Commun. Math. Phys. 109, 229–248 (1987)

Communications in
Mathematical
Physics

An L^p-Theory for the n-Dimensional, Stationary, Compressible Navier-Stokes Equations, and the Incompressible Limit for Compressible Fluids. The Equilibrium Solutions

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Abstract. In this paper we study the system (1.1), (1.3), which describes the stationary motion of a given amount of a compressible heat conducting, viscous fluid in a bounded domain Ω of R^n, $n \geqq 2$. Here $u(x)$ is the velocity field, $\rho(x)$ is the density of the fluid, $\zeta(x)$ is the absolute temperature, $f(x)$ and $h(x)$ are the assigned external force field and heat sources per unit mass, and $p(\rho, \zeta)$ is the pressure. In the physically significant case one has $g = 0$. We prove that for small data (f, g, h) there exists a unique solution (u, ρ, ζ) of problem (1.1), $(1.3)_1$, in a neighborhood of $(0, m, \zeta_0)$; for arbitrarily large data the stationary solution does not exist in general (see Sect. 5). Moreover, we prove that (for barotropic flows) the stationary solution of the Navier-Stokes equations (1.8) is the incompressible limit of the stationary solutions of the compressible Navier-Stokes equations (1.7), as the Mach number becomes small. Finally, in Sect. 5 we will study the equilibrium solutions for system (4.1). For a more detailed explanation see the introduction.

1. Introduction

In this paper we study the system

$$\begin{cases} -\mu\Delta u - \nu\nabla \operatorname{div} u + \nabla p(\rho,\zeta) = \rho[f - (u\cdot\nabla)u], \\ \operatorname{div}(\rho u) = g, \\ -\chi\Delta\zeta + c_v\rho u\cdot\nabla\zeta + \zeta p'_\zeta(\rho,\zeta)\operatorname{div} u = \rho h + \psi(u,u), \quad \text{in} \quad \Omega, \\ u_{|\Gamma} = 0, \zeta_{|\Gamma} = \zeta_0, \end{cases} \tag{1.1}$$

in a bounded open domain Ω in R^n, for arbitrarily large $n \geqq 2$. It is assumed that Ω lies (locally) on one side of its boundary Γ, a C^2 manifold. Here,

$$\psi(u,u) = \chi_0 \sum_{i,j=1}^{n} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2 + \chi_1(\operatorname{div} u)^2, \tag{1.2}$$

and $(v\cdot\nabla)u = \sum_{i=1}^{n} v_i(\partial u/\partial x_i)$. System (1.1) describes the stationary motion of a compressible, heat-conductive, viscous fluid (see Serrin [10]). In Eq. (1.1), $u(x) = (u_1(x), u_2(x), \ldots, u_n(x))$ is the velocity field, $\rho(x)$ is the density of the fluid, $\zeta(x)$ is the absolute temperature, $f(x)$ and $h(x)$ are the assigned external force field and heat sources per unit mass, and $p(\rho, \zeta)$ is the pressure. In the physically significant case one has $g = 0$; however, it is not without interest, from a mathematical point of view, to study the general case.

In order to avoid technicalities, we will assume that the coefficients $\mu > 0, \nu > -\mu$, $\chi > 0$, c_v, χ_0, χ_1, are constant. A dependence of those coefficients on u, ρ, ζ, as well as the introduction in Eq. (1.1) of other kinds of non-linearities, does not give rise to substantial difficulties. For the same reason, we assume that $\zeta_0 > 0$ is constant.

Since the total mass of fluid is given, we impose the condition

$$\frac{1}{|\Omega|}\int_\Omega \rho(x)dx = m, \quad \text{or equivalently} \quad \frac{1}{|\Omega|}\int_\Omega \sigma(x)dx = 0, \tag{1.3}$$

where $m > 0$ is given, and $\sigma(x)$ is defined by setting $\rho(x) \equiv m + \sigma(x)$.

The function $p(\rho, \zeta)$ is defined, and has Lipschitz continuous first order partial derivatives p'_ρ and p'_ζ in a neighborhood $[m - l, m + l] \times [\zeta_0 - l_1, \zeta_0 + l_1]$ of (m, ζ_0), where $0 < l \leqq m/2$, $0 < l_1 \leqq \zeta_0/2$. Consequently, we can write

$$\begin{cases} p'_\rho(m + \sigma, \zeta_0 + \alpha) = k + \omega_1(\sigma, \alpha), \\ p'_\zeta(m + \sigma, \zeta_0 + \alpha) = \omega_2(\sigma, \alpha), \end{cases} \tag{1.4}$$

where $k = p'_\rho(m, \zeta_0)$, $\omega_1(0,0) = 0$, and ω_1, ω_2 are Lipschitz continuous in $I(l, l_1) \equiv [-l, l] \times [-l_1, l_1]$. We assume that $k > 0$ (in fact, $k \neq 0$ would be sufficient here).

By setting

$$\rho = m + \sigma, \quad \zeta = \zeta_0 + \alpha,$$

we write system (1.1) in the equivalent form

$$\begin{cases} -\mu\Delta u - \nu\nabla \operatorname{div} u + k\nabla\sigma = F(f, u, \sigma, \alpha), \\ m \operatorname{div} u + u\cdot\nabla\sigma + \sigma \operatorname{div} u = g, \\ -\chi\Delta\alpha = H(h, u, \sigma, \alpha), \quad \text{in} \quad \Omega, \\ u_{|\Gamma} = 0, \quad \alpha_{|\Gamma} = 0, \end{cases} \tag{1.5}$$

where, by definition,

$$\begin{cases} F(f, u, \sigma, \alpha) = (\sigma + m)[f - (u\cdot\nabla)u] - \omega_1(\sigma, \alpha)\nabla\sigma - \omega_2(\sigma, \alpha)\nabla\alpha, \\ H(h, u, \sigma, \alpha) = (\sigma + m)h - c_v(m + \sigma)u\cdot\nabla\alpha + \psi(u, u) + \dfrac{\zeta_0 + \alpha}{m + \sigma}\omega_2(\sigma, \alpha)(u\cdot\nabla\sigma - g). \end{cases} \tag{1.6}$$

Note that Eq. $(1.5)_2$ is used in deducing the expression H.

Let $p > n$ be fixed. We prove that problem (1.1), (1.3) has a unique solution $(u, \rho, \zeta) \in W^{2,p} \times W^{1,p} \times W^{2,p}$ in a neighborhood of $(0, m, \zeta_0)$ provided the data

$(f, g, h) \in L^p \times \tilde{W}_0^{1,p} \times L^p$ belong to a suitable neighborhood of the origin. See Theorem 3.1. This results can be generalized as follows:

Theorem 1.1. *Let $p \in]1, +\infty[$ and $j \geqq -1$, verify $j+2 > n/p$. Assume that $\Gamma \in C^{3+j}$ and $p(\rho, \zeta) \in C^{3+j}(I(l, l_1))$. If $(f, g, h) \in W^{j+1,p} \times \tilde{W}_0^{j+2,p} \times W^{j+1,p}$ verify the condition*

$$\| f \|_{j+1,p} + \| g \|_{j+2,p} + \| h \|_{j+1,p} \leqq c'_0,$$

then there exists a unique solution $(u, \rho, \zeta) \in W_0^{j+3,p} \times W^{j+2,p} \times W^{j+3,p}$ of problem (1.1), (1.3) *in the ball*

$$\| u \|_{j+3,p} + \| \rho - m \|_{j+2,p} + \| \zeta - \zeta_0 \|_{j+3,p} \leqq c'_1.$$

In the above statement c'_0, c'_1 are suitable positive constants depending only on Ω, $n, p, j, \mu, v, k, m, \zeta_0, c_v, \chi, \chi_0, l, l_1, T_i, S_i (i = 1, 2)$, where $T_i = \sup |\omega_i(\sigma, \alpha)|$ for $(\sigma, \alpha) \in I(l, l_1)$, and S_i is the norm of $\omega_i(\sigma, \alpha)$ in the space $C^{2+j}(I(l, l_1))$.

We recall that for the incompressible Navier–Stokes equations a corresponding result is well known. In fact, for the linear Stokes problem (which is an elliptic system) the result is classical (see [1, 4]). For the nonlinear problem the result follows immediately by using the estimates for the linear Stokes problem together with the Schauder fixed point theorem.

The L^p-theory enables us to treat the n-dimensional case by handling only derivatives of order less than or equal to two (case $j = -1$). Nevertheless, our proof applies as well to the case $j > -1$, without further difficulties or technical calculations, as shown in the last part of Sect. 3; the details will be given in a forthcoming note, by A. Defranceschi [14]. To be more specific, here we will concentrate our attention on the case $j = -1$, since our proof turns out to be (slightly) more complicated just for this case.

Section 4 is concerned with barotropic motions, described by system (4.1). The main goal of this section is the study of the incompressible limit of a family of compressible barotropic flows. We assume that $p_\lambda(\rho)$ is a family of state functions depending on a parameter λ, such that $p'_\lambda(m) \to +\infty$ (i.e., the Mach number becomes small) as $\lambda \to +\infty$. We refer to [5, 6] for the physical justification of the assumptions done in Sect. 4. We will prove that the solution (u_∞, m) of the incompressible Navier–Stokes equations (1.8) is the limit of the solutions $(u_\lambda, \rho_\lambda)$ of the compressible Navier–Stokes systems of Eq. (1.7), as $\lambda \to +\infty$. Moreover, $p_\lambda(\rho_\lambda(x)) \to \pi(x)$, as $\lambda \to +\infty$.

More precisely, we prove in Sect. 4 the following result:

Theorem 1.2. *Let $p > n$ be fixed, and let the assumptions (done in Sect. 4) on the family of state functions $p_\lambda(\rho)$ hold. Then, there exist positive constants c'_8, c'_9, depending at most on $\Omega, n, p, \mu, v, m, l, \phi, k_0$, such that if $f \in L^p, |f|_p \leqq c'_8$, then the following statements hold:*

(i) *for each $\lambda \geqslant \lambda_0$, the problem*

$$\begin{cases} -\mu \Delta u_\lambda - v \nabla \operatorname{div} u_\lambda + \nabla p_\lambda(\rho_\lambda) = \rho_\lambda [f - (u_\lambda \cdot \nabla) u_\lambda], \\ \operatorname{div}(\rho_\lambda u_\lambda) = 0, \quad in \quad \Omega, \\ (u_\lambda)_{|\Gamma} = 0, \quad \bar{\rho}_\lambda = m, \end{cases} \tag{1.7}$$

has a unique solution $(u_\lambda, \rho_\lambda) \in W_0^{2,p} \times W^{1,p}$ *in the ball* $\|u_\lambda\|_{2,p} \leqq c_9'$, $\|\rho_\lambda - m\|_{1,p} \leqq c_9'/k_\lambda$.

(ii) *If* $\lim k_\lambda = +\infty$ *as* $\lambda \to +\infty$, *then* $u_\lambda \to u_\infty$ *weakly in* $W_0^{2,p}$, $\operatorname{div} u_\lambda \to 0$ *weakly in* $W_0^{1,p}$, $\rho_\lambda \to m$ *strongly in* $W^{1,p}$, $\nabla p_\lambda(\rho_\lambda) \to \nabla\pi$ *weakly in* L^p, *where* $(u_\infty, \nabla\pi)$ *is the unique solution of the incompressible Navier–Stokes equations*

$$\begin{cases} -\mu\Delta u_\infty + \nabla\pi(x) = m[f - (u_\infty\cdot\nabla)u_\infty], \\ \operatorname{div} u_\infty = 0, \quad in \quad \Omega, \\ (u_\infty)_{|\Gamma} = 0. \end{cases} \tag{1.8}$$

A similar result was proved for $n \leqq 3$, $j = 1$, $p = 2$, in ref. 2. The proof given here applies as well to the case $W^{j,p}$, if $j + 2 > n/p$, $\Gamma \in C^{3+j}$, $p(\rho) \in C^{3+j}$. The details will be given in the forthcoming note, referred to above.

Finally, in Sect. 5 we will study the existence of equilibrium solutions (solutions such that $u(x) = 0, \forall x \in \Omega$), for arbitrarily *large* external forces $f = \nabla F$. In particular, we will show that if $p(\rho) = R\rho^\gamma$, $R > 0, \gamma > 0, \gamma \neq 1$, the equilibrium solution does not exists in general (even for a constant external force f). If $\gamma = 1$, then the equilibrium solution exists for every bounded potential $F(x)$.

For small external forces, it was prove in [11] that the stationary solutions are stable (for the equilibrium solutions this was proved in [7]). It would be interesting to study the stability of the equilibrium solutions for large external forces, even if $n = 1$.

Some Considerations. It is worth noting that the core of this paper is the study of the linear system (2.1); see Theorems 2.1 and 3.3. In this system the unknowns are u and σ, the vector field v being fixed. Due to the term $v\cdot\nabla\sigma$, system (2.1) is not an elliptic system in the sense of Agmon, Douglis, and Nirenberg [1; part II], except if $v(x)$ vanishes identically in Ω. In fact (assume, for simplicity, that $\mu = k = m = 1$, and $\nu = 0$), consider in Ω the system $-\Delta u + \nabla\sigma = F, \operatorname{div} u + v\cdot\nabla\sigma + a(x)\sigma = g$. By using the notations of [1; part II] one has: $L(x,\xi) = v(x)\cdot\xi|\xi|^{2n}$ if $v(x) \neq 0$; $L(x,\xi) = (1 + a(x))|\xi|^{2n}$ if $v(x) = 0$. Consequently, the ellipticity condition (see [1], Eq. (1.5)) "$L(x,\xi) \neq 0$ for real $\xi \neq 0$", cannot be satisfied unless $v(x) = 0$, for all $x \in \Omega$.

If $v = 0$ and if a is small enough (for instance, if $|a|_\infty < 1$), the system is elliptic. However, this last property cannot be used to treat the term $v\cdot\nabla\sigma$ as a perturbation term. In fact, by assuming that $\sigma \in W^{1,p}$, the term $v\cdot\nabla\sigma$ belongs just to L^p. In this situation, equation $\operatorname{div} u + v\cdot\nabla\sigma + a(x)\sigma = g$ yields $\operatorname{div} u \in L^p$, and equation $-\Delta u + \nabla\sigma = F$ gives $u \in W^{1,p}$, $\sigma \in L^p$. Hence, from the point of view of regularity, we loses one derivative.

We also point out that the evolution problem is easier to solve due to the presence of the term $(\partial\sigma/\partial t) + v\cdot\nabla\sigma$ in the evolution counterpart of Eq. $(2.1)_2$.

Let us briefly explain the main ideas utilized in the sequel (see also [2]) to solve the linear system (2.1): By applying the divergence operator to both sides of Eq. $(2.1)_1$ one shows that $(\mu + \nu)\Delta \operatorname{div} u = k\Delta\sigma - \operatorname{div} F$, and by applying the Laplace operator to both sides of Eq. $(2.1)_2$ one gets

$$\frac{mk}{\mu + \nu}\Delta\sigma + v\cdot\nabla\Delta\sigma = G(F, g, \sigma), \tag{1.9}$$

where $G(F,g,\sigma)$ is given by (2.5). In order to be able to solve Eq. (1.9) for $\Delta\sigma$, we replace $G(F,g,\sigma)$ by $G(F,g,\tau)$, where $\tau\in\bar{W}^{1,p}$ is an arbitrary function. This yields Eq. (2.4), where (heuristically) λ should be regarded as $\Delta\sigma$ (actually $\lambda=\Delta\sigma$, if $\tau=\sigma$ is a solution of (2.1)). Once λ is known, we want to determine $\operatorname{div} u$. By applying the divergence operator to both sides of $(2.1)_1$, we get Eq. $(2.10)_1$, where (heuristically) λ should be regarded as $\operatorname{div} u$ (we will show that $\theta=\operatorname{div} u$, in the event that we have a fixed point $\sigma=\tau$). The remarkable property $\operatorname{div} u=0$ on Γ[1] suggest us to impose on θ the boundary condition $(2.10)_2$. Once θ is known, Eq. $(2.1)_1$ gives u and σ, by solving the Stokes linear problem (2.13). In Eq. (2.13) we replace $\theta(x)$ by $\theta_0(x)=\theta(x)-\bar{\theta}$, where $\bar{\theta}$ is the mean value of $\theta(x)$, since the compatibility condition $\bar{\theta}_0=0$ is required here.

For each fixed pair (F,g) the above sequence of maps define a map $\tau\to\sigma$. We will prove that this map is a contraction (in a suitable ball), and that the pair (u,σ), corresponding to the fixed point $\sigma=\tau$, is a solution of (2.11).

We end this section by calling the reader's attention to the following papers, in which results directly related with those of Sect. 3 can be found: M. Padula [8], A. Valli [11], A. Valli and W. Zajaczkowski [13], H. Beirão da Veiga [2], A. Valli [12].

The corresponding evolution problem has been studied by many authors. Here, we mention only a sequence of papers by Matsumura and Nishida (see [7], and references) and Valli's paper [11].

To readers interested in the incompressible limit of compressible fluids, we suggest papers by Klainerman and Majda [5], Majda [6], and Schochet [9].

2. The Linearized System

In this section we study the linear system

$$\begin{cases} -\mu\Delta u-\nu\nabla\operatorname{div} u+k\nabla\sigma=F(x), \\ m\operatorname{div} u+v\cdot\nabla\sigma+\sigma\operatorname{div} v=g, \quad \text{in} \quad \Omega, \\ u_{|\Gamma}=0, \end{cases} \tag{2.1}$$

by adapting to the L^p-case the method introduced in [2]. We start with some notation. We denote by $W^{j,p}$, j an integer, $1<p<+\infty$, the Sobolev space $W^{j,p}(\Omega)$, endowed with the usual norm $\|\quad\|_{j,p}$, and by $|\quad|_p$, $1\leqq p\leqq+\infty$, the usual norm in $L^p=L^p(\Omega)$. Hence, $\|\quad\|_{0,p}=|\quad|_p$. For convenience, we also use the symbol $W^{j,p}$ to denote the space of vector fields v in Ω such that $v_i\in W^{j,p}(\Omega)$, $i=1,2,\dots,n$. This convention applies to all the functional spaces and norms utilized here. For $j\geqq 1$ we define $W_0^{j,p}=\{v\in W^{j,p}: v=0 \text{ on } \Gamma\}$. Note that $W_0^{j,p}=W^{j,p}\cap W_0^{1,p}$, is not the closure of $\mathscr{D}(\Omega)=C_0^\infty(\Omega)$. Furthermore, we set $\bar{W}^{j,p}=\{\tau\in W^{j,p}: \bar{\tau}=0\}$, $\bar{W}_0^{j,p}=W_0^{j,p}\cap\bar{W}^{j,p}$, $j\geqq 1$, where in general $\bar{\phi}$ denotes the mean value of ϕ (x) in Ω. Finally, for vector fields, we defined $W_{0,d}^{j,p}=\{v\in W_0^{j,p}: \operatorname{div} v=0 \text{ on } \Gamma\}$, $j\geqq 2$.

With the only exception for Sect. 5, $c, c_i, i\geqq 0$, will denote positive constants depending at most on Ω, n, p. The symbol c may be utilized (even in the same equation) to indicate distinct constants. In Sect. 5, c will denote an integration constant.

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1 Equation $(1.5)_2$ implies that every stationary solution u must verify the equation $\operatorname{div} u=0$ on Γ

We denote by $q=p/(p-1)$ the dual exponent of p, and by r^* the Sobolev embedding exponent $r^*=(n-r)/nr$. Recall that $W^{1,p} \subsetneq L^\infty$, if $p>n$, and $W^{1,p} \subsetneq L^{p^*}$, if $1 \leqq p < n$.

Now we state the main result in this section (see also Theorem 3.3):

Theorem 2.1. *Let $p>n$ be fixed, and let $F \in L^p$, $g \in \bar{W}_0^{1,p}$. There exist positive constants c and γ, γ defined by Eq. (2.22), such that if $v \in W_{0,d}^{2,p}$ verifies the assumption*

$$\|v\|_{2,p} \leqq \gamma k, \tag{2.2}$$

then there exists a unique solution $(u,\sigma) \in W_{0,d}^{2,p} \times \bar{W}^{1,p}$ of problem (2.1). *Moreover,*

$$\mu \|u\|_{2,p} + k\|\sigma\|_{1,p} \leqq c\left(1+\frac{\mu+|v|}{\mu+\nu}\right)|F|_p + c\frac{\mu+|\nu|}{m}\|g\|_{1,p}. \tag{2.3}$$

Proof. Let $\tau \in \bar{W}^{1,p}$, and consider the linear problem

$$\frac{mk}{\mu+\nu}\lambda + v\cdot\nabla\lambda = G(F,g,\tau), \tag{2.4}$$

where by definition

$$G(F,g,\tau) = \Delta g + \frac{m}{\mu+\nu}\operatorname{div} F - [2\nabla v:\nabla^2\tau + \Delta v\cdot\nabla\tau + \Delta(\tau \operatorname{div} v)]. \tag{2.5}$$

Here, $\nabla v:\nabla^2\tau = \sum(\partial v_i/\partial x_k)(\partial^2\tau/\partial x_i\partial x_k)$. Note that G is a linear map from $L^p \times \bar{W}_0^{1,p} \times \bar{W}^{1,p}$ into $W^{-1,p}$ (the dual space of $W_0^{1,q}$), and that

$$\|G\|_{-1,p} \leqq \|g\|_{1,p} + \frac{m}{\mu+\nu}|F|_p + c\|v\|_{2,p}\|\tau\|_{1,p}. \tag{2.6}$$

In [3] we prove that there exists a positive constant c_1 such that if $\|v\|_{2,p} \leqq c_1\, mk/(\mu+\nu)$, then there exists a linear continuous map $L: W^{-1,p} \to W^{-1,p}$ such that $\lambda = LG$ is a weak solution of Eq. (2.4), for every $G \in W^{-1,p}$. Moreover

$$\frac{mk}{\mu+\nu}\|\lambda\|_{-1,p} \leqq c\|G\|_{-1,p}. \tag{2.7}$$

By a weak solution of (2.4) (see [3] for details), we mean a distribution $\lambda \in W^{-1,p}$ such that

$$\left\langle \frac{mk}{\mu+\nu}\phi - \operatorname{div}(\phi v), \lambda \right\rangle = \langle \phi, G\rangle, \ \forall \phi \in \mathscr{D}(\Omega). \tag{2.8}$$

Here, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $W_0^{1,p}$ and $W^{-1,p}$. The above result yields

$$\frac{mk}{\mu+\nu}\|\lambda\|_{-1,p} \leqq c\left(\frac{m}{\mu+\nu}|F|_p + \|g\|_{1,p} + \|v\|_{2,p}\|\tau\|_{1,p}\right). \tag{2.9}$$

Now let $\theta \in W_0^{1,p}$ be the solution of the Dirichlet problem

$$\begin{cases}(\mu+\nu)\Delta\theta = k\lambda - \operatorname{div} F & \text{in } \Omega,\\ \theta_{|\Gamma} = 0.\end{cases} \tag{2.10}$$

Clearly,

$$(\mu+\nu)\|\theta\|_{1,p} \leqq c|F|_p + c\frac{\mu+\nu}{m}(\|g\|_{1,p} + \|v\|_{2,p}\|\tau\|_{1,p}). \tag{2.11}$$

Next, define

$$\theta_0(x) = \theta(x) - \bar{\theta}. \tag{2.12}$$

Since $\bar{\theta}_0 = 0$, there exists a unique solution $(u,\sigma) \in W_0^{2,p} \times \bar{W}^{1,p}$ of the linear Stokes problem

$$\begin{cases} -\mu\Delta u + k\nabla\sigma = F + \nu\nabla\theta_0, \\ \operatorname{div} u = \theta_0, \quad \text{in} \quad \Omega, \\ u_{|\Gamma} = 0. \end{cases} \tag{2.13}$$

Moreover (see [1,4])

$$\mu\|u\|_{2,p} + k\|\sigma\|_{1,p} \leqq c(|F|_p + (\mu+|\nu|)\|\theta_0\|_{1,p}). \tag{2.14}$$

Since (2.2) holds, and $\|\theta_0\|_{1,p} \leqq c\|\theta\|_{1,p}$, one easily gets

$$\mu\|u\|_{2,p} + k\|\sigma\|_{1,p} \leqq \frac{k}{2}\|\tau\|_{1,p} + c\left(1 + \frac{\mu+|\nu|}{\mu+\nu}\right)|F|_p + c_2\frac{\mu+|\nu|}{m}\|g\|_{1,p}. \tag{2.15}$$

At this point, we call attention to the sequence of linear maps

$$(F,g,\tau) \to (F,G) \to (F,\lambda) \to (F,\theta) \to (F,\theta_0) \to (u,\sigma),$$

where F is left unchanged, and the elements $G, \lambda, \theta, \theta_0, (u,\sigma)$, are defined by Eqs. (2.5), (2.4), (2.10), (2.12), and (2.13), respectively. The product map is linear and continuous, by (2.15). Hence, if (u_1,σ_1) is the solution corresponding to the data (F,g,τ_1), it follows that $(u-u_1,\sigma-\sigma_1)$ is the solution corresponding to the data $(0,0,\tau-\tau_1)$. Consequently, $\|\sigma-\sigma_1\|_{1,p} \leqq (1/2)\|\tau-\tau_1\|_{1,p}$, and the map $\tau \to \sigma$ is a contraction in $\bar{W}^{1,p}$. Hence, it has a (unique) fixed point $\sigma = \tau$.

Now, we prove that the pair (u,σ), corresponding to the fixed point $\sigma = \tau$ is a solutions of (2.1). The main point is to prove $(2.1)_2$, since $(2.1)_1$ and $(2.1)_3$ follow immediately from (2.13). From $(2.10)_1$ we get

$$\lambda = \frac{\mu+\nu}{k}\Delta \operatorname{div} u + \frac{1}{k}\operatorname{div} F, \tag{2.16}$$

since $\Delta\theta = \Delta\theta_0 = \Delta \operatorname{div} u$. On the other hand, by applying the divergence operator to both sides of Eq. $(2.13)_1$, and by using $(2.13)_2$ and (2.16), we show that $\lambda = \Delta\sigma$. Now, we replace λ by the right-hand side of (2.16) in the first term on the left-hand side of (2.8), and by $\Delta\sigma$ in the second term on the left-hand side of (2.8). This yields

$$\begin{aligned} &\langle \phi, m\Delta \operatorname{div} u\rangle - \langle \operatorname{div}(\phi v), \Delta\sigma\rangle \\ &\quad = \langle \phi, \Delta g - 2\nabla v{:}\nabla^2\sigma - \Delta v\cdot\nabla\sigma - \Delta(\sigma \operatorname{div} v)\rangle, \quad \forall \phi \in \mathscr{D}(\Omega). \end{aligned} \tag{2.17}$$

We claim that

$$-\langle \operatorname{div}(\phi v), \Delta\sigma\rangle + \langle \phi, 2\nabla v{:}\nabla^2\sigma + \Delta v\cdot\nabla\sigma\rangle = \langle \phi, \Delta(v\cdot\nabla\sigma)\rangle, \forall \phi \in \mathscr{D}(\Omega), \tag{2.18}$$

where the pairing on the right-hand side denotes the duality between $\mathscr{D}(\Omega)$ and $\mathscr{D}'(\Omega)$. Let $\sigma_n \in C^3(\Omega), \sigma_n \to \sigma$ in $W^{1,p}(\Omega)$, as $n \to +\infty$. The identity (2.18), for the functions σ_n, follows from the formulae $\Delta(v \cdot \nabla \sigma_n) = v \cdot \nabla \Delta \sigma_n + 2\nabla v : \nabla^2 \sigma_n + \Delta v \cdot \nabla \sigma_n$. By passing to the limit as $n \to +\infty$, we prove (2.18), for σ.

From (2.17) and (2.18) we get

$$\Delta(m \operatorname{div} u + m\bar{\theta} + v \cdot \nabla \sigma + \sigma \operatorname{div} v - g) = 0, \tag{2.19}$$

in Ω. Assume that one has

$$m \operatorname{div} u + m\bar{\theta} + v \cdot \nabla \sigma + \sigma \operatorname{div} v - g = 0 \tag{2.20}$$

in Ω. Then, by integrating in Ω both sides of this equation, one easily concludes that $\bar{\theta} = 0$. Hence $(2.1)_2$ holds, as desired. Let us prove (2.20)[2]. Set $\Omega_\varepsilon = \{z \in \Omega : \operatorname{dist}(z, \Gamma) > \varepsilon\}$, $\Gamma_\varepsilon = \{z \in \Omega : \operatorname{dist}(z, \Gamma) = \varepsilon\}$, where $\varepsilon > 0$ is assumed to be enough small so that Γ_ε is a regular manifold. Let $g_\varepsilon(x, y)$ be the Green's function for the Dirichlet problem in Ω_ε, and let $x \in \Omega$ be *fixed*. We want to prove (2.20) at x. For $\varepsilon \in]0, (1/2)\operatorname{dist}(x, \Gamma)[$ one has $|\nabla g_\varepsilon(x, y)| \leqq K, \forall y \in \Gamma_\varepsilon$, where K does not depend on ε. Let U denote the left-hand side of (2.20). Since U is harmonic in Ω_ε, one has $U(x) = \int_{\Gamma_\varepsilon} (\partial g_\varepsilon(x, y)/\partial n_y)\, U(y) d\Gamma_\varepsilon(y)$, where n_y denotes the outward normal to Γ_ε at y. This shows that $|U(x)| \leqq K \int_{\Gamma_\varepsilon} |U(y)| d\Gamma_\varepsilon(y)$. Consequently, if we show that $\liminf \int_{\Gamma_\varepsilon} |U(y)| d\Gamma_\varepsilon(y) = 0$, as $\varepsilon \to 0$, it must be that $U(x) = 0$. Actually, it is sufficient to verify that

$$\liminf_{\varepsilon \to 0} \operatorname{ess} \int_{\Gamma_\varepsilon} |v \cdot \nabla \sigma| d\Gamma_\varepsilon = 0, \tag{2.21}$$

since the function $m \operatorname{div} u + m\bar{\theta} + \sigma \operatorname{div} v - g \in W_0^{1,p}(\Omega)$ converges uniformly to 0 as $\operatorname{dist}(y, \Gamma) \to 0$.

Assume by contradiction, that there exist positive constants ε_0 and δ such that the integral on the left-hand side of (2.21) is greater than δ, for a.a. $\varepsilon \in]0, \varepsilon_0[$. Denoting by K_0 the Lipschitz constant of v in $\bar{\Omega}$, and recalling that v vanishes on Γ, it follows that $\int |\nabla \sigma| d\Gamma_\varepsilon \geqq \delta/(K_0 \varepsilon)$, for almost all $\varepsilon \in]0, \varepsilon_0[$. This contradicts the estimate

$$\int_0^{\varepsilon_0} \left(\int_{\Gamma_\varepsilon} |\nabla \sigma| d\Gamma_\varepsilon \right) d\varepsilon = \int_{\Omega/\Omega_{\varepsilon_0}} |\nabla \sigma| dy < +\infty.$$

The existence part of Theorem 2.1 is completely proved. As in [2], Sect. 2, we show that there exists a constant c_0 such that the solution of (2.1) is unique if $\|v\|_{2,p} \leqq (m\mu_0 k)/[c_0(\mu + |v|)^2]$, where $\mu_0 = \min\{\mu, \mu + v\}$.

For convenience, we set

$$\gamma = \min\left\{ \frac{c_1 m}{\mu + v}, \frac{m}{2c_2(\mu + |v|)}, \frac{m\mu_0}{c_0(\mu + |v|)^2} \right\}. \tag{2.22}$$

---

2 In case that $j \geqq 0$ (see Theorem 1.1) the proof of (2.20) is immediate, since the left-hand side of (2.20) is harmonic in Ω and vanishes on Γ(since $\operatorname{div} u = -\bar{\theta}, v = 0, \operatorname{div} v = g = 0$, on Γ). However, if $j = -1$, this argument has to be carefully handled, since the function $\nabla \sigma$ has not a trace on Γ

Then, inequality (2.2) includes all the assumptions on $\|v\|_{2,p}$ utilized in proving Theorem 2.1.

Remark 2.2. Define the linear operator A_v by

$$A_v(u,\sigma) = (-\mu\Delta u - v\nabla \operatorname{div} u + k\nabla\sigma, m \operatorname{div} u + v\cdot\nabla\sigma + \sigma \operatorname{div} v), \tag{2.23}$$

where v is as in Theorem 2.1, and the domain of A_v is

$$D(A_v) = \{(u,\sigma)\in W^{2,p}_{0,d} \times \bar{W}^{1,p}: v\cdot\nabla\sigma\in W^{1,p}_0\}. \tag{2.24}$$

By endowing $D(A_v)$ with the norm

$$\|u\|_{2,p} + \|\sigma\|_{1,p} + \|v\cdot\nabla\sigma\|_{1,p}, \tag{2.25}$$

$D(A_v)$ becomes a Banach space. Theorem 2.1 proves that A_v maps $D(A_v)$ homeomorphically onto $L^p \times \bar{W}^{1,p}_0$.

Note that, in general, $D(A_v) \neq D(A_w)$, if $v \neq w$.

3. The Non-linear Problem

In this section we prove the existence and uniqueness Theorem 1.1 in case that $j = -1$, i.e. we prove Theorem 3.1 below. We close the Section by showing how to adapt the proofs, in case that $j \geqq 0$.

Recalling (1.4), we set

$$T_i = \sup_{I(l,l_1)} |\omega_i(\sigma,\alpha)|, \quad S_i = \sup_{I(l,l_1)} \frac{|\omega_i(\sigma,\alpha) - \omega_i(\tau,\beta)|}{|(\sigma,\alpha)-(\tau,\beta)|}, \quad i = 1,2.$$

Moreover, we set $\tilde{\omega}_1(t,s) = \sup|\omega_1(\sigma,\alpha)|$, for $|\sigma| \leqq t, |\alpha| \leqq s$. We denote by c_3 a positive constant such that

$$|\tau|_\infty \leqq c_3\|\tau\|_{1,p}, \quad |\beta|_\infty \leqq c_3\|\beta\|_{1,p}, \quad \forall\tau\in\bar{W}^{1,p}, \quad \forall\beta\in W^{1,p}_0. \tag{3.1}$$

For convenience, *in this section* we denote by $c', c'_i, i \geqq 0$, positive constants depending at most on $\Omega, n, p, \mu, v, k, m, \zeta_0, c_v, \chi, \chi_0, \chi_1, l, l_1, T_i, S_i$. The symbol c' may be utilized, even in the same equation, to denote distinct constants.

One has the following result[3]:

Theorem 3.1. *Let $p > n$ be fixed. There exist constants c'_0, c'_1, such that if $f\in L^p, g\in\bar{W}^{1,p}_0, h\in L^p$, and*

$$|f|_p + \|g\|_{1,p} + |h|_p \leqq c'_0, \tag{3.2}$$

there exists a unique solution $(u,\sigma,\alpha)\in W^{2,p}_0 \times \bar{W}^{1,p} \times W^{2,p}_0$ of problem (1.5) *in the ball*

$$\|u\|_{2,p} + \|\sigma\|_{1,p} + \|\alpha\|_{2,p} \leqq c'_1. \tag{3.3}$$

Hence, there exists a unique solution $(u,\rho,\zeta)\in W^{2,p}_0 \times W^{1,p} \times W^{2,p}$ of problem (1.1), (1.3), *in the corresponding neighborhood of* $(0,m,\zeta_0)$.

3 We assume here that $p(\rho,\zeta)$ has Lipschitz continuous first order derivatives. However, Theorem 3.1 still holds if $p(\rho,\zeta)$ is only assumed to be continuously differentiable

In order to prove Theorem 3.1 we state some auxiliary results. The verification of the following statement is immediate, and is left to the reader:

Lemma 3.1. *Let* $\tau \in \bar{W}^{1,p}, \beta \in W_0^{1,p}$, *verify the assumptions*

$$\|\tau\|_{1,p} \leqq \frac{l}{c_3}, \quad \|\beta\|_{1,p} \leqq \frac{l_1}{c_3}, \tag{3.4}$$

and let $v \in W^{2,p}$, $f \in L^p, g \in \bar{W}_0^{1,p}, h \in L^p$. *Then*

$$\begin{cases} |F(f, v, \tau, \beta)|_p \leqq \frac{3}{2} m(|f|_p + c\|v\|_{1,p}^2) + \tilde{\omega}_1(|\tau|_\infty, |\beta|_\infty)\|\tau\|_{1,p} + T_2\|\beta\|_{1,p}, \\ |H(h, v, \tau, \beta)|_p \leqq \frac{3}{2} m|h|_p + c\,c_v m\|v\|_{1,p}\|\beta\|_{1,p} \\ \quad + c(\chi_0 + \chi_1)\|v\|_{1,p}\|v\|_{2,p} + c\dfrac{\zeta_0 T_2}{m}(\|v\|_{2,p}\|\tau\|_{1,p} + |g|_p). \end{cases} \tag{3.5}$$

Theorem 2.1, Lemma 3.1, and classical results for the Dirichlet problem (3.7), yield the following statement:

Theorem 3.2. *Let* $v \in W_{0,d}^{2,p}$ *verify* (2.2), *and assume that the hypotheses in Lemma* 3.1 *hold. Then, there exists a unique solution* $(u, \sigma) \in W_{0,d}^{2,p} \times \bar{W}^{1,p}$ *of problem*

$$\begin{cases} -\mu \Delta u - \nu \nabla \operatorname{div} u + k \nabla \sigma = F(f, v, \tau, \beta), \\ m \operatorname{div} u + v \cdot \nabla \sigma + \sigma \operatorname{div} v = g, \quad \text{in} \quad \Omega, \\ u_{|\Gamma} = 0, \end{cases} \tag{3.6}$$

and a unique solution $\alpha \in W_0^{2,p}$ *of the problem*

$$\begin{cases} -\chi \Delta \alpha = H(h, v, \tau, \beta), \quad \text{in} \quad \Omega, \\ \alpha_{|\Gamma} = 0. \end{cases} \tag{3.7}$$

Moreover,

$$\mu\|u\|_{2,p} + k\|\sigma\|_{1,p} \leqq c\left(1 + \frac{\mu + |\nu|}{\mu + \nu}\right)[m|f|_p + \|v\|_{1,p}^2 + \tilde{\omega}_1(|\tau|_\infty, |\beta|_\infty)\|\tau\|_{1,p} + T_2\|\beta\|_{1,p}] + c\frac{\mu + |\nu|}{m}\|g\|_{1,p}, \tag{3.8}$$

$$\chi\|\alpha\|_{2,p} \leqq c|h|_p + cc_v m\|v\|_{1,p}\|\beta\|_{1,p} + c(\chi_0 + \chi_1)\|v\|_{2,p}^2 + c\frac{\zeta_0 T_2}{m}(\|v\|_{2,p}\|\tau\|_{1,p} + |g|_p). \tag{3.9}$$

In the sequel, the solution (u, σ, α) of problems (3.6), (3.7) with data (v, τ, β) is denoted by $(u, \sigma, \alpha) = T(v, \tau, \beta)$. Let us write (3.8), (3.9) in the abbreviated form

$$\begin{cases} \|u\|_{2,p} + \|\sigma\|_{1,p} \leqq c_2'(|f|_p + \|g\|_{1,p} + \|v\|_{2,p}^2 + \|\tau\|_{1,p}^2 + \|\beta\|_{1,p}^2) + c_3'\|\beta\|_{1,p}, \\ \|\alpha\|_{2,p} \leqq c_2'(|h|_p + \|g\|_{1,p} + \|v\|_{2,p}^2 + \|\tau\|_{1,p}^2 + \|\beta\|_{1,p}^2). \end{cases} \tag{3.10}$$

Set

$$r_0 = \min\left\{\gamma k, \frac{l}{c_3}, \left(\frac{l_1}{5c_2' c_3}\right)^{1/2}, \frac{1}{25c_2'}, \frac{1}{5c_2'(1+c_3')}\right\}, \tag{3.11}$$

let $r \in]0, r_0]$, and assume that

$$|f|_p \leqq r^2, \quad |h|_p \leqq r^2, \quad \|g\|_{1,p} \leqq r^2, \tag{3.12}$$

and that

$$\|v\|_{2,p} \leqq r, \quad \|\tau\|_{1,p} \leqq r, \quad \|\beta\|_{1,p} \leqq 5c_2' r^2. \tag{3.13}$$

Under these assumptions, the conditions (2.2), (3.4) are fulfilled. Consequently, from (3.10) it follows that $(u, \sigma, \alpha) = T(v, \tau, \beta)$ verifies

$$\|u\|_{2,p} \leqq r, \quad \|\sigma\|_{1,p} \leqq r, \quad \|\alpha\|_{1,p} \leqq 5c_2' r^2. \tag{3.14}$$

Consequently $T(B_r) \subset B_r$, where B_r is the subset of $W^{2,p}_{0,d} \times \bar{W}^{1,p} \times W^{2,p}_0$ defined by Eqs. (3.13). In order to accomplish the proof of Theorem 3.1, we will show that T is a contraction in B_r, with respect to the norm $W^{1,2}_0 \times L^2 \times W^{1,2}_0$, if r is sufficiently small. It follows then, by the contracting mapping principle, that T has a (unique) fixed point $(u, \sigma, \alpha) = (v, \tau, \beta)$ in B_r (we could also use Schauder's fixed point theorem, since B_r is a convex and compact subset with respect to the $W^{1,2}_0 \times L^2 \times W^{1,2}_0$ topology).

Let $(u, \sigma, \alpha) = T(v, \tau, \beta), (u_1, \sigma_1, \alpha_1) = T(v_1, \tau_1, \beta_1), F = F(f, v, \tau, \beta), F_1 = F(f, v_1, \tau_1, \beta_1)$, $H = H(h, v, \tau, \beta), H_1 = H(h, v_1, \tau_1, \beta_1)$. One has, in Ω,

$$\begin{cases} -\mu\Delta(u-u_1) - \nu\nabla \operatorname{div}(u-u_1) + k\nabla(\sigma-\sigma_1) = F - F_1, \\ m \operatorname{div}(u-u_1) + v_1\cdot\nabla(\sigma-\sigma_1) + (v-v_1)\cdot\nabla\sigma + \sigma_1 \operatorname{div}(v-v_1) + (\sigma-\sigma_1)\operatorname{div} v = 0, \end{cases} \tag{3.15}$$

and

$$-\chi\Delta(\alpha-\alpha_1) = H - H_1. \tag{3.16}$$

Here, we use the notation $\| \quad \|_{k,2} = \| \quad \|_k$. By multiplying both sides of Eq. $(3.15)_1$ by $m(u-u_1)$, both sides of Eq. $(3.15)_2$ by $k(\sigma-\sigma_1)$, by integrating the resulting equations in Ω, and by adding them side by side, we show that

$$\begin{aligned} m(\mu - |v|)\|\nabla(u-u_1)\|_0^2 &\leqq m\|F-F_1\|_{-1}\|u-u_1\|_1 \\ &+ ck\|\sigma_1\|_{1,p}\|v-v_1\|_1\|\sigma-\sigma_1\|_0 + ck\|v\|_{2,p}\|\sigma-\sigma_1\|_0^2. \end{aligned} \tag{3.17}$$

In proving (3.17) we utilize some well known Sobolev inequalities. Note that if $2^* = 2n/(n-2)$, one has $(1/2^*) + (1/2) + (/p) < 1$.

Arguing as on proving Eq. (3.12) in reference [2], we show that

$$\|u-u_1\|_1^2 + c_4'(1 - c_5'\|v\|_{2,p})\|\sigma-\sigma_1\|_0^2 \leqq c'\|\sigma_1\|_{1,p}^2\|v-v_1\|_1^2 + c'\|F-F_1\|_{-1}^2. \tag{3.18}$$

Moreover,

$$\|\alpha-\alpha_1\|_1 \leqq c'\|H-H_1\|_{-1}. \tag{3.19}$$

Straightforward calculations show that (see appendix)

$$\begin{aligned}\|F-F_1\|_{-1} &\leqq c'(|f|_p+\|v_1\|_{1,p}^2)\|\tau-\tau_1\|_0\\ &+c'(1+\|\tau\|_{1,p})(\|v\|_{1,p}+\|v_1\|_{1,p})\|v-v_1\|_1\\ &+c'(\|\tau\|_{1,p}+\|\tau_1\|_{1,p}+\|\beta\|_{1,p}+\|\beta_1\|_{1,p})(\|\tau-\tau_1\|_0+\|\beta-\beta_1\|_0), \qquad (3.20)\end{aligned}$$

and that

$$\begin{aligned}\|H-H_1\|_{-1} &\leqq c'(\|v\|_{1,p}+\|v_1\|_{1,p}+\|\tau\|_{1,p}+\|\beta\|_{1,p})\|v-v_1\|_1\\ &+c'[|h|_p+|g|_p+(\|\beta\|_{1,p}+\|\tau\|_{1,p})\|v\|_{1,p}+(1+\|\tau_1\|_{1,p}+\|\beta_1\|_{1,p}\\ &+\|\tau_1\|_{1,p}^3+\|\beta_1\|_{1,p}^3)\|v_1\|_{1,p}]\|\tau-\tau_1\|_0+c'(|g|_p+\|v\|_{1,p}\|\tau\|_{1,p}\\ &+\|v_1\|_{1,p}\|\tau_1\|_{1,p}+\|v_1\|_{1,p})\|\beta-\beta_1\|_0. \qquad (3.21)\end{aligned}$$

By using (3.12), (3.13) it follows that the coefficients of $\|v-v_1\|_1$, $\|\tau-\tau_1\|_0$, $\|\beta-\beta_1\|_0$, in the right-hand sides of Eqs. (3.20), (3.21) are polynomials on r, vanishing for $r=0$, and with coefficients which are positive constants of type c'. Similarly, the coefficients $c_5'\|v\|_{2,p}$ and $c'\|\sigma_1\|_{1,p}^2$ (that appear in Eq. (3.18)) are polynomials on r of the above type. Since the exact form of these polynomials is not important here, we will denote them by the symbol $\varepsilon=\varepsilon(r)$.

By using the above notation, the estimates (3.18), (3.19), (3.20), (3.21), yield

$$\|u-u_1\|_1^2+(c'-\varepsilon)\|\sigma-\sigma_1\|_0^2\leqq\varepsilon\|v-v_1\|_1^2+\varepsilon\|\tau-\tau_1\|_0^2+\varepsilon\|\beta-\beta_1\|_0^2, \qquad (3.22)$$

$$\|\alpha-\alpha_1\|_1^2\leqq\varepsilon\|v-v_1\|_1^2+\varepsilon\|\tau-\tau_1\|_0^2+\varepsilon\|\beta-\beta_1\|_0^2. \qquad (3.23)$$

Hence,

$$\begin{aligned}&\|u-u_1\|_1^2+(c'-\varepsilon)\|\sigma-\sigma_1\|_0^2+\|\alpha-\alpha_1\|_1^2\\ &\leqq\varepsilon\|v-v_1\|_1^2+\varepsilon\|\tau-\tau_1\|_0^2+\varepsilon\|\beta-\beta_1\|_0^2. \qquad (3.24)\end{aligned}$$

Hence, for a sufficiently small value of r (depending only on $\Omega, n, p, \ldots, T_i, S_i$), T is a contraction in B_r, with respect to a suitable norm in $W_0^{1,2}\times L^2\times W_0^{1,2}$. The proof of Theorem 3.1 is accomplished.

Proof of Theorem 1.1. The changes to be made on the proof of Theorem 3.1, in order to adapt it for all values of j, are quite obvious. We start by showing it for Theorem 2.1, which is the main tool in our proofs.

The proof of Theorem 2.1 was done by solving the sequence of linear problems (2.5), (2.4), (2.10), (2.12), (2.13), and by getting suitable estimates for the corresponding solutions. Let us show that the proof goes on, in case that $j\geqq 0$, exactly as for $j=-1$.

Problem (2.5). The following estimate for the solution of problem (2.5) is easily obtained by using Sobolev's inequalities:

$$\|G\|_{j,p}\leqq\|g\|_{j+2,p}+\frac{m}{\mu+\nu}\|F\|_{j+1,p}+c\|v\|_{j+3,p}\|\tau\|_{j+2,p}.$$

Problem (2.4). The following result is proved in [3]: There exist positive constants c and c_1 (here the constants depend also on j) such that if $\|v\|_{j+3,p}\leqq c_1 mk/(\mu+\nu)$, and $v=0$ on Γ, then there exists a solution λ of problem (2.4), moreover one has

$$\frac{mk}{2(\mu+\nu)}\|\lambda\|_{j,p}\leqq c\|G\|_{j,p}.$$

Problem (2.10). For the solution θ of the Dirichlet problem (2.10) the estimate

$$(\mu+\nu)\|\theta\|_{j+2,p}\leqq c(\|F\|_{j+1,p}+k\|\lambda\|_{j,p})$$

is well known.

Problem (2.12). Obviously, the function θ_0, defined by Eq. (2.12), verifies the estimate $\|\theta_0\|_{j+2,p}\leqq c\|\theta\|_{j+2,p}$.

Problem (2.13). The solution (u,σ) of this Stokes linear problem (see [1,4]) verifies the estimate

$$\mu\|u\|_{j+3,p}+k\|\sigma\|_{j+2,p}\leqq c(\|F\|_{j+1,p}+(\mu+|\nu|)\|\theta_0\|_{j+2,p}).$$

By putting together the above estimates, one easily gets (c.f.r. (2.15))

$$\mu\|u\|_{j+3,p}+k\|\sigma\|_{j+2,p}$$
$$\leqq\frac{k}{2}\|\tau\|_{j+2,p}+c\left(1+\frac{\mu+|\nu|}{\mu+\nu}\right)\|F\|_{j+1,p}+c_2\frac{\mu+|\nu|}{m}\|g\|_{j+2,p}.$$

By arguing as in Sect. 2 (recall footnote 1) one proves the following result:

Theorem 3.3. *Let* p, j and Γ *be as in Theorem 1.1 and let* $F\in W^{j+1,p}, g\in\bar{W}_0^{j+2,p}$. *Then, there exist positive constants* $c=c(\Omega,n,p,j), \gamma=\gamma(\Omega,n,p,j,\mu,\nu,m)$, *such that if* $v\in W_{0,d}^{j+3,p}$ *verifies the condition* $\|v\|_{j+3,p}\leqq\gamma k$, *then there exists a unique solution* $(u,\sigma)\in W_{0,d}^{j+3,p}\times\bar{W}^{j+2,p}$ *of problem* (2.1). *Moreover,*

$$\mu\|u\|_{j+3,p}+k\|\sigma\|_{j+2,p}\leqq c\left(1+\frac{\mu+|\nu|}{\mu+\nu}\right)\|F\|_{j+1,p}+c\frac{\mu+|\nu|}{m}\|g\|_{j+2,p}.$$

The rest of the proof of Theorem 1.1 (the "non-linear part") closely follows the proof of Theorem 3.1. For an arbitrary $j\geqq-1$, we define the set B_r by the inequalities (c.f.r. (3.13))

$$\|v\|_{j+3,p}\leqq r,\quad \|\tau\|_{j+2,p}\leqq r,\quad \|\beta\|_{j+2,p}\leqq c'r^2,$$

and we prove that (for sufficiently small values of r) $T(B_r)\subset B_r$, and that T is a contraction in B_r with respect to the norm $W_0^{1,2}\times L^2\times W_0^{1,2}$. The proofs of these two statements in case that $j>-1$, differ from the proofs given above only by the particular Sobolev's inequalities to be used.

Before ending this section we remark that in Theorem 1.1, if $h\in W^{j,p}$ (instead of $h\in W^{j+1,p}$) then $\zeta\in W^{j+2,p}$, and if $h\in W^{j+2,p}$, then $\zeta\in W^{j+4,p}$.

4. The Incompressible Limit

In this section we study the system

$$\begin{cases}-\mu\Delta u-\nu\nabla\operatorname{div}u+\nabla p(\rho)=\rho[f-(u\cdot\nabla)u],\\ \operatorname{div}(\rho u)=0,\quad\text{in}\quad\Omega,\\ u_{|\Gamma}=0,\end{cases}\tag{4.1}$$

describing the barotropic motion of a compressible, viscous fluid. Since we are interested in studying the limit of the solution u when $k \equiv p'_\rho(m) \to +\infty$, it is necessary to state an existence result for problem (4.1) in which: (i) the dependence of some suitable structural constants of the state function $p(\rho)$ in terms of k is given; (ii) the dependence on k of the constants appearing on the estimates, is shown. This is the aim of Theorem 4.2 below.

Let $p(\rho)$ be continuously differentiable in $[m-(l/k), m+(l/k)]$, where $0 < l/k < m/2$, and write p'_ρ in the form $p'_\rho(m+\sigma) = k + \omega(\sigma)$. We assume that there exists $\alpha \in]0, 1]$ such that

$$|\omega(\sigma)| \leqq S|\sigma|^\alpha, \quad \forall \sigma \in [-l/k, l/k], \tag{4.2}$$

and we denote by ϕ a positive constant for which

$$S \leqq \phi k^{1+\alpha}. \tag{4.3}$$

Arguing as in proving Theorem 3.2, we show

Theorem 4.1. *Let the above assumptions on the state function $p(\rho)$ hold, let $v \in W^{2,p}_{0,d}$ verify* (2.2), *and let*

$$\|k\tau\|_{1,p} \leqq \frac{l}{c_3}. \tag{4.4}$$

If $f \in L^p$, there exists a unique solution $(u, \sigma) \in W^{2,p}_{0,d} \times \bar{W}^{1,p}$ of the problem

$$\begin{cases} -\mu \Delta u - \nu \nabla \operatorname{div} u + k \nabla \sigma = F(f, v, \tau) \\ m \operatorname{div} u + v \cdot \nabla \sigma + \sigma \operatorname{div} v = 0, \quad \text{in} \quad \Omega, \\ u_{|\Gamma} = 0, \end{cases} \tag{4.5}$$

where $F(f, v, \tau) = (\tau + m)[f - (v \cdot \nabla)v] - \omega(\tau)\nabla\tau$. Moreover,

$$\|u\|_{2,p} + \|k\sigma\|_{1,p} \leqq c'_1 |f|_p + c'_2 \|v\|^2_{1,p} + c'_3 \phi \|k\tau\|^{1+\alpha}_{1,p}. \tag{4.6}$$

Here the positive constants c'_1, c'_2, c'_3, depend at most on $\Omega, n, p, \mu, \nu, m$ (and are distinct from the constants c'_1, c'_2, c'_3, in Sect. 3). Following Sect. 3, we denote by $(u, \sigma) = T(v, \tau)$ the solution of system (4.5). Define

$$r_0 = \min\left\{\gamma k, \frac{l}{c_3}, \frac{1}{3c'_2}, \left(\frac{1}{3c'_3\phi}\right)^{1/\alpha}\right\}, \tag{4.7}$$

and assume that $f \in L^p$ is fixed, and verifies

$$|f|_p \leqq r/(3c'_1). \tag{4.8}$$

One easily shows that $T(B_r) \subset B_r$, where B_r is defined by the inequalities

$$\|v\|_{2,p} \leqq r, \ \|k\tau\|_{1,p} \leqq r. \tag{4.9}$$

Moreover, there exist positive constants c', c'_4, c'_5, depending at most on $\Omega, n, p, \mu, \nu, m$, such that

$$\begin{aligned} &\|u - u_1\|^2_1 + c'_4(1 - c'_5\|v\|_{2,p})\|k(\sigma - \sigma_1)\|^2_0 \\ &\leqq c'\|k\sigma_1\|^2_{1,p}\|v - v_1\|^2_1 + c'\|F - F_1\|^2_{-1}, \end{aligned} \tag{4.10}$$

where $(u_1, \sigma_1) = T(v_1, \tau_1)$, $F = F(f, v, \tau)$, $F_1 = F(f, v_1, \tau_1)$. Inequality (4.10) is proved as inequality (3.18). Straightforward calculations (see (3.20)) show that

$$\|F - F_1\|_{-1} \leqq c|f|_p \|\tau - \tau_1\|_0 + c(1 + \|\tau\|_{1,p})(\|v\|_{1,p} + \|v_1\|_{1,p})\|v - v_1\|_1 + c\|v_1\|_{1,p}^2 \|\tau - \tau_1\|_0 + c_3^\alpha \phi(\|k\tau\|_{1,p}^\alpha + \|k\tau_1\|_{1,p}^\alpha)\|k(\tau - \tau_1)\|_0. \quad (4.11)$$

Arguing as in Sect. 3, one easily verifies that there exists a positive constant $r_1 \leqq r_0$, depending only on $\Omega, n, p, \mu, v, m, l, \phi, k_0$, such that for $r \leqq r_1$, T is a contraction in B_r, with respect to the $W_0^{1,2} \times L^2$ norm (for convenience, we denote by k_0 a positive constant such that $k \geqq k_0$). The above result proves the following theorem:

Theorem 4.2. *Let $p > n$ be fixed, and let the above assumptions on the state function $p(\rho)$ hold. Then, there exist positive constants c_6', c_7', depending only on $\Omega, n, p, \mu, v, m, l, \phi$ and k_0, such that if $f \in L^p$ verifies $|f|_p \leqq c_6'$, then there exists a unique solution $(u, \rho) \in W_0^{2,p} \times W^{1,p}$ of problem (4.1) $(1.3)_1$, in the ball $\|u\|_{2,p} \leqq c_7'$, $\|\rho - m\|_{1,p} \leqq c_7'/k$.*

Assumptions on the family $p_\lambda(\rho)$. Here, $p_\lambda(\rho)$ is a family of state functions depending on a parameter $\lambda \in [\lambda_0, +\infty[$, and such that $k_\lambda \to +\infty$ as $\lambda \to +\infty$. Our assumptions on $p_\lambda(\rho)$ are the following:

Let $p_\lambda'(\rho)$ denote the derivative $dp_\lambda(\rho)/d\rho$. We set $k_\lambda = p_\lambda'(m)$, we assume that $k_\lambda \geqq k_0 > 0$, and we suppose that $p_\lambda(\rho)$ is defined and continuously differentiable in the interval $[m - l/k_\lambda, m + l/k_\lambda]$, where $0 < l < (k_0 m)/2$. Moreover, we assume that $|p_\lambda'(\rho) - p_\lambda'(m)| \leqq S_\lambda |\rho - m|^\alpha$, for a fixed $\alpha \in]0, 1]$. Hence, by setting $p_\lambda'(m + \sigma) = k_\lambda + \omega_\lambda(\sigma)$, one has $\omega_\lambda(0) = 0$, and

$$|\omega_\lambda(\sigma)| \leqq S_\lambda |\sigma|^\alpha, \quad \forall \sigma \in [-l/k_\lambda, l/k_\lambda]. \quad (4.12)$$

We assume that there exists a positive constant ϕ such that

$$S_\lambda \leqq \phi k_\lambda^{1+\alpha}, \quad \forall \lambda \geqq \lambda_0. \quad (4.13)$$

These quite general assumptions contain the physically interesting cases described by Klainerman and Majda in [5]. In particular, if k, α and S are the parameters relative to a given state function $p(\rho)$, and if we define $p_\lambda(\rho) = \lambda^2 p(\rho)$, then $S_\lambda \leqq (S/k)k_\lambda$. Hence, $S_\lambda \leqq \phi k_\lambda^{1+\alpha}$, where $\phi = S/(\lambda_0^{2\alpha} k^{1+\alpha})$. In that example we may view $M = 1/\lambda$ as the Mach number (see [5]).

Proof of Theorem 1.2. Part (i) follows from Theorem 4.2. Moreover, the estimates show that there exists $u_\infty \in W_0^{2,p}$ such that $u_\lambda \to u_\infty$, and $\rho_\lambda \to m$, in the topologies indicated in the statement. The convergence of all the sequence u_λ to u_∞ follows from the uniqueness of the solution of (4.15), which holds if c_8' is sufficiently small.

From Eq. $(4.14)_2$, one easily verifies that $\operatorname{div} u_\lambda \to 0$ in L^p. Since $\|\operatorname{div} u_\lambda\|_{1,p} \leqq c_9'$, one gets the weak convergence in $W_0^{1,p}$. Clearly, $\operatorname{div} u_\infty = 0$.

Finally, we pass to the limit in Eq. $(4.14)_1$. One has $\nabla p_\lambda(\rho_\lambda) \to \mu \Delta u_\infty + m[f - (u_\infty \cdot \nabla) u_\infty]$, weakly in L^p, since: $\Delta u_\lambda \to \Delta u_\infty$ and $-v\nabla \operatorname{div} u_\lambda \to 0$ weakly in L^p; $\rho_\lambda \to m$ strongly in $W^{1,p}$; and $\rho_\lambda (u_\lambda \cdot \nabla) u_\lambda \to m(u_\infty \cdot \nabla) u_\infty$, strongly in L^p. Obviously the limit of the sequence $\nabla p_\lambda(\rho_\lambda)$ must be of the form $\nabla \pi(x)$, for $\pi \in \bar{W}^{1,p}$. ∎

5. The Equilibrium Solutions

The results described in this section are obtained by using very elementary methods. The proofs are independent of the results of the preceding sections.

By an equilibrium solutions (e.s.) of system (4.1) [respectively (1.1)] we mean a solution (u, ρ) [respectively (u, ρ, ζ)] such that $u = 0$ in Ω, and

$$0 < \operatorname*{ess\,inf}_{x\in\Omega} \rho(x), \quad \operatorname*{ess\,sup}_{x\in\Omega} \rho(x) < +\infty. \tag{5.1}$$

If $h = g = 0$ the two problems are equivalent (set $p(\rho) = p(\rho, \zeta_0)$ in system (4.1)). In the sequel we will consider the system (4.1). We call equilibrium solutions the functions $\rho(x)$, in the class (5.1), such that $(0, \rho(x))$ solves (4.1). It is easily shown that a necessary condition for the existence of the equilibrium solution is that $f = \nabla F$, for some potential F. Moreover, the equilibrium solutions are the solutions of the equation

$$\nabla p(\rho(x)) = \rho(x) \nabla F(x), \quad x \in \Omega. \tag{5.2}$$

Here, we are interested in the study of the e.s. under the effect of *arbitrarily large* external forces $f = \nabla F$. We will present useful necessary and sufficient conditions for the existence of the equilibrium solution for an arbitrary $F \in L^\infty(\Omega)$.

In the sequel, p is a continuously differentiable real function defined on $R^+ = \{s \in R : s > 0\}$, such that $p'(s) > 0$, $\forall s \in R^+$. We look for equilibrium solutions verifying (5.1) and $(1.3)_1$, for a fixed $m > 0$. We define

$$\pi(s) = \int_m^s t^{-1} p'(t)\, dt, \quad \forall s \in R^+. \tag{5.3}$$

We denote by $]a, b[$ the range of π, $]a, b[= \pi(R^+)$. One has $-\infty \leqq a < 0 < b \leqq +\infty$, since $\pi(m) = 0$. We define $\Phi = \pi^{-1}$. Clearly, $\Phi(]a, b[) = R^+$. We set $\Phi(a) = 0$, $\Phi(b) = +\infty$.

Let ρ and F be defined and measurable in Ω, and let ρ verify the assumptions (5.1) and $(1.3)_1$. Then ρ is said to be a *weak-equilibrium solution* if there exists a real constant c such that

$$\pi(\rho(x)) = F(x) + c, \quad \text{a.e. in} \quad \Omega. \tag{5.4}$$

Moreover, ρ is said to be a *strong-equilibrium solution* if $\rho \in W^{1,1}$ and if (5.2) holds a.e. in Ω. A weak-equilibrium solution is a strong e.s. if and only if $\rho \in W^{1,1}$. Furthermore, $\rho \in W^{1,1}$ if and only if $F \in W^{1,1}$. Hence it is sufficient to take in account the weak formulation (5.4). Note that Eq. (5.4) shows that ρ and F have the same regularity, if p is sufficiently smooth. For the sake of convenience we will concentrate here on the L^∞ case.

Definition 5.1. Let $F \in L^\infty$. A function ρ is called an equilibrium solution of system (4.1) *if $\rho \in L^\infty$ and if* (5.4), (5.1) *and* $(1.3)_1$ *hold.*

We set $m_0 = \operatorname{ess\,inf} F$ in Ω, $M_0 = \operatorname{ess\,sup} F$ in Ω. One has the following result:

Theorem 5.2. *Let $F \in L^\infty$, be given. There exists an equilibrium solution $\rho(x)$ if and only* *if there exists a constant*

$$c \in]a - m_0, b - M_0[, \tag{5.5}$$

such that

$$\frac{1}{|\Omega|}\int_\Omega \Phi(c + F(x))\,dx = m. \tag{5.6}$$

If such a constant exists then the (unique) equilibrium solution is given by

$$\rho(x) = \Phi(c + F(x)), \quad \forall x \in \Omega. \tag{5.7}$$

Proof. The condition (5.6) corresponds to condition $(1.3)_1$. Uniqueness is shown by noting that the left-hand side of (5.6) is a continuous and increasing function of c in the interval (5.5). The details are left to the reader.

Theorem 5.3. *Under the assumptions of Theorem 5.2, there exists an equilibrium solution $\rho(x)$ if and only if*

$$a - m_0 < b - M_0, \tag{5.8}$$

and

$$\frac{1}{|\Omega|}\int_\Omega \Phi(a - m_0 + F(x))\,dx < m < \frac{1}{|\Omega|}\int_\Omega \Phi(b - M_0 + F(x))\,dx. \tag{5.9}$$

In this case the e.s. *$\rho(x)$ is given by (5.7), where c is the (unique) solution of (5.5), (5.6).*

Proof. Condition (5.7) follows from (5.5). Moreover, (5.6) holds for some $c \in]a - m_0, b - M_0[$ if and only if

$$\lim_{c \to (a - m_0)^+} \frac{1}{|\Omega|}\int_\Omega \Phi(c + F(x))dx < m, \tag{5.10}$$

$$\lim_{c \to (b - M_0)^-} \frac{1}{|\Omega|}\int_\Omega \Phi(c + F(x))dx > m. \tag{5.11}$$

These two conditions are equivalent to (5.9), by a well known generalization of Beppo Levi's theorem to functions taking values in $[0, +\infty]$. Note in particularly that the integral on the right-hand side of (5.9) is well defined, since $0 \leqq \Phi(b - M_0 + F(x)) \leqq +\infty$, a.e. in Ω.

Corollary 5.4. *If $a = -\infty$ [respectively $b = +\infty$] then the conditions (5.8) and $(5.9)_1$ [respectively $(5.9)_2$] hold, for every $F \in L^\infty$. In particular, if $]a, b[=]-\infty, +\infty[$ the equilibrium solution exists for every $F \in L^\infty$.*

Corollary 5.5. *Let $F \in L^\infty$. Then, the equilibrium solution exists if*

$$M_0 - m_0 < \min\{-a, b\}. \tag{5.12}$$

In particular this holds if $|F|_\infty < (1/2)\min\{-a, b\}$.

Proof. If (5.12) holds then $a - m_0 + F(x) < 0$, and $b - M_0 + F(x) > 0$, a.e. in Ω. Hence $\Phi(a - m_0 + F(x)) < m < \Phi(b - M_0 + F(x))$, and assumption (5.9) holds. ■

Remark 5.6. The lack of condition $(5.9)_1$ [respectively $(5.9)_2$] corresponds to the formation of vacuum [respectively points of infinite density]. If $a = -\infty$ [respectively $b = +\infty$] the first [respectively second] phenomena does not occur, under the effect of bounded pontentials F.

Let us now consider the one dimensional case $\Omega =]0, l[, l > 0$, in the presence of external forces $f \in L^1$. The potential

$$F(x) = \int_0^x f(t)\, dt$$

is then an absolutely continuous function in $[0, l]$. Consequently, the equilibrium solution ρ (if exists) belongs to $W^{1,1}$ and verifies (5.2). The sufficient condition (5.12) can be written in the equivalent form

$$\left| \int_x^y f(t)\, dt \right| < \min\{-a, b\}, \quad \forall x, y \in [0, l]. \tag{5.13}$$

Examples. As shown before, the existence of the equilibrium solution under a given potential F, strongly depends on the values of a and b. It is of interest to consider the classical case $p(\rho) = R\rho^\gamma$, $R > 0, \gamma > 0$. It follows immediately from our results that if $\gamma > 1$ [respectively $\gamma < 1$], vacuum [respectively infinite densities] may occur. On the contrary, if $\gamma = 1$ the e.s. exists, for every bounded potential F.

Let us give an example, corresponding to the main case $\gamma > 1$. For convenience, we consider the one dimensional case $\Omega =]0, 1[$. Let $m = 1$, $\gamma = 2$, $R = 1/2$. Then $p(\rho) = (1/2)\rho^2, \pi(\rho) = \rho - 1$, $\Phi(s) = 1 + s, a = -1$, $b = +\infty$. In the light of Corollary 5.4, we have only to check the first condition (5.9), which corresponds to the formation of vacuum. By assuming that $f(x) = \beta$ is constant, this condition is $|\beta| < 2$. Formulae (5.6) and (5.7) yield $c = -\beta/2$ and $\rho(x) = 1 + \beta(x - 1/2)$, respectively. If $\beta > 0$, the forces act on the positive direction, and the point $x = 0$ is subject to the largest decompression. For $\beta = 2$ the vacuum is attained at the point $x = 0$, since $\rho(0) = 0$.

Appendix

For the reader's convenience, we will prove here the estimate (3.21). The proof of (3.20) is easier, and is left to the reader. Recall that under the assumptions made in Sect. 3 one has $|\tau|_\infty \leqq l, |\tau_1|_\infty \leqq l, |\beta|_\infty \leqq l_1, |\beta_1|_\infty \leqq l_1$.

In the sequel we use freely the inequality

$$\| \psi_1 \psi_2 \psi_3 \|_{-1} \leqq c |\psi_1|_\infty \| \psi_2 \|_0 |\psi_3|_p, \tag{1}$$

which is easily proved by using Sobolev's inequality $|\phi|_{2^*} \leqq c \| \phi \|_1, \forall \phi \in W_0^{1,2}$. One has

$$\begin{aligned} \| H(h, v, \tau, \beta) - H(h, v_1, \tau_1, \beta_1) \|_{-1} &\leqq \| (\tau - \tau_1) h \|_{-1} + c_v \| (\tau - \tau_1) v \cdot \nabla \beta \|_{-1} \\ &+ \| (m + \tau_1)(v - v_1) \cdot \nabla \beta \|_{-1} + \| (m + \tau_1) v_1 \cdot \nabla(\beta - \beta_1) \|_{-1} \\ &+ \| \psi(v, v) - \psi(v_1, v_1) \|_{-1} + \| (\beta - \beta_1)(m + \tau)^{-1} \omega_2(\tau, \beta)(v \cdot \nabla \tau - g) \|_{-1} \\ &+ \| (\zeta_0 + \beta_1)(\tau_1 - \tau)(m + \tau)^{-1}(m + \tau_1)^{-1} \omega_2(\tau, \beta)(v \cdot \nabla \tau - g) \|_{-1} \end{aligned}$$

$$
\begin{aligned}
&+ \|(\zeta_0+\beta_1)(m+\tau_1)^{-1}(\omega_2(\tau,\beta)-\omega_2(\tau_1,\beta_1))(v\cdot\nabla\tau-g)\|_{-1}\\
&+ \|(\zeta_0+\beta_1)(m+\tau_1)^{-1}\omega_2(\tau_1,\beta_1)(v-v_1)\cdot\nabla\tau\|_{-1}\\
&+ \|(\zeta_0+\beta_1)(m+\tau_1)^{-1}\omega_2(\tau_1,\beta_1)v_1\cdot\nabla(\tau-\tau_1)\|_{-1}.
\end{aligned}
\tag{2}
$$

By using adequately the inequality (1), one gets

$$
\begin{aligned}
\|H-H_1\|_{-1} \leqq{}& c|h|_p\|\tau-\tau_1\|_0 + cc_v\|v\|_{1,p}\|\beta\|_{1,p}\|\tau-\tau_1\|_0\\
&+ c(m+l)\|\beta\|_{1,p}\|v-v_1\|_0 + \|(m+\tau_1)v_1\cdot\nabla(\beta-\beta_1)\|_{-1}\\
&+ c(\chi_0+\chi_1)(\|v\|_{1,p}+\|v_1\|_{1,p})\|v-v_1\|_1\\
&+ c(2/m)T_2(\|v\|_{1,p}\|\tau\|_{1,p}+|g|_p)\|\beta-\beta_1\|_0\\
&+ c(\zeta_0+l_1)(4/m^2)T_2(\|v\|_{1,p}\|\tau\|_{1,p}+|g|_p)\|\tau-\tau_1\|_0\\
&+ c(\zeta_0+l_1)(2/m)(\|v\|_{1,p}\|\tau\|_{1,p}+|g|_p)S_2(\|\tau-\tau_1\|_0+\|\beta-\beta_1\|_0)\\
&+ c(\zeta_0+l_1)(2/m)T_2\|\tau\|_{1,p}\|v-v_1\|_0\\
&+ \|(\zeta_0+\beta_1)(m+\tau_1)^{-1}\omega_2(\tau_1,\beta_1)v_1\cdot\nabla(\tau-\tau_1)\|_{-1}.
\end{aligned}
\tag{3}
$$

We remark that each term on the right-hand side of (2) is bounded by the term on the right-hand side of (3) that occupies the same relative position. In order to estimate the fourth and the last term on the right-hand side of (3) we will use the estimate

$$
\|w\cdot\nabla\psi\|_{-1} \leqq c\left(|w|_\infty + \sum_{i,j}|D_i w_j|_p\right)\|\psi\|_0. \tag{4}
$$

Here w is a vector field and ψ a scalar field in Ω. This inequality is proved by noting that

$$
\left|\int_\Omega (w\cdot\nabla\psi)\phi\,dx\right| \leqq c(|\operatorname{div} w|_p\|\psi\|_0|\phi|_{2*} + |w|_\infty\|\psi\|_0\|\phi\|_1).
$$

By using inequality (4), one easily verifies that the fourth and the last term on the right-hand side of (3) are bounded by $c(m+l_1+\|\tau_1\|_{1,p})\|v_1\|_{1,p}\|\beta-\beta_1\|_0$, and by

$$
\begin{aligned}
c(\zeta_0+l_1+\|\beta_1\|_{1,p})&\left(\frac{2}{m}+\frac{4}{m^2}\|\tau_1\|_{1,p}\right)(T_2+S_2\|\tau_1\|_{1,p}\\
&+S_2\|\beta_1\|_{1,p})\|v_1\|_{1,p}\|\tau-\tau_1\|_0,
\end{aligned}
$$

respectively. By using these estimates together with (3), one gets (3.21).

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Communicated by C. H. Taubes

Received June 9, 1986; in revised form October 7, 1986