{4\pi} \right)^{n/4} [(t_1 - a) \cdots (t_n - t_{n-1})]^{-1/4} N_n(f) \, dt_1 \cdots dt_n$$

$$\leq \left(\frac{\lambda}{4\pi} \right)^{n/4} \cdot \frac{\left[\Gamma\left(\frac{3}{4} \right) \right]^n}{\Gamma\left(\frac{3n}{4} + 1 \right)} (b - a)^{3n/4} N_n(f) < +\infty .$$

Here we have used Dirichlet's integral [9] to evaluate the integral over Δ_n.

The finiteness of the left member justifies the use of Fubini's theorem in an argument establishing an equation similar to that above, but without absolute-value signs. Hence, using (3.1), (3.4), and (3.5), we obtain for positive λ the equation

$$\text{(3.6)} \qquad \int_{C[a,b]} F(\lambda^{-1/2}x + y) \, dx = \int \underset{\Delta_n}{\cdots} \int h(t_1, \cdots, t_n; y(t_1), \cdots, y(t_n); \lambda) dt_1 \cdots dt_n .$$

We now show by Morera's theorem that the right-hand member is analytic in λ, for $\Re \lambda > 0$.

For $\Re\lambda > 0$, we see by applying our Remark following Lemma 1 to equation (3.3) that

$$|h(t_1, \cdots, t_n; u_1, \cdots, u_n; \lambda)| \leq N_n(f)\left(\frac{|\lambda|^2}{4\pi\Re\lambda}\right)^{n/4} [(t_1 - a)\cdots(t_n - t_{n-1})]^{-1/4}$$

and hence

$$\int \underset{\Delta_n}{\cdots} \int |h(t_1, \cdots, t_n; y(t_1), \cdots, y(t_n); \lambda)| dt_1 \cdots dt_n$$

$$\leq \left(\frac{|\lambda|^2}{4\pi\Re\lambda}\right)^{n/4} \frac{\left[\Gamma\left(\frac{3}{4}\right)\right]^n}{\Gamma\left(\frac{3n}{4}+1\right)} (b - a)^{3n/4} N_n(f).$$

Thus, by dominated convergence, we see that the integral

$$\int \underset{\Delta_n}{\cdots} \int h(t_1, \cdots, t_n; y(t_1), \cdots, y(t_n); \lambda)\, dt_1 \cdots dt_n$$

is a continuous function of λ. Integrating with respect to λ this expression around a closed contour Γ in $\Re\lambda > 0$, and applying Fubini's and Morera's theorems, we deduce from the analyticity of h in the half-plane $\Re\lambda > 0$ that the right member of (3.6) is analytic for $\Re\lambda > 0$. Thus $\int_{C[a,b]} F(\lambda^{-1/2}x + y)\, dx$ has an analytic extension to $\Re\lambda > 0$, and (3.2) is established and Lemma 3 is proved.

COROLLARY TO LEMMA 3. *Under the hypothesis of Lemma* 3,

$$\left\| \int_{C[a,b]}^{\text{anw}_\lambda} F(x + (\cdot)\rho)\, dx \right\|_W \leq \rho^{-n/2} \frac{\left[\Gamma\left(\frac{3}{4}\right)\right]^n (b - a)^{3n/4} (2\pi)^{-n/4} N_n(f)}{\Gamma\left(\frac{3n}{4}+1\right)},$$

where $\|\cdots\|_W$ denotes the $L_2(C_1[a, b])$-norm.

Proof. Using Fubini's Theorem and (2.4) of Lemma 1, we see that

$$\int_{C[a,b]} \left| \int_{C[a,b]}^{\text{anw}_\lambda} F(x + \rho y)\, dx \right|^2 dy$$

$$= \int_{C[a,b]} \left| \int \underset{\Delta_n}{\cdots} \int h(t_1, \cdots, t_n; \rho y(t_1), \cdots, \rho y(t_n); \lambda)\, dt_1 \cdots dt_n \right|^2 dy$$

$$= \int_{C[a,b]} \int \underset{\Delta_n}{\cdots} \int \int \underset{\Delta_n}{\cdots} \int h(t_1, \cdots, t_n; \rho y(t_1), \cdots, \rho y(t_n); \lambda)$$

$$\overline{h}(s_1, \cdots, s_n; \rho y(s_1), \cdots, \rho y(s_n); \lambda)\, dt_1 \cdots dt_n\, ds_1 \cdots ds_n\, dy$$

$$= \int \underset{\Delta_n}{\cdots} \int \int \underset{\Delta_n}{\cdots} \int \int_{C[a,b]} h(t_1, \cdots) \overline{h}(s_1, \cdots)\, dy\, dt_1 \cdots dt_n\, ds_1 \cdots ds_n$$

$$\leq \int_{\Delta_n}^{\cdots}\int \int_{\Delta_n}^{\cdots}\int \sqrt{\int_{C[a,b]}|h(t_1, \cdots)|^2 dy}\sqrt{\int_{C[a,b]}|h(s_1, \cdots)|^2 dy}\, dt_1 \cdots dt_n ds_1 \cdots ds_n$$

$$\leq \int_{\Delta_n}^{\cdots}\int \int_{\Delta_n}^{\cdots}\int \rho^{-n}\gamma(t_1, \cdots, t_n)^{1/2}\gamma(s_1, \cdots, s_n)^{1/2}[N_n(f)]^2 dt_1 \cdots dt_n ds_1 \cdots ds_n$$

$$\leq \rho^{-n}\left(\frac{\left[\Gamma\left(\frac{3}{4}\right)\right]^n}{\Gamma\left(\frac{3n}{4}+1\right)}(b-a)^{3n/4}(2\pi)^{-n/4}N_n(f)\right)^2 .$$

THEOREM 4. *Let* q *be real* ($q \neq 0$), *let* $F \in \mathscr{S}_n$, *and let* f *be its defining kernel. Then* T_qF, *the mean Feynman transform of* F, *exists and is of the form*

$$(3.7) \qquad (T_q F)(y) \approx \int_{\Delta_n}^{\cdots}\int g(t_1, \cdots, t_n; y(t_1), \cdots, y(t_n))\, dt_1 \cdots dt_n,$$

where $g \in \mathscr{K}_n$ *and*

$$(3.8) \qquad g(t_1, \cdots, t_n; w_1, \cdots, w_n) = (-iq)^{n/2}\gamma \overset{(w)}{\int_{-\infty}^{\infty}} \overset{(n)}{\cdots} \int_{-\infty}^{\infty} f(t_1, \cdots, t_n; v_1, \cdots, v_n) \exp\left(\frac{iq}{2}\sum_{j=1}^{n}\frac{[(v_j - v_{j-1}) - (w_j - w_{j-1})]^2}{t_j - t_{j-1}}\right) dv_1 \cdots dv_n .$$

Moreover,

$$(3.9) \qquad N_n(g) = N_n(f) < +\infty .$$

Proof of Theorem 4. We define a function Φ on $\Delta_n \times C[a, b]$ by

$$(3.10) \qquad \Phi(t_1, \cdots, t_n; x) \equiv f(t_1, \cdots, t_n; x(t_1), \cdots, x(t_n)).$$

For each point $t_1, \cdots, t_n$, the function Φ satisfies the hypotheses of Theorem 2, and thus by (2.21) and (2.22), for each $(t_1, \cdots, t_n) \in \Delta_n$,

$$(3.11) \qquad \underset{\substack{\lambda \to -iq \\ \Re\lambda > 0}}{\text{l. i. m.}}\,(w_s)\int_{C[a,b]}^{\text{anw}_\lambda} \Phi(t_1, \cdots, t_n; x+y)\, dx = g(t_1, \cdots, t_n; y(t_1), \cdots, y(t_n)),$$

where g is given by (3.8) and $\|g(t_1, \cdots, t_n; \cdots)\| = \|f(t_1, \cdots, t_n; \cdots)\|$. Thus

$$N_n(g) = \sup_{\Delta_n}\|g(t_1, \cdots, t_n; \cdots)\| = \sup_{\Delta_n}\|f(t_1, \cdots, t_n; \cdots)\| = N_n(f) < +\infty .$$

Moreover, it follows from (3.8) and the measurability of f that g is Borel measurable on $\Delta_n \times \mathbb{R}^n$. Further, Lemma 1 implies that for $\Re\lambda > 0$,

$$(3.12) \qquad \int_{C[a,b]}^{\text{anw}_\lambda} \Phi(t_1, \cdots, t_n; x+y)\, dx = h(t_1, \cdots, t_n; y(t_1), \cdots, y(t_n); \lambda),$$

where h is given by equation (3.3) of Lemma 3. Thus, for each positive ρ, the function (2.27) in the proof of Theorem 2 satisfies condition (2.26), and therefore

$$\lim_{\substack{\lambda \to -iq \\ \Re\lambda > 0}} \int_{C[a,b]} |h(t_1, \cdots, t_n; \rho y(t_1), \cdots, \rho y(t_n); \lambda) - g(t_1, \cdots, t_n; \rho y(t_1), \cdots, \rho y(t_n))|^2 \, dy = 0 .$$

In order to use the dominated-convergence theorem, we note that by Lemma 1, for each $\rho > 0$,

$$\left(\int_{C[a,b]} |h(t_1, \cdots, t_n; \rho y(t_1), \cdots, \rho y(t_n); \lambda)|^2 \, dy \right)^{1/2} \leq \rho^{-n/2} \gamma^{1/2} N_n(f)$$

and

$$\left(\int_{C[a,b]} |g(t_1, \cdots, t_n; \rho y(t_1), \cdots, \rho y(t_n))|^2 \, dy \right)^{1/2}$$

$$= \left(\gamma \int_{-\infty}^{\infty} \overset{(n)}{\cdots} \int_{-\infty}^{\infty} |g(t_1, \cdots, t_n; \rho w_1, \cdots, \rho w_n)|^2 \exp\left(-\frac{1}{2} \sum_{j=1}^{n} \frac{(w_j - w_{j-1})^2}{(t_j - t_{j-1})} \right) dw_1 \cdots dw_n \right)^{1/2}$$

$$\leq \gamma^{1/2} \| g(t_1, \cdots, t_n; \cdots) \| \rho^{-n/2} \leq \gamma^{1/2} \rho^{-n/2} N_n(f) < +\infty .$$

Hence we have the inequality

$$\int_{C[a,b]} |h(t_1, \cdots, t_n; \rho y(t_1), \cdots, \rho y(t_n); \lambda) - g(t_1, \cdots, t_n; \rho y(t_1), \cdots, \rho y(t_n))|^2 \, dy$$

$$\leq 4\rho^{-n} [(2\pi)^n (t_1 - t_0) \cdots (t_n - t_{n-1})]^{-1/2} [N_n(f)]^2 < +\infty ,$$

and since the second member is integrable with respect to $t_1, \cdots, t_n$ over Δ_n, we see that

$$\lim_{\substack{\lambda \to -iq \\ \Re\lambda > 0}} \int \overset{(n)}{\underset{\Delta_n}{\cdots}} \int \int_{C[a,b]} |h(t_1, \cdots, t_n; \rho y(t_1), \cdots, \rho y(t_n); \lambda) - g(t_1, \cdots, t_n; \rho y(t_1), \cdots, \rho y(t_n))|^2 \, dy \, dt_1 \cdots dt_n = 0 .$$

Using Fubini's theorem, and keeping the same arguments for h and g, we have the equation

$$\lim_{\substack{\lambda \to -iq \\ \Re\lambda > 0}} \int_{C[a,b]} \int \overset{\cdots}{\underset{\Delta_n}{}} \int |h - g|^2 \, dt_1 \cdots dt_n \, dy = 0,$$

and hence, by Schwarz's inequality,

$$\lim_{\substack{\lambda \to -iq \\ \Re\lambda > 0}} \int_{C[a,b]} \left| \int \overset{\cdots}{\underset{\Delta_n}{}} \int (h - g) \, dt_1 \cdots dt_n \right|^2 dy = 0 .$$

Thus, by (3.2) of Lemma 3, and since f is the defining kernel of F, we have proved for each $\rho > 0$ that

$$\lim_{\substack{\lambda \to -iq \\ \Re\lambda > 0}} \int_{C[a,b]} \left| \int_{C[a,b]}^{\mathrm{anw}_\lambda} F(x + \rho y) \, dx \right.$$

$$\left. - \int \overset{(n)}{\underset{\Delta_n}{\cdots}} \int g(t_1, \cdots, t_n; \rho y(t_1), \cdots, \rho y(t_n)) \, dt_1 \cdots dt_n \right|^2 dy = 0,$$

and Theorem 4 is proved.

COROLLARY TO THEOREM 4. *Under the hypotheses of Theorem* 4,

$$(3.13) \qquad (T_q F)(y) \approx \int \overset{\cdots}{\underset{\Delta_n}{}} \int (T_q \Phi(t_1, \cdots, t_n; \cdot))(y) \, dt_1 \cdots dt_n,$$

where

$$(3.14) \qquad \Phi(t_1, \cdots, t_n; x) = f(t_1, \cdots, t_n; x(t_1), \cdots, x(t_n)) .$$

Proof. From (3.11) and the definition of the mean Feynman transform it follows that

$$g(t_1, \cdots, t_n; y(t_1), \cdots, y(t_n)) = (T_q \Phi(t_1, \cdots, t_n; \cdot))(y) .$$

Thus, by (3.7), we see that

$$(T_q F)(y) \approx \int \overset{\cdots}{\underset{\Delta_n}{}} \int (T_q \Phi(t_1, \cdots, t_n; \cdot))(y) \, dt_1 \cdots dt_n,$$

and the corollary is proved.

THEOREM 5. *Let* q *be real* ($q \neq 0$), *and let* $F \in \mathscr{S}_n'$. *Then* $T_{-q} T_q F \approx F$.

Proof. By the corollary to Theorem 4, we have (3.13) and (3.14). From Theorem 2, it follows that $T_q \Phi$ is given by

$$(T_q \Phi(t_1, \cdots, t_n; \cdot))(y) \approx g(t_1, \cdots, t_n; y(t_1), \cdots, y(t_n)),$$

where for each $t_1, \cdots, t_n$, $g \in L_q(\mathbb{R}^n)$ and

$$\| g(t_1, \cdots, t_n; \cdots) \| = \| f(t_1, \cdots, t_n; \cdots) \| \leq N_n(f) < +\infty .$$

Therefore, by virtue of (2.23), g is given by (3.8). By Theorem 4, $g \in \mathscr{K}_n$ and thus g satisfies the hypotheses imposed on f in Theorem 4. By two applications of the Corollary to Theorem 4, and by Theorem 1 and Theorem 3, we obtain the relation

$$T_{-q}T_qF \approx T_{-q}\left\{\int \overset{(n)}{\underset{\Delta_n}{\cdots}} \int T_q\Phi(t_1, \cdots, t_n; \cdot)dt_1 \cdots dt_n\right\}$$

$$\approx \int \overset{(n)}{\underset{\Delta_n}{\cdots}} \int T_{-q}T_q\Phi(t_1, \cdots, t_n; \cdot)dt_1 \cdots dt_n \approx \int \overset{(n)}{\underset{\Delta_n}{\cdots}} \int \Phi(t_1, \cdots, t_n; \cdot)dt_1 \cdots dt_n = F.$$

4. THE TRANSFORMATION T_q APPLIED TO FUNCTIONALS $F \in \mathscr{S}$

Our next lemmas establish an inequality relating the Wiener norm of a functional in $\mathscr{S}$ to the norms of its corresponding kernel sequence.

LEMMA 4. *If* $F \in \mathscr{S}_n$ *and* f *is a defining kernel for* F, *then*

$$\|F\|_W \leq \frac{\left[\Gamma\left(\frac{3}{4}\right)\right]^n}{\Gamma\left(\frac{3n}{4}+1\right)}(b-a)^{3n/4}(2\pi)^{-n/4}N_n(f).$$

Proof. By applying the Fubini Theorem, the Schwarz inequality, and the Dirichlet integral formula [9], we obtain the relations

$$\int_{C[a,b]} |F(x)|^2 dx = \int_{C[a,b]}\left| \int \underset{\Delta_n}{\cdots} \int \int \underset{\Delta_n}{\cdots} \int f(t_1, \cdots, t_n; x(t_1), \cdots, x(t_n)) \right.$$

$$\left. \bar{f}(s_1, \cdots, s_n; x(s_1), \cdots, x(s_n))\, dt_1 \cdots dt_n\, ds_1 \cdots ds_n \right| dx$$

$$\leq \int \underset{\Delta_n}{\cdots} \int \int \underset{\Delta_n}{\cdots} \int \int_{C[a,b]} |f|\ |\bar{f}|\, dx\, dt_1 \cdots dt_n\, ds_1 \cdots ds_n$$

$$= \int \underset{\Delta_n}{\cdots} \int \int \underset{\Delta_n}{\cdots} \int \left\{ \gamma(\vec{t}) \int_{-\infty}^{\infty} \overset{(n)}{\cdots} \int_{-\infty}^{\infty} |f(t_1, \cdots, t_n; u_1, \cdots, u_n)|^2 \right.$$

$$\exp\left(-\frac{1}{2}\sum_{j=1}^{n} \frac{(u_j - u_{j-1})^2}{t_j - t_{j-1}}\right) du_1 \cdots du_n\, \gamma(\vec{s}) \int_{-\infty}^{\infty} \overset{(n)}{\cdots} \int_{-\infty}^{\infty} |f(s_1, \cdots, s_n; v_1, \cdots, v_n)|^2$$

$$\left. \exp\left(-\frac{1}{2}\sum_{j=1}^{n} \frac{(v_j - v_{j-1})^2}{s_j - s_{j-1}}\right) dv_1 \cdots dv_n \right\}^{1/2} dt_1 \cdots dt_n\, ds_1 \cdots ds_n$$

$$\leq [N_n(f)]^2 \int \underset{\Delta_n}{\cdots} \int \int \underset{\Delta_n}{\cdots} \int [\gamma(\vec{t})\, \gamma(\vec{s})]^{1/2}\, dt_1 \cdots dt_n\, ds_1 \cdots ds_n$$

$$= [N_n(f)]^2 \left[\int \underset{\Delta_n}{\cdots} \int [\gamma(\vec{t})]^{1/2}\, dt_1 \cdots dt_n \right]^2$$

$$= (2\pi)^{-n/2} [N_n(f)]^2 \left[\int \underset{\Delta_n}{\cdots} \int [(t_1 - t_0) \cdots (t_n - t_{n-1})]^{-1/4} dt_1 \cdots dt_n \right]^2$$

$$= (2\pi)^{-n/2} [N_n(f)]^2 \left[\Gamma\left(\frac{3}{4}\right) \right]^{2n} \left[\Gamma\left(\frac{3n}{4} + 1\right) \right]^{-2} (b - a)^{3n/2} .$$

Thus Lemma 4 is proved.

For use in Theorem 6, we shall need to apply Lemma 4 to $F(\rho x)$, because the Wiener limit has to be scale invariant.

Remark. Under the hypotheses of Lemma 4, we have for $\rho > 0$ the inequality

$$\| F(\rho(\,\cdot\,)) \|_W \leq \rho^{-n/2} \frac{\Gamma\left(\frac{3}{4}\right)^n}{\Gamma\left(\frac{3n}{4} + 1\right)} (b - a)^{3n/4} (2\pi)^{-n/4} N_n(f) .$$

LEMMA 5. *Let* $F_n \in \mathscr{S}_n$, *and let* $f_n \in \mathscr{K}_n$ *be a kernel for* F_n *for* $n = 0, 1, \cdots$ *Suppose also that*

$$[N_n(f_n)]^{-1/n} = o(n^{3/4}) \qquad \text{as } n \to +\infty .$$

Then it follows for all positive numbers ρ *that*

$$\sum_{n=0}^{\infty} \| F_n((\,\cdot\,)\rho) \|_W < +\infty , \tag{4.1}$$

and the series $\sum_{n=0}^{\infty} F_n(\rho x)$ *converges absolutely for almost all* $x \in C[a, b]$ *and converges in the* $L_1(C[a, b])$*-mean and in the* $L_2(C[a, b])$*-mean. In fact,*

$$F \equiv \sum_{n=0}^{\infty} F_n \in \mathscr{S}.$$

Moreover, for each $y \in C[a, b]$ *and each* $\rho > 0$,

$$\sum_{n=0}^{\infty} \int_{C[a,b]} | F_n(\rho x + y) |\, dx < +\infty . \tag{4.2}$$

Proof. By the Remark preceding this lemma,

$$\| F_n(\rho(\,\cdot\,)) \|_W \leq \rho^{-n/2} \frac{\Gamma(3/4)^n}{\Gamma\left(\frac{3n}{4} + 1\right)} (b - a)^{3n/4} (2\pi)^{-n/4} N_n(f_n) ,$$

and hence, applying the root test and Stirling's theorem, we see that (4.1) holds. Since the L_1-norm over Wiener space is dominated by the L_2-norm over Wiener space,

$$\sum_{n=0}^{\infty} \int_{C[a,b]} |F_n(\rho x)|\, dx < +\infty . \tag{4.3}$$

An application of the monotone-convergence theorem implies that

$$\sum_{0}^{\infty} |F_n(\rho x)| < +\infty \qquad \text{a.e. on } C[a, b] .$$

Since the translation theorem for Wiener integrals does not allow us to proceed directly from (4.3) to (4.2), we must examine the corresponding kernels of the functionals F_n.

For each $y \in C[a, b]$ and each $\rho > 0$, let

$$H_n(x) \equiv F_n(\rho x + y) ,$$

so that

$$H_n(x) = \int \underset{\Delta_n}{\cdots} \int h_n(t_1, \cdots, t_n;\, x(t_1), \cdots, x(t_n))\, dt_1 \cdots dt_n , \tag{4.4}$$

where

$$h_n(t_1, \cdots, t_n;\, u_1, \cdots, u_n) \equiv f_n(t_1, \cdots, t_n;\, \rho u_1 + y(t_1), \cdots, \rho u_n + y(t_n)) . \tag{4.5}$$

Since by hypothesis

$$F_n(x) \equiv \int \underset{\Delta_n}{\cdots} \int f_n(t_1, \cdots, t_n;\, x(t_1), \cdots, x(t_n))\, dt_1 \cdots dt_n$$

whenever the integral on the right exists, it follows that (4.4) holds whenever the integral on the right exists. From (4.4) and (4.5) it follows that $H_n \in \mathscr{S}_n$ and that h_n is a kernel for H_n and $N_n(h_n) = \rho^{-n/2} N_n(f_n)$.

Thus H_n and h_n satisfy the hypotheses of this lemma, so that (4.3) becomes

$$\sum_{n=0}^{\infty} \int_{C[a,b]} |H_n(x)|\, dx < +\infty ;$$

thus (4.2) is established and the lemma is proved.

THEOREM 6. *Let* $F \in \mathscr{S}$, *and let* q *be real* ($q \neq 0$). *Then* $T_q F$ *exists and* $T_q F \in \mathscr{S}$ *and*

$$T_{-q} T_q F \approx F .$$

Proof. Let $\{F_n\}$ be a defining sequence for F, and let $\{f_n\}$ be the corresponding kernel sequence. By Theorem 4, $T_q F_n$ exists and

$$(T_q F_n)(y) \approx G_n(y) \equiv \int \underset{\Delta_n}{\cdots} \int g_n(t_1, \cdots, t_n; y(t_1), \cdots, y(t_n))\, dt_1 \cdots dt_n,$$

where $g_n \in \mathscr{K}_n$ and $G_n \in \mathscr{S}_n$. Moreover, $N_n(g_n) = N_n(f_n)$, and therefore

$$[N_n(g_n)]^{1/n} = o(n^{3/4}) \qquad \text{as } n \to \infty,$$

since $F \in \mathscr{S}$. By Lemma 5 applied to $\{G_n\}$, we see that $\sum_{n=0}^{\infty} G_n(y)$ converges absolutely a. e. on $C[a, b]$; and if we let

$$G(y) \equiv \sum_{n=0}^{\infty} G_n(y),$$

we see that $G \in \mathscr{S}$. We wish to show that $T_q F \approx G$. Let

$$H_n(y, \lambda) = \int_{C[a,b]}^{\text{anw}_\lambda} F_n(x + y)\, dx . \tag{4.6}$$

The right-hand side exists, by Lemma 3, and it is analytic in λ for $\Re\lambda > 0$, for each $y \in C[a, b]$. Let $\mathscr{D}_r = \{\lambda \mid |\lambda| \leq r,\ \Re\lambda \geq 1/r\}$ for each $r > 0$. By the last inequality in the statement of Lemma 3 and by (0.6), for each $r > 0$ and each $y \in C[a, b]$, the series $\sum_{n=0}^{\infty} H_n(y, \lambda)$ converges uniformly in λ for $\lambda \in \mathscr{D}_r$. We define

$$H(y, \lambda) = \sum_{n=0}^{\infty} H_n(y, \lambda)$$

for $y \in C[a, b]$ and $\Re\lambda > 0$. Then, for each $y \in C[a, b]$, the function $H(y, \lambda)$ is analytic in the half-plane $\Re\lambda > 0$. For real $\lambda > 0$, we see from (4.6) and the proof of Lemma 3 that

$$H_n(y, \lambda) = \int_{C[a,b]} F_n(\lambda^{-1/2} x + y)\, dx$$

for each y, and from (4.2) we see that

$$\begin{aligned} \int_{C[a,b]} \sum_{n=0}^{\infty} F_n(\lambda^{-1/2} x + y)\, dx &= \sum_{n=0}^{\infty} \int_{C[a,b]} F_n(\lambda^{-1/2} x + y)\, dx \\ &= \sum_{n=0}^{\infty} H_n(y, \lambda) = H(y, \lambda). \end{aligned} \tag{4.7}$$

Thus $H(y, \lambda)$ is the analytic extension of the first member of (4.7) to $\Re\lambda > 0$, and

$$H(y, \lambda) = \int_{C[a,b]}^{\text{anw}_\lambda} \sum_{n=0}^{\infty} F_n(x + y)\, dx .$$

By the proof of Theorem 1,

$$H(y, \lambda) \approx \int_{C[a,b]}^{\text{anw}_\lambda} F(x+y)\,dx\,.$$

To prove that $T_q F$ exists, we consider for each $\rho > 0$ the relations

$$\int_{C[a,b]} |H(\rho y, \lambda) - G(\rho y)|^2\,dy = \int_{C[a,b]} \left| \sum_{n=0}^{\infty} [H_n(\rho y, \lambda) - G_n(\rho y)] \right|^2 dy$$

$$\equiv \left\| \sum_{n=0}^{\infty} [H_n(\rho(\,\cdot\,), \lambda) - G_n(\rho(\,\cdot\,))] \right\|_W^2 \leq \left(\sum_{n=0}^{\infty} \|H_n(\rho(\,\cdot\,), \lambda) - G_n(\rho(\,\cdot\,))\|_W \right)^2 .$$

Each term in the series in the last member tends to zero as $\lambda \to -iq$.

By the Corollary to Lemma 3, the Remark following Lemma 4, and (3.9) of Theorem 4, we have the estimate

$$\|H_n(\rho(\,\cdot\,), \lambda) - G_n(\rho(\,\cdot\,))\|_W \leq \|H_n(\rho(\,\cdot\,), \lambda)\|_W + \|G_n(\rho(\,\cdot\,))\|_W$$

$$\leq 2\rho^{-n/2} \frac{\left[\Gamma\left(\frac{3}{4}\right)\right]^n}{\Gamma\left(\frac{3n}{4}+1\right)} (b-a)^{3n/4} (2\pi)^{-n/4} N_n(f_n)\,.$$

Hence, by the root test and (0.6), we obtain for each $\rho > 0$ the inequality

$$\sum_0^{\infty} \|H_n(\rho(\,\cdot\,), \lambda) - G_n(\rho(\,\cdot\,))\|_W$$

$$\leq \sum_0^{\infty} 2\rho^{-n/2} \frac{\left[\Gamma\left(\frac{3}{4}\right)\right]^n}{\Gamma\left(\frac{3n}{4}+1\right)} (b-a)^{3n/4} (2\pi)^{-n/4} N_n(f_n) < +\infty\,.$$

Since each term $\|H_n(\rho(\,\cdot\,), \lambda) - G_n(\rho(\,\cdot\,))\|_W$ tends to 0 as $\lambda \to -iq$ $(\Re\lambda > 0)$, we see that $\sum_0^\infty \|H_n((\,\cdot\,)\rho, \lambda) - G_n((\,\cdot\,)\rho)\|_W \to 0$ as $\lambda \to -iq$ $(\Re\lambda > 0)$. Thus, $T_q F$ exists and $T_q F \approx G$.

Thus we have shown that $T_q \sum_0^\infty F_n \approx \sum_0^\infty T_q F_n$, and since $T_q F \in \mathscr{S}$, we see, using Theorem 5, that

$$T_{-q} T_q F = T_{-q} T_q \sum_0^\infty F_n \approx T_{-q} \sum_0^\infty T_q F_n \approx \sum_0^\infty T_{-q} T_q F_n \approx \sum_0^\infty F_n = F,$$

and the theorem is proved.

5. THE TRANSFORMATION T_q APPLIED TO ENTIRE FUNCTIONS OF INTEGRALS

We now prove that under mild conditions, functionals of the form (0.7) are elements of $\mathscr{S}$.

THEOREM 7. *Let* $\Phi(z) = \sum_{n=0}^{\infty} a_n z^n$ *be an entire function of order less than four, let* $\theta \in \mathscr{K}_1$, *and let*

$$F(x) \equiv \Phi\left[\int_a^b \theta(t, x(t))\, dt\right].$$

Then $F \in \mathscr{S}$, *and therefore for real* q $(q \neq 0)$, $T_q F \in \mathscr{S}$ *and* $T_{-q} T_q F \approx F$.

Proof. The general term of the series defining F is

$$F_n(x) \equiv a_n\left(\int_a^b \theta(t, x(t))\, dt\right)^n = a_n \int_a^b \overset{(n)}{\cdots} \int_a^b \prod_{j=1}^{n} \theta(t_j, x(t_j))\, dt_1 \cdots dt_n$$

$$= a_n\, n! \int \underset{\Delta_n}{\cdots} \int \prod_{j=1}^{n} \theta(t_j, x(t_j))\, dt_1 \cdots dt_n,$$

where we have used (0.5) and the fact that the product of the functions θ is a symmetric function of $t_1, \cdots, t_n$. Thus

$$F_n(x) = \int \underset{\Delta_n}{\cdots} \int f_n(t_1, \cdots, t_n; x(t_1), \cdots, x(t_n))\, dt_1 \cdots dt_n,$$

where

$$f_n(t_1, \cdots, t_n; u_1, \cdots, u_n) = a_n\, n! \prod_{j=1}^{n} \theta(t_j, u_j).$$

Now

$$N_n(f_n) = \sup_{(t_1, \cdots, t_n) \in \Delta_n} \left\| a_n\, n! \prod_{j=1}^{n} \theta(t_j, u_j)\right\|$$

$$\leq \sup_{(t_1, \cdots, t_n) \in \Delta_n} |a_n|\, n! \prod_{j=1}^{n} \|\theta(t_j, \cdot)\| \leq n!\, |a_n|\, [N_1(\theta)]^n < +\infty.$$

If $\varepsilon > 0$, then

$$\frac{n \log n}{\log 1/|a_n|} < 4 - \varepsilon$$

for all sufficiently large n (since Φ is of order less than 4); therefore

$$|a_n| < n^{-n/(4-\varepsilon)}.$$

Using Stirling's formula, we deduce that

$$|N_n(f_n)|^{1/n} \leq [n!\,|a_n|]^{1/n} N_1(\theta) \leq [n!]^{1/n} n^{-1/(4-\varepsilon)} N_1(\theta) = o(n^{3/4}) \quad \text{as } n \to +\infty .$$

Since $F(x) = \sum_{n=0}^{\infty} F_n(x)$, it follows that $F \in \mathscr{S}$, and Theorem 7 now follows from Theorem 6.

Since there is considerable interest in function-space integrals of functionals of the form

$$\exp\left(\int_a^b \theta(t, x(t))\, dt \right),$$

we point out that Theorem 7 applies to functionals of this type.

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