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AN L^q-ANALYSIS OF VISCOUS FLUID FLOW PAST A ROTATING OBSTACLE

Dedicated to Professor Hermann Sohr on his sixty-fifth birthday

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Abstract. Consider the problem of time-periodic strong solutions of the Stokes and Navier-Stokes system modelling viscous incompressible fluid flow *past* or *around* a rotating obstacle in Euclidean three-space. Introducing a rotating coordinate system attached to the body, a linearization yields a system of partial differential equations of second order involving an angular derivative *not* subordinate to the Laplacian. In this paper we find an explicit solution for the linear whole space problem when the axis of rotation is parallel to the velocity of the fluid at infinity. For the analysis of this solution in L^q-spaces, $1 < q < \infty$, we will use tools from harmonic analysis and a special maximal operator reflecting paths of fluid particles past or around the obstacle.

1. Introduction. In recent years the analysis of the Navier-Stokes equations and of models of non-Newtonian fluids describing the flow around or past a rotating body has attracted much attention. Here we consider the Navier-Stokes equations modelling viscous flow either *past* a rotating body K in Euclidean 3-space $\boldsymbol{R}^3$ with axis of rotation $\omega = \tilde{\omega} e_3 = \tilde{\omega}(0, 0, 1)^T$, $\tilde{\omega} \neq 0$, and with velocity $u_\infty = ke_3 \neq 0$ at infinity or *around* a rotating body K which is moving in the direction of its axis of rotation. In each case a coordinate transform and a linearization yield the system of partial differential equations

$$\begin{aligned} u_t - \nu\Delta u + k\partial_3 u - (\omega \wedge x) \cdot \nabla u + \omega \wedge u + \nabla p = f \,, \\ \operatorname{div} u = 0 \end{aligned} \tag{1.1}$$

in a time-independent exterior domain $\Omega = \boldsymbol{R}^3 \backslash K$ together with the initial-boundary condition

$$u(x, t) = \omega \wedge x - u_\infty, \quad u(x, 0) = u_0, \quad u \to 0 \text{ as } |x| \to \infty \,.$$

Here $u = (u_1, u_2, u_3)^T$ and p denote the velocity and pressure of the fluid, resp., f is a given external force, and $\nu > 0$ is the constant coefficient of viscosity. In the stationary case to be analyzed in this paper, we are led to an elliptic equation in the sense of Agmon-Douglis-Nirenberg in which the term $(\omega \wedge x) \cdot \nabla u$ is *not* subordinate to $-\nu\Delta u$ in the exterior domain Ω. Note that a stationary solution (u, p) of (1.1) will lead to a time-periodic solution of the original linearized problem.

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To be more precise, consider the Navier-Stokes equations

$$(1.2)\qquad \begin{aligned} v_t - \nu\Delta v + v\cdot\nabla v + \nabla q &= \tilde f && \text{in } \Omega(t),\ t>0\,,\\ \operatorname{div} v &= 0 && \text{in } \Omega(t),\ t>0\,,\\ v(y,t) &= \omega\wedge y && \text{on } \partial\Omega(t),\ t>0\,,\\ v(y,t) &\to u_\infty \neq 0 && \text{as } |y|\to\infty \end{aligned}$$

with an initial value $v(y,0)=v_0(y)$ and $\omega=\tilde\omega e_3\neq 0$ in the time-dependent exterior domain

$$\Omega(t) = O_\omega(t)\Omega\,,$$

where $O_\omega(t)$ denotes the orthogonal matrix

$$O_\omega(t) = \begin{pmatrix} \cos\tilde\omega t & -\sin\tilde\omega t & 0\\ \sin\tilde\omega t & \cos\tilde\omega t & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Then, introducing

$$(1.3)\qquad x = O_\omega^T(t)y\,,\quad u(x,t) = O_\omega^T(t)(v(y,t)-u_\infty)\,,\quad p(x,t)=q(y,t)\,,$$

(u,p) will satisfy the modified Navier-Stokes system

$$(1.4)\qquad \begin{aligned} u_t - \nu\Delta u + u\cdot\nabla u + (O_\omega^T(t)u_\infty)\cdot\nabla u &\\ -(\omega\wedge x)\cdot\nabla u + \omega\wedge u + \nabla p &= f && \text{in } \Omega\times(0,\infty)\,,\\ \operatorname{div} u &= 0 && \text{in } \Omega\times(0,\infty)\,,\\ u(x,t) &= \omega\wedge x - O_\omega^T(t)u_\infty && \text{on } \partial\Omega\times(0,\infty)\,,\\ u(x,t) &\to 0 && \text{as } |x|\to\infty\,. \end{aligned}$$

For details of the elementary calculation, see [9] when $u_\infty = 0$; for $u_\infty\neq 0$ the additional term $u_\infty\cdot\nabla_y u(O_\omega^T(t)y,t) = (O_\omega^T(t)u_\infty)\cdot\nabla_x u$ will appear. In the case, $u_\infty \parallel \omega$, say $u_\infty = ke_3$, to be considered here, $O_\omega^T(t)u_\infty = ke_3$ for all $t>0$. Thus (1.4) will lead to the system

$$(1.5)\qquad \begin{aligned} u_t - \nu\Delta u + u\cdot\nabla u + k\partial_3 u &\\ -(\omega\wedge x)\cdot\nabla u + \omega\wedge u + \nabla p &= f && \text{in } \Omega\,,\\ \operatorname{div} u &= 0 && \text{in } \Omega\,,\\ u &= \omega\wedge x - ke_3 && \text{on } \partial\Omega\,,\\ u &\to 0 && \text{as } |x|\to\infty\,, \end{aligned}$$

a stationary solution of which corresponds to a time-periodic solution of the original system (1.2). However, if u_∞ is not parallel to the axis of rotation e_3, the term $O_\omega^T(t)u_\infty$ depends on t. Therefore, in this paper, we will study the linearized and stationary version of (1.4) only when u_∞ is parallel to ω.

Finally, we may consider the problem of a rotating body with axis of rotation ω and with an additional translational velocity $-u_\infty$. In this case, $\Omega(t) = O_\omega(t)\Omega - u_\infty t$ and $v(y,t)\to 0$ as $|y|\to\infty$ in (1.1). Then the transformation

$$x = O_\omega^T(t)(y+u_\infty t),\quad u(x,t) = O_\omega^T(t)v(y,t),\quad p(x,t)=q(y,t)$$

again will lead to (1.4) (observer invariance) with the same fundamental difference between the cases $u_\infty \parallel \omega$ and $u_\infty \wedge \omega \neq 0$.

The mathematical analysis of viscous flow past or around rotating obstacles started with [1], where weak instationary solutions have been constructed in an even more general setting allowing for time-dependent functions $\omega(t)$ and $u_\infty(t)$. Decay results for this problem in the whole space are discussed in [2]. Using semigroup theory, local mild and unique solutions are constructed in [9], [10] when $u_\infty = 0$; since the corresponding semigroup is strongly continuous, but *not* analytic, it is not clear whether the mild solution is a strong one. A different approach in homogeneous Besov spaces is used in [13], where the term $(\omega \wedge x) \cdot \nabla u$ has been replaced by the more general term $(Mx) \cdot \nabla u$ with an arbitrary traceless 3×3-matrix M; here a local classical and unique solution is found for nondecaying initial data. Several linear and stationary auxiliary problems in the whole space and in exterior domains have been analyzed in [11]; to some extent the results are generalized to the nonlinear case and the problem including the term $(\omega \wedge x) \cdot \nabla u$ in [12]. Several advanced *a priori* estimates of stationary and instationary solutions can be found in [7], including even non-Newtonian fluids; in particular L^2-estimates for (1.1) are established. Pointwise estimates yielding decay rates such as $|v(x)| \leq c(1+|x|)^{-1}$ are obtained in [8] for the stationary nonlinear problem when $u_\infty = 0$. With regard to further developments, e.g., to the discussion of stability, L^q-estimates, $1 < q < \infty$, are presented in [3] for the linearized whole space problem (1.1) when $u_\infty = ke_3 = 0$. For the physical background and for applications to the free fall of particles in fluids, see [7] and references therein. In [17] the time-dependent fundamental solution (Green's function) $\Gamma(z, y; t)$ is calculated for the case $u_\infty \neq 0$, and several pointwise estimates are given for $t \to 0$, $t \to \infty$ and for small and large spatial data z, y.

The main results of this paper are the following.

THEOREM 1.1. (1) *Let* $1 < q < \infty$, $f \in L^q(\boldsymbol{R}^3)^3$ *and* $g \in W^{1,q}(\boldsymbol{R}^3)$ *such that even* $|(x_1, x_2)|g \in L^q(\boldsymbol{R}^3)$. *Furthermore, let* $\nu > 0$, $k \in \boldsymbol{R}$ *and* $\omega = (0, 0, \tilde{\omega})^T \in \boldsymbol{R}^3\backslash\{0\}$. *Then the linear problem in* $\boldsymbol{R}^3$,

$$-\nu\Delta u + k\partial_3 u - (\omega \wedge x) \cdot \nabla u + \omega \wedge u + \nabla p = f\,, \quad \operatorname{div} u = g\,, \tag{1.6}$$

has a solution $(u, p) \in \hat{W}^{2,q}(\boldsymbol{R}^3)^3 \times \hat{W}^{1,q}(\boldsymbol{R}^3)$ *satisfying the a priori estimates*

$$\|\nu\nabla^2 u\|_q + \|\nabla p\|_q \leq c(\|f\|_q + \|\nu\nabla g + (\omega \wedge x)g - kge_3\|_q)\,, \tag{1.7}$$

$$\begin{aligned} &\|k\partial_3 u\|_q + \|(\omega \wedge x) \cdot \nabla u - \omega \wedge u\|_q \\ &\quad \leq c\left(1 + \frac{k^4}{\nu^2|\omega|^2}\right)(\|f\|_q + \|\nu\nabla g + (\omega \wedge x)g - kge_3\|_q) \end{aligned} \tag{1.8}$$

with a constant $c > 0$ *independent of* ν, k *and* ω.

(2) *In addition to the assumptions in* (1) *and given a solution* $(u, p) \in \hat{W}^{2,q}(\boldsymbol{R}^3)^3 \times \hat{W}^{1,q}(\boldsymbol{R}^3)$ *of* (1.6), *suppose that* $f \in L^r(\boldsymbol{R}^3)^3$, $g \in W^{1,r}(\boldsymbol{R}^3)$, $|(x_1, x_2)|g \in L^r(\boldsymbol{R}^3)$, *and let* $(u_1, p_1) \in \hat{W}^{2,r}(\boldsymbol{R}^3)^3 \times \hat{W}^{1,r}(\boldsymbol{R}^3)$ *be another solution of* (1.6). *Then* $p - p_1$ *is constant and* $u - u_1$ *equals* $\alpha e_3 + \beta\omega \wedge x$, $\alpha, \beta \in \boldsymbol{R}$.

COROLLARY 1.2. (1) *Let* $1 < q < 4$, $f \in L^q(\boldsymbol{R}^3)^3$ *and* $g \in W^{1,q}(\boldsymbol{R}^3)$ *such that* $|(x_1, x_2)|g \in L^q(\boldsymbol{R}^3)$, *and let* $(u, p) \in \hat{W}^{2,q}(\boldsymbol{R}^3)^3 \times \hat{W}^{1,q}(\boldsymbol{R}^3)$ *be the solution of* (1.6). *Then there exists* $\beta \in \boldsymbol{R}$ *such that*

$$\nabla'(u - \beta\omega \wedge x) \in L^r(\boldsymbol{R}^3)^6 \quad \textit{for all} \quad r > 1,\ \frac{1}{r} \in \frac{1}{q} - \left[\frac{1}{4}, \frac{1}{3}\right].$$

Moreover,

$$\|\nabla'(u - \beta\omega \wedge x)\|_r \le C(\|f\|_q + \|\nu\nabla g + (\omega \wedge x)g - kge_3\|_q)$$

with a constant $C = C(\nu, k, \omega; r) > 0$.

(2) *In* (1) *assume that even* $1 < q < 2$. *Then there exist* $\alpha, \beta \in \boldsymbol{R}$ *such that*

$$u - \beta\omega \wedge x - \alpha e_3 \in L^s(\boldsymbol{R}^3)^3 \quad \textit{for all} \quad s > 1,\ \frac{1}{s} \in \frac{1}{q} - \left[\frac{1}{2}, \frac{2}{3}\right].$$

Moreover,

$$\|u - \beta\omega \wedge x - \alpha e_3\|_s \le C(\|f\|_q + \|\nu\nabla g + (\omega \wedge x)g - kge_3\|_q)$$

with a constant $C = C(\nu, k, \omega; s) > 0$.

REMARK 1.3. (1) In Theorem 1.1 fix f and g and let $(u_{\nu,k,\omega}, p_{\nu,k,\omega})$ denote a solution of (1.6) for $\nu > 0$, $k \neq 0$ and $\omega = \tilde{\omega}e_3$, $\tilde{\omega} \neq 0$. Furthermore, let $\nu_0 > 0$, $k_0 \in \boldsymbol{R}$ and $\tilde{\omega}_0 = \tilde{\omega}_0 e_3$, $\tilde{\omega}_0 \in \boldsymbol{R}$. Then

$$u_{\nu,k,\omega} \rightharpoonup u_{\nu_0,k_0,\omega_0} \ \text{ in } \ \hat{W}^{2,q}(\boldsymbol{R}^3)^3, \quad p_{\nu,k,\omega} \rightharpoonup p_{\nu_0,k_0,\omega_0} \ \text{ in } \ \hat{W}^{1,q}(\boldsymbol{R}^3)$$

weakly as $(\nu, k, \omega) \to (\nu_0, k_0, \omega_0)$, where $(u_{\nu_0,k_0,\omega_0}, p_{\nu_0,k_0,\omega_0})$ solves (1.6) with ν_0 replacing ν, k_0 replacing k and ω_0 replacing ω. This result extends to the case of f, g depending on ν, k, ω such that $f_{\nu,k,\omega} \rightharpoonup f_{\nu_0,k_0,\omega_0}$ and $g_{\nu,k,\omega} \rightharpoonup g_{\nu_0,k_0,\omega_0}$ in suitable weak topologies.

(2) Compared to the case $k = 0$ considered in [3], the results in Theorem 1.1 are stronger. The uniqueness assertion does not allow for a term $\gamma(x_1, x_2, -2x_3)^T$, $\gamma \in \boldsymbol{R}$, as in [3] due to the term $k\partial_3 u$.

(3) In (1.6) it is not possible to estimate the terms $(\omega \wedge x) \cdot \nabla u$ and $\omega \wedge u$ separately in L^q unless f and g satisfy an infinite set of compatibility conditions. The argument is based on the simple identity

$$(\omega \wedge x) \cdot \nabla u - \omega \wedge u = \tilde{\omega} O_{e_3}(\theta)\partial_\theta(O_{e_3}^T(\theta)u)\,;$$

for more details see Remark 2.3, Proposition 2.4 in [3] when $k = 0$.

(4) The fundamental solution of (1.5) which will be computed “explicitly" in Section 2 below will not lead to a classical Calderón-Zygmund integral operator, when considering Δu in terms of f (and g). See Section 2 in [3] for more details when $k = 0$; in this case the fundamental solution has a slightly simpler form.

(5) It is *not* evident that for the solution u of (1.6) both lower order terms $k\partial_3 u$ and $(\omega \wedge x) \cdot \nabla u - \omega \wedge u$ can be estimated in L^q-norms from the right-hand side. On the other hand, it is remarkable that the *a priori* estimate (1.8) depends on $(k/\sqrt{\nu|\omega|})^4$. The proof in Section 2 using Marcinkiewicz’ multiplier theorem implies that for $q = 2$ the term

$C(1 + (k/\sqrt{\nu|\omega|})^2)$ will suffice. Thus complex interpolation will slightly improve (1.8). Finally, an explicit example will indicate that the term $C(1 + (k/\sqrt{\nu|\omega|})^2)$ is optimal, see the end of Section 2.

In this paper we use standard notation for Lebesgue spaces and Sobolev spaces, namely $L^q(\Omega)$ and $W^{k,q}(\Omega)$, $1 \le q \le \infty$, for bounded and unbounded domains $\Omega \subset \boldsymbol{R}^3$. To control problems also in unbounded domains we need the space $L^q_{\mathrm{loc}}(\bar{\Omega})$ of functions which are L^q-integrable on every compact subset of $\bar{\Omega}$ and homogeneous Sobolev spaces

$$\hat{W}^{k,q}(\Omega) = \{u \in L^1_{\mathrm{loc}}(\bar{\Omega})/\Pi_{k-1}\,;\ \partial^\alpha u \in L^q(\Omega) \ \text{ for all } \ \alpha \in \boldsymbol{N}_0^n, |\alpha| = k\}\,,$$

where $\partial^\alpha = \partial_1^{\alpha_1} \cdot \dots \cdot \partial_n^{\alpha_n}$ for a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \boldsymbol{N}_0^n$ and Π_{k-1} denotes the set of all polynomials on $\boldsymbol{R}^n$ of degree $\le k-1$. The space $\hat{W}^{k,q}(\Omega)$ consists of equivalence classes of L^1_{loc}-functions being unique only up to elements from Π_{k-1} and is equipped with the norm $\sum_{|\alpha|=k} \|\partial^\alpha u\|_q$. Since $\hat{W}^{k,q}(\Omega)$ can be considered as a closed subspace of $L^q(\Omega)^N$ for some $N = N(k,n) \in \boldsymbol{N}$, it is reflexive and separable for every $q \in (1,\infty)$. For more details on these spaces see Chapter II in [6], Chapter III in [14] and also [4], [5]. However, sometimes being less careful, we will consider $v \in \hat{W}^{k,q}(\Omega)$ as a function (representative) rather than an equivalence class of functions, i.e., $v \in L^1_{\mathrm{loc}}(\Omega)$ such that $\partial^\alpha v \in L^q(\Omega)$ for every multi-index α with $|\alpha| = k$.

The Fourier transform on $\boldsymbol{R}^3$ of a function or distribution u is denoted by $\mathcal{F}u = \hat{u}$, i.e., formally

$$\hat{u}(\xi) = (2\pi)^{-3/2} \int_{\boldsymbol{R}^3} e^{-ix\cdot\xi} u(x)dx\,, \quad \xi \in \boldsymbol{R}^3\,.$$

The Fourier transform and its inverse $\mathcal{F}^{-1}$ will be needed in particular on Schwartz's space $\mathcal{S}(\boldsymbol{R}^3)$ of rapidly decreasing functions and on its dual space $\mathcal{S}'(\boldsymbol{R}^3)$ of tempered distributions. Furthermore, $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$, and $\mathcal{D}'(\Omega)$ denotes the set of all distributions on Ω. The application of a distribution T on a test function u (or of a functional $T \in X'$ in a dual space X' of a given Banach space X on an element $u \in X$) is denoted by $\langle T, u\rangle$. Given $q \in (1,\infty)$, let $\hat{W}^{-1,q}(\Omega)$ be the dual space of $\hat{W}^{1,q'}(\Omega)$, $q' = q/(q-1)$.

Finally $B_r(y) = \{x \in \boldsymbol{R}^3\,;\ |x-y| < r\}$, $r > 0$, denotes a ball in $\boldsymbol{R}^3$ with respect to the Euclidean norm $|\cdot|$; moreover, $B_r = B_r(0)$ and $B_r^c = \boldsymbol{R}^3 \setminus B_r$. The vector $y = (y_1, y_2) \in \boldsymbol{R}^2$ rotated through $+\pi/2$ is denoted by $y^\perp = (-y_2, y_1)$. If $y = (y_1, y_2, y_3) \in \boldsymbol{R}^3$, then $y' = (y_1, y_2)$, and $\nabla' = (\partial_1, \partial_2)$ is the corresponding partial gradient on $\boldsymbol{R}^3$. As usual, c denotes a generic positive constant which may change its value from line to line.

2. The whole space problem. To solve the whole space problem (1.6), i.e.,

$$(2.1) \qquad -\nu\Delta u + k\partial_3 u - (\omega \wedge x)\cdot\nabla u + \omega \wedge u + \nabla p = f\,, \quad \operatorname{div} u = g \quad \text{in } \boldsymbol{R}^3\,,$$

we eliminate the pressure term. Applying div to $(2.1)_1$, p is seen to be a weak solution of the equation

$$\Delta p = \operatorname{div} f + \nu\Delta g + (\omega \wedge x)\cdot\nabla g - k\partial_3 g\,,$$

that is,

$$(\nabla p, \nabla\varphi) = (f, \nabla\varphi) + (\nu\nabla g + (\omega \wedge x)g, \nabla\varphi) - k(g, \partial_3\varphi) \tag{2.2}$$

for all test functions $\varphi \in C_0^\infty(\boldsymbol{R}^3)$. By the assumptions of Theorem 1.1 the right-hand side defines a functional F on $\hat{W}^{1,q'}(\boldsymbol{R}^3)$ satisfying the estimate

$$\|F\|_{\hat{W}^{-1,q}(\boldsymbol{R}^3)} \leq \|f\|_q + \|\nu\nabla g + (w \wedge x)g - kge_3\|_q \, .$$

Since the operator $-\Delta$ is well-known to be an isomorphism from $\hat{W}^{1,q}(\boldsymbol{R}^3)$ onto $\hat{W}^{-1,q}(\boldsymbol{R}^3)$, there exists a unique $p \in \hat{W}^{1,q}(\boldsymbol{R}^3)$ solving (2.2) and satisfying the *a priori* estimate

$$\|\nabla p\|_q \leq \|f\|_q + \|\nu\nabla g + (\omega \wedge x)g - kge_3\|_q \, . \tag{2.3}$$

Then u in (2.1) is a solution of the equation

$$-\nu\Delta u + k\partial_3 u - (\omega \wedge x) \cdot \nabla u + \omega \wedge u = f - \nabla p \, . \tag{2.4}$$

A uniqueness argument below will prove that even $\operatorname{div} u = g$.

To find an explicit solution of (2.4) we omit the term ∇p and write f instead of $f - \nabla p$; furthermore, we assume $\tilde{\omega} > 0$, divide by $\tilde{\omega}$ and get that

$$-\frac{\nu}{\tilde{\omega}}\,\Delta u + \frac{k}{\tilde{\omega}}\,\partial_3 u - (e_3 \wedge x) \cdot \nabla u + e_3 \wedge u = \frac{1}{\tilde{\omega}} f \, . \tag{2.5}$$

Next introduce cylindrical coordinates $(r, \theta, x_3) \in \overline{\boldsymbol{R}_+} \times [0, 2\pi) \times \boldsymbol{R}$, $r = \sqrt{x_1^2 + x_2^2}$, for $x = (x_1, x_2, x_3)^T$ and observe that

$$\partial_\theta u = (e_3 \wedge x) \cdot \nabla u \, .$$

To apply the Fourier transform $\mathcal{F} = \hat{}$ to (2.5) we use cylindrical coordinates $(s, \varphi, \xi_3) \in \overline{\boldsymbol{R}_+} \times [0, 2\pi) \times \boldsymbol{R}$, $s = \sqrt{\xi_1^2 + \xi_2^2}$, for $\xi = (\xi_1, \xi_2, \xi_3)^T$ as well and note that $\widehat{\partial_\theta u} = \partial_\varphi \hat{u}$. Thus $\hat{u}$ satisfies the equation

$$\frac{1}{\tilde{\omega}}(\nu|\xi|^2 + ik\xi_3)\hat{u} - \partial_\varphi \hat{u} + e_3 \wedge \hat{u} = \frac{1}{\tilde{\omega}}\,\hat{f} \, ,$$

and $\hat{v}(\varphi) = O_{e_3}^T(\varphi)\hat{u}(s, \varphi, \xi_3)$ solves the problem

$$\frac{1}{\tilde{\omega}}(\nu|\xi|^2 + ik\xi_3)\hat{v} - \partial_\varphi \hat{v} = \frac{1}{\tilde{\omega}} O_{e_3}^T(\varphi)\hat{f} \, . \tag{2.6}$$

This inhomogeneous, linear ordinary differential equation of first order with respect to φ has a unique 2π-periodic solution

$$\hat{v}(\varphi) = \frac{1/\tilde{\omega}}{1 - e^{-2\pi(\nu|\xi|^2 + ik\xi_3)/\tilde{\omega}}} \int_0^{2\pi} e^{-(\nu|\xi|^2 + ik\xi_3)t/\tilde{\omega}} O_{e_3}^T(\varphi + t)\hat{f}(O_{e_3}(t)\xi)\,dt \, ,$$

where $\xi \,\hat{=}\, (s, \varphi, \xi_3)$. Consequently,

$$\hat{u}(\xi) = \frac{1}{1 - e^{-2\pi(\nu|\xi|^2 + ik\xi_3)/\tilde{\omega}}} \int_0^{2\pi/\tilde{\omega}} e^{-(\nu|\xi|^2 + ik\xi_3)t} O_\omega^T(t)\hat{f}(O_\omega(t)\xi)\,dt \, , \tag{2.7}$$

or, using the geometric series and the $2\pi/\tilde{\omega}$-periodicity of the map $t \mapsto O_\omega^T(t)\hat{f}(O_\omega(t)\xi)$,

$$\hat{u}(\xi) = \int_0^\infty e^{-(\nu|\xi|^2+ik\xi_3)t} O_\omega^T(t)\hat{f}(O_\omega(t)\xi)dt \,. \tag{2.8}$$

Since the term $e^{itk\xi_3} \in \mathcal{S}'(\boldsymbol{R}^3)$ is the Fourier transform of the shift operator $f \mapsto f(\cdot - kte_3)$ on $\mathcal{S}'(\boldsymbol{R}^3)$, we may also write

$$\hat{u}(\xi) = \int_0^\infty e^{-\nu|\xi|^2 t} O_\omega^T(t)(\mathcal{F}f(O_\omega(t)\cdot - kte_3))(\xi)dt \,. \tag{2.9}$$

Finally, note that $e^{-\nu|\xi|^2 t}$ is the Fourier transform of the heat kernel

$$E_t(x) = \frac{1}{(4\pi\nu t)^{3/2}} e^{-|x|^2/4\nu t} \,,$$

yielding

$$u(x) = \int_0^\infty E_t * O_\omega^T(t) f(O_\omega(t)\cdot - kte_3)(x)dt \,. \tag{2.10}$$

The fundamental solution given by (2.10) coincides—up to the Helmholtz projection—with $\int_0^\infty \Gamma(z, y; t)dt$, where $\Gamma(z, y; t)$ denotes the time-dependent fundamental solution in [17].

To prove the a priori estimate (1.7) of $\nabla^2 u$ it suffices—due to the well-known estimate $\|\partial_i\partial_j u\|_q \le c\|\Delta u\|_q$ for all $1 \le i, j \le n$, $1 < q < \infty$—to consider Δu only. Given $f \in \mathcal{S}(\boldsymbol{R}^3)^3$, by (2.9)

$$\begin{aligned} \widehat{-\nu\Delta u}(\xi) = \nu|\xi|^2\hat{u}(\xi) &= \int_0^\infty \hat{\psi}_{\nu t}(\xi) O_\omega^T(t)\mathcal{F}f(O_\omega(t)\cdot - kte_3)(\xi)\frac{dt}{t} \\ &= \int_0^\infty \hat{\psi}_t(\xi) O_{\omega/\nu}^T(t)\mathcal{F}f\left(O_{\omega/\nu}(t)\cdot - \frac{k}{\nu}te_3\right)(\xi)\frac{dt}{t} \,, \end{aligned}$$

where

$$\hat{\psi}(\xi) = \frac{1}{(2\pi)^{3/2}}|\xi|^2 e^{-|\xi|^2} \quad \text{and} \quad \hat{\psi}_t(\xi) = \hat{\psi}(\sqrt{t}\xi) \quad \text{for} \quad t > 0$$

are the Fourier transforms of a function $\psi \in \mathcal{S}(\boldsymbol{R}^3)$ and of $\psi_t(x) = t^{-3/2}\psi(x/\sqrt{t})$, $t > 0$, respectively. Next we decompose $\hat{\psi}_t$ by choosing a cut-off function $\tilde{\chi} \in C_0^\infty(1/2, 2)$ such that $0 \le \tilde{\chi} \le 1$ and $\sum_{j=-\infty}^\infty \tilde{\chi}(2^{-j}r) = 1$ for all $r > 0$. Then let $\hat{\chi}_j(\xi) := \tilde{\chi}(2^{-j}|\xi|)$, yielding $\chi_j \in \mathcal{S}(\boldsymbol{R}^3)$ with

$$\operatorname{supp} \hat{\chi}_j \subset A(2^{j-1}, 2^{j+1}) := \{\xi \in \boldsymbol{R}^3\,;\ 2^{j-1} \le |\xi| \le 2^{j+1}\}\,.$$

Finally, define $\psi^j \in \mathcal{S}(\boldsymbol{R}^3)$ by

$$\psi^j = (2\pi)^{-3/2}\chi_j * \psi \quad \text{or equivalently} \quad \hat{\psi}^j = \hat{\chi}_j \cdot \hat{\psi}, \ j \in \boldsymbol{Z}\,,$$

yielding $\psi = \sum_{j=-\infty}^{+\infty} \psi^j$. Using ψ^j, we define the operator

$$T_j f(x) = \int_0^\infty \psi_t^j * O_{\omega/\nu}^T(t) f(O_{\omega/\nu}(t)\cdot - \frac{k}{\nu}te_3)(x)\frac{dt}{t} \,. \tag{2.11}$$

Now we have to prove that the series $\sum_{j=-\infty}^\infty T_j$ converges in the operator norm topology on $L^q(\boldsymbol{R}^3)^3$.

LEMMA 2.1. *The function* ψ_t^j, $j \in \boldsymbol{Z}$, $t > 0$, *has the following properties*:

(1) $\operatorname{supp} \psi_t^j \subset A(2^{j-1}/\sqrt{t}, 2^{j+1}/\sqrt{t})$.

(2) $\|\psi^j\|_1 \le c2^{-2|j|}$ *and* $|\psi^j(x)| \le c2^{-2|j|}h_{2^{-2|j|}}(x)$ *for all* $x \in \boldsymbol{R}^3$, *where* $h(x) = (1+|x|^2)^{-2}$, $h_t(x) = t^{-3/2}h(x/\sqrt{t})$ *and* $c > 0$ *is a constant independent of* $j \in \boldsymbol{Z}$ *and of* $x \in \boldsymbol{R}^3$.

PROOF. See Lemma 3.1 in [3]. □

LEMMA 2.2. *For* $j \in \boldsymbol{Z}$ *let* $\mathcal{M}_j$ *denote the maximal operator*

$$\mathcal{M}_j g(x) = \sup_{r>0} \int_{A_r} (|\psi_t^j| * |g|)\left(O_{\omega/\nu}^T(t)x + \frac{k}{\nu}te_3\right)\frac{dt}{t}, \tag{2.12}$$

where $A_r = [r/16, 16r]$. *Then for* $q \in (2,\infty)$ *the operator* T_j *satisfies the estimate*

$$\|T_j f\|_q \le c2^{-|j|}\|\mathcal{M}_j\|_{(q/2)'}^{1/2}\|f\|_q \tag{2.13}$$

with a constant $c > 0$ *independent of* $j \in \boldsymbol{Z}$. *The term* $\|\mathcal{M}_j\|_{(q/2)'}$ *denotes the operator norm of the sublinear operator* $\mathcal{M}_j$ *on* $L^{(q/2)'}(\boldsymbol{R}^3)^3$, *the dual of* $L^{q/2}(\boldsymbol{R}^3)^3$.

PROOF. To estimate $\|T_j f\|_q$ we use the Littlewood-Paley decomposition of $T_j f$, i.e., $\|T_j f\|_q$ will be replaced by the equivalent L^q-norm

$$\left\|\left(\int_0^\infty |\varphi_s * T_j f(\cdot)|^2 \frac{ds}{s}\right)^{1/2}\right\|_q, \tag{2.14}$$

where $\varphi \in \mathcal{S}(\boldsymbol{R}^3)$, $\varphi_s(x) = s^{-3/2}\varphi(x/\sqrt{s})$ for $s > 0$, $\hat{\varphi}_s(\xi) = \tilde{\varphi}(\sqrt{s}|\xi|)$ and where $\tilde{\varphi} \in C_0^\infty(1/2, 2)$ satisfies $0 \le \tilde{\varphi} \le 1$ and $\int_0^\infty \tilde{\varphi}(s)^2(ds/s) = 1/2$, see I §8.23 in [16]. Thus there exists $0 \le g \in L^{(q/2)'}(\boldsymbol{R}^3)$ with $\|g\|_{(q/2)'} = 1$ such that

$$\begin{aligned}\|T_j f\|_q^2 &\sim \left\|\int_0^\infty |\varphi_s * T_j f(\cdot)|^2 \frac{ds}{s}\right\|_{q/2} = \int_{\boldsymbol{R}^3} g(x)\int_0^\infty |\varphi_s * T_j f(x)|^2 \frac{ds}{s}dx \\ &= \int_0^\infty \left(\int_{\boldsymbol{R}^3} |\varphi_s * T_j f(x)|^2 g(x)dx\right)\frac{ds}{s}.\end{aligned}$$

By (2.11) and the radial symmetry of φ_s and of ψ_t^j

$$\begin{aligned}\varphi_s * T_j f(x) &= \int_0^\infty O_{\omega/\nu}^T(t)\varphi_s * \psi_t^j * \left[f(O_{\omega/\nu}(t)\cdot - \frac{k}{\nu}te_3)\right](x)\,\frac{dt}{t} \\ &= \int_{A(s,j)} O_{\omega/\nu}^T(t)(\varphi_s * \psi_t^j * f)\left(O_{\omega/\nu}(t)x - \frac{k}{\nu}te_3\right)\frac{dt}{t},\end{aligned} \tag{2.15}$$

since

$$\varphi_s * \psi_t^j = 0 \quad \text{unless} \quad t \in A(s,j) := [2^{2j-4}s, 2^{2j+4}s].$$

Note that $\int_{A(s,j)}(dt/t) = \log 2^8$ for every $j \in \boldsymbol{Z}$ and $s > 0$. Thus the inequality of Cauchy-Schwarz, the associativity of convolutions, the inequality

$$|\psi_t^j * (\varphi_s * f)(y)|^2 \le \|\psi_t^j\|_1(\psi_t^j * |\varphi_s * f|^2)(y)$$

and Lemma 2.1 imply that

$$
\begin{aligned}
|\varphi_s * T_j f(x)|^2 &\le c \int_{A(s,j)} \left| (\psi_t^j * (\varphi_s * f))\Big(O_{\omega/\nu}(t)x - \frac{k}{\nu} te_3 \Big) \right|^2 \frac{dt}{t} \\
&\le c_j \int_{A(s,j)} (|\psi_t^j| * |\varphi_s * f|^2)\Big(O_{\omega/\nu}(t)x - \frac{k}{\nu} te_3 \Big) \frac{dt}{t} ,
\end{aligned} \tag{2.16}
$$

where $c_j = c2^{-2|j|}$. Consequently, by (2.15) and (2.16)

$$
\begin{aligned}
\|T_j f\|_q^2 &\le c_j \int_0^\infty \int_{A(s,j)} \int_{\boldsymbol{R}^3} |\psi_t^j| * |\varphi_s * f|^2 \Big(O_{\omega/\nu}(t)x - \frac{k}{\nu} te_3 \Big) g(x) dx \frac{dt}{t} \frac{ds}{s} \\
&= c_j \int_0^\infty \int_{A(s,j)} \int_{\boldsymbol{R}^3} |\psi_t^j| * |\varphi_s * f|^2(x) g\Big(O^T_{\omega/\nu}(t)x + \frac{k}{\nu} te_3 \Big) dx \frac{dt}{t} \frac{ds}{s} \\
&= c_j \int_{\boldsymbol{R}^3} \int_0^\infty |\varphi_s * f|^2(y) \int_{A(s,j)} |\psi_t^j| * g\Big(O^T_{\omega/\nu}(t)y + \frac{k}{\nu} te_3 \Big) \frac{dt}{t} \frac{ds}{s} dy \\
&\le c_j \int_{\boldsymbol{R}^3} \Big(\int_0^\infty |\varphi_s * f|^2(y) \frac{ds}{s} \Big) \mathcal{M}_j g(y) dy ,
\end{aligned}
$$

where also the identity $\int_{\boldsymbol{R}^3} |\psi_t^j| * v(x) h(x) dx = \int_{\boldsymbol{R}^3} v(y) |\psi_t^j| * h(y) dy$ has been used. Now, by Hölder's inequality

$$
\|T_j f\|_q^2 \le c2^{-2|j|} \left\| \int_0^\infty |\varphi_s * f|^2(\cdot) \frac{ds}{s} \right\|_{q/2} \|\mathcal{M}_j g\|_{(q/2)'} ,
$$

and the Littlewood-Paley decomposition of f, cf. (2.14) for $T_j f$, completes the proof. □

LEMMA 2.3. *The maximal operator $\mathcal{M}_j$, cf.* (2.12), *satisfies on $L^p(\boldsymbol{R}^3)^3$, $1 < p < \infty$, the operator norm estimate*

$$
\|\mathcal{M}_j\|_p \le c2^{-2|j|}, \quad j \in \boldsymbol{Z} ,
$$

where $c = c(p)$ is independent of $j \in \boldsymbol{Z}$.

PROOF. By Lemma 2.1 together with the trivial inequality $h_{t2^{-2|j|}}(x) \le c h_{s2^{-2|j|}}(x)$ for all $j \in \boldsymbol{Z}$, $t \in A_s = [s/16, 16s]$, $s > 0$, and $x \in \boldsymbol{R}^3$, where $c > 0$ is independent of j, t, s and x,

$$
\begin{aligned}
\mathcal{M}_j g(x) &\le c2^{-2|j|} \sup_{s>0} h_{s2^{-2|j|}} * \int_{A_s} |g| \Big(O^T_{\omega/\nu}(t)x + \frac{k}{\nu} te_3 \Big) \frac{dt}{t} \\
&\le c2^{-2|j|} \sup_{r>0} h_r * \sup_{s>0} \frac{1}{s} \int_{A_s} |g| \Big(O^T_{\omega/\nu}(t)x + \frac{k}{\nu} te_3 \Big) dt .
\end{aligned}
$$

Next we will use the classical Hardy-Littlewood maximal operator $\mathcal{M}$ on $L^p(\boldsymbol{R}^3)$ defined by

$$
\mathcal{M}g(x) := \sup_{s>0} \frac{1}{|B_s(x)|} \int_{B_s(x)} |g(y)| dy
$$

and a “helical" maximal operator

$$\mathcal{M}_{\text{hel}}\, g(\theta, x_3) := \sup_{s>0} \frac{1}{s} \int_{A_s} |g| \Big(\theta - \frac{\omega}{\nu} t, x_3 + \frac{k}{\nu} t\Big) dt$$

for functions g depending on (θ, x_3) which are 2π-periodic in θ. Since $0 \le h \in L^1(\boldsymbol{R}^3)$ is radially symmetric and strictly decreasing,

$$\sup_{r>0} h_r * u(x) \le c\mathcal{M} u(x)\,,$$

cf. II §2.1 in [16]. Hence

$$\mathcal{M}_j\, g(x) \le c 2^{-2|j|} \mathcal{M}(\mathcal{M}_{\text{hel}}\, g_r(\cdot,\cdot))(x)\,,$$

where $g_r(\theta, x_3) = g(r, \theta, x_3) = g(x)$ is considered as a function of θ, x_3 only, and

$$\|\mathcal{M}_j\, g\|_p \le c 2^{-2|j|} \|\mathcal{M}_{\text{hel}}\, g_r(\cdot,\cdot)\|_{L^p(\boldsymbol{R}^3)} \tag{2.17}$$

due to the L^p-continuity of $\mathcal{M}$. To estimate $\mathcal{M}_{\text{hel}}\, g_r(\cdot,\cdot)$ in $L^p(\boldsymbol{R}^3)$, fix $r > 0$ and use the 2π-periodicity of g_r with respect to θ to get that

$$\begin{aligned}
&\int_{\boldsymbol{R}} \int_0^{2\pi} |\mathcal{M}_{\text{hel}}\, g_r(\theta, x_3)|^p \,d\theta\, dx_3 \\
&\quad\le \int_{\boldsymbol{R}} \int_0^{2\pi} \left| \sup_{s>0} \frac{1}{s} \int_{-16s}^{16s} |g_r| \Big(\theta - \frac{\omega}{k}\Big(x_3 + \frac{k}{\nu} t\Big), x_3 + \frac{k}{\nu} t\Big) dt \right|^p d\theta\, dx_3 \\
&\quad= \int_{\boldsymbol{R}} \int_0^{2\pi} \left| \sup_{s>0} \frac{1}{s} \int_{-16s}^{16s} \gamma_{r,\theta}\Big(x_3 + \frac{k}{\nu} t\Big) dt \right|^p d\theta\, dx_3\,,
\end{aligned}$$

where $\gamma_{r,\theta}(y_3) = |g_r|(\theta - (\omega/k) y_3, y_3)$. Thus a variant of the Hardy-Littlewood maximal operator on $\boldsymbol{R}^1$ applied to $\gamma_{r,\theta}(\cdot)$ yields a constant $c > 0$ independent of r, θ and of k/ν such that

$$\begin{aligned}
\int_{\boldsymbol{R}} \int_0^{2\pi} |\mathcal{M}_{\text{hel}}\, g_r(\theta, x_3)|^p \,d\theta\, dx_3 &\le c \int_0^{2\pi} \int_{\boldsymbol{R}} \big|\gamma_{r,\theta}(x_3)\big|^p dx_3\, d\theta \\
&= c \int_0^{2\pi} \int_{\boldsymbol{R}} \big|g_r(\theta, x_3)\big|^p dx_3\, d\theta\,.
\end{aligned}$$

Now a further integration with respect to the measure $r\,dr$, $r \in (0, \infty)$, proves the L^p-estimate

$$\|\mathcal{M}_{\text{hel}}\, g_r(\cdot,\cdot)\|_{L^p(\boldsymbol{R}^3)} \le c\|g\|_{L^p(\boldsymbol{R}^3)}\,. \tag{2.18}$$

Combining (2.17) and (2.18) completes the proof. □

PROOF OF THEOREM 1.1 (1). By Lemma 2.2 and Lemma 2.3 the operator T_j, see (2.11), satisfies the estimate

$$\|T_j f\|_q \le c 2^{-2|j|} \|f\|_q\,, \quad j \in \boldsymbol{Z}\,,\ f \in \mathcal{S}(\boldsymbol{R}^3)^3\,,$$

for $q \in (2,\infty)$ and with a constant $c = c(q) > 0$. Hence $T = \sum_{j=-\infty}^{\infty} T_j$ converges in the operator norm on $L^q(\boldsymbol{R}^3)^3$ and $\|Tf\|_q \le c\|f\|_q$, i.e., for every $f \in \mathcal{S}(\boldsymbol{R}^3)^3$ the equation (2.4) with $p = 0$ has a solution $u \in \mathcal{S}'(\boldsymbol{R}^3)^3 \cap \hat{W}^{2,q}(\boldsymbol{R}^3)^3$ satisfying the inequality

$$\|\nu\nabla^2 u\|_q \le c\|\nu\Delta u\|_q \le c\|f\|_q, \quad f \in \mathcal{S}(\boldsymbol{R}^3)^3 . \tag{2.19}$$

To prove an analogous inequality for $\partial_3 u$ we use a representation of $k\partial_3 u$ induced by (2.7), i.e., using

$$k' = k/\tilde{\omega}, \quad \nu' = \nu/\tilde{\omega} \quad \text{and} \quad D(\xi) = 1 - e^{-2\pi(\nu'|\xi|^2 + ik'\xi_3)},$$

we have

$$\widehat{k\partial_3 u}(\xi) = \frac{ik'\xi_3}{D(\xi)} \int_0^{2\pi} e^{-(\nu'|\xi|^2 + ik'\xi_3)t} O_{e_3}^T(t) \hat{f}(O_{e_3}(t)\xi)\,dt \tag{2.20}$$

for $f \in \mathcal{S}(\boldsymbol{R}^3)^3$. Choose a cut-off function $\eta \in C_0^\infty(B_1(0))$ with $\eta(\xi) = 1$ for $\xi \in B_{1/2}(0)$ and recall the effect of the multiplicative term $e^{-ik't\xi_3}$ in (2.7) through (2.9). Thus we may write

$$\widehat{k\partial_3 u}(\xi) = m_0(\xi)\hat{I}_0(\xi) + m_1(\xi)\hat{I}_1(\xi) , \tag{2.21}$$

where, using $\eta_{\nu'}(\xi) = \eta(\sqrt{\nu'}\xi)$,

$$m_0(\xi) = \frac{ik'\xi_3\eta_{\nu'}(\xi)}{D(\xi)}, \quad m_1(\xi) = \frac{k'}{\sqrt{\nu'}} \frac{1 - \eta_{\nu'}(\xi)}{D(\xi)} \tag{2.22}$$

and

$$I_0(x) = \int_0^{2\pi} E'_t * O_{e_3}^T(t) f(O_{e_3}(t)\cdot - k'te_3)(x)\,dt ,$$

$$I_1(x) = \sqrt{\nu'} \int_0^{2\pi} \partial_3 E'_t * O_{e_3}^T(t) f(O_{e_3}(t)\cdot - k'te_3)(x)\,dt ,$$

where $E'_t(\cdot)$ denotes the heat kernel with ν' replacing ν. Since $\|E'_t\|_1 = 1$, $|\partial_3 E'_t(x)| \le (c/\sqrt{\nu' t})E'_t(x/2)$ and $\|f(O_{e_3}(t)\cdot - k'te_3)\|_p = \|f\|_p$, Young's inequality yields

$$\|I_0\|_q \le 2\pi\|f\|_q \quad \text{and} \quad \|I_1\|_q \le c\|f\|_q , \tag{2.23}$$

where $c > 0$ is independent of k, ω and ν. Furthermore, an elementary, but lengthy calculation will show that m_0, m_1 satisfy the following pointwise estimates

$$\max_{j=0,1} \max_{\alpha} \sup_{\xi \ne 0} |\xi^\alpha D_\xi^\alpha m_j(\xi)| \le c\left(1 + \left(\frac{k'}{\sqrt{\nu'}}\right)^4\right) = c\left(1 + \frac{k^4}{\nu^2|\omega|^2}\right) \tag{2.24}$$

with a constant $c > 0$ independent of ν, ω and k; here $\alpha \in \boldsymbol{N}_0^3$ runs through the set of all multi-indices $\alpha \in \{0,1\}^3$.

The proof of (2.24) for m_1 is immediate, since $m_1(\xi) = 0$ unless $\nu'|\xi|^2 \ge 1/2$ yielding a uniform pointwise lower bound of the denominator $D(\xi)$; hence, e.g.,

$$\xi_3\partial_3 m_1(\xi) = \frac{k'\xi_3}{\sqrt{\nu'}}\left(-\frac{\sqrt{\nu'}}{D(\xi)}(\partial_3\eta)(\sqrt{\nu'}\xi) - (1-\eta_{\nu'}(\xi))\frac{2\pi}{D(\xi)^2}(2\nu'\xi_3 + ik')e^{-2\pi(\nu'|\xi|^2 + ik'\xi_3)}\right),$$

and consequently,

$$|\xi_3\partial_3 m_1(\xi)| \le c\left(\frac{k'}{\sqrt{\nu'}} + \left(\frac{k'}{\sqrt{\nu'}}\right)^2\right) \le c\left(1 + \left(\frac{k'}{\sqrt{\nu'}}\right)^2\right).$$

Concerning m_0 note that $m_0(\xi) = 0$ unless $\nu'|\xi|^2 \le 1$. However, since $k'\xi_3$ may take arbitrary values, the denominator D needs a more careful analysis. For each $\xi \in \boldsymbol{R}^3$ there exists an $n \in \boldsymbol{Z}$ such that $|k'\xi_3 - n| \le 1/2$, yielding due to Taylor's expansion of $1 - e^{-z}$

$$D(\xi) \sim 2\pi(\nu'|\xi|^2 + i(k'\xi_3 - n)).$$

If $n = 0$, i.e., $|k'\xi_3| \le 1/2$, then

$$|m_0(\xi)| \le c\frac{|ik'\xi_3|}{|\nu'|\xi|^2 + ik'\xi_3|} \le c.$$

If $n \ne 0$ and $|k'\xi_3 - n| \le 1/4$, then

$$|m_0(\xi)| \le c\frac{|ik'\xi_3|}{|\nu'|\xi|^2 + i(k'\xi_3 - n)|} \le c\frac{|k'\xi_3|}{\nu'|\xi|^2} \le c\left|\frac{k'}{\nu'\xi_3}\right| \le c\left(\frac{k'}{\sqrt{\nu'}}\right)^2,$$

since $|k'\xi_3| \ge 3/4$. Finally, if $1/4 < |k'\xi_3 - n| \le 1/2$ and $n \ne 0$, then $|D|$ has a uniform positive lower bound and

$$|m_0(\xi)| \le c|k'\xi_3| \le c\frac{|k'|}{\sqrt{\nu'}},$$

since $\nu'\xi_3^2 \le 1$. Summarizing, we get that

$$\|m_0\|_\infty \le c\left(1 + \left(\frac{k'}{\sqrt{\nu'}}\right)^2\right). \tag{2.25}$$

However, the derivative $\xi_3\partial_3 m_0(\xi)$ yields a term

$$2\pi\frac{ik'\xi_3}{D(\xi)^2}(2\nu'\xi_3^2 + ik'\xi_3)e^{-2\pi(\nu'|\xi|^2 + ik'\xi_3)},$$

which can be estimated by the fourth order term $c(1 + (k'/\sqrt{\nu'})^4)$. Since the application of the derivatives $\xi_1\partial_1$ and $\xi_2\partial_2$ does not require further powers of $k'/\sqrt{\nu'}$, the inequality (2.24) is proved.

Now, Marcinkiewicz' multiplier theorem [15] and (2.21) through (2.24) yield for every $q \in (1,\infty)$ the *a priori* estimate

$$\|k\partial_3 u\|_q \le c\left(1 + \frac{k^4}{\nu^2|\omega|^2}\right)\|f\|_q, \quad f \in \mathcal{S}(\boldsymbol{R}^3)^3, \tag{2.26}$$

with a constant $c > 0$ independent of f, k, ν and ω.

To extend (2.19) and (2.26) to arbitrary $f \in L^q(\boldsymbol{R}^3)^3$, $q > 2$, and to get a vector field $u \in L^1_{\mathrm{loc}}(\boldsymbol{R}^3)^3$ with $\nabla^2 u, \partial_3 u \in L^q(\boldsymbol{R}^3)$ solving

$$Lu := -\nu\Delta u + k\partial_3 u - (\omega \wedge x)\cdot\nabla u + \omega \wedge u = f,$$

choose a sequence $(f_j) \subset \mathcal{S}(\boldsymbol{R}^3)^3$ such that $f_j \to f$ in $L^q(\boldsymbol{R}^3)^3$ as $j \to \infty$. Let $(u_j) \subset L^1_{\mathrm{loc}}(\boldsymbol{R}^3)^3$ denote the corresponding solutions of $Lu_j = f_j$ satisfying

$$\sup_j (\|\nu\nabla^2 u_j\|_q + \|k\partial_3 u_j\|_q) < \infty\,. \tag{2.27}$$

Now there are constant vectors $c_j, d_{1j}, d_{2j} \in \boldsymbol{R}^3$ such that

$$\int_{B_1} (\partial_1 u_j - d_{1j})dx = \int_{B_1} (\partial_2 u_j - d_{2j})dx = \int_{B_1} (u_j - (c_j + d_{1j}x_1 + d_{2j}x_2))dx = 0\,.$$

Then Poincaré's inequality and the *a priori* estimate (2.27) imply that for all $m \in \boldsymbol{N}$

$$\sup_j (\|\nabla^2(u_j - r_j)\|_q + \|\partial_3(u_j - r_j)\|_q + \|u_j - r_j\|_{L^q(B_m)}) \le C_m$$

for some constant $C_m > 0$; here r_j denotes the linear polynomial $r_j(x) = c_j + d_{1j}x_1 + d_{2j}x_2$. Using concepts of weak convergence and compact embeddings, we find a subsequence—again denoted by $(u_j - r_j)$—and $\tilde{u} \in L^1_{\mathrm{loc}}(\boldsymbol{R}^3)^3$ such that

$$\begin{aligned} \|\nabla^2(u_j - r_j) - \nabla^2\tilde{u}\|_q + \|\partial_3(u_j - r_j) - \partial_3\tilde{u}\|_q \to 0\,, \\ u_j - r_j \to \tilde{u} \quad \text{in } L^q(B_m) \quad \text{for all} \quad m \in \boldsymbol{N} \end{aligned} \tag{2.28}$$

as $j \to \infty$. In particular, $L(u_j - r_j) \to L\tilde{u}$ in the sense of distributions, and since $Lu_j = f_j$, also $Lr_j \to L\tilde{u} - f$ in $\mathcal{D}'(\boldsymbol{R}^3)^3$. Since the space of linear polynomials Π_1 and also $L(\Pi_1^3) \subset \Pi_1^3$ are finite-dimensional, $Lr_j \to Lr$ for some $r \in \Pi_1^3$ as $j \to \infty$. Hence, with $u = \tilde{u} + r$, we get that $Lu = f$ and by (2.28) that

$$\|\nu\nabla^2 u\|_q \le c\|f\|_q, \quad \|k\partial_3 u\|_q \le c\left(1 + \frac{k^4}{\nu^2|\omega|^2}\right)\|f\|_q \tag{2.29}$$

when $q > 2$.

To prove (2.29) also for $q \in (1, 2)$, we use a standard duality argument. The adjoint T^* of T is given by

$$T^*g(x) = \int_0^\infty (\psi_t * O_{\omega/\nu}(t)g)\left(O^T_{\omega/\nu}(t)x + \frac{k}{\nu}te_3\right)\frac{dt}{t}, \quad g \in \mathcal{S}(\boldsymbol{R}^3)^3\,.$$

Checking the proofs of Lemmata 2.2 and 2.3, we easily see that $\|T^*g\|_{q'} \le c\|g\|_{q'}$ in the dual space $L^{q'}(\boldsymbol{R}^3)^3$. Thus $\|Tf\|_q \le c\|f\|_q$ for all $q \in (1, 2)$ and $f \in \mathcal{S}(\boldsymbol{R}^3)^3$. Since (2.26) has been proved for all $q \in (1, \infty)$, we get (2.29) for $q \in (1, 2)$, $f \in L^q(\boldsymbol{R}^3)^3$ and a solution u of $Lu = f$.

The remaining case $q = 2$ can be proved by complex interpolation or by Plancherel's Theorem.

Now the proof of part (1) is complete (except for the equation $\operatorname{div} u = g$, see below). □

PROOF OF THEOREM 1.1 (2). To prove this uniqueness and regularity assertion it suffices to consider a solution $(u, p) \in \mathcal{S}'(\boldsymbol{R}^3)^4$ of (1.6) when $f = 0$ und $g = 0$. Then it has to be shown that u equals a linear polynomial $\alpha e_3 + \beta\omega \wedge x$ for suitable $\alpha, \beta \in \boldsymbol{R}$ and that p is constant. In Fourier space (1.6) yields the equations

$$\nu|\xi|^2\hat{u} + ik\xi_3\hat{u} - (\omega \wedge \xi) \cdot \nabla_\xi \hat{u} + \omega \wedge \hat{u} + i\xi\hat{p} = 0, \quad i\xi \cdot \hat{u} = 0\,,$$

and hence $|\xi|^2\hat{p} = 0$. Thus $\operatorname{supp}\hat{p} \subset \{0\}$ and p is a polynomial. Since ∇p is assumed to be contained in $L^q(\boldsymbol{R}^3)^3 + L^r(\boldsymbol{R}^3)^3$ for $1 < q, r < \infty$, p must be constant. To analyze $\hat{u}$ we introduce cylindrical coordinates (s, φ, ξ_3) for ξ and let $\hat{v}(\varphi) := O_{e_3}^T(\varphi)\hat{u}(s, \varphi, \xi_3)$ as before. Since $\hat{p} = \hat{p}_0\delta_0$, where $\hat{p}_0 \in \boldsymbol{C}$ and δ_0 denotes Dirac's δ-distribution,

$$\frac{1}{\tilde{\omega}}(\nu|\xi|^2 + ik\xi_3)\hat{v} - \partial_\varphi\hat{v} = -\frac{i\hat{p}_0}{\tilde{\omega}}\delta_0 \quad \text{in } \mathcal{S}(\boldsymbol{R}^3)^3\,, \tag{2.30}$$

cf. (2.6). Now consider an arbitrary test function $\psi \in C_0^\infty(\boldsymbol{R}^3\backslash\{0\})^3$, and let

$$\psi_0(s, \varphi, \xi_3) := e^{-(\nu'|\xi|^2 + ik'\xi_3)\varphi} \int_{-\infty}^{\varphi} e^{(\nu'|\xi|^2 + ik'\xi_3)\varphi'}\psi(s, \varphi', \xi_3)\,d\varphi'\,,$$

where $\nu' = \nu/\tilde{\omega}$, $k' = k/\tilde{\omega}$. Obviously, $\psi_0 \in C_0^\infty(\boldsymbol{R}^3\backslash\{0\})^3$ and $(\nu'|\xi|^2 + ik'\xi_3)\psi_0 + \partial_\varphi\psi_0 = \psi$. Consequently,

$$\langle\hat{v}, \psi\rangle = \langle\hat{v}, (\nu'|\xi|^2 + ik'\xi_3 + \partial_\varphi)\psi_0\rangle = \langle(\nu'|\xi|^2 + ik\xi_3 - \partial_\varphi)\hat{v}, \psi_0\rangle = 0$$

due to (2.30) and since $\operatorname{supp}\psi_0 \subset \boldsymbol{R}^3\backslash\{0\}$. Hence $\operatorname{supp}\hat{v} \subset \{0\}$ and also $\operatorname{supp}\hat{u} \subset \{0\}$, implying that u is a polynomial. Since by assumption $\nabla^2 u$ are contained in $L^q(\boldsymbol{R}^3) + L^r(\boldsymbol{R}^3)$, $\nabla^2 u = 0$ and u is a linear polynomial, say, $u(x) = a + Bx$ with $a \in \boldsymbol{R}^3$ and a real 3×3-matrix $B = (b_{ij})_{1 \le i,j \le 3}$. Then an elementary calculation will show that $a = \alpha e_3$, $\alpha \in \boldsymbol{R}$, and $B_{ij} = 0$ except for $B_{21} = -B_{12} = \beta \in \boldsymbol{R}$. Hence $u(x) = \alpha e_3 + \beta\omega \wedge x$. □

PROOF OF COROLLARY 1.2 (1). Since $\nabla^2 u, \partial_3 u \in L^q(\boldsymbol{R}^3)$, $1 < q < 4$, there exists a unique real 3×3-matrix $B = \big(b_{ij}\big)_{1 \le i,j \le 3}$ such that

$$\nabla(u - Bx) \in L^r(\boldsymbol{R}^3)^9 \quad \text{for all} \quad r > 1,\ \frac{1}{r} \in \frac{1}{q} - \left[\frac{1}{4}, \frac{1}{3}\right]\,, \tag{2.31}$$

see Theorem 2.3 in [4] or Chapter VII.4 in [6]. Moreover, $\operatorname{div} u \in L^q(\boldsymbol{R}^3)$ by assumption and $\operatorname{div} u - \operatorname{tr} B \in L^r(\boldsymbol{R}^3)$, yielding $\operatorname{tr} B = 0$. Analogously, the assumption $\partial_3 u \in L^q(\boldsymbol{R}^3)^3$ implies that the coefficients b_{13}, b_{23}, and b_{33} vanish. Concerning u_3 the inequality (1.8) shows that $-x_2\partial_1 u_3 + x_1\partial_2 u_3 \in L^q(\boldsymbol{R}^3)$, yielding

$$\frac{1}{|x|}(-x_2\partial_1 u_3 + x_1\partial_2 u_3) \in L^q(B_1^c)\,;$$

on the other hand, $\nabla' u_3 - (b_{31}, b_{32})^T \in L^r(B_1^c)^2$. Hence

$$\frac{1}{|x|}(-x_2 b_{31} + x_1 b_{32}) \in L^q(B_1^c) + L^r(B_1^c)\,,$$

which is possible if and only if $b_{31} = b_{32} = 0$. Summarizing the previous results, we conclude from (2.31) the existence of constants $\beta, \gamma, \delta \in \boldsymbol{R}$ such that

$$v' := u' - \beta\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} - \gamma\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} - \delta\begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$$

satisfies $\nabla v' \in L^r(\boldsymbol{R}^3)^6$ for all $r > 1,\ 1/r = 1/q - [1/4, 1/3]$. Furthermore, by (1.8)

$$L'u' := -\frac{1}{|x|}(-x_2\partial_1 u' + x_1\partial_2 u') + \frac{1}{|x|}(-u_2, u_1)^T \in L^q(B_1^c)^2 .$$

Note that

$$L'\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} = \frac{2}{|x|}\begin{pmatrix} x_2 \\ x_1 \end{pmatrix} \quad \text{and} \quad L'\begin{pmatrix} x_2 \\ x_1 \end{pmatrix} = -\frac{2}{|x|}\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix},$$

but that

$$L'\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} = 0 .$$

Case 1. $1 < q < 12/7$. In this case, r defined by $1/r = 1/q - 1/4$ satisfies $1 < r < 3$. Hence the result $\nabla' v' \in L^r(B_1^c)^4$ and Theorem II5.1 in [6] yield the existence of a vector $v'_\infty \in \boldsymbol{R}^2$ such that

$$\frac{v' - v'_\infty}{|x|} \in L^r(B_1^c)^2 .$$

Thus

$$L'(u' - (v' - v'_\infty)) = \frac{2\gamma}{|x|}\begin{pmatrix} x_2 \\ x_1 \end{pmatrix} - \frac{2\delta}{|x|}\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} + \frac{1}{|x|}v'^{\perp}_\infty \in L^q(B_1^c)^2 + L^r(B_1^c)^2 ,$$

proving that $\gamma = \delta = 0$ and $v'_\infty = 0$. Consequently, $v = u - \tilde{\beta}\omega \wedge x$ with $\tilde{\beta} = \beta/\tilde{\omega}$ satisfies $\nabla v \in L^r(\boldsymbol{R}^3)^9$.

Case 2. $12/7 \le q < 4$. If $q < 3$, define $r > 1$ by $1/r = 1/q - 1/3$, yielding $r \ge 4$; however, if $3 \le q < 4$, let $1/r = 1/q - 1/4$, yielding $r \ge 12$. Hence $r > 3$ and $v' \in \hat{W}^{1,r}(\boldsymbol{R}^3)^2 \subset C^{0,\alpha}_{\text{loc}}(\boldsymbol{R}^3)^2$, where $\alpha = 1 - 3/r > 0$. Then, by Lemma 2.4 below,

$$\frac{v' - v'_0}{|x|} \in L^r(\boldsymbol{R}^3)^2 ,$$

where $v'_0 \in \boldsymbol{R}^2$ is well-defined. Arguing as in Case 1, we get that

$$L'(u' - (v' - v'_0)) = \frac{2\gamma}{|x|}\begin{pmatrix} x_2 \\ x_1 \end{pmatrix} - \frac{2\delta}{|x|}\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} + \frac{1}{|x|}v'^{\perp}_0 \in L^q(B_1^c)^2 + L^r(B_1^c)^2 .$$

Although $(1/|x|)v'^{\perp}_0 \in L^r(B_1^c)^2$, we may conclude that $\gamma = \delta = 0$. Consequently, $v = u - \tilde{\beta}\omega \wedge x$ with $\tilde{\beta} = \beta/\tilde{\omega}$ satisfies $\nabla v \in L^r(\boldsymbol{R}^3)^9$.

(2). By Theorem 2.3 in [4] there exists $a \in \boldsymbol{R}^3$ such that

$$v = u - a \in L^s(\boldsymbol{R}^3) \quad \text{for all} \quad s > 1, \frac{1}{s} \in \frac{1}{q} - \left[\frac{1}{2}, \frac{2}{3}\right] ;$$

here we assume for simplicity that $\beta \in \boldsymbol{R}$ from part (1) vanishes, yielding $\nabla v \in L^r(\boldsymbol{R}^3)^9$ for all $r > 1, 1/r = 1/q - [1/4, 1/3]$. Let

$$\tilde{L}u' := -(-x_2\partial_1 u' + x_1\partial_2 u') + u'^{\perp} = -\partial_\theta u' + u'^{\perp} .$$

Since $\tilde{L}v' = -\partial_\theta v' + v'^{\perp} = \tilde{L}u' - a'^{\perp}$, we get from an integration with respect to $\theta \in [0, 2\pi]$ that

$$2\pi a'^{\perp} = \int_0^{2\pi} \tilde{L}u' d\theta - \int_0^{2\pi} v'^{\perp} d\theta \in L^q(\boldsymbol{R}^3)^2 + L^s(\boldsymbol{R}^3)^2 .$$

Hence $a' = 0$. □

LEMMA 2.4. *Let $r > n$ and $v \in \hat{W}^{1,r}(\boldsymbol{R}^n)$. To be more precise, choose $v \in L^r_{\mathrm{loc}}(\boldsymbol{R}^n)$ such that $\nabla v \in L^r(\boldsymbol{R}^n)^n$. Then $v_0 = v(0)$ is well-defined and*

$$\int_{\boldsymbol{R}^n} \frac{|v - v_0|^r}{|x|^r} dx \leq c \int_{\boldsymbol{R}^n} |\nabla v|^r dx\,.$$

PROOF. Since $r > n$, $v \in C^{1,\alpha}_{\mathrm{loc}}$, $\alpha = 1 - n/r$, and there exists a constant $c > 0$ independent of $\varepsilon > 0$ such that $|v(x) - v_0| \leq c|x|^\alpha \|\nabla v\|_{L^r(B_\varepsilon)}$ for all $x \in B_\varepsilon$. Then, for all $0 < \varepsilon < R < \infty$,

$$\begin{aligned}
\int_{B_R \setminus B_\varepsilon} \frac{|v - v_0|^r}{|x|^r} dx &= \frac{1}{n - r} \int_{B_R \setminus B_\varepsilon} |v - v_0|^r \operatorname{div}\left(\frac{x}{|x|^r}\right) dx \\
&= -\frac{r}{n - r} \int_{B_R \setminus B_\varepsilon} (v - v_0)|v - v_0|^{r-2} \nabla v \cdot \frac{x}{|x|^r} dx \\
&\quad + \frac{R^{1-r}}{n - r} \int_{\partial B_R} |v - v_0|^r do - \frac{\varepsilon^{1-r}}{n - r} \int_{\partial B_\varepsilon} |v - v_0|^r do\,.
\end{aligned}$$

Since $n - r < 0$, we omit the integral on ∂B_R and get from the Hölder continuity of $v(x)$ and from Hölder's inequality that

$$\begin{aligned}
&\int_{B_R \setminus B_\varepsilon} \frac{|v - v_0|^r}{|x|^r} dx \\
&\quad \leq \frac{r}{r - n} \left(\int_{B_R \setminus B_\varepsilon} \frac{|v - v_0|^r}{|x|^r} dx \right)^{1/r'} \left(\int_{B_R \setminus B_\varepsilon} |\nabla v|^r dx \right)^{1/r} + c\|\nabla v\|^r_{L^r(B_\varepsilon)}\,.
\end{aligned}$$

As $\varepsilon \to 0$ and $R \to \infty$, we get the assertion with the constant $c = (r/(r - n))^r$. □

PROOF OF REMARK 1.3 (1). For $(\nu, k, \tilde{\omega}) \in (\boldsymbol{R}^*_+ \times \boldsymbol{R} \times \boldsymbol{R})$ let $L_{\nu,k,\omega}$ denote the operator

$$L_{\nu,k,\omega}(u, p) = (-\nu\Delta u + k\partial_3 u - (\omega \wedge x) \cdot \nabla u + \omega \wedge u + \nabla p, \operatorname{div} u)\,.$$

Consider any sequence $(\nu_j, k_j, \tilde{\omega}_j) \subset \boldsymbol{R}^*_+ \times \boldsymbol{R} \times \boldsymbol{R}$ such that $(\nu_j, k_j, \tilde{\omega}_j) \to (\nu_0, k_0, \omega_0) \in \boldsymbol{R}^*_+ \times \boldsymbol{R} \times \boldsymbol{R}$ as $j \to \infty$. Given $f \in L^q(\boldsymbol{R}^3)^3$ and $g \in W^{1,q}(\boldsymbol{R}^3)$ with $|(x_1, x_2)|g \in L^q(\boldsymbol{R}^3)$, let $(u_j, p_j) \in \hat{W}^{2,q}(\boldsymbol{R}^3)^3 \times \hat{W}^{1,q}(\boldsymbol{R}^3)$ denote a solution of the equation $L_j(u_j, p_j) = (f, g)$ where $L_j = L_{\nu_j,k_j,\omega_j}$. If $k_j = 0$ and/or $\omega_j = 0$, we refer to [3] or to classical results for the Oseen or Stokes system in $\boldsymbol{R}^3$ [4, 5, 6, 14]. Thus we get the *a priori* estimate

$$\|\nabla^2 u_j\|_q + \|\nabla p_j\|_q \leq C(\|f\|_q + \|g\|_{W^{1,q}(\boldsymbol{R}^3)} + \|(x_1, x_2)g\|_q)$$

with a constant C independent of $j \in \boldsymbol{N}$, i.e., the sequence (u_j, p_j) is bounded in the reflexive space $\hat{W}^{2,q}(\boldsymbol{R}^3)^3 \times \hat{W}^{1,q}(\boldsymbol{R}^3)$. Hence there exists a subsequence of (u_j, p_j)—again denoted by (u_j, p_j)—and a pair $(u, p) \in \hat{W}^{2,q}(\boldsymbol{R}^3)^3 \times \hat{W}^{1,q}(\boldsymbol{R}^3)$ such that $u_j \rightharpoonup u$ in $\hat{W}^{2,q}(\boldsymbol{R}^3)^3$ and

$p_j \rightharpoonup p$ in $\hat{W}^{1,q}(\boldsymbol{R}^3)$ as $j \to \infty$. Furthermore, we find polynomials $r_j \in \Pi_1^3$ and constants $\pi_j \in \boldsymbol{R}$ such that

$$u_j - r_j \to u \ \text{ in } \ W^{1,q}(B_m)\,, \quad p_j - \pi_j \to p \ \text{ in } \ L^q(B_m)$$

for all $m \in \boldsymbol{N}$, cf. (2.27) through (2.28). Then, in the sense of distributions,

$$L_j(u_j - r_j, p_j - \pi_j) \to L_{\nu_0,k_0,\omega_0}(u, p) \quad \text{as } j \to \infty\,.$$

Thus $L_j(r_j, \pi_j) \to (f, g) - L_{\nu_0,k_0,\omega_0}(u, p)$ in the sense of distributions. Since the sequence $(L_j(r_j, \pi_j))$ runs in the finite-dimensional space Π_1^4, we conclude that $(f, g) - L_{\nu_0,k_0,\omega_0}(u, p) \in L^q(\boldsymbol{R}^3)^4 \cap \Pi_1^4$ and consequently that $L_{\nu_0,k_0,\omega_0}(u, p) = (f, g)$. Given any other weakly convergent subsequence of (u_j, p_j) with weak limit $(\tilde{u}, \tilde{p})$, it is straightforward to see that $\tilde{u} = u$ in $\hat{W}^{2,q}(\boldsymbol{R}^3)^3$ and $\tilde{p} = p$ in $\hat{W}^{1,q}(\boldsymbol{R}^3)$. Hence $u_{\nu,k,\omega}$ converges weakly to u_{ν_0,k_0,ω_0}. Furthermore, the proof may easily be extended to weakly convergent right-hand sides $(f_{\nu,k,\omega}, g_{\nu,k,\omega})$ with limit (f, g). □

PROOF OF THE EQUATION $\operatorname{div} u = g$ IN (1.6). The solution (u, p) constructed so far satisfies (2.2) and (2.4). Applying div to (2.4), we get that $v = \operatorname{div} u - g$ solves the equation

$$-\nu\Delta v + k\partial_3 v - (\omega \wedge x) \cdot \nabla v = 0 \quad \text{in } \boldsymbol{R}^3\,.$$

The arguments of the proof of uniqueness in Theorem 1.1(2) imply that

$$v = \text{const}\,,$$

since $\nabla v \in L^q(\boldsymbol{R}^3)^3$. Analogously, a solution $u \in \hat{W}^{2,q}(\boldsymbol{R}^3)^3$ of (2.4) is uniquely determined up to the affine term $\alpha e_3 + \beta\omega \wedge x + \gamma(x_1, x_2, 0)^T$, where $\alpha, \beta, \gamma \in \boldsymbol{R}$. Note that $\gamma = 0$ in the proof of Theorem 1.1(2), where $\operatorname{div} u = 0$ has been used. Here $\operatorname{div} u$ is uniquely determined up to $\operatorname{div}(\gamma(x_1, x_2, 0)^T) = 2\gamma$. Hence

$$v = \operatorname{div} u - g = 0\,,$$

when replacing u by $u - \gamma(x_1, x_2, 0)^T$ for a suitable constant $\gamma \in \boldsymbol{R}$. □

REMARK 1.3 (5). Finally, we discuss the term $c(1 + (k/\sqrt{\nu|\omega|})^4)$ in the *a priori* estimate (1.8). For $q = 2$ the properties of the multiplier m_1, see (2.22), and the estimate (2.25) prove that (1.8) holds with $c(1 + (k/\sqrt{\nu}|\omega|)^2)$.

Under the additional assumption that $O_\omega^T(t) f(O_\omega(t)x)$ and consequently also its Fourier transform $O_\omega^T(t)\hat{f}(O_\omega(t)\xi)$ are t-independent, the formula (2.20) simplifies to

$$\widehat{k\partial_3 u}(\xi) = \frac{ik'\xi_3}{\nu'|\xi|^2 + ik'\xi_3}\hat{f}(\xi)\,.$$

Hence, in this special case, by Marcinkiewicz' multiplier theorem, $\|k\partial_3 u\|_q \le c\|f\|_q$, $1 < q < \infty$, with a constant $c > 0$ independent of ν, k and ω.

A final example will show that even in the L^2-case the constant c in (1.8) needs the term $k^2/(\nu|\omega|)$. We start with a function $f = (f', 0) \in L^2(\boldsymbol{R}^3)^3$ such that in Fourier space

$$\hat{f}'(\xi) = \begin{cases} \xi'^{\perp}, & 0 < \varphi < \pi\,, \\ -\xi'^{\perp}, & \pi < \varphi < 2\pi\,, \end{cases}$$

when $\nu'|\xi|^2 \le 1/4$ but $\hat{f}'(\xi) = 0$ when $\nu'|\xi|^2 > 1/4$, where φ is the angular part of ξ in cylindrical coordinates. Note that f is solenoidal in the weak sense; hence the pressure p satisfying (2.2) vanishes (when $g = 0$) and $f = f - \nabla p$. Then (2.20) and an elementary integration imply that for $0 < \varphi < \pi$

$$\widehat{k\partial_3 u'}(\xi) = \hat{f}'(\xi)\frac{ik'\xi_3}{\nu'|\xi|^2 + ik'\xi_3}\left(1 - 2e^{-(\pi-\varphi)(\nu'|\xi|^2+ik'\xi_3)}\frac{1 - e^{-\pi(\nu'|\xi|^2+ik'\xi_3)}}{1 - e^{-2\pi(\nu'|\xi|^2+ik'\xi_3)}}\right).$$

For fixed k and ν, choose $|\omega| > 0$ sufficiently small and consider $\xi = (\xi', \xi_3)$ such that

$$\left||\xi'| - \frac{\omega}{k}\right| \le \frac{\omega}{2k} \quad \text{and} \quad \left|\xi_3 - \frac{\omega}{k}\right| \le \frac{\omega}{4k}\frac{\nu\omega}{k^2} \le \frac{\omega}{2k},$$

yielding $|k'\xi_3 - 1| \le \nu\omega/(4k^2) \sim \nu'|\xi|^2 \le 1/4$ and $|\nu'|\xi|^2 + ik'\xi_3| \sim |k'\xi_3| \sim 1$, but $|\nu'|\xi|^2 + i(k'\xi_3 - 1)| \sim \nu'|\xi|^2$. Hence, for these $\xi \in \boldsymbol{R}^3$ satisfying $\nu'|\xi|^2 \sim \nu\omega/k^2$

$$|\widehat{k\partial_3 u'}(\xi)| \sim |\hat{f}'(\xi)|\frac{1}{|\nu'|\xi|^2 + ik'(\xi_3 - 1)|} \sim |\hat{f}'(\xi)|\frac{1}{\nu'|\xi|^2} \sim |\hat{f}'(\xi)|\frac{k^2}{\nu\omega}.$$

This rough estimate and Plancherel's theorem show that

$$\|k\partial_3 u'\|_2 \sim \frac{k^2}{\nu\omega}\|f'\|_2. \qquad \square$$

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