An LMI Approach to Networked Control Systems with Data Packet Dropout and Transmission Delays *

Mei Yu, Long Wang, Tianguang Chu

Center for Systems and Control, Department of Mechanics and Engineering Science,
Peking University, Beijing, 100871, P. R. China. Fax: (8610)62754388,
E-mail: yumei@pku.edu.cn, longwang@mech.pku.edu.cn, chutg@pku.edu.cn

Abstract: This paper mainly deals with the problems of data packet dropout and transmission delays induced by communication channel in networked control systems (NCSs). Two approaches are introduced to deal with the problems. One is the delay system approach, the other is the switched system approach. Sufficient conditions on the stabilization of the NCSs are established in terms of linear matrix inequality (LMI), which can be easily solved with the efficient LMI toolbox. Attention is focused on the design of memoryless state feedback controller that guarantee stability of the closed-loop systems in spite of data packet loss and transmission delays.

Keywords: Networked control systems, data packet dropout, transmission delays, LMIs.

1 Introduction

Networks have received increasing attentions in recent years because of the popularization and advantages of using network cables in control systems [5, 16, 21, 27, 34]. Since the traditional point-to-point controller architectures are being replaced by those based on serial communication channel and the price of microcontroller in such architecture has dropped, networks can bring new functionalities that were not available in the past: the cost for cabling, installation, and maintenance can be reduced drastically. Examples include industrial automation, intelligent vehicle systems and advanced aircraft and spacecraft, etc.

Networked control systems (NCSs) are sampled-date systems which are composed of digital controllers interfaced to continuous-time physical plants, but NCSs do not satisfy the condition that the plant output and control inputs are delivered at the same time, which is true for the conventional sampled-data system. Due to network channel nondeterminism, the controller may not be able to receive all of the plant output updates at the time of the control calculation. NCSs are nonlinear systems, but they are different to the conventional

*Supported by the National Natural Science Foundation of China (No. 10372002 and No. 60274001), the National Key Basic Research and Development Program (No. 2002CB312200). Corresponding author: Professor Long Wang.

nonlinear systems because of their discontinuous, nondeterminism, and the variety of control mechanism. Thus, the insertion of communication network in the feedback control loop complicates the application of standard results in analysis and design of an NCS because many ideal assumptions made on the traditional control theories can't be applied to NCSs directly (see, e.g., [29]-[32] and the references therein).

When designing an NCS, however, a new constraint must be accommodated-the limited bandwidth of the communication network that are affected mainly by four factors: the sampling rates at which the devices on the network send information, the message size of the information, the network protocol that controls the transmission and the topology of the network. We assume that the actuator and sensor used to measure the process' output are connected through a communication channel with finite bandwidth, which is shared by several NCSs. The problem of stabilization with finite communication bandwidth was introduced by Wong and Brockett [30, 31] and further pursued by [9, 10]. In [11, 12], a dwell-time switching method was proposed to reduce the data rate of the network. Moreover, [11, 13] introduced signal quantization into the design of the systems, thus reduced the data rate transmitted by the communication channel. [20] made use of a model of the plant to reduce the network usage. A new framework for distributed control systems was introduced in [32], which used estimators at each node to achieve a significant savings in the required bandwidth.

One of the issues raised in NCSs is the unreliable transmission paths because of limited bandwidth and large amount of data packet transmitted over one line, which may result in data packet dropout. This may happen while exchanging data among devices, and can degrade performance and destabilize the system even without network inserted. The augmented state space method is the significant methods proposed in the literature to deal with the problem of data packet dropout [35]. The performance of real-time NCSs with data dropouts was considered in [17]. [1] used an uncertainty threshold principle to show that under certain conditions there was a rate dropped packets for which an undisturbed NCS was mean square stable. A packet dropping network was modelled as an erasure channel in [7]. However, the research mentioned above is concerned primarily with analysis issues rather than control design.

Another challenge in the NCSs is the network-induced delay effect on the control loop, which may degrade the performance of the NCSs and may result in instability, too. So far, various methodologies have been formulated to deal with the problem of network delays. An augmented state vector method was proposed in [25] to control a linear system over a periodic delay network. Queuing mechanisms was developed in [6, 18], which utilized some deterministic or probabilistic information of NCSs for the control. Random delays were treated in [23] via an optimal stochastic control methodology. See [3, 15] and the references therein for related works. However, no method has been given in the above references on how to estimate the maximum allowable value of the network-induced delay that does not violate the stability of the NCS.

Because data packet dropout and transmission delays may be potential sources to the instability and poor performance of NCSs, this paper considers stabilization of NCSs with the problem under consideration. Two kinds of approaches are introduced to deal with the problem considered in this paper. One is the delay system approach, the other is the switched system approach. We mean that the NCSs with data packet dropout and transmission delays are modelled as those two kinds of systems, and different techniques will be used to deal with them.

First, the NCSs (for the continuous case and discrete case) with such problems are modelled as linear systems with time-varying delays (continuous and discrete, respectively), which may subject to fast time-varying. This paper construct the feedback controller via solving a set of linear matrix inequalities (LMIs) which can be easily solved using the LMI toolbox [2]. The admissible bound of data packet loss and of delays can be obtained in terms of LMIs. Then, the NCSs (continuous case and discrete case) with data packet dropout and transmission delays are modelled as linear switched systems (continuous and discrete, respectively), similar results are obtained, what makes difference is that the transmission delays are constant in this part.

One technical comment should mentioned. In the first part, the continuous-time NCS with data packet dropout is modelled as a linear system with time-varying input delay. There are two main methods to deal with the continuous-time linear systems with time-varying delays: one is the Krasovskii-based method and the other is the Razumikhin-based method. Usually, Krasovskii-based method requires the varying delay satisfy $\dot{\tau}(t) < 1$, where $\tau(t)$ is the delay part, while $\tau(t)$ modelled from the real practice in this paper does not satisfy this condition. For the Razumikhin-based method, there are results to design feedback controller to stabilize a time-varying input delay system that satisfies above condition [22], unfortunately, the equations can not be directly solved and the procedure requires the tuning of parameters. The controller constructed in this paper can be directly obtained via solving the quasi-convex optimization problem [2].

2 Delay system approach

In this section, for continuous-time case and discrete-time case, we model NCSs with data packet dropout and transmission delays as linear systems with input delay, thus the theory and conclusion for the delay system can be applied to the design of such NCSs. The primary objective of this section is to design explicit express of stabilizing state feedback controllers and find the admissible upper bound of the dropped packets and transmission delays for the NCSs transmitted in single-packet manner and multiple-packet manner.

2.1 The continuous-time case

Data packet dropout in an NCS is unavoidable because of limited bandwidth. When packet collision occurs, it might be more advantageous to drop the old packet and transmit a new one than repeated retransmission attempt. We first consider the case where controller and the actuator are combined into one node and there are no transmission delays between the sensor and the combined node. An NCS with the possibility of dropping data packet can be described as in Fig. 1. The model consists of a continuous plant

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

and a continuous controller (realized by a zero-order hold (ZOH))

$$u(t) = F\bar{x}(t), \qquad t \in [t_k, t_{k-1}), \qquad k = 1, 2, \cdots, \tag{2}$$

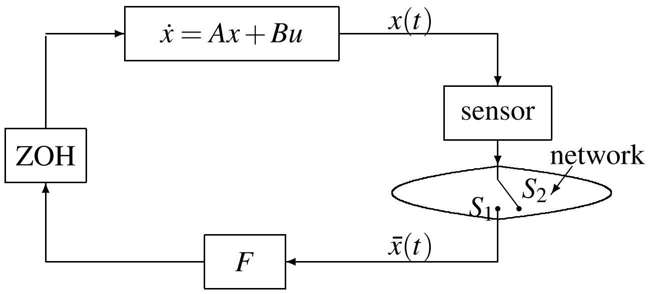

Figure 1: An NCS with data packet dropout

where $x(t) \in \mathbf{R}^n$, $u(t) \in \mathbf{R}^m$ are the plant state and the plant input respectively. F is the state feedback gain matrix to be designed, A, B are known real constant matrices with appropriate dimensions, and it is assumed in this paper that the pair (A,B) is controllable. We assume the sampling period is a positive constant scalar h. The sensor shown in Fig. 1 takes on the work of sampling and transmission, thus the output of network is the sampling value of the state. t_k is the sampling instant, and $\bar{x}(t)$ is the output of the network.

The network is modelled as a switch. When the switch is closed (in position S_1), network packet (containing $x(t_k)$) is transmitted, and the controller utilizes the updated data, whereas when it is open (in position S_2), the output of the switch is held at the previous value and the packet is lose, and the controller uses the old data. For a fixed sampling period, what we are concerned is the maximum quantity of packet loss, which will not destabilize the closed-loop system. The dynamics of the switch can be expressed as follows:

The NCS (1) with no packet dropout at time t_k: $\bar{x}(t) = x(t_k)$;

The NCS (1) with one packet dropout at time t_k: $\bar{x}(t) = x(t_k - h)$;

$$\vdots$$

The NCS (1) with $d(k)$ packets dropout at time t_k: $\bar{x}(t) = x(t_k - d(k)h)$.
The quantity of dropped packets is accumulated from the latest time when $\bar{x}(t)$ has been updated.

Thus the closed-loop system with the network packet loss effect is described as

$$\dot{x}(t) = Ax(t) + BFx(t_k - d(k)h), \qquad t \in [t_k, t_{k+1}).$$

Let $\tau(t) = t - t_k + d(k)h$, then the system can be expressed as:

$$\dot{x}(t) = Ax(t) + BFx(t - \tau(t)), \qquad t \in [t_k, t_{k+1}). \tag{3}$$

The quantity $d(k) \in Z^+$ may vary with time t and it is assumed that

$$0 \le d(k) \le d_k, \qquad d_k \in Z^+.$$

Hence the delay $\tau(t)$ satisfies

$$0 \le \tau(t) = t - t_k + d(k)h \le (\bar{d} + 1)h, \qquad t \in [t_k, t_{k+1}),$$

where $\bar{d} = \max_k \{d_k\} < \infty$. Also it is assumed that $(\bar{d}+1)h \le \bar{\tau}$, where $\bar{\tau}$ is a positive scalar.

Remark 1. *The NCS (1)–(2) with data packet dropout is modelled as a linear system (3) with time-varying input delay, and this enables us to apply the theory of delay systems to the analysis and design of such NCSs.*

Remark 2. *It can be easily verified that $\tau(t)$ does not satisfying the condition: $\dot{\tau}(t) < 1$, and thus the Razumikhin-based approach will be use to deal with the problem under consideration.*

2.1.1 Stabilization of the NCSs with data packet dropout

The following lemma will be used in the proof of our main result.

Lemma 1. *[24] For any positive definite matrix $Q \in \mathbf{R}^{n\times n}$, the following inequality holds:*

$$2x^T y \leq x^T Q^{-1} x + y^T Q y,$$

where $x \in \mathbf{R}^n$, $y \in \mathbf{R}^n$.

The following result will ensure the stabilization of NCS (1)–(2) with data packet dropout.

Theorem 1. *If there exist symmetric positive definite matrices X, R_1, $R_2 \in \mathbf{R}^{n\times n}$, a symmetric positive definite matrix $Q \in \mathbf{R}^{m\times m}$, a matrix $Y \in \mathbf{R}^{m\times n}$ satisfying the following LMIs*

$$\begin{bmatrix} \frac{1}{\bar{\tau}}(XA^T + AX + Y^T B^T + BY) + 2X & BQ \\ QB^T & -Q \end{bmatrix} < 0, \tag{4}$$

$$\begin{bmatrix} -X & Y^T B^T \\ BY & -R_2 \end{bmatrix} \leq 0, \tag{5}$$

$$\begin{bmatrix} -X & XA^T \\ AX & -R_1 \end{bmatrix} \leq 0, \tag{6}$$

$$\begin{bmatrix} Q & Y \\ Y^T & X \end{bmatrix} \leq 0, \tag{7}$$

$$R_1 + R_2 - X \leq 0. \tag{8}$$

then system (1) can be asymptotically stabilized via state feedback

$$u(t) = YX^{-1}x(t)$$

whenever the data packet dropout satisfying $0 \leq d_k \leq \frac{\bar{\tau}}{h} - 1$.

Proof. From the last section, the NCS (1)–(2) can be modelled as (3), thus we turn to consider system (3). By Newton-Leibniz formula, equation (3) can be written as

$$\dot{x} = (A + BF)x(t) - BF \int_{t-\tau(t)}^{t} (Ax(\theta) + BFx(\theta - \tau(\theta)))d\theta \tag{9}$$

with the initial condition $x(t_0+\theta) = \phi(\theta)$, $\forall\, \theta \in \xi_{t_0,2\tau}$, where ϕ is a continuous norm-bounded initial function and $\xi_{t_0,2\tau} = \{t \in R : t = \eta - 2\tau(\eta) \le t_o,\ \eta \ge t_o\}$. As is shown in [19], asymptotic stability of system (3) can be guaranteed by asymptotic stability of system (9). Next we proceed to analyze system (9).

Consider the following Lyapunov function

$$V(x_t) = x^T(t)Px(t),$$

where P is a symmetric positive definite matrix. The time derivative of $V(x_t)$ along the trajectory of system (9) is given by

$$\dot{V}(x_t) = x^T(t)(P(A+BF)+(A+BF)^TP)x(t) - 2x^T(t)PBF\int_{t-\tau(t)}^{t}(Ax(\theta)+BFx(\theta-\tau(\theta)))d\theta. \qquad (10)$$

Let $X = P^{-1}$, $F = YP$. Pre- and post-multiplying (5) and (6) by block-diag $[P\ P]$, using the standard Schur complement, we can get the following inequalities:

$$A^TR_1^{-1}A \le P, \qquad (BF)^TR_2^{-1}BF \le P. \qquad (11)$$

To use Razumikhin-type theorem [19], we evaluate $\dot{V}(x_t)$ for the case

$$V(x_\eta) < \delta V(x_t),\ \ t-2\tau \le \eta \le t, \qquad (12)$$

where $\delta > 1$. From (12) and Lemma 1, it follows that

$$\begin{aligned}
-2x^T(t)PBF\textstyle\int_{t-\tau(t)}^{t}Ax(\theta)d\theta \le\ & \textstyle\int_{t-\tau(t)}^{t}x^T(\theta)A^TR_1^{-1}Ax(\theta)d\theta + \tau(t)x^T(t)PBFR_1F^TB^TPx(t)\\
\le\ & \textstyle\int_{t-\tau(t)}^{t}x^T(\theta)Px(\theta)d\theta + \tau(t)x^T(t)PBFR_1F^TB^TPx(t)\\
\le\ & x^T(t)\tau(t)(\delta P + PBFR_1F^TB^TP)x(t),
\end{aligned} \qquad (13)$$

$$\begin{aligned}
-2x^T(t)PBF\textstyle\int_{t-\tau(t)}^{t}BFx(\theta-\tau(\theta))d\theta \le\ & \textstyle\int_{t-\tau(t)}^{t}x^T(\theta-\tau(\theta))F^TB^TR_2^{-1}BFx(\theta-\tau(\theta))d\theta\\
& +\tau(t)x^T(t)PBFR_2F^TB^TPx(t)\\
\le\ & x^T(t)\tau(t)(\delta P + PBFR_2F^TB^TP)x(t).
\end{aligned} \qquad (14)$$

Inserting (13) and (14) into (11) yields

$$\dot{V}(x_t) \le x^T(t)(P(A+BF)+(A+BF)^TP+\tau(t)(2\delta P + PBF(R_1+R_2)F^TB^TP))x(t).$$

Note that LMIs (7), (8) can be easily transformed into the following inequalities,

$$P^{-1}-(R_1+R_2) \ge 0, \qquad Q - FP^{-1}F^T \ge 0.$$

Hence $\dot{V}(x_t) < 0$ for $V(x_\eta) < \delta V(x_t)$ $(t-2\tau \le \eta \le t)$ if

$$P(A+BF)+(A+BF)^TP+\tau(t)(2\delta P + PBQB^TP) < 0. \qquad (15)$$

Pre- and post-multiplying (4) by P, and by Schur complement, we can get that (4) is equivalent to

$$P(A+BF)+(A+BF)^TP+\tau(t)(2P + PBQB^TP) < 0.$$

Using the continuity of (15) in δ, (4) guarantees that there exists a $\delta > 1$ sufficiently small such that (15) holds for $\tau(t) \le \bar{\tau}$. Furthermore, the explicit expression of feedback gain F can be given as YX^{-1}. From the modelling process we know that to guarantee the stabilization of NCS (1), data packet dropout should satisfy $0 \le d_k \le \frac{\bar{\tau}}{h} - 1$. □

Theorem 1 provides a method of designing a state feedback controller to stabilize an NCS with the possibility of data packet loss, and the bound $\bar{\tau}$, which is related to the allowable data packet drops that does't destabilize the NCS, can be solved via the quasi-convex optimization algorithm [2].

Remark 3. *Although high sampling rates improve performance in conventional control systems, this also induce high traffic load on the network medium. High traffic loads may lead to high packet drop rate, which increases time delays and degrades control performance. With a certain performance, there must be a trade-off between the sampling rate and the data packet dropout, which will be our future research work.*

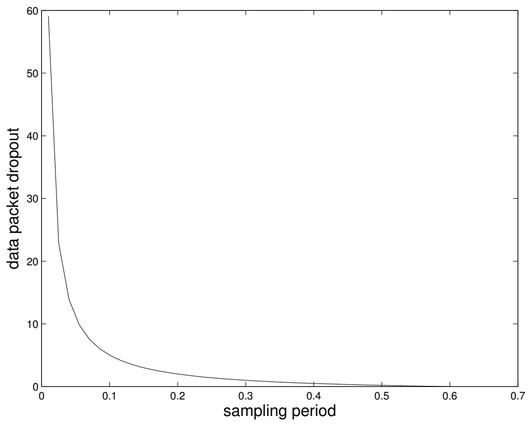

Figure 2: The relationship between sampling rate and data packet dropout

Example 1. *Consider the state-space plant model*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -0.8 & -0.01 \\ 1 & 0.1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.4 \\ 0.1 \end{bmatrix} u$$

The feedback controller takes the form $u = -F\bar{x}(t)$ with F be designed.

Based on Theorem 1, we can solve the corresponding LMIs (4)–(8) and obtain the feedback gain $F = [-4.7298 \ -4.7820]$. By solving the quasi-convex optimization problem using LMI toolbox, we get the admissible bound $\bar{\tau}$ 0.6011. Therefore, if the sampling period $h = 0.2s$, then the closed-loop system is still stable even in the case of two data packets dropped in every three packets. For the system under consideration, the relationship between sampling period and data packet loss is illustrated in Fig. 2, from which we can see that with the sampling period increased, the data packet dropout have to be decreased in order to guarantee the stability of the NCS.

2.1.2 Stabilization of NCSs with transmission delays and data packet dropout

The NCS model with transmission delays and data packet dropout is shown in Fig. 3. Here the controller and the actuator are assumed to be separated. We consider two kinds of delays induced by the network: sensor-to-controller delay τ_{sc} and controller-to-actuator delay τ_{ca}, which may be less than or greater than one sampling

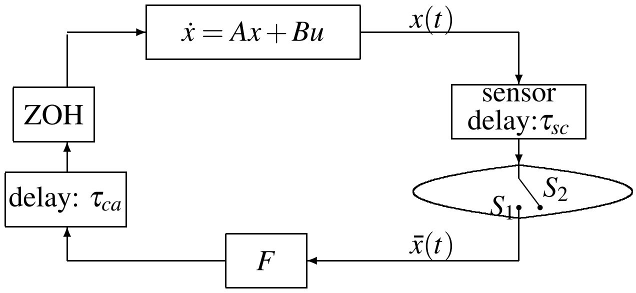

Figure 3: An NCS with delay and data packet dropout

period h. In fact, if the feedback controller is static, these two delays can be lumped together. The model consists of a continuous plant

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{16}$$

and a continuous controller (realized by a ZOH)

$$u(t) = -F\bar{x}(t - \tau_{ca}), \qquad t \in [t_k, t_{k-1}), \qquad k = 1, 2, \cdots. \tag{17}$$

The network can also be modelled as a switch and the dynamic of the switch with transmission delay can be described as follows:

The NCS (16)–(17) with no packet dropout at time t_k: $\bar{x}(t) = x(t_k - \tau_{sc})$;

The NCS (16)–(17) with one packet dropout at time t_k: $\bar{x}(t) = x(t_k - \tau_{sc} - h)$;

$$\vdots$$

The NCS (16)–(17) with $d(k)$ packets dropout at time t_k: $\bar{x}(t) = x(t_k - \tau_{sc} - d(k)h)$.
In this mode of switch, the quantity of dropped packet is accumulated from the latest time when $\bar{x}(t)$ has been updated.

Thus the closed-loop system with transmission delays and network packet loss effect is described as

$$\begin{aligned} \dot{x}(t) &= Ax(t) - BF\bar{x}(t - \tau_{ca}) \\ &= Ax(t) - BFx(t_k - \tau_{ca} - \tau_{sc} - d(k)h), \qquad t \in [t_k, t_{k+1}). \end{aligned}$$

Let $\tau(t) = t - t_k + \tau_{ca} + \tau_{sc} + d(k)h$, then the closed-loop system can be given as:

$$\dot{x}(t) = Ax(t) - BFx(t - \tau(t)). \qquad t \in [t_k, t_{k-1}) \tag{18}$$

The quantity $d(k) \in Z^+$ may vary with time t and it is assumed that

$$0 \le d(k) \le d_k, \qquad d_k \in Z^+.$$

Hence the delay $\tau(t)$ satisfies

$$0 \le \tau(t) = t - t_k + \tau_{ca} + \tau_{sc} + d(k)h \le (\bar{d} + 1)h + \tau_{ca} + \tau_{sc}, \qquad t \in [t_k, t_{k+1}),$$

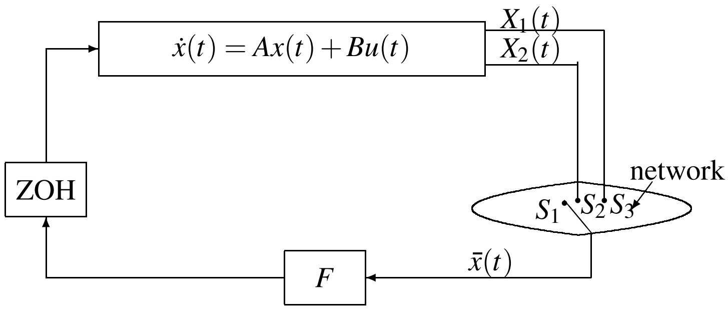

Figure 4: network with multiple packet transmission

where $\bar{d} = \max_k \{d_k\} < \infty$.

Again, NCS (16)–(17) with transmission delays and data packet dropout is modelled as a linear system (18) with time-varying input delay, so we can use the existing theory of delay systems to the system (18).

Now, we develop a sufficient condition for the stabilization of the NCS with transmission delays and data packet dropout.

Theorem 2. *If there exist symmetric positive definite matrices X, R_1, $R_2 \in \mathbf{R}^{n\times n}$, a symmetric positive definite matrix $Q \in \mathbf{R}^{m\times m}$ and a matrix $Y \in \mathbf{R}^{m\times n}$ satisfying the following LMIs:*

$$\begin{bmatrix} \frac{1}{\bar{\tau}}(XA + AX + Y^T B^T + BY) + 2X & BQ \\ QB^T & -Q \end{bmatrix} < 0,$$

$$\begin{bmatrix} -X & Y^T B^T \\ BY & -R_2 \end{bmatrix} \le 0,$$

$$\begin{bmatrix} -X & XA^T \\ AX & -R_1 \end{bmatrix} \le 0,$$

$$\begin{bmatrix} Q & Y \\ Y^T & X \end{bmatrix} \le 0,$$

$$R_1 + R_2 - X \le 0.$$

then NCS (16)–(17) is asymptotically stable via a state feedback controller

$$u(t) = YX^{-1}x(t)$$

whenever the delay and data packet dropout satisfying $0 \le (d_k + 1)h + \tau_{ca} + \tau_{sc} \le \bar{\tau}$.

Proof. The proof of Theorem 2 is analogous to that of Theorem 1 and is thus omitted. □

2.1.3 Stabilization of NCSs with multiple-packet transmission

An NCS with multiple-packet transmission and data packet dropout can also be modelled as a linear multiple delay system. In multiple-packet transmission mode, plant state or output are split into separate packets. Here for simplicity we consider the case that the actuator and controller are combined into one node and there are no transmission delays.

The NCS model of multiple-packet transmission with the possibility of dropping data packets is shown in Fig. 4. The model consists of a continuous plant (1) and a continuous controller (2) (realized by a ZOH). Here, matrix B is assumed to be of full column rank. For simplicity, we assume that $x(t) = \left[X_1^T(t) \;\; X_2^T(t)\right]^T$, where $X_1(t) = [x_1(t) \;\; \cdots \;\; x_r(t)]$ and $X_2(t) = [x_{r+1}(t) \;\; \cdots \;\; x_n(t)]$, $r < n$. Then the output of network is given by $\bar{x}(t) = \left[\bar{X}_1^T(t) \;\; \bar{X}_2^T(t)\right]^T$, where $\bar{X}_1(t) = [\bar{x}_1(t) \;\; \cdots \;\; \bar{x}_r(t)]$ and $\bar{X}_2(t) = [\bar{x}_{r+1}(t) \;\; \cdots \;\; \bar{x}_n(t)]$. When the switch being in position S_1, the output of S_1 and S_2 are held at the previous value and these two packets are lose; When the switch being in position S_2, network packet containing $X_1(t_k)$ is transmitted, and $\bar{X}_1(t_k)$ is updated, whereas the output of switch S_2 is held at the previous value; similarly analysis can be undergone when the switch being in position S_3. The dynamics of the network are described as

$$\bar{X}_1(t) = \bar{X}_1(t_k - h), \qquad \bar{X}_2(t) = \bar{X}_2(t_k - h) \qquad \text{if the switch is in position } S_1;$$

$$\bar{X}_1(t) = \bar{X}_1(t_k - h), \qquad \bar{X}_2(t) = X_2(t_k) \qquad \text{if the switch is in position } S_2;$$

$$\bar{X}_1(t) = X_1(t_k), \qquad \bar{X}_2(t) = \bar{X}_2(t_k - h) \qquad \text{if the switch is in position } S_3.$$

So, the output of the network can be given by:

$$\bar{x}(t_k) = \begin{bmatrix} \bar{X}_1(t) \\ \bar{X}_2(t) \end{bmatrix} = \begin{bmatrix} X_1(t_k - d_1(k)h) \\ X_2(t_k - d_2(k)h) \end{bmatrix} = \begin{bmatrix} X_1(t - \tau_1(t)) \\ X_2(t - \tau_2(t)) \end{bmatrix},$$

where $\tau_1(t) = t - t_k + d_1(k)h, \; \tau_2(t) = t - t_k + d_2(k)h$. Thus the closed-loop system with the network packet loss effect is modelled as

$$\dot{x} = Ax(t) + BF\bar{x}(t) = Ax(t) + BFC_1x(t - \tau_1(t)) + BFC_2x(t - \tau_2(t)), \tag{19}$$

where

$$C_1 = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \; C_2 = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix}.$$

The quantity $d_i(k) \in Z^+$ $(i = 1,2)$ may vary with time t and it is assumed that

$$0 \le d_i(k) \le d_{ik}, \qquad d_{ik} \in Z^+.$$

Hence the delay $\tau_i(t)$ $(i = 1,2)$ satisfy the following property:

$$0 \le \tau_i(t) = t - t_k + d_i(k)h \le (\bar{d}_i + 1)h, \qquad t \in [t_k, t_{k+1}),$$

where $\bar{d}_i = \max_k\{d_{ik}\} < \infty$.

Remark 4. *The multiple-packet transmission NCS (1)–(2) with data packet dropout can also be modelled as a linear system with time-varying multiple delay (19) to which the existing theory of delay systems applies.*

The following result gives a sufficient condition for the stabilization of the NCS (19) with data packet dropout in multiple-packet transmission.

Theorem 3. *If there exist symmetric positive definite matrices P, $U_i \in \mathbf{R}^{n\times n}$ $(i = 1,\cdots,6)$, a matrix $M \in \mathbf{R}^{m\times m}$ and a matrix $N \in \mathbf{R}^{m\times n}$ satisfying*

$$PB = BM, \tag{20}$$

and the following LMIs:

$$A^T U_1 A \leq P, A^T U_4 A \leq P, \tag{21}$$

$$U_2 \leq P,\ U_3 \leq P,\ U_5 \leq P,\ U_6 \leq P, \tag{22}$$

$$\begin{bmatrix} P & C_1^T N^T B^T \\ BNC_1 & P \end{bmatrix} \geq 0, \tag{23}$$

$$\begin{bmatrix} P & C_2^T N^T B^T \\ BNC_2 & P \end{bmatrix} \geq 0, \tag{24}$$

$$\begin{bmatrix} \Lambda & \bar{\tau}_1 BNC_1 & \bar{\tau}_1 BNC_1 & \bar{\tau}_1 BNC_1 & \bar{\tau}_2 BNC_2 & \bar{\tau}_2 BNC_2 & \bar{\tau}_2 BNC_2 \\ * & -\bar{\tau}_1 U_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & -\bar{\tau}_1 U_2 & 0 & 0 & 0 & 0 \\ * & * & * & -\bar{\tau}_1 U_3 & 0 & 0 & 0 \\ * & * & * & * & -\bar{\tau}_2 U_4 & 0 & 0 \\ * & * & * & * & * & -\bar{\tau}_2 U_5 & 0 \\ * & * & * & * & * & * & -\bar{\tau}_2 U_6 \end{bmatrix} < 0. \tag{25}$$

where $\Lambda = PA + A^T P + BNC_1 + BNC_2 + C_1^T N^T B^T + C_2^T N^T B^T + 3(\bar{\tau}_1 + \bar{\tau}_2)P$, then the NCS (24) *with B being full column rank is asymptotically stable via a state feedback controller*

$$u(t) = YX^{-1}x(t)$$

for data packet dropout satisfying $0 \leq d_{ik} \leq \frac{\bar{\tau}_i}{h} - 1$.

Proof. As shown in Theorem 1, (19) can be written as

$$\begin{aligned} \dot{x} = \ & (A + BFC_1 + BFC_2)x(t) - BFC_1 \textstyle\int_{t-\tau_1(t)}^{t} (Ax(\theta) + BFC_1 x(\theta - \tau_1(\theta)) + BFC_2 x(\theta - \tau_2(\theta))d\theta \\ & -BFC_2 \textstyle\int_{t-\tau_2(t)}^{t} (Ax(\theta) + BFC_1 x(\theta - \tau_1(\theta)) + BFC_2 x(\theta - \tau_2(\theta))d\theta \end{aligned} \tag{26}$$

with the initial condition $x(t_0 + \theta) = \phi(\theta)$, $\forall\ \theta \in \xi_{t_0}$, where ϕ is a continuous norm-bounded initial function and $\xi_{t_0} = \bigcup_{i=1}^{2} \xi_{t_0}^{i} = \bigcup_{i=1}^{2} \{t \in R : t = \eta - 2\tau_i(\eta) \leq t_o,\ \eta \geq t_o\}$. From [19], global uniform asymptotic stability of system (19) will be guaranteed by global uniform asymptotic stability of system (26).

Consider the following Lyapunov function

$$V(x_t) = x^T(t)Px(t),$$

where P is a symmetric positive definite matrix. The time derivative of $V(x_t)$ along the trajectory of system (26) is given by

$$\begin{aligned}\dot{V}(x_t) = \; & x^T(P(A+BFC_1+BFC_2)+(A+BFC_1+BFC_2)^TP)x \\ & -2x^TPBFC_1\int_{t-\tau_1(t)}^{t}(Ax(\theta)+BFC_1x(\theta-\tau_1(\theta))+BFC_2x(\theta-\tau_1(\theta)))d\theta \\ & -2x^TPBFC_2\int_{t-\tau_2(t)}^{t}(Ax(\theta)+BFC_1x(\theta-\tau_2(\theta))+BFC_2x(\theta-\tau_2(\theta)))d\theta.\end{aligned} \tag{27}$$

Let $R_i = U_i^{-1}$, $i = 1,\cdots,6$, from (21)-(22), we obtain

$$A^TR_1^{-1}A \le P,\; A^TR_4^{-1}A \le P, \tag{28}$$

$$R_2^{-1} \le P,\; R_3^{-1} \le P,\; R_5^{-1} \le P,\; R_6^{-1} \le P. \tag{29}$$

Because B is full column rank, then it follows from (20) that M is also full rank, and thus invertible. Using this and Schur complement, we can deduce from (23)-(24) and (29) that

$$C_1^TF^TB^TR_2^{-1}BFC_1 \le C_1^TF^TB^TPP^{-1}PBFC_1 = C_1^TN^TB^TP^{-1}BNC_1 \le P, \tag{30}$$

$$C_2^TF^TB^TR_3^{-1}BFC_2 \le C_2^TF^TB^TPP^{-1}PBFC_2 = C_2^TN^TB^TP^{-1}BNC_2 \le P, \tag{31}$$

$$C_1^TF^TB^TR_5^{-1}BFC_1 \le C_1^TF^TB^TPP^{-1}PBFC_1 = C_1^TN^TB^TP^{-1}BNC_1 \le P, \tag{32}$$

$$C_2^TF^TB^TR_6^{-1}BFC_2 \le C_2^TF^TB^TPP^{-1}PBFC_2 = C_2^TN^TB^TP^{-1}BNC_2 \le P, \tag{33}$$

where $N = MF$. To use the Razumikhin-type theorem [19], we evaluate $\dot{V}(x_t)$ for the case

$$V(x_\eta) < \delta V(x_t),\;\; t-2\tau \le \eta \le t, \tag{34}$$

where $\delta > 1$. From this and Lemma 1, it follows from (28) and (30)-(34) that

$$\begin{aligned}-2x^TPBFC_1\int_{t-\tau_1(t)}^{t}Ax(\theta)d\theta \le \; & \int_{t-\tau_1(t)}^{t}x^T(\theta)A^TR_1^{-1}Ax(\theta)d\theta \\ & +\tau_1(t)x^T(t)PBFC_1R_1C_1^TF^TB^TPx(t) \\ \le \; & \int_{t-\tau_1(t)}^{t}x^T(\theta)Px(\theta)d\theta \\ & +\tau_1(t)x^T(t)PBFC_1R_1C_1^TF^TB^TPx(t) \\ \le \; & x^T(t)\tau_1(t)(\delta P+PBFC_1R_1C_1^TF^TB^TP)x(t),\end{aligned}$$

$$\begin{aligned}-2x^TPBFC_1\int_{t-\tau_1(t)}^{t}BFC_1x(\theta-\tau_1(\theta))d\theta \le \; & \int_{t-\tau_1(t)}^{t}x^T(\theta-\tau_1(\theta))C_1^TF^TB^TR_2^{-1}BFC_1x(\theta-\tau_1(\theta))d\theta \\ & +\tau_1(t)x^T(t)PBFC_1R_2C_1^TF^TB^TPx(t) \\ \le \; & x^T(t)\tau_1(t)(\delta P+PBFC_1R_2C_1^TF^TB^TP)x(t),\end{aligned}$$

$$\begin{aligned}-2x^TPBFC_1\int_{t-\tau_1(t)}^{t}BFC_2x(\theta-\tau_2(\theta))d\theta \le \; & \int_{t-\tau_1(t)}^{t}x^T(\theta-\tau_2(\theta))C_2^TF^TB^TR_3^{-1}BFC_2x(\theta-\tau_2(\theta))d\theta \\ & +\tau_1(t)x^T(t)PBFC_1R_3C_1^TF^TB^TPx(t) \\ \le \; & x^T(t)\tau_1(t)(\delta P+PBFC_1R_3C_1^TF^TB^TP)x(t),\end{aligned}$$

$$\begin{aligned}-2x^TPBFC_2\int_{t-\tau_2(t)}^{t}Ax(\theta)d\theta \le \; & \int_{t-\tau_2(t)}^{t}x^T(\theta)A^TR_4^{-1}Ax(\theta)d\theta \\ & +\tau_2(t)x^T(t)PBFC_2R_4C_2^TF^TB^TPx(t) \\ \le \; & x^T(t)\tau_2(t)(\delta P+PBFC_2R_4C_2^TF^TB^TP)x(t),\end{aligned}$$

$$
\begin{aligned}
-2x^T PBFC_2 \int_{t-\tau_2(t)}^{t} BFC_1 x(\theta-\tau_1(\theta))d\theta \leq\ & \int_{t-\tau_2(t)}^{t} x^T(\theta-\tau_1(\theta))C_1^T F^T B^T R_5^{-1} BFC_1 x(\theta-\tau_1(\theta))d\theta \\
& +\tau_2(t)x^T(t)PBFC_2R_5C_2^T F^T B^T Px(t) \\
\leq\ & x^T(t)\tau_2(t)(\delta P+PBFC_2R_5C_2^T F^T B^T P)x(t), \\
-2x^T PBFC_2 \int_{t-\tau_2(t)}^{t} BFC_2 x(\theta-\tau_2(\theta))d\theta \leq\ & \int_{t-\tau_2(t)}^{t} x^T(\theta-\tau_2(\theta))C_2^T F^T B^T R_6^{-1} BFC_2 x(\theta-\tau_2(\theta))d\theta \\
& +\tau_2(t)x^T(t)PBFC_2R_6C_2^T F^T B^T Px(t) \\
\leq\ & x^T(t)\tau_2(t)(\delta P+PBFC_2R_6C_2^T F^T B^T P)x(t).
\end{aligned}
$$

From above, we obtain that

$$
\begin{aligned}
\dot{V}(x_t) \leq\ & x^T(P(A+BFC_1+BFC_2)+(A+BFC_1+BFC_2)^T P \\
& +\tau_1(t)(3\delta P+PBFC_1(R_1+R_2+R_3)C_1^T F^T B^T P) \\
& +\tau_2(t)(3\delta P+PBFC_2(R_4+R_5+R_6)C_2^T F^T B^T P))x \\
=\ & x^T(PA+BNC_1+BNC_2+A^T P+C_1^T N^T B^T+C_2^T N^T B^T \\
& +\tau_1(t)(3\delta P+BNC_1(R_1+R_2+R_3)C_1^T N^T B^T) \\
& +\tau_2(t)(3\delta P+BNC_2(R_4+R_5+R_6)C_2^T N^T B^T))x.
\end{aligned}
$$

Hence, $\dot{V}(x_t)<0$ for $V(x_\eta)<\delta V(x_t)$ $(t-2\tau\leq\eta\leq t)$ if

$$
\begin{pmatrix}
PA+BNC_1+BNC_2+A^T P+C_1^T N^T B^T+C_2^T N^T B^T \\
+\tau_1(t)(3\delta P+BNC_1(R_1+R_2+R_3)C_1^T N^T B^T) \\
+\tau_2(t)(3\delta P+BNC_2(R_4+R_5+R_6)C_2^T N^T B^T)
\end{pmatrix} < 0. \tag{35}
$$

Using Schur complement, (25) is equivalent to

$$
\begin{pmatrix}
PA+BNC_1+BNC_2+A^T P+C_1^T N^T B^T+C_2^T N^T B^T \\
+\bar{\tau}_1(3P+BNC_1(R_1+R_2+R_3)C_1^T N^T B^T) \\
+\bar{\tau}_2(3P+BNC_2(R_4+R_5+R_6)C_2^T N^T B^T)
\end{pmatrix} < 0. \tag{36}
$$

Using the continuity of (36) in δ, (25) guarantees that there exists a $\delta>1$ sufficient small such that (36) holds for $\tau_i(t)\leq\bar{\tau}_i,\ i=1,2$, and the conclusion thus follows. $\square$

For the NCSs with multiple-packet transmission, the feedback gain can be obtained by Theorem 3. If $\bar{\tau}_1$ is fixed, the bound $\bar{\tau}_2$, which is related to the allowable data packet drops that does't destabilize the NCS, can be solved via the quasi-convex optimization algorithm [2]. It is the same the other way round.

2.2 Discrete-time case

Now, we extend continuous-time case to the discrete-time case. The manner of the modelling and the conclusions in this part are similar to that of the continuous case, while the proofs are quite different.

An NCS shown as in Fig. 5 consists of a discrete plant and a discrete controller

$$
\begin{aligned}
& x(k+1)=Ax(k)+Bu(k) \\
& u(k)=F\bar{x}(k), \qquad k=1,2,\cdots,
\end{aligned} \tag{37}
$$

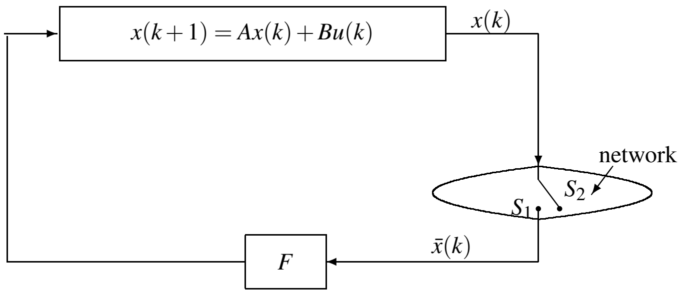

Figure 5: An NCS with data packet dropout

where $x(k) \in \mathbf{R}^n$ and $u(k) \in \mathbf{R}^m$ are the plant state and the plant input, respectively. $\bar{x}(k)$ is the output of the network. F is the state feedback gain matrix to be designed, A, B are known real constant matrices with appropriate dimensions. We first consider the case that controller and actuator are combined onto a single node and there are no transmission delays between the sensor and the combined node. The network is modelled as a switch. When the switch is closed (in position S_1), network packet (containing $x(k)$) is transmitted and the actuator utilizes the fresh data, whereas when it is open (in position S_2), the output of the switch is held at the previous value (the packet is lose), and the actuator uses the old data. If $\bar{x}(k)$ has not updated for $d(k)$ times at step k, the dynamics of the switch can be described as: $\bar{x}(k) = x(k-d(k))$. Thus NCS (37) with the packet loss effect can be modelled as

$$x(k+1) = Ax(k) + BF(k-d(k)). \tag{38}$$

The quantity $d(k) \in Z^+$ may vary with time step k and it is assumed that

$$0 \leq d(k) \leq d_k \leq \bar{m}, \qquad d_k \in Z^+.$$

where $\bar{m} = \max_k \{d_k\} < \infty$.

Remark 5. *The NCS (37) with data packet dropout is modelled as a linear time delay system (38), and this enables us to apply the theory of delay systems to the analysis of such NCSs [26, 33].*

2.2.1 Stabilization of the NCS with data packet dropout

We present the following result that ensures stabilization of NCS (37) with data packet dropout.

Theorem 4. *For a given positive integer $\bar{m} > 0$, NCS (37) can be asymptotically stabilized for data packet dropout satisfying $0 \leq d_k \leq \bar{m}$ if there exist symmetric positive definite matrices X, $Z \in \mathbf{R}^{n\times n}$ and a matrix $Y \in \mathbf{R}^{m\times n}$ satisfying the following LMI:*

$$\begin{bmatrix} -X & 0 & \cdots & 0 & XA^T & \bar{m}Y^T \\ 0 & -Z & \cdots & 0 & ZB^T & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -Z & ZB^T & 0 \\ AX & BZ & \cdots & BZ & -X & 0 \\ \bar{m}Y & 0 & \cdots & 0 & 0 & -\bar{m}Z \end{bmatrix} < 0. \tag{39}$$

Furthermore the feedback control law is given by $u(kh) = YX^{-1}\bar{x}(kh)$.

Proof. From analysis in the previous subsection, the closed-loop system of the system (37) with $u(k) = F\bar{x}(kh)$ can be written as (dropping h for simplicity in the following):

$$\Sigma_{c1}: \qquad x(k+1) = Ax(k) + BF\sum_{i=1}^{\bar{m}} \delta(d(k)-i)x(k-i),$$

where

$$\delta(n) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

Define a Lyapunov function $V(k)$ as follows:

$$V(k) = x^T(k)Px(k) + \sum_{i=1}^{\bar{m}} \sum_{j=k-i}^{k-1} x(j)^T F^T QFx(j),$$

where P and Q are positive definite matrices, then

$$\begin{aligned}
\triangle V(k) &= V(k+1) - V(k) \\
&= x^T(k)[A^T PA + \bar{m}F^T QF - P]x(k) + 2x^T(k)A^T P\textstyle\sum_{i=1}^{\bar{m}} \delta(d(k)-i)BFx(k-i) \\
&\quad + \textstyle\sum_{i=1}^{\bar{m}}\sum_{j=1}^{\bar{m}} \delta(d(k)-i)\delta(d(k)-j)x^T(k-i)F^T B^T PBFx(k-i) \\
&\quad - \textstyle\sum_{i=1}^{\bar{m}} x^T(k-i)F^T QFx(k-i).
\end{aligned}$$

Since

$$\sum_{i=1}^{\bar{m}} x^T(k-i)F^T QFx(k-i) \geq \sum_{i=1}^{\bar{m}} \delta(d(k)-i)\delta(d(k)-i)x^T(k-i)F^T QFx(k-i),$$

the following inequality holds,

$$\begin{aligned}
\triangle V(k) \leq\ & x^T(k)[A^T PA + \bar{m}F^T QF - P]x(k) + 2x^T(k)A^T P\textstyle\sum_{i=1}^{\bar{m}} \delta(d(k)-i)BFx(k-i) \\
& + \textstyle\sum_{i=1}^{\bar{m}}\sum_{j=1}^{\bar{m}} \delta(d(k)-i)\delta(d(k)-j)x^T(k-i)F^T B^T PBFx(k-i) \\
& - \textstyle\sum_{i=1}^{\bar{m}} \delta(d(k)-i)\delta(d(k)-i)x^T(k-i)F^T QFx(k-i). \\
=\ & W(k)^T \begin{bmatrix} A^T PA + \bar{m}F^T QF - P & A^T PB & \cdots & A^T PB \\ B^T PA & -Q + B^T PB & \cdots & B^T PB \\ \vdots & \vdots & \vdots & \vdots \\ B^T PA & B^T PB & \cdots & -Q + B^T PB \end{bmatrix} W(k),
\end{aligned}$$

where $W(k) = \left[x(k)^T \;\; x(k-1)^T F^T \delta(d(k)-1) \;\; \cdots \;\; x(k-\bar{m})^T F^T \delta(d(k)-\bar{m})\right]^T$. Hence $\triangle V(k) < 0$, if

$$\begin{bmatrix} A^T PA + \bar{m}F^T QF - P & A^T PB & \cdots & A^T PB \\ B^T PA & -Q + B^T PB & \cdots & B^T PB \\ \vdots & \vdots & \vdots & \vdots \\ B^T PA & B^T PB & \cdots & -Q + B^T PB \end{bmatrix} < 0,$$

which can be written as

$$\begin{bmatrix} A^T \\ B^T \\ \vdots \\ B^T \end{bmatrix} P[A \;\; B \;\; \cdots \;\; B] + \begin{bmatrix} \bar{m}F^TQF-P & 0 & 0 & 0 \\ 0 & -Q & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -Q \end{bmatrix} < 0.$$

By Schur complement, it is equivalent to

$$\begin{bmatrix} \bar{m}F^TQF-P & 0 & \cdots & 0 & A^T \\ 0 & -Q & \cdots & 0 & B^T \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -Q & B^T \\ A & B & \cdots & B & -P^{-1} \end{bmatrix} < 0.$$

Pre- and Post- multiplying both sides of the above inequality by diag $[P^{-1} \;\; Q^{-1} \;\; \cdots \;\; Q^{-1} \;\; I]$, letting $X = P^{-1}$, $Y = FX$, $Z = Q^{-1}$ and applying Schur complement again yield matrix inequality (39). □

Remark 6. *For NCS (37) with transmission delays and data packet dropout, we can get analogous conclusion as that of Theorem 2 by combining the result of Theorem 4 and the modelling technique as that of continuous.*

To illustrate the feasibility of the proposed method, we give an illustrative example.

Example 2. *Consider the state-space plant model*

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 0.6 & 0.5 \\ 0 & 0.71 \end{bmatrix} x(k) + \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} u(k), \\ u(k) &= F\,\bar{x}(k). \end{aligned} \tag{40}$$

with F to be designed.

For simplicity, we only consider NCS (40) with data packet dropout effect. First, we obtain the upper bound of data packet dropout $\bar{m} = 3$ by search the largest $\bar{m}$ that does not violate condition in Theorem 4. Then, solving the LMI in Theorem 4 with LMI toolbox [2], we have

$$P = \begin{bmatrix} 5.0447 & 0.1629 \\ 0.1629 & 14.1394 \end{bmatrix}, \;\; Q = 1.8780, \;\; F = [0.7001 \;\; 0].$$

So the NCS (40) with up to 75% arbitrary data packet dropout can be asymptotically stabilized.

2.2.2 Stabilization of the NCS with multiple-packet transmission

For the NCS transmitted in a multiple-packet manner we also consider the case that the actuator and controller are combined into one node and there are no transmission delays.

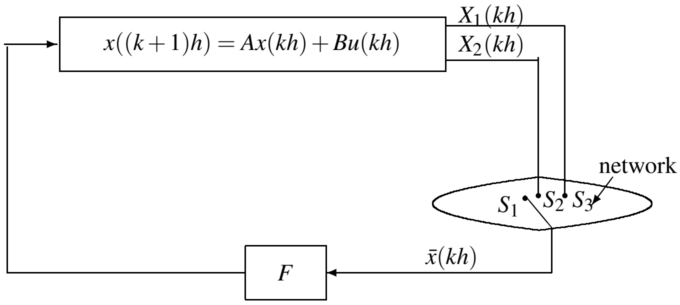

Figure 6: An NCS with multiple-packet transmission

First, we assume that $x(k) = \left[X_1^T(k) \;\; X_2^T(k)\right]^T$, where $X_1(k) = [x_1(k) \;\; \cdots \;\; x_r(k)]^T$ and $X_2(k) = [x_{r+1}(k) \;\; \cdots \;\; x_n(k)]^T$. Fig. 6 illustrates the case where the plant state is transmitted in two packets. When the switch in the position S_1, the output of positions S_1 and S_2 are held at the previous value and these two packets are lose; When the switch in the position S_2, network packet containing $X_2(k)$ is transmitted, whereas the output of the switch S_1 is held at the previous value; otherwise, network packet containing $X_1(k)$ is transmitted, and the output of the switch S_2 is held at the previous value. The dynamics of the switch can be modelled as

$$\begin{array}{lll}
\bar{X}_1(k) = \bar{X}_1(k-1), & \bar{X}_2(k) = \bar{X}_2(k-1) & \text{if the switch is in the position } S_1; \\
\bar{X}_1(k) = \bar{X}_1(k-1), & \bar{X}_2(k) = X_2(k) & \text{if the switch is in the position } S_2; \\
\bar{X}_1(k) = X_1(k), & \bar{X}_2(k) = \bar{X}_2(k-1) & \text{if the switch is in the position } S_3.
\end{array}$$

Thus the output of the network can be given as:

$$\bar{x}(k) = \begin{bmatrix} \bar{X}_1(k) \\ \bar{X}_2(k) \end{bmatrix} = \begin{bmatrix} X_1(k-d_1(k)) \\ X_2(k-d_2(k)) \end{bmatrix},$$

and NCS (37) transmitted in two packets with the network packet lose effect can be described as

$$x(k+1) = Ax(k) + BF\bar{x}(k) = Ax(k) + BFC_1x(k-d_1(k)) + BFC_2x(k-d_2(k)), \tag{41}$$

where

$$C_1 = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \; C_2 = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix}.$$

The quantity $d_i(k) \in Z^+$ $(i = 1, 2)$ may vary with k and it is assumed that

$$0 \leq d_i(k) \leq d_{ik} \leq \bar{m}_i, \qquad d_{ik} \in Z^+,$$

where $\bar{m}_i = \max_k \{d_{ik}\} < \infty$.

The following result gives a sufficient condition on the stabilization of NCS (37) transmitted in two packets with data packet dropout effect.

Theorem 5. *For given positive integers $\bar{m}_i > 0$ (i=1,2), NCS (37) transmitted in two packets can be asymptotically stabilized for data packet dropout satisfying $0 \leq d_{ik} \leq \bar{m}_i$ if there exist symmetric positive definite* *matrices* $X,\ Z_i \in \mathbf{R}^{n\times n}\quad (i=1,2)$ *and a matrix* $Y \in \mathbf{R}^{m\times n}$ *satisfying the following LMI:*

$$\begin{bmatrix} -X & 0 & \cdots & 0 & 0 & \cdots & 0 & XA^T & \bar{m}_1 Y^T C_1^T & \bar{m}_2 Y^T C_2^T \\ 0 & -Z_1 & \cdots & 0 & 0 & \cdots & 0 & Z_1 B^T & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -Z_1 & 0 & \cdots & 0 & Z_1 B^T & 0 & 0 \\ 0 & 0 & \cdots & 0 & -Z_2 & \cdots & 0 & Z_2 B^T & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & -Z_2 & Z_2 B^T & 0 & 0 \\ AX & BZ_1 & \cdots & BZ_1 & BZ_2 & \cdots & BZ_2 & -X & 0 & 0 \\ \bar{m}_1 C_1 Y & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & -\bar{m}_1 Z_1 & 0 \\ \bar{m}_2 C_2 Y & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & -\bar{m}_2 Z_2 \end{bmatrix} < 0. \tag{42}$$

Furthermore the feedback control law is given by $u(k) = YX^{-1}\bar{x}(k)$*.*

Proof. From above analysis, the closed-loop system of the system (1) with $u(k) = F\bar{x}(kh)$ can be written as:

$$\Sigma_{c2}: \quad x((k+1)) = Ax(k) + BFC_1 \sum_{i=1}^{\bar{m}_1} \delta(d_1(k)-i)x(k-i) + BFC_2 \sum_{i=1}^{\bar{m}_2} \delta(d_2(k)-i)x(k-i),$$

where

$$\delta(n) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

Define a Lyapunov function $V(k)$ as follows:

$$V(k) = x^T(k)Px(k) + \sum_{\kappa=1}^{2} \sum_{i=1}^{\bar{m}_\kappa} \sum_{j=k-i}^{k-1} x(j)^T F^T C_\kappa^T Q_\kappa C_\kappa F x(j),$$

where P and Q_κ are positive definite matrices. Let $X = P^{-1}$, $Y = FX$, $Z = Q^{-1}$ and use similar technique as that of Theorem 4, the conclusion can be easily obtained. □

For the NCS (37) transmitted in two packets, we can find the upper bound of data packet dropout of each transmission channel by search the largest $\bar{m}_i$ that does not violate the condition of Theorem 5.

Remark 4. Similarly, an NCS transmitted in S packets with data packet dropout effect can be modelled as

$$x(k+1) = Ax(k) + BF\bar{x}(k) = Ax(k) + \sum_{i=1}^{S} BFC_i x(k-d_i(k)), \tag{43}$$

where

$$C_i = \begin{bmatrix} 0 & & & & 0 \\ & \ddots & & & \\ & & I_i & & \\ & & & \ddots & \\ 0 & & & & 0 \end{bmatrix} \leftarrow \text{the } ith \text{ block.}$$

and $d(k)$ is assumed to satisfy $d_i(k) \le d_{ik} \le \bar{m}_i$ for $i = 1, 2, \cdots, S$, where $d_{ik} \in Z^+$, $\bar{m}_i = \max_k \{d_{ik}\} < \infty$.

We briefly summarize the stabilization conclusion.

Theorem 6. *For given positive integers $\bar{m}_i > 0$ $(i = 1, \cdots, S)$, NCS (37) transmitted in S packets can be stabilized for data packet dropout satisfying $0 \le d_{ik} \le \bar{m}_i$ if there exist symmetric positive definite matrices X, $Z \in \mathbf{R}^{n\times n}$ and a matrix $Y \in \mathbf{R}^{m\times n}$ satisfying the following LMI:*

$$\begin{bmatrix}
-X & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & XA^T & \bar{m}_1 Y^T C_1^T & \cdots & \bar{m}_S Y^T C_S^T \\
0 & -Z_1 & \cdots & 0 & \cdots & 0 & \cdots & 0 & Z_1 B^T & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & -Z_1 & \cdots & 0 & \cdots & 0 & Z_1 B^T & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & -Z_S & \cdots & 0 & Z_S B^T & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & -Z_S & Z_S B^T & 0 & \cdots & 0 \\
AX & BZ_1 & \cdots & BZ_1 & \cdots & BZ_S & \cdots & BZ_S & -X & 0 & \cdots & 0 \\
\bar{m}_1 C_1 Y & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 & -\bar{m}_1 Z_1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\bar{m}_S C_S Y & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & -\bar{m}_S Z_S
\end{bmatrix} < 0.$$

Furthermore the feedback control law is given by $u(k) = YX^{-1}\bar{x}(k)$.

Remark 7. *The result of this section may be applied in lossless network to save network bandwidth or to obtain the maximum delay between state update.*

3 Switched system approach

In this section, we propose an iterative approach to model NCSs with data packet dropout as switched linear systems for continuous-time case and discrete-time case, thus the existing theory and conclusion for the switched systems can be applied to the design of such NCSs. We focus our attention on the designing of explicit express of stabilizing state feedback controllers and find the admissible upper bound of the dropped packets for the NCSs. For the discrete-time NCSs with constant transmission delays and data packet dropout, similar model and results can be obtained.

3.1 Continuous-time case

We consider the set up with a clock-driven sensor, and both the controller and the actuator are combined into one event-driven node. That is, network communication only occurs between the sensor and the controller through a communication channel with finite bandwidth. We assume that there are no transmission delays between the sensor and the combined node.

An NCS with the possibility of data packet dropout is shown in Fig. 1. The sensor shown in Fig. 1 takes on the work of sampling and transmission, and the output of network is the sampling value of the state. It is assumed that the maximum data packet loss between two successive transmission within a bound d, where d is a positive integer.

The controlled plant is given by

$$\dot{x}(t) \quad = \quad Ax(t) + Bu(t), \tag{44}$$

where $x(t) \in \mathbf{R}^n$, $u(t) \in \mathbf{R}^m$ are the plant state and the plant input, respectively. A, B are known real constant matrices with appropriate dimensions, and the pair (A, B) is stabilizable. The sampling period h is a positive constant scalar and $t_k = kh$ ($k = 0, 1, \cdots$) is the sampling instant. The output of the network is

$$\bar{x}(t) = \begin{cases} x(t_k), \ t \in [t_k, t_{k+1}), & \text{if } x(t_k) \text{ is transmitted successfully;} \\ x(t_{k-1}), \ t \in [t_k, t_{k+1}), & \text{otherwise,} \end{cases} \tag{45}$$

which is used to construct the state feedback controller.

With a set of candidate state feedback gains $\{F_0,\ F_1, \cdots, F_d\}$ to be designed, we give Feedback Mechanism 1 to stabilize the NCS:

For sampling instant t_k ($k = 0, 1, \cdots$), if the register updates successfully, i.e., there is no date dropout, then we choose F_0 as the feedback gain; otherwise, if the register has not update for i times, we choose F_i as the feedback gain.

Now, we propose the iterative approach. First, we assume that the packet containing $x(0)$ is transmitted to the controller safely, that is $\bar{x}(0) = x(0)$, then

$$\begin{aligned} x(t) &= \Phi(t)x(0) + \Gamma(t)F_0x(0) \qquad t \in [0, h), \\ x(h) &= \Phi(h)x(0) + \Gamma(h)F_0x(0), \end{aligned}$$

where $\Phi(t) \triangleq e^{At}$, and $\Gamma(t) \triangleq \int_0^t e^{As} ds B$. In the next sampling period, if the packet containing $x(h)$ is transmitted to the controller successfully, we obtain

$$\begin{aligned} x(t) &\triangleq x(h+\tau) = \Phi(\tau)x(h) + \Gamma(\tau)F_0x(h) \qquad \tau \in [0, h), \\ x(2h) &= \Phi(h)x(h) + \Gamma(h)F_0x(h); \end{aligned}$$

otherwise

$$\begin{aligned} x(t) &\triangleq x(h+\tau) = \Phi(\tau)x(h) + \Gamma(\tau)F_1x(0) \qquad \tau \in [0, h), \\ x(2h) &= \Phi(h)x(h) + \Gamma(h)F_1x(0). \end{aligned}$$

Suppose that $t_{k_1} = k_1 h$ is the latest update time of $\bar{x}(t)$ from the initial time, then the states of plant (44) at the sampling instants from $t_{k_0} = 0$ to $t_{k_1} = k_1 h$ can be described as

$$x(ih) = (\Phi^i(h) + \Phi^{i-1}(h)\Gamma(h)F_0 + \cdots + \Gamma(h)F_{i-1})x(0); \ \ i = 1, \cdots, k_1,$$

and the state of plant (44) between two successive sampling instants wh and $(w+1)h$ $(0 \leq w < k_1)$ can be expressed as

$$\begin{aligned} x(t) &\stackrel{\Delta}{=} x(wh+\tau) = \Phi(\tau)x(wh) + \Gamma(\tau)F_{w-1}x(0) \\ &= \Phi(\tau)(\Phi^{w}(h) + \Phi^{w}(h)\Gamma(h)F_0 + \cdots + \Gamma(h)F_{w-1})x(0) \\ &\quad +\Gamma(\tau)F_{w-1}x(0), \qquad \tau \in [0,h). \end{aligned}$$

Suppose that the successive update time of $\bar{x}(t)$ is $t_{k_0} = 0$, $t_{k_i} = k_i h$, $i = 1,2,\cdots$. In this pattern of transmission, the states of plant (44) at these update instants can be described as follows:

$$x(k_j h) = (\Phi^{k_j-k_{j-1}}(h) + \Phi^{k_j-k_{j-1}-1}(h)\Gamma(h)F_0 + \cdots + \Gamma(h)F_{k_j-k_{j-1}-1})x(k_{j-1}h), \ \ j = 1,2,\cdots$$

The state of the plant (44) between two successive transmission instants $t_{k_{j-1}}$ and t_{k_j} can be described as

$$\begin{aligned} x(t) &\stackrel{\Delta}{=} x(wh+\tau) = \Phi(\tau)x(wh) + \Gamma(\tau)F_{w-k_{j-1}}x(k_{j-1}h) \\ &= \Phi(\tau)(\Phi^{w-k_{j-1}}(h) + \Phi^{w-k_{j-1}-1}(h)\Gamma(h)F_0 + \cdots + \Gamma(h)F_{w-k_{j-1}-1})x(k_{j-1}h) \\ &\quad +\Gamma(\tau)F_{w-k_{j-1}-1}x(k_{j-1}h), \end{aligned} \tag{46}$$

where $t \in [wh,(w+1)h)$, $\quad w \in [k_{j-1}, k_j)$, $\quad \tau \in [0,h)$.

We refer to the time interval between two successive update instants t_{k_i} and $t_{k_{i+1}}$ as one transmission period. Because the maximum data packet loss between two transmission instants is d, then the maximum transmission period of the sensor is $d+1$ sampling period. Let us define another sequence

$$z(0) = x(0),\ z(1) = x(k_1 h),\ \cdots, z(j) = x(k_j h), \cdots, \tag{47}$$

it follows that

$$z(j) \stackrel{\triangle}{=} A(j)y(j-1),\ j = 1,2,\cdots, \tag{48}$$

where

$$A(j) = \Phi^{k_j-k_{j-1}}(h) + \Phi^{k_j-k_{j-1}-1}(h)\Gamma(h)F_0 + \cdots + \Gamma(h)F_{k_j-k_{j-1}-1}. \tag{49}$$

Then it must be true that

$$A(j) \in \Omega, \quad \Omega = \{\bar{A}_0,\ \bar{A}_1, \cdots, \bar{A}_d\}, \tag{50}$$

where

$$\bar{A}_i = \Phi^{i+1}(h) + \Phi^{i}(h)\Gamma(h)F_0 + \cdots + \Gamma(h)F_i,\ \ i = 0,1,2,\cdots,d. \tag{51}$$

It is easy to see that the behavior of plant (44) at the transmission instants with arbitrary but finite data packet dropout can be described by the following switched system

$$z(k+1) = \bar{A}_i\, z(k), \quad i = 0,\cdots,d \tag{52}$$

for arbitrary switching, where $\bar{A}_i \in \Omega$.

3.1.1 Stabilization of the NCS with data packet dropout

Without loss of generality, we assume that 0 is an equilibrium of plant (44), and plant (44) starts at $t_0 = 0$ with the initial condition $x(0)$. The following result will ensure the asymptotically stability of the NCS (44) with data packet dropout. It is a consequence of the state boundedness between transmission instants and Theorem 2.3 in [4].

Definition 1. *[28] A function $\phi : R_+ \to R_+$ is of* **class K** *if it is continuous, strictly increasing, and $\phi(0) = 0$.*

Lemma 2. *If there exists a continuous differentiable, locally positive definite function $V : \mathbf{R}^n \mapsto \mathbf{R}_+$ and functions α, β, γ of* **class** *K such that for all $x \in B_r$ ($B_r \triangleq \{x : \|x\| \leq r\}$)*

$$\alpha(\|x\|) \leq V(x) \leq \beta(\|x\|), \tag{53}$$

and

$$\Delta V_{k_j} \triangleq V(x(t_{k_{j+1}})) - V(x(t_{k_j})) \leq -\gamma(\|x(t_{k_j})\|), \tag{54}$$

then NCS (44) is uniformly asymptotically stable.

Proof. See Appendix A. □

Using the method of multiple Lyapunov functions, Lemma 2 is only concerned with the Lyapunov function's decreasing at transmission instants; it does not require the Lyapunov function to be strictly decreasing over time, i.e. $\dot{V}(x(t)) < 0$.

From Lemma 2 and the discussion in the last section, the asymptotic stability of plant (44) with arbitrary but finite data packet dropout can be guaranteed by that of switched system (52) for arbitrary switching. Now we extend the stability result to the problem of stabilization.

Theorem 7. *If there exist a symmetric positive definite matrices $Q \in \mathbf{R}^{n\times n}$ and matrices $Y_i \in \mathbf{R}^{m\times n}$ satisfying the following LMIs*

$$\begin{bmatrix} -Q & X_i^T \\ X_i & -Q \end{bmatrix} < 0, \tag{55}$$

for $i = 0, 1, \cdots, d$, where $X_i = \Phi^{i+1}(h)Q + \Phi^i(h)\Gamma(h)Y_0 + \cdots + \Gamma(h)Y_i$, then the NCS (37) can be asymptotically stabilized via Feedback Mechanism 1 with

$$F_i = Y_i Q^{-1}, \quad i = 0, 1, \cdots, d.$$

Proof. To prove the stabilization of the NCS (44), we only need to find a Lyapunov function and a state feedback controller that satisfy the conditions of Lemma 2. Consider Lyapunov function

$$V(x(t)) = x(t)^T P x(t) \tag{56}$$

where P is a symmetric positive definite matrix. The difference of function V along the trajectory of the NCS (44) at the update instants is given by

$$\begin{aligned}\Delta V_j &\triangleq V(x(t_{k_{j+1}})) - V(x(t_{k_j})) \\ &= z(j)^T(A(j)^T PA(j) - P)z(j),\end{aligned}$$

where we used (47) and (48) in the last equality. Thus $\Delta V_j < 0$ if

$$A(j)^T PA(j) - P < 0. \tag{57}$$

Using (50), (51), and by Schur complement, (57) is equivalent to

$$\begin{bmatrix} -P & \bar{A}_i^T P \\ P\bar{A}_i & -P \end{bmatrix} < 0, \tag{58}$$

for $i = 0, 1, \cdots, d$. We set $P^{-1} = Q$. Pre- and post-multiplying (58) by block-diag $[Q\ Q]$, and substituting $Y_i = F_i Q$ into $\bar{A}_i$ given by (51), (58)is equivalent to (55). Furthermore, when the LMIs (55) are feasible, the explicit express of the desired state feedback gains are given by $F_i = Y_i Q^{-1}$. The conclusion follows. $\square$

This theorem provides a method of designing a controller to stabilize plant (44) for arbitrary data packet dropout within the bound d. The explicit express of the feedback gains can be easily obtained by solving a set of LMIs via the efficient LMI toolbox [2].

The upper bound of data packet dropout d can be obtained as follows.

Remark 8. *For the plant (44), we can find the upper bound of data packet dropout by search the largest d that does not violate the condition of Theorem 7. That is,*

$$\max \ d$$

subject to: $\exists\, Q > 0$ and Y_i $(i = 0, \cdots, d)$ satisfying (55).

Example 3. *Consider the state-space plant model (Example 5 in [35])*

$$\begin{aligned}\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u(t), \\ y(t) &= [1\ \ 0]\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},\end{aligned} \tag{59}$$

with the state feedback gains F_i $(i = 0, \cdots, d)$ be designed.

With sampling period $h = 1s$ ($h = 0.3s$ in [35]), we obtain

$$\Phi(1) = \begin{bmatrix} 1.0000 & 0.9516 \\ 0 & 0.9048 \end{bmatrix}, \quad \Gamma(1) = \begin{bmatrix} 10 \times exp(-1/10) - 9 \\ -exp(-1/10) + 1 \end{bmatrix}$$

With $d = 4$, solving the LMIs problem in Theorem 7 with LMI toolbox [2], we have

$$Y_0 = \begin{bmatrix} -5.2294 \\ -12.7666 \end{bmatrix}^T, Y_1 = \begin{bmatrix} 3.4450 \\ 1.6938 \end{bmatrix}^T, Y_2 = \begin{bmatrix} 1.1501 \\ 0.5655 \end{bmatrix}^T,$$

$$Y_3 = \begin{bmatrix} 0.3884 \\ 0.1910 \end{bmatrix}^T, Y_4 = \begin{bmatrix} 0.1446 \\ 0.0711 \end{bmatrix}^T, Q = \begin{bmatrix} 0.9017 & -0.0839 \\ -0.0839 & 1.0311 \end{bmatrix}.$$

Therefore, the NCS with only 20% data packet transmission can be asymptotically stabilized with the following feedback gains:

$$F_0 = \begin{bmatrix} -7.0049 & -12.9520 \end{bmatrix}, F_1 = \begin{bmatrix} 4.0039 & 1.9686 \end{bmatrix}, F_2 = \begin{bmatrix} 1.3367 & 0.6572 \end{bmatrix},$$

$$F_3 = \begin{bmatrix} 0.4514 & 0.2220 \end{bmatrix}, F_4 = \begin{bmatrix} 0.1681 & 0.0826 \end{bmatrix},$$

and feedback mechanism: *if the data packet (containing the information of state measurement) is transmitted instantly, we use the feedback gain F_0 and the fresh state measurement to construct the controller; otherwise, if the register has not update for i times, we choose F_i as the feedback gain for $i = 0, 1, \cdots, 4$.* This means that an effective sampling period of $h = 5s$ can be used in this system.

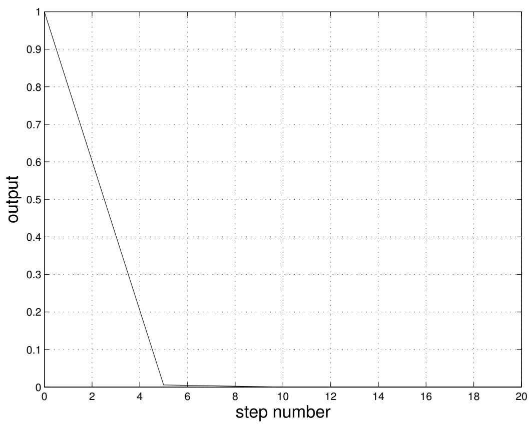

Figure 7: The output of NCS (59) with 80% data packet dropout

Suppose data packet dropout of the system within the bound $d = 4$ is distributed in a random sequence 4, 3, 1, 0, 2, 0, $\cdots$, that is, the update instants of plant (59) are $t_0 = 0$, $t_1 = 5h$, $t_2 = 9h$, $t_3 = 11h$, $\cdots$. In this pattern of transmission, with above feedback mechanism and the initial condition $x_0 = [1\ -1]^T$, the output of plant (59) at the update instants is shown in Fig. 7. It can be seen from Fig. 7 that plant (59) with 80% data packet dropout can be asymptotically stabilized in 9 steps. In [35], the system is stabilized in 25 steps though their sampling period is much smaller than ours. Obviously, our result is less conservation than the one based on the method in [35].

The setup in Fig. 1 has also been studied in [29]. Using the approach in [29], the maximum allowable transfer interval can be computed as 4.5×10^{-4}s. This will consume a lot of network bandwidth if it is implemented in a real application.

The result in the former can be generalized to include rate constraints on the convergence of the trajectory to the origin. This introduces some primitive performance measure into the framework.

It is well known that if the Lyapunov function satisfying

$$V(x(t_{k_{j+1}})) < \alpha^2 V(x(t_{k_j})) \tag{60}$$

for $0 < \alpha \leq 1$, then

$$\|x(t_{k_j})\| \leq \alpha^j \|x(0)\|. \tag{61}$$

We can now extend Theorem 7 by applying the same arguments to inequality (60). Here we briefly summarize the conclusion.

Corollary 1. *If there exist a symmetric positive definite matrix $P \in \mathbf{R}^{n\times n}$ and matrices $Y_i \in \mathbf{R}^{m\times n}$ satisfying the following LMIs*

$$\begin{bmatrix} -\alpha^2 Q & X_i^T \\ X_i & -Q \end{bmatrix} < 0, \tag{62}$$

for $i = 0, \cdots, d$, where X_i is defined as that of Theorem 7, then the NCS (44) can be asymptotically stabilized with a decay rate α via Feedback Mechanism 1 with

$$F_i = Y_i Q^{-1}, \quad i = 0, 1, \cdots, d.$$

3.2 Discrete-time case

In this part the NCSs with packet dropout and constant transmission delays will be studied.

3.2.1 Stabilization with packet dropout and transmission delays via state feedback

Depending on the medium access protocol of the control network, network induced delays can be constant or time varying. Here, we consider the constant case, which is often appeared in the scheduling networks. We also assume that the maximum data packet loss between two successive safe transmission within a bound d, where d is a positive integer. For simplicity, the feedback gain is designed to be constant in this part.

First, we consider NCS (37) with one step delay. We assume that the packet containing $x(0)$ with one step delay is transmitted to the controller successfully, then

$$\begin{aligned} x(1) &= Ax(0), \\ x(2) &= Ax(1) + BFx(0). \end{aligned}$$

Suppose that the successive transmitted state measurements are $x(0)$, $x(k_1), \cdots, x(k_i), \cdots$, the evolution of these states can be described as follows:

$$
\begin{aligned}
x(k_1) &= (A^{k_1}+A^{k_1-2}BF+\cdots+BF)x(0),\\
x(k_2) &= (A^{k_2-k_1}+A^{k_2-k_1-2}BF+\cdots+BF)x(k_1)+A^{k_2-k_1-1}BFx(0),\\
&\vdots\\
x(k_j) &= (A^{k_j-k_{j-1}}+A^{k_j-k_{j-1}-2}BF+\cdots+BF)x(k_{j-1})+A^{k_j-k_{j-1}-1}BFx(k_{j-2}),\\
&\vdots
\end{aligned}
$$

Let us define another sequence

$$z(0)=x(0),\ z(1)=x(k_1),\ \cdots,z(j)=x(k_j),\cdots. \tag{63}$$

It follows that

$$
\begin{aligned}
z(j) &= (A^{k_j-k_{j-1}}+A^{k_i-k_{j-1}-2}BF+\cdots+BF)\,z(j-1)+A^{k_i-k_{j-1}-1}BFz(j-2)\\
&\triangleq A(j)z(j-1)+B(j)z(j-2),\ j=1,2,\cdots,
\end{aligned}
$$

where

$$A(j)=A^{k_j-k_{j-1}}+A^{k_j-k_{j-1}-2}BF+\cdots+BF,$$

$$B(j)=A^{k_i-k_{j-1}-1}BF.$$

Let $w(j)=[z^T(j)\,z^T(j-1)]^T$ be the augmented state vector; the evolution of the NCS (37) at the transmission instants with the effect of network packet dropout and one step delay can be described by

$$w(j+1)=\begin{bmatrix} A(j) & B(j)\\ I & 0\end{bmatrix}w(j)\triangleq\Lambda(j)w(j),\ \ j=1,2,\cdots,$$

it must be true that

$$\Lambda(j)\in\Omega,\quad \Omega=\{\Lambda_0,\ \Lambda_1,\cdots,\Lambda_d\},$$

where

$$\Lambda_i=\begin{bmatrix} A^{i+1}+A^{i-1}BF+\cdots+BF & A^iBF\\ I & 0\end{bmatrix}. \tag{64}$$

Similarly to the discussion of continuous case, it can be easily verified that the asymptotic stability of the NCS (37) with one step delay and arbitrary but finite data packet dropout can be guaranteed by the asymptotic stability of the following switched linear system

$$w(k+1)=\Lambda_i w(k),\ \ k=1,2,\cdots. \tag{65}$$

for arbitrary switching, where $\Lambda_i\in\Omega$. Therefore, we next proceed to analyze the switched system (65) for arbitrary switching. The following result gives a sufficient condition on the stability of the switched system (65), which is a special case of [14].

Lemma 3. *[14] If there exists a symmetric positive definite matrix $P \in \mathbf{R}^{n\times n}$ satisfying the following LMIs*

$$\begin{bmatrix} -P & \Lambda_i^T P \\ P\Lambda_i & -P \end{bmatrix} < 0, \ \forall\, i = 0, 1, \cdots, d, \tag{66}$$

then the switched system (65) is asymptotically stable.

We now present a solution to the problem of stabilization of the system (37) with the effect of one step delay and arbitrary but finite data packet dropout.

Theorem 8. *If there exist a symmetric positive definite matrix $Q \in \mathbf{R}^{n\times n}$ and a matrix $W \in \mathbf{R}^{m\times n}$ satisfying the following LMIs*

$$\begin{bmatrix} \begin{bmatrix} -Q & 0 \\ 0 & -Q \end{bmatrix} & \Psi_i^T \\ \Psi_i & \begin{bmatrix} -Q & 0 \\ 0 & -Q \end{bmatrix} \end{bmatrix} < 0, \tag{67}$$

for $i = 0, 1, \cdots, d$, where

$$\Psi_i = \begin{bmatrix} A^{i+1}Q + A^{i-1}BW + \cdots + BW & A^iBW \\ Q & 0 \end{bmatrix},$$

then the NCS (37) with one step delay and data packet dropout within the bound d can be asymptotically stabilized via the state feedback

$$u(k) = WQ^{-1}\bar{x}(k).$$

Proof. Noting that the asymptotic stability of the NCS (37) can be ensured by that of the switched system (65), by Lemma 3, we only need to prove that (66) holds. Set

$$P^{-1} = \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix} \triangleq \bar{Q}, \tag{68}$$

where Q is a symmetric positive definite matrix. Pre- and post-multiplying inequality (66) by block-diag $[\bar{Q}\ \bar{Q}]$, it is easy to see that (66) is equivalent to

$$\begin{bmatrix} -\bar{Q} & \bar{Q}\Lambda_i^T \\ \Lambda_i\bar{Q} & -\bar{Q} \end{bmatrix} < 0, \ i = 0, 1, \cdots, d. \tag{69}$$

Let $W = FQ$, by the definition of Λ_i in (64), we know that (69) is equivalent to (67). The conclusion follows. $\square$

Above result can further be generalized to the case of s step delays.

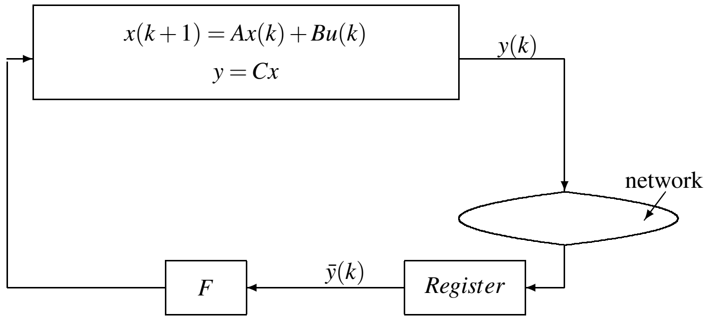

Figure 8: An NCS via static output feedback

Corollary 2. *If there exist a symmetric positive definite matrix $Q \in \mathbf{R}^{n\times n}$ and a matrix $W \in \mathbf{R}^{m\times n}$ satisfying the following LMIs*

$$\begin{bmatrix} \begin{bmatrix} -Q & 0 \\ 0 & -Q \end{bmatrix} & \Theta_i^T \\ \Theta_i & \begin{bmatrix} -Q & 0 \\ 0 & -Q \end{bmatrix} \end{bmatrix} < 0,$$

for $i = 0, 1, \cdots, d$, where

$$\Theta_i = \begin{bmatrix} A^{i+1}Q + A^{i-s}BW + \cdots + BW & A^iBW + \cdots + A^{i-s+1}BW \\ Q & 0 \end{bmatrix},$$

then the NCS (37) with s step delays and data packet dropout within the bound d can be asymptotically stabilized via the state feedback

$$u(k) = WQ^{-1}\bar{x}(k).$$

3.2.2 Stabilization with packet dropout via Output feedback

In this section, we consider the problem of stabilization of an NCS with data packet dropout via static output feedback. As shown in Fig. 8, the NCS consists of a discrete plant and a discrete static output feedback controller

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k), \\ u(k) &= F\bar{y}(k), \qquad k = 1, 2, \cdots, \end{aligned} \tag{70}$$

where $y(k) \in \mathbf{R}^m$ is the output of the plant. $F \in \mathbf{R}^{m\times n}$ is the output feedback gain matrix to be designed, C is a known real constant matrix with appropriate dimensions and $\bar{y}(k)$ is the successfully transmitted data that will be used to construct the controller. Similarly, we propose the following iterative approach for the case without delays.

Suppose that the successive update steps of $\bar{y}(k)$ are 0, $k_1, \cdots, k_i, \cdots$. In this pattern of transmission, the states of the NCS at the update steps can be described as follows:

$$x(k_j) \quad = \quad (A^{k_j-k_{j-1}} + A^{k_j-k_{j-1}-1}BFC + \cdots + BFC)x(k_{j-1}), \quad j = 1, \cdots.$$

Let us define another sequence

$$z(0) = x(0),\ z(1) = x(k_1),\ \cdots, z(j) = x(k_j), \cdots. \tag{71}$$

It follows that

$$z(j) = (A^{k_j-k_{j-1}} + A^{k_j-k_{j-1}-1}BFC + \cdots + BFC)\, z(j-1) \stackrel{\triangle}{=} A(j)z(j-1), \tag{72}$$

where $A(j) = A^{k_j-k_{j-1}} + A^{k_j-k_{j-1}-1}BFC + \cdots + BFC$. Because the upper bound of data packet dropout is d, it must be true that

$$A(j) \in \Omega, \qquad \Omega = \{\bar{A}_0,\ \bar{A}_1, \cdots, \bar{A}_d\}, \tag{73}$$

where

$$\bar{A}_i = A^{i+1} + A^i BFC + \cdots + BFC. \tag{74}$$

As the discussion in the previous sections, the asymptotic stability of the NCS (70) with arbitrary but finite data packet dropout will be guaranteed by the asymptotic stability of the following switched system

$$z(k+1) = \bar{A}_i z(k) \tag{75}$$

for arbitrary switching, where $\bar{A}_i \in \Omega$.

Now, we are ready to present the following result.

Theorem 9. *We assume that C is of full row rank. If there exist a symmetric positive definite matrix $Q \in \mathbf{R}^{n\times n}$, and matrices $W \in \mathbf{R}^{m\times m}, M \in \mathbf{R}^{m\times m}$ satisfying*

$$CQ = MC, \tag{76}$$

and the following LMIs

$$\begin{bmatrix} -Q & Q(A^{i+1})^T + C^T W^T B^T (A^i)^T + \cdots + C^T W^T B^T \\ A^{i+1}Q + A^i BWC + \cdots + BWC & -Q \end{bmatrix} < 0, \tag{77}$$

for $i = 0, 1, \cdots, d$, then the NCS (70) can be asymptotically stabilized for data packet dropout within the bound d via the static output feedback

$$u(k) = WM^{-1}\bar{y}(k). \tag{78}$$

Proof. As the proof of Theorem 8, we only need to prove that the given conditions guarantee that (66) holds. Consider the following Lyapunov function

$$V(x(k)) = x(k)^T P x(k) \tag{79}$$

where P is a symmetric positive definite matrix. The difference of function V along the trajectory of the system (70) at the update steps is given by

$$\begin{aligned} \Delta V_j &\triangleq V(x(k_{j+1})) - V(x(k_j)) \\ &= z(j)^T (A(j)^T P A(j) - P) z(j), \end{aligned}$$

where we used (71) and (72) in the last equality. Thus $\Delta V_j < 0$ if

$$A(j)^T P A(j) - P < 0. \tag{80}$$

Using Schur complement together with (73), (80) is equivalent to

$$\begin{bmatrix} -P & \bar{A}_i^T P \\ P\bar{A}_i & -P \end{bmatrix} < 0, \quad \forall\, i = 0, 1, \cdots, d. \tag{81}$$

Set $P^{-1} = Q$. Pre- and post-multiplying (81) by block-diag $[Q\ Q]$, it follows from (74) that (81) is equivalent to

$$\begin{bmatrix} -Q & Q(A^i)^T + QC^T F^T B^T (A^{i-1})^T + \cdots + QC^T F^T B^T \\ A^i Q + A^{i-1} BFCQ + \cdots + BFCQ & -Q \end{bmatrix} < 0. \tag{82}$$

Because C is of full row rank, we obtain from (76) that M is also of full rank, and thus invertible. Let $W = FM$, it follows from (76) that (77) is equivalent to (82). The conclusion follows. □

The following remark gives an approach to finding the maximum data packet dropout bound.

Remark 9. *For the NCS (70) stabilized via the static output feedback* (78)*, we can find the maximum bound of data packet dropout by search the largest d that does not violate the conditions of Theorem 9. That is,*

$$\max \ d$$

subject to $\exists\, Q > 0$, *W and M satisfying (76)–(77).*

Example 4. *Consider the state-space plant model*

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 2 & 0.8 \\ 0 & 0.99 \end{bmatrix} x(k) + \begin{bmatrix} 0.2 \\ 1 \end{bmatrix} u(k), \\ y(k) &= [2\ \ 1] x(k), \end{aligned} \tag{83}$$

the feedback controller takes the form $u = F\bar{y}(k)$ *with* F *to be designed.*

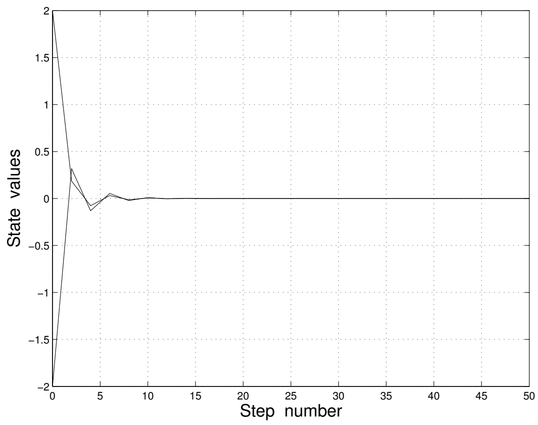

Figure 9: The step response of NCS (83) via output feedback

For simplicity, we consider the case that the transmission periods are identical. By Remark 9, we obtain $d = 2$. Using Theorem 9, we have

$$Q = \begin{bmatrix} 2.6700 & -3.3603 \\ -3.3603 & 4.2835 \end{bmatrix}, \; W = -0.0143, \; M = 0.3044, \; F = -0.0469.$$

With the initial state $x(0) = [-2 \;\; 2]^T$, for one data packet dropped in every two packets, the step response of the NCS (83) is shown in Fig. 9, from which we know that the system can be stabilized in 12 steps.

4 Conclusions

This paper mainly considers the problems induced by communication channel in networked control systems: data packet dropout and transmission delays. We modelled NCSs with these problem into delay time systems and linear switched systems. We construct the feedback controllers via solving the corresponding linear matrix inequalities (LMIs) which can be easily solved using the LMI toolbox [2]. The admissible bound of data packet loss and transmission delays can also be obtained in terms of LMIs.

References

[1] Azimi-Sadjadi, B., Stability of networked control systems in the presence of packet losses, *Proceedings of the 42nd IEEE Conference on Decision and Control*, 2003, 1: 676–681.

[2] Boyd, S., El Ghaou, L., Feron, E. and Balakrishnan, V., Linear matrix inequalities in systems and control theory, Philadelphia: SIAM, 1993.

[3] Branicky, M. S., Phillips, S. M. and Zhang Wei, Stability of networked control systems: explicit analysis of delay, *Proceedings of the American Control Conference*, 2000, 4: 2352–2357.

[4] Branicky, M. S., Multiple Lyapunov functions and other analysis tools for switched and hybrid systems, *IEEE Transactions on Automatic Control*, 1998, 43: 475–482.

[5] Brockett, R., Stabilization of motor networks , *Proceedings of the 34th IEEE Conference on Decision and Control* 1995, 2: 1484–1488.

[6] Chan, H. and Özgüner, Ü. Closed-loop control of systems over a communications network with queues, *International Journal of Control* 1995, 65: 493–510.

[7] Hadjicostis, C. N. and Touri, R., Feedback control utilizing packet dropping network links, *Proceedings of the 41nd IEEE Conference on Decision and Control*, 2002, 2: 1205–1210.

[8] Hale, J. K. and Lunel, S. M. V., Introduction to functional differential equations, Springer–Verlag, New York, 1993.

[9] Hristu, D. and Morgansen, K., Limited communication control, *Systems and Control Letters* 1999, 37: 193–205.

[10] Hristu, D., Stabilization of LTI systems with communication constraints, *Proceedings of the American Control Conference*, 2000, 4: 2342–2346.

[11] Ishii, H. and Francis, B. A., Limited Data Rate in Control Systems with Networks, Lecture Notes in Control and Information Sciences, vol. 275, Springer, Berlin, 2002.

[12] Ishii, H. and Francis, B. A., Stabilizing a linear system by switching control with dwell time, *IEEE Transactions on Automatic Control*, 2002, pp. 1962–1973.

[13] Ishii, H. and Francis, B. A., Quadratic stabilization of sampled-data systems with quantization, *Automatica*, 2003, pp. 1793–1800.

[14] Jamal, D., Pierre, R. and Claude, I. Stability Analysis and control synthesis for switched systems: a switched Lyapunov function approach, *IEEE. Trans. Automa. Contr.*, 2002, 47 (11): 1883–1887.

[15] Lian, F-Li, Moyne, J. R., and Tilbury, D. M., Analysis and modeling of networked control systems: MIMO case with multiple time delays, *Proceedings of the American Control Conference*, 2001, 6: 4306–4312.

[16] Lian, F-Li, Moyne, J. R., Tilbury, D. M., Performace evalation of control networks, *IEEE. Control Systems Magazine*, 2001, 21 (1): 66–83.

[17] Ling, Q. and Lemmon, M. D., Robust performance of soft real-time networked control systems with data dropouts, *Proceedings of the Conference on Decision and Control*, 2002, 1225–1230.

[18] Luck, R. Ray, A. An observer-based compensator for distributed delays, *Automatica,* 1990, 26 (5): 903–908.

[19] Hale, J. K. and Lunel, S. M. V., Introduction to functional differential equations , Springer–Verlag, New York, 1993.

[20] Montestruque, L. A. and Antsaklis, P. J., On the model-based control of networked systems, *Automatica*, 2003, 39: 1837–1843.

[21] Mu, S., Chu, T., Hao, F. and Wang, L., "Output feedback control of networked control systems," *Proceedings of the IEEE International Conference on Systems, Man & Cybernetics,* Hyatt Regency, Washington, D.C., USA. 2003, pp. 211-216.

[22] Niculescu, S.-J. Fu, M. and Li, H., Delay-dependent closed-loop stability of linear systems with input delay: an LMI approach , *Proceedings of the 36th IEEE Conference on Decision and Control*, 1997, 2: 1623–1628.

[23] Nilsson, J., Real-time control systems with delays, PhD thesis, Lund Institute of Technology, 1998.

[24] Petersen, I. R., A stabilization algorithm for a class of uncertain linear systems, *Syst. Control Lett.,* 1987, 8(4): 493–510.

[25] Ray, A. and Galevi, Y., Intergrated communication and control systems: Part II—design considerations, *Journal of Dyanmic Systems, Measurement, and Control*, 1988, 110: 374–381.

[26] Song, S.-H., Kim, J.-K., Yim, C.-H. and Kim, H.-C., H_∞ control of discrete-time linear systems with time-varying delays in state, *Automatica*, 1999, 35 (9): 1587–1591.

[27] Tzes, A., Nikolakopoulos, G. and Koutroulis, I., Development and experimental verification of a mobile client-centric networked controlled system, *Proceedings. European Control Conference,* University of Patras, Greece, 2003.

[28] Vidyasagar, M., Nonlinear System Analysis, Prentice Hall, Englewood Cliffs, NJ, Second Edition, 1993.

[29] Walsh, G. C., Hong Ye and Bushnell, L. G., Stability analysis of networked control systems, *IEEE Transactions on Control Systems Technology*, 2002, 10: 438–446.

[30] Wong, W. S. and Brockett, R. W., Systems with finite communication bandwidth constraints— I. State estimation problems, *IEEE. Trans. Automa. Contr.*, 1997, 42: 1294–1299.

[31] Wong, W. S. and Brockett, R. W., Systems with Finite Communication Bandwidth Constraints—II. Stabilization with Limited Information Feedback, *IEEE. Trans. Automa. Contr.*, 1999, 44: 1049–1053.

[32] Yook, J. K., Tilbury, D. M. and Soparkar, N. R., Trading computation for bandwidth: reducing communication in distributed control systems using state estimators, *IEEE Transactions on Control Systems Technology*, 2002, 10 (4): 503–518.

[33] Yu, L. and Gao, F., Optimal guaranteed cost control of discrete-time uncertain systems with both state and input delays, *Journal of the Franklin Institute*, 2001, 338: 101–110.

[34] Yu, M., Wang, L., Chu, T. and Hao, F. An LMI Approach to Networked Control Systems with Data Packet Dropout and Transmission Delays, accepted by *International Journal of Hybrid Systems.*

[35] Zhang Wei, Branicky, M. S. and Phillips, S. M., Stability of networked control systems, *IEEE. Control Systems Magazine,* 2001, 21: 84–99.

Appendix A

Proof. The state of the plant (44) between two successive transmission instants $t_{k_{j-1}}$ and t_{k_j} can be described as

$$\begin{aligned} x(t) &\stackrel{\Delta}{=} x(wh+\tau) = \Phi(\tau)x(wh) + \Gamma(\tau)F_{w-k_{j-1}-1}x(k_{j-1}h) \\ &= \Phi(\tau)(\Phi^{w-k_{j-1}}(h) + \Phi^{w-k_{j-1}-1}(h)\Gamma(h)F_0 + \cdots + \Gamma(h)F_{w-k_{j-1}-1})x(k_{j-1}h) \\ &\quad + \Gamma(\tau)F_{w-k_{j-1}-1}x(k_{j-1}h), \end{aligned} \tag{84}$$

where $t \in [wh,(w+1)h)$, $w \in [k_{j-1},k_j)$, $\tau \in [0,h)$, then the state of the NCS, $x(t)$, between successive transmission instants is bounded by

$$\begin{aligned} \|x(t)\| &= \|(\Phi(\tau)(\Phi^{w-k_{j-1}}(h) + \Phi^{w-k_{j-1}-1}(h)\Gamma(h)BF_0 + \cdots + \Gamma(h)F_{w-k_{j-1}-1} + \Gamma(\tau)F_{w-k_{j-1}-1})x(k_{j-1}h)\| \\ &\leq c\|x(k_{j-1}h)\|, \end{aligned} \tag{85}$$

where

$$c = \max_{i=0,1,\cdots,d,\tau\in[0,h)} \{\|(\Phi(\tau)(\Phi^{i+1}(h) + \Phi^i(h)\Gamma(h)F_0 + \cdots + \Gamma(h)F_i + \Gamma(\tau)F_i)\|\}.$$

First we prove that the NCS is uniformly stable. Given any $\varepsilon > 0$, let $\bar{\varepsilon} = \min\{\varepsilon, r\}$, because $x(t)$ and $V(x(t))$ is a continuous function of time and $V(0) = 0$, we can find a $\delta(\varepsilon) > 0$ such that

$$\bar{\beta}(\delta) = \sup_{\|x(t)\|\leq\delta} V(x(t)) < \alpha(\bar{\varepsilon}/c) \leq \alpha(\varepsilon/c). \tag{86}$$

Now we claim that for all $\|x(0)\| \leq \delta$, $\|x(t)\| < \varepsilon$, $\forall t \geq 0$. This can be proved by contradiction. Suppose at $t^* > 0$, $\|x(t^*)\| = \varepsilon$. Again this is sufficient because $x(t)$ is continuous. If $t^* \in [t_{k_j}, t_{k_{j+1}})$ then by the boundedness of $x(t)$ during transmissions given by (85), we have

$$\|x(t_{k_j})\| \geq \varepsilon/c.$$

By (54) and the fact that α is a function of class K, we have

$$V(x(t_{k_j})) \geq \alpha(\|x(t_{k_j})\|) \geq \alpha(\varepsilon/c).$$

However, since V decreases at updates, and by (86),

$$V(x(t_{k_j})) \leq V(x(0)) \leq \bar{\beta}(\delta) < \alpha(\varepsilon/c).$$

Now we have a contradiction, and hence the NCS is uniformly stable.

To show asymptotic stability, observe that the sequence $V(x(t_{k_j}))(j=1,2,\cdots)$ is decreasing and positive, and therefore has a limit $L \geq 0$. We have

$$\begin{aligned} 0 = L - L &= \lim_{j\to\infty} V(x(t_{k_{j+1}})) - \lim_{j\to\infty} V(x(t_{k_j})) \\ &= \lim_{j\to\infty} [V(x(t_{k_{j+1}})) - V(x(t_{k_j}))]. \end{aligned}$$

Because $\gamma(.)$ is a function of **class** K, it follows from (54) that

$$V(x(t_{k_{j+1}})) - V(x(t_{k_j})) \leq -\gamma(\|x(t_{k_j}\|) \leq 0,$$

which implies that

$$\lim_{k\to\infty} \gamma(\|x(t_{k_j})\|) = 0.$$

Thus, we have

$$\lim_{j\to\infty} \|x(t_{k_j})\| = 0.$$

From (85) we finally obtain that

$$\lim_{k\to\infty} \|x(t)\| = 0.$$

This completes the proof. □