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An LMI Approach to Structured Sparse Feedback Design in Linear Control Systems

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***Abstract*— Consider the classical state feedback design in the linear system $\dot{x} = Ax + Bu$ subject to performance specifications with an additional requirement that the control input vector $u = Kx$ has as many zero entries as possible. The corresponding gain K is referred to as a *row-sparse controller*. We propose an approach to approximate solution of this kind of nonconvex problems by formulating the proper convex surrogate,—the minimization of a certain matrix norm subject to LMI constraints. The novelty of the paper is the problem formulation itself and the construction of the surrogate. The two main contributions are the design of low-dimensional output to be used in static output feedback, and suboptimal design illustrated via LQR. The results of preliminary numerical experiments are twofold. First, in many test problems, the number of controls was considerably reduced without significant loss in performance. Second, the number of nonzero entries obtained by our method is either very close to or coincide with the minimum possible amount. The approach can be further extended to handle numerous problems of optimal and robust control in sparse formulation.**

I. INTRODUCTION

In the recent years, the sparsification concept became popular in many fields of system theory and practice. In the optimization context, this concept arises naturally in problems with integer-valued cost, while the constraints are continuous. Such problems are known to be nonconvex and NP-hard; a possible approach is to formulate a related *convex surrogate problem* and adopt its solution as an approximation to that of the original one. As a rule, no strict assertions can be made about the accuracy of the estimates, but usually there is a sound heuristic behind such an approach.

One of the first and most striking areas of application of sparsity is the theory and practice of l_1-optimization, the framework formulated in [15] and later extended and generalized in many directions, just to mention compressed sensing [2], l_1-filtering [6] and many other fields.

The idea behind this approach is to give the "simplest" explanation of the observed data using functions in a given basis, which reduces to minimizing the number of nonzero entries of a vector subject to convex constraints. This hard problem is substituted by its convex surrogate, the minimization of the l_1-norm, which is shown to be an *efficient heuristic*, the term proposed in [3], where similar ideology was demonstrated to be fruitful in many problems in system theory.

Here we consider yet another heuristic; it is related to the minimization of the number of nonzero rows or columns of a matrix. Such matrices will be referred to as *row-sparse* and *column-sparse*; it is what we mean by structured sparse gain matrix or structured sparse feedback.

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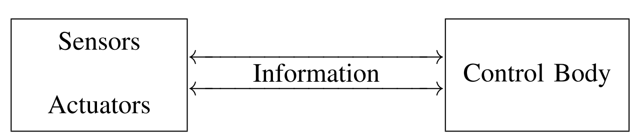
Such minimization problems arise in multi-task learning and simultaneous sparse approximation, where it is required to approximate several data arrays simultaneously by a small number of elementary signals, [16]. A nice convex surrogate for this combinatorial problem is a special matrix norm often referred to as $l_{1,\infty}$-norm. It penalizes the sum of maximum absolute values of each row, and efficient numerical methods for optimization of $l_{1,\infty}$-regularized convex objectives have been proposed; e.g., see [13]. In the recent paper [12], this norm was used in the sparse identification framework.

While in signal processing, image recognition, optimization, natural language processing, computer vision, etc., the sparsity concept is widely used, there seem to be very little related *control literature*. Among the few recent papers on sparse feedback we mention [9], [10], where the sparsity structure was specified in advance thus leading to nonconvex optimization problems; the emphasis in these works was put on algorithmic optimization issues.

In this paper, we exploit the sparse ideology in the design of linear control systems via a nonstandard integer-valued criterion, e.g., the number of the nonzero components of the control input vector. In contrast to the classical optimal control problems such as the linear quadratic regulation (LQR), this performance index is hard to optimize, since it involves the combinatorial search. We show that solving the convex surrogate based on $l_{1,\infty}$-norm heuristic leads to a suboptimal solution in a straightforward way. Similarly, special formulations of the static output feedback (SOF) design related to the *construction* of the appropriate output vector can be solved easily with this heuristic.

A transparent motivation for use of sparsity in control design can be found in the so-called C^3 paradigm, which considers the triad *Control, Communication, Computation* as an integrated compound to be analyzed from a unified point of view, [5], [4], [11]. Within this concept, reducing the number of states required to control the plant is synonymous with the number of *sensors*; the number of controls is associated with the number of *actuators*, while reducing the number of outputs is equivalent to minimizing the amount of *information* transmitted through control channels.



We propose a regular suboptimal approach to control problems of this sort by formulating an appropriate convex surrogate and show via examples that the exploited heuristic is efficient. Overall, our method is a fusion of the quadratic stabilization concept and the ideas of l_1-optimization in the matrix formulation.

The approach is characterized by simplicity (the surrogates

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are formulated as low-dimensional semidefinite programs, and the standard MATLAB tools are used in the numerical implementation), versatility (continuous and discrete time problems are solved along the same lines; static feedback laws can be designed both from the state and output signals), and, most importantly, extendability to various optimal control problems such as LQR, H_∞-optimization, optimal rejection of exogenous bounded disturbances, etc. Robust formulations are also made possible.

To the best of our knowledge, the very statement of the problem is new, thus constituting the first contribution of this paper. Second, we propose a simple and numerically efficient approximate solution method based on the associated heuristic. Third, as shown via examples, the method exposes quite a nice performance.

II. Efficient Matrix Norm Heuristics

We start with recalling the well-known problem of minimizing the l_0 *vector* (quasi)norm defined as

$$\|x\|_0 \doteq \sum_i |\mathrm{sign}\, x_i|,$$

which is the number of nonzero entries of x. The l_0-norm is not a convex function, hence, is hard to optimize. Instead, a convex surrogate based on the vector l_1-norm is used. Specifically, the following classical result from linear algebra is in the basis of the l_1 approach to sparse optimization.

Theorem 1 (Vector norm heuristic): If the problem

$$\min \|x\|_1 \quad \text{s. t.} \quad Ax = b,$$

where $A \in \mathbb{R}^{m,n}$, $b \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, $m < n$, is feasible, then there exists a solution $\widehat{x}$ having no more than m nonzero components. □

The vector l_1-norm is known to represent an *efficient heuristic* for the l_0-quasinorm, meaning that the minimum of $\|x\|_1$ over a set specified by convex constraints that differ from the linear ones as in the theorem above, is a "good" approximation for sparse solutions.

To formulate the main result of this section, a matrix analog of Theorem 1, we introduce special matrix norms.

First, as counterparts of the vector l_0-quasinorm, we introduce the matrix quasinorms, $\|X\|_{r_0}$ and $\|X\|_{c_0}$, which are defined as the number of nonzero rows and columns of the matrix X, respectively.

We next consider the *row* and *column norms* of $X \in \mathbb{R}^{n,p}$:

$$\|X\|_{r_1} = \sum_{i=1}^{n} \max_{1\le j\le p} |x_{ij}|, \qquad \|X\|_{c_1} = \sum_{j=1}^{p} \max_{1\le i\le n} |x_{ij}|.$$

The former is sometimes referred to as rx-norm or $l_{1,\infty}$-norm. Its main feature is the ability to recover row-sparse solutions of matrix equations, [16], [13]. Similarly, the c_1-norm recovers column-sparse solutions.

The theorem below presents an *efficient heuristic* for use in all the constructions in this paper.

Theorem 2 (Matrix norm heuristic): If the problem

$$\min \|X\|_{r_1} \quad \text{s. t.} \quad AX = B,$$

with $A\in\mathbb{R}^{m,n}$, $m < n$, $B\in\mathbb{R}^{m,p}$, $X\in\mathbb{R}^{n,p}$ is feasible, then there exists a solution having at most m nonzero rows. □

The proof follows from the two lemmas below.

Lemma 1: The vertices of the set

$$\mathcal{Q} = \{X \in \mathbb{R}^{n,p}\colon \|X\|_{r_1} \le 1\}$$

are defined by the matrices with all but one zero rows, with the nonzero row having all entries equal to unity in absolute value. □

Proof: For given c_{ij}, consider the problem

$$\max \sum_{ij} x_{ij} c_{ji} \quad \text{s. t.} \quad -t_i \le x_{ij} \le t_i, \;\; \sum_i t_i \le 1,$$

which is finding the maximum of a linear function over the set $\mathcal{Q}$. Clearly, for the fixed values of the t_i's, the solution is given by $x^*_{ij} = t_i \,\mathrm{sign}\, c_{ij}$. Hence, the problem reduces to the minimization of a linear function on the unit simplex:

$$\max \sum_i a_i t_i \quad \text{s. t.} \quad \sum_i t_i \le 1, \;\; t_i \ge 0,$$

where $a_i = \sum_j |c_{ij}|$. The solution t^* of this problem is attained at the point with $t^*_k = 1$, $k = \arg\max_i a_i$, and the rest of the components being zeros. This point corresponds to a vertex of $\mathcal{Q}$. ■

Lemma 2: Let Q be a polytope in $\mathbb{R}^n$. Consider the following linear program in the variables $t \in \mathbb{R}$, $x \in \mathbb{R}^n$:

$$\min t \quad \text{s. t.} \quad x \in tQ, \;\; t \ge 0, \;\; Ax = b,$$

where $A \in \mathbb{R}^{m,n}$, $b \in \mathbb{R}^m$, $m < n$. Then there exists a solution (t^*, x^*) such that x^* belongs to the convex hull of m vertices of the polytope t^*Q. □

Proof: The polytope can be represented as the convex hull of its vertices x^i; hence, introducing the new variable $\alpha = 1/t$ we arrive at the equivalent LP w.r.t. $\alpha\in\mathbb{R}$, $\lambda\in\mathbb{R}^n$:

$$\max \alpha \quad \text{s. t.} \quad A\sum_i \lambda_i x^i = \alpha b, \;\; \sum_i \lambda_i \le 1, \;\; \lambda_i \ge 0.$$

The solution is attained at a point (α^*, λ^*) having no more than $m+1$ nonzero components (the number of equality constraints in the LP above). Since $\alpha^* \ne 0$, we have no more than m nonzero components in λ^*, which means that

$$\alpha^* x^* \doteq \sum_{i=1}^{n} \lambda^*_i x^i$$

belongs to the convex hull of no more than m points x^i. In other words, x^* belongs to the convex hull of no more than m vertices of the polytope t^*Q. ■

Theorem 2 can be formulated for the c_1-norm and zero columns. If X is a column vector, we arrive at Theorem 1.

In the sections to follow we show how a systematic use of r_1- or c_1-norm facilitates the sparsification of controllers. In other words, in the control setup, $\|\cdot\|_{r_1}$ will be shown to be *an efficient heuristic* for $\|\cdot\|_{r_0}$ in the sense that the minimization of $\|\cdot\|_{r_1}$ subject to certain convex inequality constraints leads to the reduction of $\|\cdot\|_{r_0}$; the same for the column norm.

Note that both norms are linear functions of the entries of a matrix, and minimizing either of them (call it r_1-*optimization* and c_1-*optimization*) subject to LMI constraints is a standard semidefinite program, SDP.

III. Design of Sparse Controllers

A. Reduced Number of Control Inputs

We start with the continuous time linear system

$$\dot{x} = Ax + Bu, \quad u = Kx, \tag{1}$$

with state vector $x \in \mathbb{R}^n$ and control $u \in \mathbb{R}^m$, so that $A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$; the pair (A, B) is assumed to be controllable. The goal is to find a stabilizing control which is *sparse* in the sense that it has zero components. This is equivalent to finding a *row-sparse* stabilizing gain matrix $K \in \mathbb{R}^{m,n}$.

Remark 1: A generic controllable system can be stabilized by a scalar control, so that the optimal solution can be found by setting all but one rows of K to zeros, and there is no need in using our technique. This simple problem is discussed here in order to introduce the required machinery in a transparent way and illustrate the principle underlying our approach. In Sections IV and V we show how Theorem 2 can be used in the nontrivial problems where the straightforward approach unavoidably requires the combinatorial search, while our method will be shown to be simple yet efficient.

We follow the Lyapunov approach to the state feedback design, e.g., see [1]. Namely, the closed-loop system is stable iff there exist K and $Q \succ 0$ such that

$$(A + BK)^\top Q + Q(A + BK) \prec 0.$$

Pre- and post-multiplying this inequality by $P = Q^{-1}$ and introducing the new variable $Y = KP$, we arrive at the LMI

$$AP + PA^\top + BY + Y^\top B^\top \prec 0, \quad P \succ 0, \tag{2}$$

in the matrix variables $P = P^\top \in \mathbb{R}^{n,n}$, $Y \in \mathbb{R}^{m,n}$. A stabilizing controller for system (1) is then given by

$$\widehat{K} = \widehat{Y}\widehat{P}^{-1}, \tag{3}$$

where $\widehat{P}, \widehat{Y}$ is a solution of (2).

Assume now that $\widehat{Y}$ is *row-sparse*, then the corresponding controller $\widehat{K}$ is row-sparse as well, since post-multiplication preserves the *zero-row structure*. Hence, we force Y to be row-sparse by minimizing its r_1-norm.

Assertion 1: The solution $\widehat{P}, \widehat{Y}$ of the SDP

$$\min \|Y\|_{r_1} \quad \text{s. t.} \quad (2)$$

in the matrix variables P, Y, defines a row-sparse stabilizing $K_{\rm sp} = \widehat{Y}\widehat{P}^{-1}$ and hence, zero components of u. □

We thus detected the control inputs which are sufficient for stabilization; these active controls correspond to the indices of nonzero rows of $K_{\rm sp}$. There is no guarantee that the solution is sparse, and we give no claims on *how sparse* this $K_{\rm sp}$ might be; however we *expect it to be sparse* because of Theorem 2.

B. State Feedback from Incomplete State Vector

We consider now the system in the simplified form

$$\dot{x} = Ax + u \tag{4}$$

and design a state feedback $u = Kx$ from an incomplete state vector. In this case, the *zero-column* structure of the gain matrix defines the redundant components of x; i.e., we are interested in finding a *column-sparse* stabilizing K.

Again, the fulfillment of the Lyapunov inequality

$$(A + K)^\top Q + Q(A + K) \prec 0, \quad Q \succ 0,$$

is necessary and sufficient for K to be stabilizing. With the new variable $Y = QK$, we arrive at the LMI

$$A^\top Q + QA + Y + Y^\top \prec 0, \quad Q \succ 0, \tag{5}$$

in the matrix variables $Q = Q^\top, Y$, so that its solution $\widehat{Q}, \widehat{Y}$ defines the stabilizing controller

$$\widehat{K} = \widehat{Q}^{-1}\widehat{Y}.$$

Clearly, if there are *zero columns* in the solution $\widehat{Y}$ of this LMI, then the corresponding controller $\widehat{K}$ is *column-sparse*.

Assertion 2: The solution $\widehat{Q}, \widehat{Y}$ of the SDP

$$\min \|Y\|_{c_1} \quad \text{s. t.} \quad (5)$$

in the variables Q, Y, gives a column-sparse controller

$$K_{\rm sp} = \widehat{Q}^{-1}\widehat{Y}$$

which implements the state feedback $u = K_{\rm sp}x$ from an incomplete state vector of system (4). □

In other words, we detected the states which are sufficient for use in the state feedback design; these correspond to the indices of nonzero columns of $K_{\rm sp}$.

Remark 2 (Weights): Assume that some specific components of x are expensive to measure. Then instead of the c_1-norm, a *weighted c_1-norm*

$$\|X\|_{c_1,w} = \sum_{j=1}^{n} w_j \max_{1\le i\le m} |x_{ij}|, \qquad w_j \ge 0,$$

should be used, where the higher values of the weights correspond to more expensive components.

C. Static Output Feedback

As a ramification of the result in Section III-B, we now show how the use of the Lyapunov matrix Q can be exploited in the SOF problem. Consider the system

$$\dot{x} = Ax + u, \qquad y = Cx \tag{6}$$

and note that if the pair (A, C) is observable, then the static output stabilization is possible, i.e., there exists K such that $A + KC$ is stable (e.g., see [14]).

Again, within the sparsification framework, our goal is to find a static output controller from an incomplete *output* vector. With the Lyapunov approach we require that there exist $Q \succ 0$ and K such that

$$(A + KC)^\top Q + Q(A + KC) \prec 0.$$

Introducing the new variable $Y = QK$ leads to the LMI

$$A^\top Q + QA + YC + C^\top Y^\top \prec 0, \quad Q \succ 0, \tag{7}$$

and we arrive at

Assertion 3: The solution $\widehat{Q}, \widehat{Y}$ of the SDP

$$\min \|Y\|_{c_1} \quad \text{s.t.} \quad (7),$$

in the variables Q, Y, defines the column-sparse controller $K_{\rm sp} = \widehat{Q}^{-1}\widehat{Y}$ which implements the SOF $u = K_{\rm sp}y$ from an incomplete output vector of system (6). □

We thus detected the outputs which are sufficient for use in the static output feedback design. Clearly, with $C = I$ we arrive at the problem in the previous subsection.

IV. Main Result I: Design of a Low-dimensional Output

A. Low-dimensional Output

In the problems considered above, the structure of the system was fixed, i.e., the matrices A, B, C were given. Assume now that the exact measurements of the full state vector x are available and the goal is to *construct* a linear *low-dimensional* output $y = Cx$ from which an SOF $u = Ky$ can be designed.

To solve this problem, consider in more detail the structure of the static *state* controller $\widehat{K}$ (3). By changing the r_1-norm in the objective function in Assertion 1 to the c_1-norm, we expect to obtain zero columns in $\widehat{Y}$:

$$u = \widehat{Y}\widehat{P}^{-1}x = \begin{pmatrix} \times & 0 & \times & 0 & \times \\ \times & 0 & \times & 0 & \times \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \times & 0 & \times & 0 & \times \end{pmatrix} \begin{pmatrix} \times & \times & \cdots & \times \\ \cdots & \cdots & \cdots & \cdots \\ \times & \times & \cdots & \times \\ \cdots & \cdots & \cdots & \cdots \\ \times & \times & \cdots & \times \end{pmatrix} x.$$

Let us now shape the two matrices: $\widetilde{K}$ which is composed of the nonzero columns of $\widehat{Y}$, and $\widetilde{C}$ composed of the rows of $\widehat{P}^{-1}$ having the same indices. We then have

$$u = Kx = \widetilde{K}\widetilde{C}x = \widetilde{K}y.$$

Assertion 4: Let $\widehat{P}, \widehat{Y}$ be the solution of the SDP

$$\min \|Y\|_{c_1} \quad \text{s.t.} \quad (2), \tag{8}$$

in the variables P, Y. Denote by $\widetilde{K}$ the matrix composed of the nonzero columns of $\widetilde{Y}$, and by $\widetilde{C}$ the matrix composed of the rows of $\widetilde{P}^{-1}$ with the same indices. Then the quantity $y = \widetilde{C}x$ represents a low-dimensional output of the system $\dot{x} = Ax + Bu$, for which the stabilizing output feedback is given by $u = \widetilde{K}y$.

B. Example

We illustrate the efficiency of our approach to SOF design via a benchmark example from [14], where the system

$$\dot{x} = Ax + Bu, \qquad y = Cx,$$

with matrices

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 13 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 5 & -1 \\ -1 & -1 & 0 \end{pmatrix},$$

was considered. Using the nontrivial quantifier elimination considerations, the authors of [14] obtained a parametrization of SOF controllers in the form

$$K_{\rm sof} = \begin{pmatrix} 2 & k \end{pmatrix}, \quad k > 46. \tag{9}$$

We follow the approach formalized in Assertion 4. Instead of inequality (2) in Assertion 1, we will use the LMI

$$AP + PA^\top + BY + Y^\top B^\top \preccurlyeq -2\sigma P, \qquad P \succ 0, \tag{10}$$

with a user-specified $\sigma > 0$, to guarantee the desired degree of stability of the closed-loop system.

Letting $\sigma = 0.05$, we solve the SDP (8) to obtain

$$\widehat{Y} = \begin{pmatrix} 1.7400 & -15.6830 & 0 \end{pmatrix},$$

$$\widehat{P}^{-1} = \begin{pmatrix} 0.3997 & 0.1531 & 0.0118 \\ 0.1531 & 0.8991 & 0.2375 \\ 0.0118 & 0.2375 & 0.0652 \end{pmatrix};$$

hence, we adopt

$$\widetilde{K} = \begin{pmatrix} 1.7400 & -15.6830 \end{pmatrix}$$

as the gain matrix and

$$\widetilde{C} = \begin{pmatrix} 0.3997 & 0.1531 & 0.0118 \\ 0.1531 & 0.8991 & 0.2375 \end{pmatrix}$$

as the output matrix, which is composed of the first two rows of $\widehat{P}^{-1}$. We thus *constructed* the output and designed the associated SOF; with this controller $\widetilde{K}$, the degree of stability of the closed loop system is $\widetilde{\sigma} \approx 0.0509$. Using controllers of the form (9), the same degree of stability is attained with

$$K_{\rm sof} = \begin{pmatrix} 2 & 49 \end{pmatrix}$$

having three times higher magnitude.

Next, noting that the first entry of the $\widehat{Y}$ matrix is much less in absolute value than the second entry, we may consider discarding this small entry:

$$\widehat{Y} = \begin{pmatrix} 0 & -15.6830 & 0 \end{pmatrix}$$

and adopting $\widetilde{K} = -15.6830$ as a candidate SOF gain with

$$\widetilde{C} = \begin{pmatrix} 0.1531 & 0.8991 & 0.2375 \end{pmatrix}$$

composed of only the second row of $\widehat{P}^{-1}$. This leads to the stable closed loop system with $\widetilde{\sigma} \approx 0.0603$.

Therefore, we designed a synthetic *scalar* output and the associated stabilizing controller.

C. Row-sparse Output Feedback

We can go further in the constructions of the previous subsection by "trimming" the output controller $\widetilde{K}$ obtained above to have zero rows. Similarly to Subsection III-A this is accomplished via r_1-optimization of $\widetilde{K}$ and gives a reduced number of *controls* which are sufficient for stabilization.

Assertion 5: Let P_r, Y_r be the solution of the following SDP in the variables P and Y:

$$\min \|Y\|_{r_1} \quad \text{s.t.} \quad (2),$$

where the variable Y is restricted to have the zero-column structure of the solution $\widehat{Y}$ of (8). Denote by $\widetilde{K}_{\rm sp}$ the matrix composed of the nonzero columns of Y_r and by $\widetilde{C}_r$ the matrix composed of the rows of P_r^{-1} having the same indices. Then the quantity $y = C_r x$ represents a low-dimensional output of the system $\dot{x} = Ax + Bu$, for which the stabilizing output feedback is given by $u = \widetilde{K}_{\rm sp}y$, where the controller $\widetilde{K}_{\rm sp}$ is row-sparse so that the control vector u has zero entries.

V. Main Result II: Optimal Control

We now consider the application of our approach to the standard LQR problem for system (1) with initial conditions $x(0) = x_0$; in this case, the method consists of three steps.

First, the optimal value of the quadratic functional

$$J = \int_0^\infty (x^\top Rx + u^\top Su)\, dt$$

on the system's trajectories over all stabilizing controllers K can be obtained by solving the SDP

$$\min \gamma \tag{11}$$

subject to

$$\begin{pmatrix} AP + PA^\top + BY + Y^\top B^\top & P & Y^\top \\ * & -R^{-1} & 0 \\ * & * & -S^{-1} \end{pmatrix} \preccurlyeq 0 \tag{12}$$

and

$$\begin{pmatrix} \gamma & x_0^\top \\ x_0 & P \end{pmatrix} \succcurlyeq 0, \tag{13}$$

in the matrix variables P, Y and the scalar variable γ, e.g., see [1]. The solution $\widehat{P}, \widehat{Y}, \widehat{\gamma}$, defines the optimal gain

$$\widehat{K} = \widehat{Y}\widehat{P}^{-1}$$

and the associated optimal value of the functional:

$$J^* = \widehat{\gamma} = x_0^\top \widehat{P}^{-1} x_0.$$

Though the controller $\widehat{K}$ is seen to be x_0-dependent, there are ways to get rid of this dependence, for example, by using the "averaging" arguments as suggested in [8]. Alternatively, a different LMI formulation can be proposed, which is the subject of a separate prospective paper by the authors. To save space and retain the uniformity of the exposition, we do not discuss here these issues.

Having obtained the optimal value of J^*, at the second step of the method we introduce a scalar relaxation coefficient $\alpha > 1$ and consider the SDP

$$\min \|Y\|_{r_1} \quad \text{s. t.} \quad (12), \ (13), \ \gamma \preccurlyeq \alpha J^* \tag{14}$$

in the variables P, Y, γ. Since r_1-norm is an efficient heuristic, we expect appearance of zero rows in the solution $\widehat{Y}$. Hence, this *zero row detection* step of the method leads to a zero-row structure of the new *sparse variable* Y_0 to be used at the third *optimization* step.

Specifically, we solve the original SDP (11)–(13) with the variables P, Y_0, γ, where Y_0 is restricted to have zero rows at the same positions as $\widehat{Y}$ does.

Overall, the three-stage procedure has the following form:

1) solve the original LQR problem to obtain the optimal value of the performance index;
2) solve the r_1-optimization problem with a relaxed bound on the performance to detect candidate zero rows;
3) solve the original LQR problem over the set of row-structured controllers.

To obtain a sparse solution in the LQR problem with brute force, one has to solve it for *all possible* zero-row structures of the gain matrix K and choose the one with the acceptable performance. In contrast, our method yields quite sparse controllers at the expense of a small loss in the performance as demonstrated via examples below.

VI. Preliminary Numerical Experiments

We consider Problem HE3 from COMP$l_e ib$, a collection of test examples for control system engineering applications, see [7]. Problem HE3 relates to the dynamics of the Bell201A-1 helicopter and its linearized state space model of eighth order with four inputs and six outputs.

Due to space limitations we do not present here the data matrices A, B, C; the interested reader is referred to [7]. By the same reason, we do not discuss various numerical implementation issues and other examples (borrowed from [7] as well as many randomly generated ones) which we tested; the results were of the same flavor.

A. Example 1: Reduced Number of Controls

We first solve the illustrative problem in Assertion 1, i.e., reduce the number of controls required for static state stabilization.

Taking $\sigma = 0.1$ and minimizing $\|Y\|_{r_1}$ subject to (10), we obtain

$$\widehat{Y}^\top = \begin{pmatrix} -0.0000 & -4.9404 & 0.7277 & 0.0000 \\ 0.0000 & -4.9251 & 1.1654 & 0.0000 \\ -0.0004 & 4.9573 & -1.1798 & 0.0003 \\ 0.0000 & 0.9207 & 1.1351 & -0.0000 \\ -0.0001 & -4.5201 & -1.1806 & -0.0006 \\ -0.0000 & 4.9471 & -1.1793 & -0.0001 \\ -0.0003 & 4.9572 & -1.1805 & -0.0003 \\ 0.0001 & -4.9568 & -1.1806 & -0.0007 \end{pmatrix}.$$

The first and the last rows of the solution are candidates for being zeros, so we *put* them to be zeros and arrive at the row-sparse controller

$$K_{\text{sp}}^\top = \begin{pmatrix} 0 & -0.0620 & 0.0217 & 0 \\ 0 & 0.0027 & 0.0123 & 0 \\ 0 & 6.1903 & -1.0867 & 0 \\ 0 & -0.0643 & -0.0132 & 0 \\ 0 & -3.0816 & -1.1929 & 0 \\ 0 & -0.2488 & -0.1320 & 0 \\ 0 & 3.4015 & -1.3406 & 0 \\ 0 & -5.0415 & -1.1951 & 0 \end{pmatrix},$$

which leads to the desired degree of stability $\sigma_{\text{sp}} \approx 0.1$. Hence, only two out of the four control inputs are used.

B. Example 2: Design of the Output

We illustrate the design of low-dimensional output as proposed in Section IV; LMI (10) will be used instead of (2).

The solution of the c_1-optimization problem in Assertion 4 gives a $\widehat{P}$ and a sparse $\widehat{Y}$ which has only the 3rd and 8th nonzero columns:

$$\widehat{Y}^\top = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -6.7592 & 6.7606 & -4.1366 & 6.7562 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2.3352 & -2.3668 & -2.3677 & -2.3674 \end{pmatrix}.$$

Hence, we accept

$$\widetilde{K} = \begin{pmatrix} -6.7592 & -2.3352 \\ 6.7606 & -2.3668 \\ -4.1366 & -2.3677 \\ 6.7562 & -2.3674 \end{pmatrix}$$

as the SOF controller for the designed two-dimensional output $y = \widetilde{C}x$, where the matrix

$$\widetilde{C}^\top = \begin{pmatrix} -0.0080 & -0.0044 \\ 0.0000 & -0.0007 \\ 0.7356 & -0.0809 \\ 0.0125 & 0.0185 \\ -0.0433 & 0.4307 \\ -0.0088 & 0.0310 \\ 0.3908 & 0.0603 \\ -0.0809 & 0.6480 \end{pmatrix}$$

is composed of the 3rd and 8th rows of the matrix $\widehat{P}^{-1}$.

We now move to the second step and make the SOF controller $\widehat{K}$ row-sparse. Namely, with the zero-column structure of the variable Y being fixed as dictated by $\widehat{Y}$ (i.e., only the 3rd and the 8th columns are nonzero), we turn to the r_1-optimization problem in Assertion 5 to obtain

$$Y_r^\top = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.0005 & 7.8141 & -2.8073 & 0.0014 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.0001 & -7.8134 & -2.9244 & -0.0022 \end{pmatrix}$$

and a certain P_r. So we discard the 1st and 4th rows of Y_r (put them to be zeros) and adopt

$$\widetilde{K}_{\rm sp} = \begin{pmatrix} 0 & 0 \\ 7.8141 & -7.8134 \\ -2.8073 & -2.9244 \\ 0 & 0 \end{pmatrix}$$

as a row-sparse output controller, where the output matrix in $y = C_r x$ is composed of the 3rd and 8th rows of P_r^{-1}. A direct test shows that this controller *is* stabilizing with the desired degree of stability.

Overall, with our method, we designed a two-dimensional output for use in static output feedback with the row-sparse gain matrix having only two (out of the four) nonzero rows. Note that, in the original form, the output controller is represented by a full 4×6 matrix.

C. Example 3: LQR

Solution of the LQR problem with $R = S = I$ and initial $x_0 = (1\,1 \dots 1)^\top$ gives the optimal controller

$$K_{\rm opt}^\top = \begin{pmatrix} -0.0832 & -0.9284 & 0.1896 & 0.0176 \\ 0.6154 & -0.0447 & 0.0225 & 0.0652 \\ -1.0050 & 18.9601 & -1.3921 & 1.2860 \\ 0.0259 & -0.1157 & -0.6600 & -0.4619 \\ -0.1685 & 0.1597 & -6.2891 & -2.9303 \\ -0.0213 & 0.1222 & -0.7592 & 0.0900 \\ -0.9011 & 20.6687 & -4.7282 & -0.1442 \\ 0.2389 & -3.7814 & -11.2563 & -6.3896 \end{pmatrix}.$$

At stage 2 of the method, we let $\alpha = 1.25$ and solve the r_1-optimization problem (14) to obtain a $\widehat{Y}$ matrix with the 1st and 4th zero rows. With these two rows being fixed at zeros, we pass on to stage 3 and re-solve the original LQR problem; this gives the row-sparse controller

$$K_{\rm sp}^\top = \begin{pmatrix} 0 & -0.9634 & 0.1979 & 0 \\ 0 & -0.0317 & 0.0401 & 0 \\ 0 & 19.5523 & -0.6174 & 0 \\ 0 & -0.1405 & -0.7753 & 0 \\ 0 & -0.1253 & -7.5970 & 0 \\ 0 & 0.1533 & -0.7700 & 0 \\ 0 & 21.2335 & -4.8240 & 0 \\ 0 & -4.3546 & -13.6827 & 0 \end{pmatrix}$$

yielding the performance $J_{\rm sp} \approx 1.0529 J^*$, i.e., the degradation is about 5%.

Note that sparsity did not change significantly neither the magnitude of the controller ($\|K_{\rm opt}\| \approx 28.5695$, $\|K_{\rm sp}\| \approx 29.2932$), nor the degree of stability of the closed loop system ($\sigma_{\rm opt} \approx 0.4002$, $\sigma_{\rm sp} \approx 0.3976$).

By way of comparison, we performed stage 3 with the Y variable having all two-zero-row structures different from the one obtained by the method (1st and 4th zero rows); the resulting sparse controllers exposed the degradation ranging from 73% to 1600%. This testifies to a reasonable behavior of the method.

VII. Conclusions

We presented a new framework for control system design with the emphasis on sparsity, which can be broadly interpreted as reduction of the control resource required for handling a system. The method is very transparent and is easy to implement; it leans on the LMI technique combined with the use of special matrix norms, so that the design reduces to solving simple semidefinite programs. As per our computational experience, the results are pretty much promising.

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