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An L^q-approach to Stokes and Navier–Stokes equations in general domains

by

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1. Introduction

Throughout this paper, $\Omega\subseteq\mathbf{R}^3$ means a general 3-dimensional domain with uniform C^2-boundary $\partial\Omega\neq\varnothing$, where the main interest is focussed on domains with noncompact boundary $\partial\Omega$. As is well known, the standard approach to the Stokes equations in L^q-spaces, $1<q<\infty$, cannot be extended to general unbounded domains in L^q, $q\neq2$; for counterexamples concerning the Helmholtz decomposition, see [7] and [24]. However, to develop a complete and analogous theory of the Stokes equations for arbitrary domains, we replace the space $L^q(\Omega)$ by

$$\tilde{L}^q(\Omega)=\begin{cases} L^2(\Omega)\cap L^q(\Omega), & 2\leqslant q<\infty,\\ L^2(\Omega)+L^q(\Omega), & 1<q<2.\end{cases}$$

First we prove the existence of the Helmholtz projection P for the space $\tilde{L}^q(\Omega)$ yielding the decomposition $f=f_0+\nabla p$, $f_0=Pf$, with properties corresponding to those in $L^q(\Omega)$.

In the next step we consider in $\tilde{L}^q(\Omega)$ the usual resolvent equation

$$\lambda u-\Delta u+\nabla p=f,\quad \operatorname{div} u=0\ \text{ in } \Omega,\quad u|_{\partial\Omega}=0, \tag{1.1}$$

with λ in the sector $\mathcal{S}_\varepsilon:=\{0\neq\lambda\in\mathbf{C}:|\arg\lambda|<\frac12\pi+\varepsilon\}$, $0<\varepsilon<\frac12\pi$. We prove an $\tilde{L}^q$-estimate similar to that in $L^q(\Omega)$, i.e.,

$$|\lambda|\,\|u\|_{\tilde{L}^q}+\|\nabla^2u\|_{\tilde{L}^q}+\|\nabla p\|_{\tilde{L}^q}\leqslant C\|f\|_{\tilde{L}^q},\quad 1<q<\infty, \tag{1.2}$$

at least when $|\lambda|\geqslant\delta>0$ and $C=C(\Omega,q,\varepsilon,\delta)>0$.

The Stokes operator $A=-P\Delta$ is well defined in $\tilde{L}^q_\sigma(\Omega)$, $1<q<\infty$, and the semigroup $\{e^{-At}:t\geqslant0\}$ is (locally in time) bounded and analytic in some sector $\{0\neq t\in\mathbf{C}: |\arg t|<\varepsilon'\}$, $0<\varepsilon'<\frac12\pi$, of the complex plane.

Further, we prove the maximal regularity estimate of the nonstationary Stokes system

$$\begin{aligned} u_t - \Delta u + \nabla p = f, \quad \operatorname{div} u = 0 \quad \text{in } \Omega \times (0,T), \\ u|_{\partial\Omega} = 0, \quad u(0) = u_0, \end{aligned} \tag{1.3}$$

with $0<T<\infty$. To be more precise, if $u_0=0$ for simplicity, then

$$\|u_t\|_{Y_q} + \|u\|_{Y_q} + \|\nabla^2 u\|_{Y_q} + \|\nabla p\|_{Y_q} \leqslant C\|f\|_{Y_q}, \tag{1.4}$$

where $Y_q = L^q(0,T;\tilde{L}^q(\Omega))$ and $C=C(T,q,\alpha,\beta,K)>0$ depends on T, q and the type α,β,K of Ω, see §2.3.

As an application of these linear results we obtain the existence of a so-called suitable weak solution u of the Navier–Stokes system

$$\begin{aligned} u_t - \Delta u + u\cdot\nabla u + \nabla p = f, \quad \operatorname{div} u = 0 \quad \text{in } \Omega \times (0,T), \\ u|_{\partial\Omega} = 0, \quad u(0) = u_0, \end{aligned} \tag{1.5}$$

with special regularity properties which are new up to now for general domains, see the conjecture in [8, p. 780]. In particular, we get for general domains the regularity property

$$\nabla p \in L^{5/4}_{\mathrm{loc}}((0,T)\times\bar{\Omega}), \tag{1.6}$$

which is needed in the partial regularity theory of the Navier–Stokes equations. Moreover, u satisfies the local energy inequality, see (2.26) below and [8, (2.5)], as well as the strong energy inequality

$$\frac{1}{2}\|u(t)\|_2^2 + \int_s^t \|\nabla u\|_2^2 \, d\tau \leqslant \frac{1}{2}\|u(s)\|_2^2 + \int_s^t \langle f, u\rangle \, d\tau \tag{1.7}$$

for a.a. $s\in[0,T)$ including $s=0$ and all t with $s\leqslant t<T$, see [25]. This result is essentially known for domains with compact boundaries; see [32, Chapter V, Theorems 3.6.2 and 3.4.1] for bounded domains, and [14], [26], [29], [33] and [37] for exterior domains.

2. Preliminaries and main results

2.1. Sum and intersection spaces

We recall some properties of sum and intersection spaces known from interpolation theory, cf. [4], [5], [27] and [36].

Consider two (complex) Banach spaces X_1 and X_2 with norms $\|\cdot\|_{X_1}$ and $\|\cdot\|_{X_2}$, respectively, and assume that both X_1 and X_2 are subspaces of a topological vector space V with continuous embeddings $X_1 \subseteq V$ and $X_2 \subseteq V$. Further, we assume that the intersection $X_1 \cap X_2$ is a dense subspace of both X_1 and X_2 in the corresponding norms.

Then the sum space

$$X_1 + X_2 := \{u_1 + u_2 : u_1 \in X_1,\ u_2 \in X_2\} \subseteq V$$

is a well-defined Banach space with the norm

$$\|u\|_{X_1+X_2} := \inf\{\|u_1\|_{X_1} + \|u_2\|_{X_2} : u = u_1 + u_2,\ u_1 \in X_1,\ u_2 \in X_2\}.$$

Another formulation of that norm is given by

$$\|u_1 + u_2\|_{X_1+X_2} = \inf\{\|u_1 - v\|_{X_1} + \|u_2 + v\|_{X_2} : v \in X_1 \cap X_2\}.$$

The intersection space $X_1 \cap X_2$ is a Banach space with norm

$$\|u\|_{X_1 \cap X_2} = \max\{\|u\|_{X_1}, \|u\|_{X_2}\},$$

which is equivalent to $\|u\|_{X_1} + \|u\|_{X_2}$. Note that the space $X_1 + X_2$ can be identified isometrically with the quotient space $(X_1 \times X_2)/D$, where $D = \{(-v, v) : v \in X_1 \cap X_2\}$, identifying $u = u_1 + u_2 \in X_1 + X_2$ with the equivalence class $[(u_1, u_2)] = \{(u_1 - v, u_2 + v) : v \in X_1 \cap X_2\}$.

Next we consider the dual spaces X_1' and X_2' of X_1 and X_2, respectively, with norms

$$\|f\|_{X_i'} = \sup\left\{\frac{|\langle u, f\rangle|}{\|u\|_{X_i}} : 0 \neq u \in X_i\right\}, \quad i = 1, 2.$$

In both cases, $\langle u, f\rangle$ denotes the value of some functional f at some element u, and $\langle\cdot,\cdot\rangle$ is called the natural pairing between the space X_i and its dual space X_i'. Note that $\|u\|_{X_i} = \sup\{|\langle u, f\rangle| / \|f\|_{X_i'} : 0 \neq f \in X_i'\}$.

Since $X_1 \cap X_2$ is dense in X_1 and in X_2, we can identify two elements $f_1 \in X_1'$ and $f_2 \in X_2'$, writing $f_1 = f_2$, if and only if $\langle u, f_1\rangle = \langle u, f_2\rangle$ holds for all $u \in X_1 \cap X_2$. In this way the intersection $X_1' \cap X_2'$ is a well-defined Banach space with norm $\|f\|_{X_1' \cap X_2'} = \max\{\|f\|_{X_1'}, \|f\|_{X_2'}\}$. The dual space $(X_1 + X_2)'$ of $X_1 + X_2$ is given by $X_1' \cap X_2'$, and we get

$$(X_1 + X_2)' = X_1' \cap X_2'$$

with the natural pairing

$$\langle u, f\rangle = \langle u_1, f\rangle + \langle u_2, f\rangle$$

for all $u = u_1 + u_2 \in X_1 + X_2$ and $f \in X_1' \cap X_2'$. Thus it holds that

$$\|u\|_{X_1+X_2} = \sup\left\{\frac{|\langle u_1, f\rangle + \langle u_2, f\rangle|}{\|f\|_{X_1' \cap X_2'}} : 0 \neq f \in X_1' \cap X_2'\right\}$$

and
$$\|f\|_{X_1'\cap X_2'} = \sup\left\{\frac{|\langle u_1, f\rangle + \langle u_2, f\rangle|}{\|u\|_{X_1+X_2}} : 0 \neq u = u_1+u_2 \in X_1+X_2\right\};$$
see [5, p. 32] and [36, p. 69]. Therefore, $|\langle u, f\rangle| \leqslant \|u\|_{X_1+X_2}\|f\|_{X_1'\cap X_2'}$.

By analogy, we obtain that
$$(X_1\cap X_2)' = X_1' + X_2'$$
with the natural pairing $\langle u, f_1+f_2\rangle = \langle u, f_1\rangle + \langle u, f_2\rangle$.

Consider closed subspaces $L_1 \subseteq X_1$ and $L_2 \subseteq X_2$ equipped with norms $\|\cdot\|_{L_1} = \|\cdot\|_{X_1}$ and $\|\cdot\|_{L_2} = \|\cdot\|_{X_2}$, and assume that $L_1\cap L_2$ is dense in both L_1 and L_2 in the corresponding norms. Then $\|u\|_{L_1\cap L_2} = \|u\|_{X_1\cap X_2}$, $u\in L_1\cap L_2$, and an elementary argument, using the Hahn–Banach theorem, shows that also
$$\|u\|_{L_1+L_2} = \|u\|_{X_1+X_2}, \quad u\in L_1+L_2. \tag{2.1}$$

In particular, we need the following special case. Let $B_1 : D(B_1)\to X_1$ and B_2: $D(B_2)\to X_2$ be closed linear operators with dense domains $D(B_1)\subseteq X_1$ and $D(B_2)\subseteq X_2$ equipped with graph norms
$$\|u\|_{D(B_1)} = \|u\|_{X_1} + \|B_1 u\|_{X_1} \quad \text{and} \quad \|u\|_{D(B_2)} = \|u\|_{X_2} + \|B_2 u\|_{X_2}.$$
We assume that $D(B_1)\cap D(B_2)$ is dense in both $D(B_1)$ and $D(B_2)$ in the corresponding graph norms. Each functional $F\in D(B_i)'$, $i=1,2$, is given by some pair $f, g\in X_i'$ in the form $\langle u, F\rangle = \langle u, f\rangle + \langle B_i u, g\rangle$. Using (2.1) with $L_i = \{(u, B_i u) : u\in D(B_i)\}\subseteq X_i\times X_i$, $i=1,2$, and the equality of norms $\|\cdot\|_{(X_1\times X_1)+(X_2\times X_2)}$ and $\|\cdot\|_{(X_1+X_2)\times(X_1+X_2)}$ on $(X_1\times X_1)+(X_2\times X_2)$, we conclude that for each $u\in D(B_1)+D(B_2)$ with decomposition $u=u_1+u_2$, $u_1\in D(B_1)$, $u_2\in D(B_2)$,
$$\|u\|_{D(B_1)+D(B_2)} = \|u_1+u_2\|_{X_1+X_2} + \|B_1u_1 + B_2u_2\|_{X_1+X_2}. \tag{2.2}$$

Suppose that X_1 and X_2 are reflexive Banach spaces implying that each bounded sequence in X_1 (and X_2) has a weakly convergent subsequence. This argument yields the following property: Given $u\in X_1+X_2$ there exist $u_1\in X_1$ and $u_2\in X_2$ with $u=u_1+u_2$ such that
$$\|u\|_{X_1+X_2} = \|u_1\|_{X_1} + \|u_2\|_{X_2}. \tag{2.3}$$

2.2. Function spaces

In the following let $D_j = \partial/\partial x_j$, $j=1,2,3$, $x=(x_1,x_2,x_3)\in\Omega\subset\mathbf{R}^3$, $\nabla = (D_1, D_2, D_3)$ and $\nabla^2 = (D_jD_k)_{j,k=1,2,3}$. The spaces of smooth functions on Ω are denoted as usual by $C^k(\Omega)$, $C^k(\overline{\Omega})$ and $C_0^k(\Omega)$ with $k\in\mathbf{N}_0 = \mathbf{N}\cup\{0\}$ or $k=\infty$. We set
$$C_{0,\sigma}^\infty(\Omega) = \{u = (u_1,u_2,u_3)\in C_0^\infty(\Omega) : \operatorname{div} u = 0\}.$$

Let $1<q<\infty$ and $q'=q/(q-1)$ such that $1/q+1/q'=1$. Then $L^q(\Omega)$ with norm $\|u\|_{L^q}=\|u\|_q=\|u\|_{q,\Omega}$ denotes the usual Lebesgue space for scalar or vector fields. Each $f=(f_1,f_2,f_3)\in L^{q'}(\Omega)=L^q(\Omega)'$ will be identified with the functional $\langle\cdot,f\rangle\colon u\mapsto\langle u,f\rangle=\langle u,f\rangle_\Omega=\int_\Omega u\cdot f\,dx$ on $L^q(\Omega)$. Let $L^q_\sigma(\Omega)=\overline{C_{0,\sigma}(\Omega)}^{\|\cdot\|_q}\subset L^q(\Omega)$ denote the subspace of divergence-free vector fields $u=(u_1,u_2,u_3)$ with normal component $N\cdot u|_{\partial\Omega}=0$ at $\partial\Omega$; here N means the outer normal at $\partial\Omega$. The usual Sobolev spaces $W^{k,q}(\Omega)$ are mainly used for $k=1,2$ with norms $\|u\|_{W^{1,q}}=\|u\|_{1,q}=\|u\|_{1,q,\Omega}=\|u\|_q+\|\nabla u\|_q$ and $\|u\|_{W^{2,q}}=\|u\|_{2,q}=\|u\|_{2,q,\Omega}=\|u\|_{1,q}+\|\nabla^2u\|_q$, respectively. Further, we need the subspaces

$$W_0^{1,q}(\Omega)=\overline{C_0^\infty(\Omega)}^{\|\cdot\|_{1,q}}\subset W^{1,q}(\Omega)\quad\text{and}\quad W_{0,\sigma}^{1,q}(\Omega)=\overline{C_{0,\sigma}^\infty(\Omega)}^{\|\cdot\|_{1,q}}\subset W^{1,q}(\Omega).$$

For simplicity, we will write C^k, L^q, $W^{1,q}_\sigma$, etc. instead of $C^k(\Omega)$, $L^q(\Omega)$, $W^{1,q}_\sigma(\Omega)$, respectively, when the underlying domain is known from the context. Moreover, we will use the same notation for spaces of scalar-, vector- and matrix-valued functions.

The sum space L^2+L^q is well defined when V in §2.1 is the space of distributions with the usual topology. We obtain that

$$(L^2+L^q)'=L^2\cap L^{q'}\quad\text{and}\quad(L^2\cap L^q)'=L^2+L^{q'},$$

where $\|u\|_{L^2\cap L^q}=\max\{\|u\|_2,\|u\|_q\}$ and

$$\begin{aligned}\|u\|_{L^2+L^q}&=\inf\{\|u_1\|_2+\|u_2\|_q:u=u_1+u_2,\ u_1\in L^2,\ u_2\in L^q\}\\&=\sup\left\{\frac{|\langle u_1+u_2,f\rangle|}{\|f\|_{L^2\cap L^{q'}}}:0\neq f\in L^2\cap L^{q'}\right\}.\end{aligned}$$

For the nonstationary problem on some time interval $[0,T)$, $0<T\leqslant\infty$, we need the usual Banach space $L^s(0,T;X)$ of measurable X-valued (classes of) functions u with norm

$$\|u\|_{L^s(0,T;X)}=\left(\int_0^T\|u(t)\|_X^s\,dt\right)^{1/s},\quad 1\leqslant s<\infty,$$

where X is a Banach space. For $s=\infty$ let

$$\|u\|_{L^\infty(0,T;X)}=\operatorname{ess\,sup}\{\|u(t)\|_X:0\leqslant t<T\}.$$

If X is reflexive and $1<s<\infty$, then the dual space of $L^s(0,T;X)$ is given by $L^s(0,T;X)'=L^{s'}(0,T;X')$, $s'=s/(s-1)$, with the natural pairing $\langle u,f\rangle_T=\int_0^T\langle u(t),f(t)\rangle\,dt$; see [20].

Let $X=L^q(\Omega)$, $1<q<\infty$. Then we use the notation

$$\|u\|_{L^s(0,T;L^q)}=\|u\|_{L^s(L^q)}=\left(\int_0^T\|u\|_q^s\,dt\right)^{1/s}.$$

Moreover, the pairing of $L^s(0,T;L^q)$ with its dual $L^{s'}(0,T;L^{q'})$ is given by $\langle u,f\rangle_T = \langle u,f\rangle_{\Omega,T} = \int_0^T\int_\Omega u\cdot f\,dx\,dt$ if $1<s<\infty$.

Let $Y_1 = L^s(0,T;L^2)$ and $Y_2 = L^s(0,T;L^q)$ with $1<q,s<\infty$. Then we see that

$$(Y_1+Y_2)' = Y_1'\cap Y_2' = L^{s'}(0,T;L^2\cap L^{q'}) = L^s(0,T;L^2+L^q)',$$

and therefore $Y_1+Y_2 = L^s(0,T;L^2+L^q)$; the pairing between Y_1+Y_2 and $Y_1'\cap Y_2'$ is given by $\langle u_1+u_2,f\rangle_T = \langle u_1,f\rangle_T + \langle u_2,f\rangle_T$ for $u_1\in Y_1$, $u_2\in Y_2$ and $f\in Y_1'\cap Y_2'$. Furthermore, we can choose the decomposition $u=u_1+u_2\in L^s(0,T;L^2+L^q)$ in such a way that

$$\|u\|_{Y_1+Y_2} = \|u_1\|_{Y_1} + \|u_2\|_{Y_2}. \tag{2.4}$$

We conclude that

$$\|u_1+u_2\|_{Y_1+Y_2} = \sup\left\{\frac{|\langle u_1+u_2,f\rangle_T|}{\|f\|_{Y_1'\cap Y_2'}} : 0\neq f\in L^{s'}(0,T;L^2\cap L^{q'})\right\}. \tag{2.5}$$

2.3. Structure properties of the boundary $\partial\Omega$

We recall some well-known technical details on the *uniform C^2-domain* $\Omega\subseteq\mathbf{R}^3$, see, e.g., [1, p. 67], [18, p. 645] and [32, p. 26]. By definition, this means that there are constants $\alpha,\beta,K>0$ with the following properties:

For each $x_0\in\partial\Omega$ we can choose a Cartesian coordinate system with origin x_0 and coordinates $y=(y_1,y_2,y_3)=(y',y_3)$, $y'=(y_1,y_2)$, obtained by some translation and rotation, as well as some C^2-function $h(y')$, $|y'|\leqslant\alpha$, with C^2-norm $\|h\|_{C^2}\leqslant K$, such that the neighborhood

$$U_{\alpha,\beta,h}(x_0) := \{(y',y_3) : h(y')-\beta < y_3 < h(y')+\beta,\ |y'|<\alpha\}$$

of x_0 satisfies

$$U^-_{\alpha,\beta,h}(x_0) := \{(y',y_3) : h(y')-\beta < y_3 < h(y'),\ |y'|<\alpha\} = \Omega\cap U_{\alpha,\beta,h}(x_0),$$

and

$$\partial\Omega\cap U_{\alpha,\beta,h}(x_0) = \{(y',y_3) : h(y') = y_3,\ |y'|<\alpha\}.$$

Without loss of generality we may assume that the axes of $y'=(y_1,y_2)$ are contained in the tangential plane at x_0. Thus at $y'=(0,0)$ we have $h(y')=0$ and $\nabla'h(y') = (\partial h/\partial y_1, \partial h/\partial y_2) = (0,0)$. Therefore, for any given constant $M_0>0$, we may choose $\alpha>0$ sufficiently small such that a *smallness condition* of the form

$$\|\nabla'h\|_{C^0} = \max\{|\nabla'h(y')| : |y'|\leqslant\alpha\} \leqslant M_0$$

is satisfied. It is important to note that the constants $\alpha, \beta, K>0$ do not depend on $x_0\in\Omega$. We call α, β, K the *type* of Ω.

Let $\bar{\Omega}$ be the closure of Ω and let $B_r(x)=\{w\in\mathbf{R}^3 : |w-x|<r\}$ be the open ball with center $x\in\mathbf{R}^3$ and radius $r>0$. Then we can choose some fixed $r\in(0,\alpha)$ depending only on α, β, K, balls $B_j=B_r(x_j)$ with centers $x_j\in\bar{\Omega}$, and C^2-functions $h_j(y')$, $|y'|\leqslant\alpha$, where $j=1,2,...,N$ if Ω is bounded and $j\in\mathbf{N}$ if Ω is unbounded, such that

$$\begin{gathered}\bar{\Omega}\subseteq\bigcup_{j=1}^{N}B_j \quad\text{and}\quad \bar{\Omega}\subseteq\bigcup_{j=1}^{\infty}B_j, \quad\text{respectively,}\\ \overline{B}_j\subseteq U_{\alpha,\beta,h_j}(x_j)\ \text{ if } x_j\in\partial\Omega, \quad \overline{B}_j\subseteq\Omega \ \text{ if } x_j\in\Omega.\end{gathered}\tag{2.6}$$

Moreover, we can construct this covering in such a way that not more than a fixed finite number $N_0=N_0(\alpha,\beta,K)\in\mathbf{N}$ of these balls $B_1, B_2, ...$ can have a nonempty intersection. Thus if we choose any N_0+1 different balls $B_1, B_2, ...$, then their common intersection is empty. If Ω is bounded, let $N_0=N$.

Concerning $\{B_j\}$, there exists a partition of unity $\varphi_j\in C_0^\infty(\mathbf{R}^3)$ with $0\leqslant\varphi_j\leqslant1$, $\operatorname{supp}\varphi_j\subseteq B_j$, $j=1,...,N$ or $j\in\mathbf{N}$, satisfying

$$\sum_{j=1}^{N}\varphi_j(x)=1 \quad\text{and}\quad \sum_{j=1}^{\infty}\varphi_j(x)=1, \quad\text{respectively, for all } x\in\bar{\Omega},\tag{2.7}$$

and the pointwise estimates $|\nabla\varphi_j(x)|$, $|\nabla^2\varphi_j(x)|\leqslant C$ uniformly with respect to x and j, where $C=C(\alpha,\beta,K)$.

If Ω is unbounded, we can represent Ω as a union of countably many bounded C^2-subdomains $\Omega_j\subseteq\Omega$, $j\in\mathbf{N}$, such that

$$\Omega_j\subseteq\Omega_{j+1} \ \text{ for all } j\in\mathbf{N}, \quad \Omega=\bigcup_{j=1}^{\infty}\Omega_j,\tag{2.8}$$

and such that each Ω_j has some fixed type $\alpha', \beta', K'>0$. Without loss of generality we may assume that $\alpha=\alpha'$, $\beta=\beta'$ and $K=K'$: each subdomain Ω_j, $j\in\mathbf{N}$, has the same type α, β, K as Ω, see [18, p. 665]. Obviously each compact subset $\Omega_0\subseteq\Omega$ is contained in some Ω_j, and therefore in each Ω_k, $k\geqslant j$; see [32, p. 56, Remark 1.4.2].

Finally we need a technical property in subsequent proofs. Given a ball $B_r(x)\subset\mathbf{R}^3$ consider some Cartesian coordinate system with origin x and coordinates $y=(y', y_3)$. Then $B_r^-(x):=\{y=(y',y_3) : |y|<r,\ y_3<0\}$ is called a half-ball with center x and radius r. We may assume without loss of generality that there are appropriate half-balls $B_j^-=B_r^-(x_j)$ of the balls B_j in (2.6) and (2.7) such that

$$\operatorname{supp}\varphi_j\subseteq B_j^- \quad\text{if } x_j\in\Omega, \ \text{where } j=1,...,N \text{ or } j\in\mathbf{N}.\tag{2.9}$$

2.4. Main results on the Stokes equations

We can extend several important L^q-properties of the Stokes equations known for special domains, such as bounded or exterior domains, to general domains Ω if we replace the usual L^q-space by the space

$$\tilde{L}^q = \tilde{L}^q(\Omega) = L^2(\Omega) \cap L^q(\Omega) \quad \text{for } 2 \leqslant q < \infty,$$

and by the space

$$\tilde{L}^q = \tilde{L}^q(\Omega) = L^2(\Omega) + L^q(\Omega) \quad \text{for } 1 < q < 2.$$

Note that $\tilde{L}^q$ is smaller than L^q when $q>2$, and larger than L^q when $1<q<2$, but that $\tilde{L}^2 = L^2$. Analogously, we define the subspace $\tilde{L}^q_\sigma = \tilde{L}^q_\sigma(\Omega) \subset \tilde{L}^q(\Omega)$ by setting $\tilde{L}^q_\sigma = L^2_\sigma(\Omega) \cap L^q_\sigma(\Omega)$ for $2 \leqslant q < \infty$, and $\tilde{L}^q_\sigma = L^2_\sigma(\Omega) + L^q_\sigma(\Omega)$ for $1<q<2$.

In the same way we modify the L^q-Sobolev spaces $W^{k,q}(\Omega)$ and the spaces

$$\begin{aligned} G^q(\Omega) &= \{\nabla p \in L^q : p \in L^q_{\mathrm{loc}}(\Omega)\}, \qquad && \|\nabla p\|_{G^q} = \|\nabla p\|_{L^q}, \\ D^q(\Omega) &= L^q_\sigma(\Omega) \cap W^{1,q}_0(\Omega) \cap W^{2,q}(\Omega), && \|u\|_{D^q} = \|u\|_{W^{2,q}}, \end{aligned}$$

$1<q<\infty$, as follows: For $2 \leqslant q < \infty$ let

$$\begin{aligned} \widetilde{W}^{k,q}(\Omega) &= W^{k,2}(\Omega) \cap W^{k,q}(\Omega), \\ \widetilde{G}^q(\Omega) &= G^2(\Omega) \cap G^q(\Omega), \\ \widetilde{D}^q(\Omega) &= D^2(\Omega) \cap D^q(\Omega), \end{aligned}$$

and for $1<q<2$ let

$$\begin{aligned} \widetilde{W}^{k,q}(\Omega) &= W^{k,2}(\Omega) + W^{k,q}(\Omega), \\ \widetilde{G}^q(\Omega) &= G^2(\Omega) + G^q(\Omega), \\ \widetilde{D}^q(\Omega) &= D^2(\Omega) + D^q(\Omega), \end{aligned}$$

$k=1,2$. Then the norms $\|\cdot\|_{\widetilde{W}^{k,q}}$, $\|\cdot\|_{\widetilde{G}^q}$ and $\|\cdot\|_{\widetilde{D}^q}$ are well defined. If Ω is bounded, then $\tilde{L}^q = L^q$, $\tilde{L}^q_\sigma = L^q_\sigma$, $\widetilde{G}^q = G^q$, $\widetilde{D}^q = D^q$ and $\widetilde{W}^{k,q} = W^{k,q}$ hold with equivalent norms. Thus the introduction of "$\sim$"-spaces is reasonable only for unbounded domains.

Our first result yields the existence of the Helmholtz projection in $\tilde{L}^q(\Omega)$. The counterexamples in [7] and [24] show that the usual L^q-theory for special domains cannot be extended to Ω for arbitrary $q \neq 2$. It is important to note that the constants $C = C(q, \alpha, \beta, K) > 0$ below only depend on q and the type α, β, K of the domain Ω.

THEOREM 2.1. (Helmholtz decomposition.) *Let $\Omega\subseteq\mathbf{R}^3$ be a uniform C^2-domain of type $\alpha,\beta,K>0$ and let $1<q<\infty$, $q'=q/(q-1)$. Then for each $f\in\tilde{L}^q$ there exists a unique decomposition $f=f_0+\nabla p$ with $f_0\in\tilde{L}^q_\sigma$ and $\nabla p\in\widetilde{G}^q$ satisfying the estimate*

$$\|f_0\|_{\tilde{L}^q}+\|\nabla p\|_{\tilde{L}^q}\leqslant C\|f\|_{\tilde{L}^q},\quad C=C(q,\alpha,\beta,K)>0. \tag{2.10}$$

The Helmholtz projection $P=\widetilde{P}_q$ defined by $\widetilde{P}_qf=f_0$ is a bounded operator from $\tilde{L}^q$ onto $\tilde{L}^q_\sigma$ satisfying $\widetilde{P}_qf=f$ if $f\in\tilde{L}^q_\sigma$, and $\widetilde{P}_q(\nabla p)=0$ if $\nabla p\in\widetilde{G}^q$. Moreover, $\langle\widetilde{P}_qf,g\rangle=\langle f,\widetilde{P}_{q'}g\rangle$ for all $f\in\tilde{L}^q$ and $g\in\tilde{L}^{q'}$.

Remark 2.2. By Theorem 2.1 we conclude that $\widetilde{P}'_q=\widetilde{P}_{q'}$ for the dual operator $\widetilde{P}'_q=(\widetilde{P}_q)'$ of $\widetilde{P}_q$, $1<q<\infty$, and $(\tilde{L}^q_\sigma)'=\tilde{L}^{q'}_\sigma$ with pairing $\langle\cdot,\cdot\rangle$. We also get that the norm defined by

$$\|u\|^*_{\tilde{L}^q_\sigma}=\sup\left\{\frac{|\langle u,f\rangle|}{\|f\|_{\tilde{L}^{q'}_\sigma}}:0\neq f\in\tilde{L}^{q'}_\sigma\right\},\quad u\in\tilde{L}^q_\sigma, \tag{2.11}$$

is equivalent to the norm $\|u\|_{\tilde{L}^q_\sigma}=\|u\|_{\tilde{L}^q}$ in the sense that $\|u\|^*_{\tilde{L}^q_\sigma}\leqslant\|u\|_{\tilde{L}^q_\sigma}\leqslant C\|u\|^*_{\tilde{L}^q_\sigma}$ with $C=C(q,\alpha,\beta,K)>0$ from (2.10).

The usual L^q-Stokes operator $A=A_q$ with domain

$$D(A_q)=D^q=L^q_\sigma\cap W^{1,q}_0\cap W^{2,q}\subset L^q_\sigma$$

and range $R(A_q)\subseteq L^q_\sigma$ defined by $A_qu=-P_q\Delta u$ is meaningful if the Helmholtz projection $P_q\colon L^q\to L^q_\sigma$ is well defined. Thus, because of the counterexamples, see [7] and [24], we cannot expect that this theory is extendable to general domains Ω for $q\neq2$ without modification of the L^q-space.

Next we will show that the usual Stokes estimate, at least for $|\lambda|\geqslant\delta>0$, remains valid for Ω when we replace the L^q-theory by the $\tilde{L}^q$-theory. More precisely, let the Stokes operator $A=\tilde{A}_q$ be defined as an operator with domain $D(\tilde{A}_q)=\widetilde{D}^q\subseteq\tilde{L}^q_\sigma$ into $\tilde{L}^q_\sigma$, by setting

$$\tilde{A}_qu=-\widetilde{P}_q\Delta u,\quad u\in\widetilde{D}^q.$$

Let I be the identity and $\mathcal{S}_\varepsilon=\{0\neq\lambda\in\mathbf{C}:|\arg\lambda|<\frac12\pi+\varepsilon\}$, $0<\varepsilon<\frac12\pi$.

THEOREM 2.3. (Stokes resolvent.) *Let $\Omega\subseteq\mathbf{R}^3$ be a uniform C^2-domain of type $\alpha,\beta,K>0$ and let $1<q<\infty$, $q'=q/(q-1)$, $0<\varepsilon<\frac12\pi$ and $\delta>0$. Then*

$$\tilde{A}_q=-\widetilde{P}_q\Delta\colon D(\tilde{A}_q)\longrightarrow\tilde{L}^q_\sigma,\quad D(\tilde{A}_q)\subset\tilde{L}^q_\sigma,$$

is a densely defined closed operator, the resolvent $(\lambda I+\tilde{A}_q)^{-1}\colon\tilde{L}^q_\sigma\to\tilde{L}^q_\sigma$ is well defined for all $\lambda\in\mathcal{S}_\varepsilon$, and for $u=(\lambda I+\tilde{A}_q)^{-1}f$, $f\in\tilde{L}^q_\sigma$, the estimate

$$|\lambda|\,\|u\|_{\tilde{L}^q_\sigma}+\|u\|_{\widetilde{W}^{2,q}}\leqslant C\|f\|_{\tilde{L}^q_\sigma},\quad|\lambda|\geqslant\delta, \tag{2.12}$$

with $C=C(q,\varepsilon,\delta,\alpha,\beta,K)>0$, is satisfied. Further, the following duality relation holds:

$$\langle\tilde{A}_qu,v\rangle=\langle u,\tilde{A}_{q'}v\rangle,\quad u\in D(\tilde{A}_q),\ v\in D(\tilde{A}_{q'}). \tag{2.13}$$

Remark 2.4. (a) From (2.12) we conclude that $-\tilde{A}_q$ generates a C^0-semigroup $\{e^{-t\tilde{A}_q}: t \geqslant 0\}$ which has an analytic extension to some sector $\{0 \neq t \in \mathbf{C}: |\arg t| < \varepsilon'\}$, $0<\varepsilon'<\frac{1}{2}\pi$, satisfying the estimate

$$\|e^{-t\tilde{A}_q} f\|_{\tilde{L}^q_\sigma} \leqslant M e^{\delta t} \|f\|_{\tilde{L}^q_\sigma}, \quad f \in \tilde{L}^q_\sigma,\ t \geqslant 0, \tag{2.14}$$

with $M=M(q,\delta,\alpha,\beta,K)>0$. Note that $\delta>0$ may be chosen arbitrarily small, but we cannot prove up to now whether (2.14) holds with $\delta=0$ for the general domain Ω.

(b) Let $f \in \tilde{L}^q$, $1<q<\infty$, $\lambda \in \mathcal{S}_\varepsilon$ and $|\lambda|>\delta$, and set $u=(\lambda I+\tilde{A}_q)^{-1}\widetilde{P}_q f$ and $\nabla p=(I-\widetilde{P}_q)(f+\Delta u)$. Then we get a unique solution pair $u \in D(\tilde{A}_q)$, $\nabla p \in \widetilde{G}^q$ of the equation $\lambda u - \Delta u + \nabla p = f$, and by (2.12),

$$|\lambda|\, \|u\|_{\tilde{L}^q} + \|\nabla^2 u\|_{\tilde{L}^q} + \|\nabla p\|_{\tilde{L}^q} \leqslant C \|f\|_{\tilde{L}^q}, \tag{2.15}$$

where $C=C(q,\varepsilon,\delta,\alpha,\beta,K)>0$.

(c) Due to (2.15) the graph norm $\|u\|_{D(\tilde{A}_q)} = \|u\|_{\tilde{L}^q_\sigma} + \|\tilde{A}_q u\|_{\tilde{L}^q_\sigma}$ on the Banach space $D(\tilde{A}_q)$ satisfies the estimate

$$C\|u\|_{\widetilde{W}^{2,q}} \leqslant \|u\|_{D(\tilde{A}_q)} \leqslant C' \|u\|_{\widetilde{W}^{2,q}}, \quad u \in D(\tilde{A}_q), \tag{2.16}$$

with constants $C=C(q,\alpha,\beta,K)>0$ and $C'=C'(q,\alpha,\beta,K)>0$. Hence the norms $\|u\|_{\widetilde{W}^{2,q}}$ and $\|u\|_{D(\tilde{A}_q)}$ are equivalent.

Another important property is the maximal regularity estimate of the nonstationary Stokes equation (1.3), which can be written, applying the Helmholtz projection, in the form

$$u_t + \tilde{A}_q u = f, \quad u(0) = u_0. \tag{2.17}$$

For simplicity, we do not use the weakest possible norm for the initial value u_0, see Remark 2.6 (a).

THEOREM 2.5. (Nonstationary Stokes system.) *Let $\Omega \subseteq \mathbf{R}^3$ be a uniform C^2-domain of type $\alpha,\beta,K>0$, and let $0<T<\infty$, $Y_q=L^q(0,T;\tilde{L}^q_\sigma)$, $1<q<\infty$.*

Then for each $f \in Y_q$ and each $u_0 \in D(\tilde{A}_q)$ there exists a unique solution

$$u \in L^q(0,T;D(\tilde{A}_q)), \quad u_t \in Y_q,$$

of the evolution system (2.17), satisfying the estimate

$$\|u_t\|_{Y_q} + \|u\|_{Y_q} + \|\tilde{A}_q u\|_{Y_q} \leqslant C(\|u_0\|_{D(\tilde{A}_q)} + \|f\|_{Y_q}) \tag{2.18}$$

with $C=C(q,T,\alpha,\beta,K)>0$.

Remark 2.6. (a) The assumption $u_0 \in D(\tilde{A}_q)$ in this theorem is not optimal and may be replaced by the weaker properties $u_0 \in \tilde{L}^q_\sigma$ and $\int_0^T \|\tilde{A}_q e^{-t\tilde{A}_q} u_0\|^q_{\tilde{L}^q_\sigma}\, dt < \infty$. Then the term $\|u_0\|_{D(\tilde{A}_q)}$ in (2.18) may be substituted by the weaker norm

$$\left(\int_0^T \|\tilde{A}_q e^{-t\tilde{A}_q} u_0\|^q_{\tilde{L}^q_\sigma}\, dt\right)^{1/q}, \quad 1<q<\infty. \tag{2.19}$$

Furthermore, by (2.16), the estimate (2.18) implies that

$$\|u_t\|_{Y_q} + \|u\|_{L^q(0,T;\widetilde{W}^{2,q})} \leqslant C(\|u_0\|_{D(\tilde{A}_q)} + \|f\|_{Y_q}), \tag{2.20}$$

where $C=C(q,T,\alpha,\beta,K)>0$.

(b) Let $f \in Y_q = L^q(0,T;\tilde{L}^q_\sigma)$ in Theorem 2.5 be replaced by $f \in \widehat{Y}_q = L^q(0,T;\tilde{L}^q)$, $1<q<\infty$. Then $u \in L^q(0,T;D(\tilde{A}_q))$, defined by $u_t + \tilde{A}_q u = \widetilde{P}_q f$, and ∇p, defined by $\nabla p(t) = (I - \widetilde{P}_q)(f + \Delta u)(t)$, is a unique solution pair of the system

$$u_t - \Delta u + \nabla p = f, \quad u(0) = u_0,$$

satisfying

$$\|u_t\|_{Y_q} + \|u\|_{Y_q} + \|\nabla^2 u\|_{\widehat{Y}_q} + \|\nabla p\|_{\widehat{Y}_q} \leqslant C(\|u_0\|_{D(\tilde{A}_q)} + \|f\|_{\widehat{Y}_q}) \tag{2.21}$$

with $C=C(q,T,\alpha,\beta,K)>0$.

Using (2.3) we see that in the case $1<q<2$, the solution pair $u, \nabla p$ possesses a decomposition $u = u^{(1)} + u^{(2)}$, $\nabla p = \nabla p^{(1)} + \nabla p^{(2)}$ such that

$$\begin{aligned} u^{(1)} &\in L^q(0,T;W^{2,2}), & u_t^{(1)} &\in L^q(0,T;L^2_\sigma), \\ u^{(2)} &\in L^q(0,T;W^{2,q}), & u_t^{(2)} &\in L^q(0,T;L^q_\sigma), \\ \nabla p^{(1)} &\in L^q(0,T;L^2), & \nabla p^{(2)} &\in L^q(0,T;L^q), \end{aligned} \tag{2.22}$$

and

$$\begin{aligned} \|u_t\|_{Y_q} + \|u\|_{Y_q} + \|\nabla^2 u\|_{\widehat{Y}_q} + \|\nabla p\|_{\widehat{Y}_q} = {} & \|u_t^{(1)}\|_{\widehat{Y}_q^{(1)}} + \|u^{(1)}\|_{\widehat{Y}_q^{(1)}} + \|\nabla^2 u^{(1)}\|_{\widehat{Y}_q^{(1)}} + \|\nabla p^{(1)}\|_{\widehat{Y}_q^{(1)}} \\ & + \|u_t^{(2)}\|_{\widehat{Y}_q^{(2)}} + \|u^{(2)}\|_{\widehat{Y}_q^{(2)}} \\ & + \|\nabla^2 u^{(2)}\|_{\widehat{Y}_q^{(2)}} + \|\nabla p^{(2)}\|_{\widehat{Y}_q^{(2)}}, \end{aligned}$$

where $\widehat{Y}_q^{(1)} = L^q(0,T;L^2)$ and $\widehat{Y}_q^{(2)} = L^q(0,T;L^q)$.

2.5. Applications

As an application we construct a so-called *suitable weak solution* u of the instationary Navier–Stokes system

$$\begin{gathered} u_t - \Delta u + u\cdot\nabla u + \nabla p = f, \quad \operatorname{div} u = 0 \ \text{ in } \Omega\times(0,T), \\ u|_{\partial\Omega} = 0, \quad u(0) = u_0, \end{gathered} \tag{2.23}$$

for the general domain $\Omega\subset\mathbf{R}^3$ with important additional properties. In particular, we are interested in estimate (2.21) for $q=\frac{5}{4}$. The reason is that the energy properties $u\in L^\infty(0,T;L^2_\sigma)$ and $\nabla u\in L^2(0,T;L^2)$ imply that $u\cdot\nabla u\in L^q(0,T;L^q)$ with $q=\frac{5}{4}$. Hence, shifting $u\cdot\nabla u$ in (2.23) to the right-hand side and considering for simplicity $u_0=0$, we get from (2.21) that $\nabla p\in L^q(0,T;L^2+L^q)$ and $\nabla p\in L^q_{\mathrm{loc}}((0,T)\times\overline{\Omega})$. This property is needed in the local regularity theory as well as in the proof of the local energy estimate. It was conjectured in [8, p. 780], and open up to now for general domains.

Moreover, we prove that u satisfies the strong energy inequality, see [14], [26], [32] and [33], which was open for general domains as well. A consequence is Leray's structure theorem [23] for general domains; note that the proof in [23] concerns the entire space $\mathbf{R}^3$ only.

We recall some definitions, see, e.g., [32] and [35]. The space $C_0^\infty([0,T);C^\infty_{0,\sigma})$ consists of smooth solenoidal vector fields v defined on $[0,T)\times\Omega$ with compact support $\operatorname{supp} v\subseteq[0,T)\times\Omega$.

Let $f\in L^{5/4}(0,T;L^2)$, $0<T\leqslant\infty$, and $u_0\in L^2_\sigma$. Then a function

$$u\in L^\infty(0,T;L^2_\sigma)\cap L^2_{\mathrm{loc}}([0,T);W^{1,2}_{0,\sigma})$$

is called a *weak solution* of (2.23) if and only if

$$-\langle u,v_t\rangle_{\Omega,T}+\langle\nabla u,\nabla v\rangle_{\Omega,T}+\langle u\cdot\nabla u,v\rangle_{\Omega,T}=\langle u_0,v(0)\rangle_\Omega+\langle f,v\rangle_{\Omega,T} \tag{2.24}$$

is satisfied for all $v\in C_0^\infty([0,T);C^\infty_{0,\sigma})$. We may assume without loss of generality that u is weakly continuous as a function from $[0,T)$ to L^2_σ.

We know that for each weak solution u there exists a distribution p in $(0,T)\times\Omega$ such that $u_t-\Delta u+u\cdot\nabla u+\nabla p=f$ holds in the sense of distributions, see [19], [28] and [32]; p is called an associated pressure of u. However, for general Ω it is crucial whether p is contained in any L^q-type space; the problem in this context is the validity of the maximal regularity estimate (2.21) for $q=\frac{5}{4}$.

The following result is essentially known for domains with compact boundaries; see [32, Chapter V, Theorem 3.6.2] for bounded domains, and [26] and [33] for exterior domains.

THEOREM 2.7. (Suitable weak solution.) *Let $\Omega\subseteq\mathbf{R}^3$ be a uniform C^2-domain of type α,β,K, and let $0<T\leqslant\infty$, $q=\frac{5}{4}$, $f\in L^q(0,T;L^2)$ and $u_0\in L^2_\sigma$. Then there exists a weak solution $u\in L^\infty(0,T;L^2_\sigma)\cap L^2_{\mathrm{loc}}([0,T);W^{1,2}_{0,\sigma})$ (called a suitable weak solution) of the system (2.23) and an associated pressure p with the following additional properties*:

(a) *Regularity*:

$$u_t,u,\nabla u,\nabla^2u,\nabla p\in L^q(\varepsilon,T';L^2+L^q) \tag{2.25}$$

for all $0<\varepsilon<T'<T$. If $u_0\in D(\tilde{A}_q)$, then (2.25) holds for $\varepsilon=0$ and all $0<T'<T$.

(b) *Local energy inequality*:

$$\frac{1}{2}\|\phi u(t)\|_2^2+\int_s^t\|\phi\nabla u\|_2^2\,d\tau\leqslant\frac{1}{2}\|\phi u(s)\|_2^2+\int_s^t\langle\phi f,\phi u\rangle\,d\tau \\ -\frac{1}{2}\int_s^t\langle\nabla|u|^2,\nabla\phi^2\rangle\,d\tau+\int_s^t\left\langle\frac{1}{2}|u|^2+p,u\cdot\nabla\phi^2\right\rangle d\tau \tag{2.26}$$

for a.a. $s\in[0,T)$, *all* $t\in[s,T)$ *and all* $\phi\in C_0^\infty(\mathbf{R}^3)$.

(c) *Strong energy inequality*:

$$\frac{1}{2}\|u(t)\|_2^2+\int_s^t\|\nabla u\|_2^2\,d\tau\leqslant\frac{1}{2}\|u(s)\|_2^2+\int_s^t\langle f,u\rangle\,d\tau \tag{2.27}$$

for a.a. $s\in[0,T)$ *including* $s=0$, *and all* $t\in[s,T)$.

Remark 2.8. (a) From (2.25) we obtain the existence of some pressure p satisfying

$$p\in L^q(\varepsilon,T';L^r_{\mathrm{loc}}(\bar\Omega)),\quad 0<\varepsilon<T'<T,\ q=\tfrac{5}{4},\ r=\tfrac{15}{7}, \tag{2.28}$$

and we get that $u\in L^2(0,T';L^6(\Omega))$, $0<T'<T$. This shows that (2.26) is well defined. As in (2.22) we obtain decompositions $u=u^{(1)}+u^{(2)}$ and $p=p^{(1)}+p^{(2)}$ satisfying

$$u_t^{(1)},\,u^{(1)},\,\nabla u^{(1)},\,\nabla^2u^{(1)},\,\nabla p^{(1)}\in L^q(\varepsilon,T';L^2)\quad\text{for }0<\varepsilon<T'<T \tag{2.29}$$

and

$$u_t^{(2)},\,u^{(2)},\,\nabla u^{(2)},\,\nabla^2u^{(2)},\,\nabla p^{(2)}\in L^q(\varepsilon,T';L^q)\quad\text{for }0<\varepsilon<T'<T, \tag{2.30}$$

which holds with $\varepsilon=0$ if additionally $u_0\in D(\tilde A_q)$. Note that we may choose $T'=T$ in (2.25) if $T<\infty$.

(b) To obtain *Leray's structure theorem* for Ω, see [23] for the case $\mathbf{R}^3$, let $T=\infty$ and assume for simplicity that $f=0$. Then u in Theorem 2.7, also called a *turbulent weak solution* of (2.23), has the following properties: There exists a countable disjoint family $\{I_k\}_{k=0}^\infty$ of intervals in $(0,\infty)$ such that

(1) $I_1=(0,T_1)$ and $I_0=[T_\infty,\infty)$ with some $0<T_1\leqslant T_\infty<\infty$;

(2) $\left|(0,\infty)\setminus\bigcup_{k=0}^\infty I_k\right|=0$ and $\sum_{k=1}^\infty|I_k|^{1/2}<\infty$, where $|\cdot|$ denotes the Lebesgue measure;

(3) $u(\cdot,t)\in C^\infty(\Omega)$ for every $t\in I_k$, $k=0,1,\dots$.

These properties imply that the $\frac{1}{2}$-dimensional Hausdorff measure of the singular set $\sigma=\{t\in(0,\infty):u(\cdot,t)\notin C^\infty(\Omega)\}$ is zero, see [8].

3. Proofs

3.1. Preliminary local results

Using the structure properties of the given uniform C^2-domain $\Omega\subseteq\mathbf{R}^3$ of type $\alpha,\beta,K>0$, see §2.3, we are able to reduce our results by the localization principle to a standard domain of the form

$$H=H_{\alpha,\beta,r,h}=\{(y',y_3):h(y')-\beta<y_3<h(y'),\ |y'|<\alpha\}\cap B_r; \tag{3.1}$$

here $h:y'\mapsto h(y')$, $|y'|\leqslant\alpha$, is a C^2-function and $B_r=B_r(0)$ a ball with radius $0<r=r(\alpha,\beta,K)<\alpha$ such that

$$\overline{B}_r\subseteq\{(y',y_3):h(y')-\beta<y_3<h(y')+\beta,\ |y'|<\alpha\}.$$

Further, we may assume that $h(0)=0$, $\nabla'h(0)=(0,0)$, $h(y')=0$ for $r\leqslant|y'|\leqslant\alpha$, and that h satisfies the smallness condition

$$\|\nabla'h\|_{C^0}=\max\{|\nabla'h(y')|:|y'|\leqslant\alpha\}\leqslant M_0, \tag{3.2}$$

where $M_0>0$ is a given constant. Recall that $\nabla'=(D_1,D_2)$.

In the subsequent proofs we can treat each problem for the standard domain (3.1) as a problem in the domain

$$H_h=\{(y',y_3)\in\mathbf{R}^3:y_3<h(y'),\ y'\in\mathbf{R}^2\}$$

with $h\in C_0^2(\mathbf{R}^2)$; H_h is called a *bent half-space*, see [9]. Then, using the smallness condition (3.2), an equation in H_h is considered as a perturbation of some equation in the half-space $H_0=\{(y',y_3)\in\mathbf{R}^3:y_3<0\}$.

The following estimates in $H=H_{\alpha,\beta,h,r}$ are well known. However, we have to check that the constants in these estimates depend only on q, α, β and K; here we need the smallness condition (3.2) on h.

Let $1<q<\infty$. First we consider the Helmholtz decomposition in H. Let $f\in L^q(H)$, $f_0\in L^q_\sigma(H)$ and $p\in W^{1,q}(H)$ satisfy $f=f_0+\nabla p$ and $\operatorname{supp} f_0\cup\operatorname{supp} p\subseteq B_r$. Then

$$\|f_0\|_{L^q(H)}+\|\nabla p\|_{L^q(H)}\leqslant C\|f\|_{L^q(H)},\quad C=C(q,\alpha,\beta,K)>0, \tag{3.3}$$

cf. [30, p. 12 and Lemma 3.8 (a)].

Next let $f\in L^q(H)$, $u\in L^q_\sigma(H)\cap W^{1,q}_0(H)\cap W^{2,q}(H)$ and $p\in W^{1,q}(H)$ satisfy

$$\lambda u-\Delta u+\nabla p=f$$

with $\lambda \in \mathcal{S}_\varepsilon$, see Theorem 2.3, and with $\operatorname{supp} u \cup \operatorname{supp} p \subseteq B_r$. Then there are constants $\lambda_0 = \lambda_0(q, \alpha, \beta, K) > 0$ and $C = C(q, \alpha, \beta, K) > 0$ such that

$$|\lambda| \, \|u\|_{L^q(H)} + \|u\|_{W^{2,q}(H)} + \|\nabla p\|_{L^q(H)} \leqslant C \|f\|_{L^q(H)} \tag{3.4}$$

if $|\lambda| \geqslant \lambda_0$. To prove this estimate we use [9, p. 624] and apply [9, Theorem 3.1 (i) and (1.2)].

The next estimate concerns the nonstationary Stokes equation in H. As usual the Stokes operator is defined by $A_q = -P_q \Delta$ with domain

$$D(A_q) = L^q_\sigma(H) \cap W^{1,q}_0(H) \cap W^{2,q}(H).$$

Let $0 < T < \infty$, $u_0 \in D(A_q)$ and $f \in L^q(0, T; L^q(H))$, and let $u \in L^q(0, T, D(A_q))$ and $p \in L^q(0, T; W^{1,q}(H))$ satisfy $\operatorname{supp} u_0 \cup \operatorname{supp} u(t) \cup \operatorname{supp} p(t) \subseteq B_r$ for a.a. $t \in [0, T]$. Moreover, assume that

$$u_t - \Delta u + \nabla p = f, \;\; u(0) = u_0 \quad \text{and} \quad -u_t - \Delta u + \nabla p = f, \;\; u(T) = u_0,$$

respectively. Then there is a constant $C = C(q, \alpha, \beta, K, T) > 0$ such that

$$\begin{aligned} \|u_t\|_{L^q(0,T;L^q(H))} + \|u\|_{L^q(0,T;W^{2,q}(H))} + \|\nabla p\|_{L^q(0,T;L^q(H))} \\ \leqslant C(\|u_0\|_{W^{2,q}(H)} + \|f\|_{L^q(0,T;L^q(H))}). \end{aligned} \tag{3.5}$$

In the case $u(0) = u_0$ this estimate follows from [34, Theorem 4.1, (4.2) and (4.21′)]. The second case $-u_t - \Delta u + \nabla p = f$, $u(T) = u_0$, can be reduced to the first case by the transformation $\tilde{u}(t) = u(T-t)$, $\tilde{f}(t) = f(T-t)$, $\tilde{p}(t) = p(T-t)$. The relatively strong assumption $u_0 \in D(A_q)$ is used for simplicity and can be weakened as in Remark 2.6 (a). Note that the conditions $u(0) = u_0$ and $u(T) = u_0$, respectively, are well defined since $u_t \in L^q(0, T; L^q_\sigma)$.

Finally, we consider the divergence problem

$$\operatorname{div} u = f \;\text{ in } H, \quad u|_{\partial H} = 0,$$

and let $L^q_0(H) = \{f \in L^q(H) : \int_H f \, dx = 0\}$. Then from [6] and [12, III, Theorem 3.2], we obtain the existence of some linear operator $R: L^q_0(H) \to W^{1,q}_0(H)$ satisfying $\operatorname{div} Rf = f$ and

$$\begin{aligned} \|Rf\|_{W^{1,q}(H)} &\leqslant C \|f\|_{L^q(H)} && \text{if } f \in L^q_0(H), \\ \|Rf\|_{W^{2,q}(H)} &\leqslant C \|f\|_{W^{1,q}(H)} && \text{if } f \in L^q_0(H) \cap W^{1,q}_0(H), \end{aligned} \tag{3.6}$$

with $C = C(q, \alpha, \beta, K) > 0$; moreover, $Rf \in W^{2,q}_0(H)$ if $f \in L^q_0(H) \cap W^{1,q}_0(H)$.

The dual operator R' of R maps $W^{-1,q'}(H)$ into $L_0^{q'}(H)$. Thus for each $p \in L^{q'}(H)$ we find a unique constant $M=M(p)$ satisfying $p-M=R'(\nabla p) \in L_0^{q'}(H)$ and the estimate

$$\|p-M\|_{L^{q'}(H)} \leqslant C\|\nabla p\|_{W^{-1,q'}(H)} = C\sup\left\{\frac{|\langle p, \operatorname{div} v\rangle|}{\|\nabla v\|_q} : 0 \neq v \in W_0^{1,q}(H)\right\} \tag{3.7}$$

with $C=C(q,\alpha,\beta,K)>0$.

Now let $\Omega \subseteq \mathbf{R}^3$ be a *bounded* C^2-domain with boundary $\partial\Omega$. Obviously, such a domain is of type α,β,K. We collect several results on the Helmholtz projection $P=P_q$ and the Stokes operator $A=A_q$, $1<q<\infty$. In this case the constant C below may depend also on Ω except for $q=2$ where Hilbert space arguments are applicable.

It is known, see [11], [30] and [34], that each $f \in L^q$ has a unique decomposition $f=f_0+\nabla p$, $f_0 \in L_\sigma^q$, $\nabla p \in G^q$, and that $P_q\colon L^q \to L_\sigma^q$ defined by $P_q f=f_0$ satisfies the estimate $\|P_q f\|_{L^q}+\|\nabla p\|_{L^q} \leqslant C\|f\|_{L^q}$ with $C=C(q,\Omega)>0$; however, it is not clear whether C depends only on the type α,β,K. We obtain that $(P_q)'=P_{q'}$ and $\langle P_q f, g\rangle=\langle f, P_{q'}g\rangle$ for all $f \in L^q$ and $g \in L^{q'}$. If $q=2$, a Hilbert space argument yields the estimate

$$\|P_2 f\|_{L^2}+\|\nabla p\|_{L^2} \leqslant 2\|f\|_{L^2}, \quad f \in L^2,\ \nabla p \in G^2, \tag{3.8}$$

with $C=C(2,\Omega)=2$ *not* depending on Ω.

The Stokes operator $A_q=-P_q\Delta\colon D(A_q) \to L_\sigma^q$, where $D(A_q)=L_\sigma^q \cap W_0^{1,q} \cap W^{2,q}$, satisfies the resolvent estimate

$$|\lambda|\,\|u\|_{L^q}+\|A_q u\|_{L^q} \leqslant C\|f\|_{L^q}, \quad C=C(\varepsilon,q,\Omega)>0,$$

where $u \in D(A_q)$, $\lambda u+A_q u=f$, $\lambda \in \mathcal{S}_\varepsilon$ and $0<\varepsilon<\frac{1}{2}\pi$, and the estimate

$$\|u\|_{W^{2,q}} \leqslant C\|A_q u\|_{L^q}, \quad C=C(q,\Omega).$$

Furthermore, $A_q'=A_{q'}$ implying that $\langle A_q u, v\rangle=\langle u, A_{q'}v\rangle$ for all $u \in D(A_q)$ and $v \in D(A_{q'})$; see [2], [3], [9], [13], [15], [16], [17], [21], [22] and [34]. If $q=2$, we obtain by a Hilbert space argument that $u \in D(A_2)$, with $\lambda u+A_2 u=f \in L_\sigma^2$, $\lambda \in \mathcal{S}_\varepsilon$, satisfies the estimate

$$|\lambda|\,\|u\|_{L^2}+\|A_2 u\|_{L^2} \leqslant C\|f\|_{L^2}, \quad C=1+\frac{2}{\cos\varepsilon}, \tag{3.9}$$

with C independent of Ω. Moreover, since A_2 is selfadjoint,

$$\langle A_2 u, u\rangle = \|A_2^{1/2}u\|_{L^2}^2 = \|\nabla u\|_{L^2}^2, \quad u \in D(A_2). \tag{3.10}$$

Let $1<q,r<\infty$, $0<T<\infty$ and $f \in L^r(0,T;L_\sigma^q)$, $u_0 \in D(A_q)$. Then the semigroup operators e^{-tA_q} and the operators $\mathcal{J}_{q,r}$ and $\mathcal{J}_{q,r}'$ given by

$$(\mathcal{J}_{q,r})f(t)=\int_0^t e^{-(t-\tau)A_q}f(\tau)\,d\tau \quad \text{and} \quad (\mathcal{J}_{q,r}'f)(t)=\int_t^T e^{-(\tau-t)A_q}f(\tau)\,d\tau$$

are well defined for $0 \leqslant t \leqslant T$, see [9] and [15]. Setting $u(t) = e^{-tA_q} u_0 + (\mathcal{J}_{q,r} f)(t)$ we obtain the unique solution $u \in L^r(0,T;D(A_q))$, $u_t \in L^r(0,T;L^q_\sigma)$, of the nonstationary Stokes system $u_t + A_q u = f$, $u(0) = u_0$, satisfying the estimate

$$\|u_t\|_{L^r(L^q)} + \|u\|_{L^r(L^q)} + \|A_q u\|_{L^r(L^q)} \leqslant C(\|u_0\|_{D(A_q)} + \|f\|_{L^r(L^q)}) \tag{3.11}$$

with $C = C(q,r,T,\Omega) > 0$. For our application it is important that $C = C(2,r,T,\Omega) = C(r,T)$ does *not* depend on Ω if $q=2$, see [31] and [32, IV.1.6]. Analogously, $u(t) = e^{-(T-t)A_q} u_0 + (\mathcal{J}'_{q,r} f)(t)$ is the unique solution of the system $-u_t + A_q u = f$, $u(T) = u_0$, in $L^r(0,T;D(A_q))$ with $u_t \in L^r(0,T;L^q_\sigma)$ satisfying the estimate (3.11) with the same constant C; this result follows from the transformation $\tilde{u}(t) = u(T-t)$, $\tilde{f}(t) = f(T-t)$. Further, we obtain the duality relation

$$(\mathcal{J}_{q,r})' = \mathcal{J}'_{q',r'}. \tag{3.12}$$

Finally we mention some well-known embedding estimates for Sobolev spaces on *bounded* C^2-domains Ω of type α, β, K, see [1, IV, Theorem 4.28], [10] and [32, II.1.3]. Given $1 < q < \infty$ and $0 < M \leqslant 1$, there exists some $C = C(q, M, \alpha, \beta, K) > 0$ such that

$$\|\nabla u\|_{L^q} \leqslant M \|\nabla^2 u\|_{L^q} + C\|u\|_{L^q} \tag{3.13}$$

for all $u \in W^{2,q}$. If $2 \leqslant q < \infty$ and $0 < M \leqslant 1$, then there exists some $C = C(q, M, \alpha, \beta, K) > 0$ such that

$$\|u\|_{L^q} \leqslant M \|\nabla^2 u\|_{L^2} + C\|u\|_{L^2} \tag{3.14}$$

for all $u \in W^{2,2}$. Finally, let $1 < q, \gamma < \infty$, $1 < r \leqslant 3$ and $0 \leqslant \alpha \leqslant 1$ such that

$$\alpha\left(\frac{1}{r} - \frac{1}{3}\right) + (1-\alpha)\frac{1}{\gamma} = \frac{1}{q}.$$

Then

$$\|u\|_{L^q} \leqslant C \|\nabla u\|^{\alpha}_{L^r} \|u\|^{1-\alpha}_{L^\gamma} \tag{3.15}$$

for all $u \in W^{1,r}_0 \cap L^\gamma$ with $C = C(r, q, \gamma) > 0$.

3.2. Helmholtz projection in $\tilde{L}^q$; Proof of Theorem 2.1

The proofs of the main theorems rest on the localization principle using the structure of the domain Ω of the type $\alpha, \beta, K > 0$, see §2.3, and the local estimates in §3.1. In the first step of each proof we assume that Ω is bounded. In this case cover $\overline{\Omega}$ by domains of the form

$$U_j = U^-_{\alpha,\beta,h_j}(x_j) \cap B_j, \quad j = 1, 2, \ldots, N, \tag{3.16}$$

with $B_j = B_r(x_j)$, $0<r=r(\alpha,\beta,K)<\alpha$, $x_j\in\bar{\Omega}$ and functions $h_j\in C^2$, where $h_j\equiv 0$ if $x_j\in\Omega$, and use the cut-off functions φ_j as in (2.6) and (2.7). We may assume that each U_j has the standard form $H=H_{\alpha,\beta,r,h}$, see (3.1) and (2.9). In the second step of each proof we consider the sequence of bounded subdomains $\Omega_j\subseteq\Omega$ of the same type α,β,K, see (2.8), and treat the limit $j\to\infty$.

Step 1. *Ω bounded.* Let $f\in L^q$, $2\leqslant q<\infty$, $f_0=P_q f\in L^q_\sigma$ and $\nabla p=f-f_0\in G^q$. Then $f\in L^2$, and we obtain, see §3.1, that

$$\|f_0\|_{L^2\cap L^q}+\|\nabla p\|_{L^2\cap L^q}\leqslant C\|f\|_{L^2\cap L^q} \tag{3.17}$$

with $C=C(q,\Omega)>0$. First we show that the constant C in (3.17) can be chosen depending only on q, α, β and K. For this purpose consider in U_j the local equation

$$\varphi_j f=\varphi_j f_0+\nabla(\varphi_j(p-M_j))-(\nabla\varphi_j)(p-M_j)$$

with the constant $M_j=M_j(p)$ such that $p-M_j=R'(\nabla p)\in L^q_0(U_j)$, see (3.7). Furthermore, we use the solution $w=R((\nabla\varphi_j)\cdot f_0)\in W^{1,q}_0(U_j)$ of the equation $\operatorname{div} w=\operatorname{div}(\varphi_j f_0)=(\nabla\varphi_j)\cdot f_0\in L^q_0(U_j)$, see (3.6). Then

$$\varphi_j f+(\nabla\varphi_j)(p-M_j)-w=(\varphi_j f_0-w)+\nabla(\varphi_j(p-M_j))$$

is the Helmholtz decomposition of $\varphi_j f+(\nabla\varphi_j)(p-M_j)-w$ in $L^q(U_j)$, and we may use estimate (3.3).

First let $2\leqslant q\leqslant 6$. Then (3.6), (3.15) with $r=\gamma=2$, and Poincaré's inequality imply that $\|w\|_{L^q(U_j)}\leqslant C\|f_0\|_{L^2(U_j)}$ with $C=C(q,\alpha,\beta,K)>0$. Further, considering $p-M_j$, we apply (3.7), (3.15) and Poincaré's inequality to obtain with $\nabla p=f-f_0$ that

$$\|p-M_j\|_{L^q(U_j)}\leqslant C(\|f\|_{L^q(U_j)}+\|f_0\|_{L^2(U_j)}),$$

where $C=C(q,\alpha,\beta,K)>0$. Combining these estimates we get the inequality

$$\|\varphi_j f_0\|^q_{L^q(U_j)}+\|\varphi_j\nabla p\|^q_{L^q(U_j)}\leqslant C(\|f\|^q_{L^q(U_j)}+\|f_0\|^q_{L^2(U_j)}) \tag{3.18}$$

with $C=C(q,\alpha,\beta,K)>0$. Next we will take the sum for $j=1,\dots,N$, and use the number $N_0=N_0(\alpha,\beta,K)\in\mathbf{N}$ introduced in §2.3, Hölder's inequality and the reverse Hölder inequality $\left(\sum_{j=1}^N |a_j|^q\right)^{1/q} \leqslant \left(\sum_{j=1}^N |a_j|^2\right)^{1/2}$. This leads to the crucial estimate

$$\begin{aligned}
\|f_0\|_{L^q(\Omega)}^q + \|\nabla p\|_{L^q(\Omega)}^q &= \int_\Omega \left(\sum_{j=1}^N \varphi_j |f_0|\right)^q dx + \int_\Omega \left(\sum_{j=1}^N \varphi_j |\nabla p|\right)^q dx \\
&\leqslant \int_\Omega N_0^{q/q'} \left(\sum_{j=1}^N |\varphi_j f_0|^q\right) dx + \int_\Omega N_0^{q/q'} \left(\sum_{j=1}^N |\varphi_j \nabla p|^q\right) dx \\
&= N_0^{q/q'} \left(\sum_{j=1}^N \|\varphi_j f_0\|_{L^q(U_j)}^q + \sum_{j=1}^N \|\varphi_j \nabla p\|_{L^q(U_j)}^q\right) \qquad (3.19) \\
&\leqslant C_1 \left(\sum_{j=1}^N \|f\|_{L^q(U_j)}^q + \left(\sum_{j=1}^N \|f_0\|_{L^2(U_j)}^2\right)^{q/2}\right) \\
&\leqslant C_2 (\|f\|_{L^q(\Omega)}^q + \|f_0\|_{L^2(\Omega)}^q)
\end{aligned}$$

with $C_i = C_i(q, \alpha, \beta, K) > 0$, $i = 1, 2$, and $2 \leqslant q \leqslant 6$; this kind of estimate will be used in an analogous way also in the subsequent proofs in §3.3 and §3.4.

In the case $6 < q < \infty$ we obtain the estimate (3.19) in the same way as above with $\|f_0\|_{L^2(\Omega)}^q$ replaced by $\|f_0\|_{L^6(\Omega)}^q$. Now we use the elementary interpolation estimate

$$\|f_0\|_{L^6(\Omega)} \leqslant \gamma \left(\frac{1}{\varepsilon}\right)^{1/\gamma} \|f_0\|_{L^2(\Omega)} + (1-\gamma)\varepsilon^{1/(1-\gamma)} \|f_0\|_{L^q(\Omega)},$$

where $0 < \gamma < 1$ is defined by

$$\frac{1}{6} = \frac{\gamma}{2} + \frac{1-\gamma}{q},$$

and where $\varepsilon > 0$ is chosen sufficiently small. Then the absorption principle yields the estimate

$$\|f_0\|_{L^q(\Omega)} + \|\nabla p\|_{L^q(\Omega)} \leqslant C(\|f\|_{L^q(\Omega)} + \|f_0\|_{L^2(\Omega)}), \quad C = C(q, \alpha, \beta, K) > 0, \qquad (3.20)$$

also for $q > 6$. Therefore, (3.20) holds for all $2 \leqslant q < \infty$. Combining (3.20) with (3.8) we get (3.17) with $C = C(q, \alpha, \beta, K) > 0$ for all $2 \leqslant q < \infty$.

Next we consider the case $f \in L^2 + L^q$, $1 < q < 2$. Choose $f_1 \in L^2$ and $f_2 \in L^q$ with $f = f_1 + f_2$, $\|f\|_{L^2 + L^q} = \|f_1\|_{L^2} + \|f_2\|_{L^q}$, and define

$$f_0 = P_2 f_1 + P_q f_2 \in L_\sigma^2 + L_\sigma^q \quad \text{and} \quad \nabla p = (I - P_2) f_1 + (I - P_q) f_2 \in G^2 + G^q$$

yielding $f=f_0+\nabla p$. Then we use the dual representation of the norm $\|f_0\|_{L^2+L^q}$, see §2.2, and obtain with (3.17), $q'>2$, that

$$\begin{aligned}\|f_0\|_{L^2+L^q} &= \sup\left\{\frac{|\langle P_2 f_1+P_q f_2, g\rangle|}{\|g\|_{L^2\cap L^{q'}}} : 0\neq g\in L^2\cap L^{q'}\right\}\\ &= \sup\left\{\frac{|\langle f_1+f_2, P_{q'}g\rangle|}{\|g\|_{L^2\cap L^{q'}}} : 0\neq g\in L^2\cap L^{q'}\right\}\\ &\leqslant \sup\left\{\frac{(\|f_1\|_{L^2}+\|f_2\|_{L^q})\|P_{q'}g\|_{L^2\cap L^{q'}}}{\|g\|_{L^2\cap L^{q'}}} : 0\neq g\in L^2\cap L^{q'}\right\}\\ &\leqslant C\|f\|_{L^2+L^q}\end{aligned} \tag{3.21}$$

with the same $C=C(q,\alpha,\beta,K)>0$ as valid for (3.17). It follows that

$$\|f_0\|_{L^2+L^q}+\|\nabla p\|_{L^2+L^q}\leqslant C\|f\|_{L^2+L^q}$$

with $C=C(q,\alpha,\beta,K)>0$.

Summarizing we obtain for every $1<q<\infty$ and $f\in\tilde{L}^q$ the estimate

$$\|f_0\|_{\tilde{L}^q}+\|\nabla p\|_{\tilde{L}^q}\leqslant C\|f\|_{\tilde{L}^q},\quad C=C(q,\alpha,\beta,K)>0, \tag{3.22}$$

where $\widetilde{P}_q f=f_0$ is defined by $f_0=P_q f$ if $f\in\tilde{L}^q=L^2\cap L^q$, $2\leqslant q<\infty$, and by $f_0=P_2 f_1+P_q f_2$ if $f=f_1+f_2\in\tilde{L}^q=L^2+L^q$, $1<q<2$. Moreover, $\nabla p=(I-\widetilde{P}_q)f\in\widetilde{G}^q=G^2\cap G^q$ if $2\leqslant q<\infty$, and $\nabla p=\nabla p_1+\nabla p_2=(I-P_2)f_1+(I-P_q)f_2\in\widetilde{G}^q=G^2+G^q$ when $1<q<2$. Thus we proved (2.10) for bounded domains Ω, and we may conclude that $\widetilde{P}_q f=P_q f$ holds for $1<q<\infty$. Therefore, the other assertions of Theorem 2.1 are obvious for bounded domains. Note that the choice of $C=C(q,\alpha,\beta,K)$ in (2.10) is the only new property in this case.

Step 2. *Ω unbounded.* Let $f\in\tilde{L}^q(\Omega)$, $1<q<\infty$, and let $f_j=f|_{\Omega_j}\in\tilde{L}^q(\Omega_j)$, $j\in\mathbf{N}$, be the restriction to the subdomain $\Omega_j\subseteq\Omega$, see (2.8). Our aim is to construct a unique solution pair $f_0\in\tilde{L}^q_\sigma(\Omega)$, $\nabla p\in\widetilde{G}^q(\Omega)$ satisfying $f=f_0+\nabla p$. For this purpose we use Step 1 with the decomposition

$$f_j=f_{j,0}+\nabla p_j,\quad \text{where } f_{j,0}=\widetilde{P}_q f_j \text{ and } \nabla p_j\in\widetilde{G}^q(\Omega_j),$$

and the uniform estimate

$$\|f_{j,0}\|_{\tilde{L}^q(\Omega_j)}+\|\nabla p_j\|_{\tilde{L}^q(\Omega_j)}\leqslant C\|f_j\|_{\tilde{L}^q(\Omega_j)}\leqslant C\|f\|_{\tilde{L}^q(\Omega)} \tag{3.23}$$

with $C>0$ as in (3.22). Here consider $\tilde{L}^q(\Omega_j)$ as a subspace of $\tilde{L}^q(\Omega)$ by extending each function on Ω_j by zero to get a function on Ω. Since $(\tilde{L}^q)'=\tilde{L}^{q'}$ and $(\tilde{L}^{q'})'=\tilde{L}^q$, cf. §2.2, we may assume, suppressing subsequences, that there exist weak limits

$$f_0=\underset{j\to\infty}{\text{w-lim}}\, f_{j,0}\in\tilde{L}^q_\sigma(\Omega)\quad\text{and}\quad \nabla p=\underset{j\to\infty}{\text{w-lim}}\,\nabla p_j\in\widetilde{G}^q(\Omega)$$

satisfying $f_0+\nabla p=f$. Note that ∇p_j treated as an element of $\tilde{L}^q(\Omega)$ when extended by zero need not be a gradient; however, by de Rham's argument, cf. [35, Chapter I, (1.29)] or [32, p. 73], we see that w-$\lim_{j\to\infty}\nabla p_j$ is indeed a gradient. From (3.23) we obtain the estimate

$$\|f_0\|_{\tilde{L}^q(\Omega)}+\|\nabla p\|_{\tilde{L}^q(\Omega)}\leqslant C\|f\|_{\tilde{L}^q(\Omega)} \tag{3.24}$$

with C as in (3.23). To prove the uniqueness of the decomposition $f=f_0+\nabla p$ assume that $f_0+\nabla p=0$, $f_0\in\tilde{L}^q_\sigma(\Omega)$, $\nabla p\in\tilde{G}^q(\Omega)$. Then we use the construction above for any $g=g_0+\nabla h\in\tilde{L}^{q'}(\Omega)$, $g_0\in\tilde{L}^{q'}_\sigma(\Omega)$, $\nabla h\in\tilde{G}^{q'}(\Omega)$, and obtain that $\langle f_0,g\rangle=-\langle\nabla p,g_0\rangle=0$. Hence $f_0=\nabla p=0$, and $\tilde{P}_qf=f_0\in\tilde{L}^q_\sigma$ is well defined. Now the assertions of Theorem 2.1 and of Remark 2.2 are easy consequences. This completes the proof.

3.3. The Stokes operator in $\tilde{L}^q$; Proof of Theorem 2.3

Step 1. *Ω bounded.* First we consider the Stokes equation $-\Delta u+\nabla p=f$ with $f\in L^q_\sigma$ and $u\in D(A_q)=L^q_\sigma\cap W^{1,q}_0\cap W^{2,q}$, $1<q<\infty$, which is equivalent to the equation $A_qu=f$, and prove the preliminary estimate

$$\|\nabla^2u\|_{L^q(\Omega)}+\|\nabla p\|_{L^p(\Omega)}\leqslant C(\|f\|_{L^q(\Omega)}+\|u\|_{L^q(\Omega)}) \tag{3.25}$$

with $C=C(q,\alpha,\beta,K)>0$ depending only on q and the type α,β,K.

This estimate has the important implication that the graph norm $\|u\|_{D(A_q)}=\|u\|_{L^q}+\|A_qu\|_{L^q}$ is equivalent to the norm $\|u\|_{W^{2,q}}$ on $D(A_q)$ with constants only depending on q, α, β and K. More precisely,

$$C_1\|u\|_{W^{2,q}}\leqslant\|u\|_{D(A_q)}\leqslant C_2\|u\|_{W^{2,q}},\quad u\in D(A_q), \tag{3.26}$$

with $C_1=C_1(q,\alpha,\beta,K)>0$ and $C_2=C_2(q,\alpha,\beta,K)>0$.

To prove (3.25) we use U_j and φ_j, $j=1,\dots,N$, as in §3.2, and consider in U_j the local equation

$$\begin{aligned}\lambda_0(\varphi_ju-w)&-\Delta(\varphi_ju-w)+\nabla(\varphi_j(p-M_j))\\&=\varphi_jf+\Delta w-2\nabla\varphi_j\cdot\nabla u-(\Delta\varphi_j)u+(\nabla\varphi_j)(p-M_j)+\lambda_0(\varphi_ju-w).\end{aligned}$$

Here λ_0 means the constant in (3.4), $M_j=M_j(p)$ is a constant such that $p-M_j=R'(\nabla p)\in L^q_0(\Omega)$, see (3.7), and $w=R((\nabla\varphi_j)\cdot u)\in W^{2,q}_0(U_j)$ is the solution of the equation $\operatorname{div}w=\operatorname{div}(\varphi_ju)=(\nabla\varphi_j)\cdot u$, see (3.6). Then we apply (3.4) with $\lambda=\lambda_0$, and use the estimates

$$\begin{aligned}\|w\|_{W^{1,q}(U_j)}&\leqslant C\|u\|_{L^q(U_j)},\\\|w\|_{W^{2,q}(U_j)}&\leqslant C\|u\|_{W^{1,q}(U_j)},\\\|p-M_j\|_{L^q(U_j)}&\leqslant C(\|f\|_{L^q(U_j)}+\|\nabla u\|_{L^q(U_j)}),\end{aligned}$$

with $C=C(q,\alpha,\beta,K)>0$, following from (3.6) and (3.7) applied to $\nabla p=f+\Delta u$ in U_j. Combining these estimates we are led to the local inequalities

$$\|\varphi_j\nabla^2 u\|^q_{L^q(U_j)}+\|\varphi_j\nabla(p-M_j)\|^q_{L^q(U_j)}\leqslant C(\|f\|^q_{L^q(U_j)}+\|u\|^q_{W^{1,q}(U_j)}) \tag{3.27}$$

with $C=C(q,\alpha,\beta,K)>0$. Taking the sum over $j=1,\dots,N$ in the same way as in (3.19), and using the absorption argument to remove $\|\nabla u\|^q_{L^q(\Omega)}$ with (3.13), we obtain the desired inequality (3.25).

Next we consider the resolvent equation

$$\lambda u+A_q u=\lambda u-\Delta u+\nabla p=f \quad \text{in } \Omega$$

with $f\in L^q_\sigma$, where $1<q<\infty$ and $\lambda\in\mathcal{S}_\varepsilon$, $0<\varepsilon<\frac{1}{2}\pi$. Our first purpose is to prove for $u\in D(A_q)$ and $\nabla p=(I-P_q)\Delta u$, $2\leqslant q<\infty$, the estimate

$$|\lambda|\;\|u\|_{L^2\cap L^q}+\|\nabla^2 u\|_{L^2\cap L^q}+\|\nabla p\|_{L^2\cap L^q}\leqslant C\|f\|_{L^2\cap L^q} \tag{3.28}$$

with $|\lambda|\geqslant\delta>0$, where $\delta>0$ is given, and $C=C(q,\varepsilon,\delta,\alpha,\beta,K)>0$. Note that this estimate is well known for bounded domains with $C=C(q,\varepsilon,\delta,\Omega)>0$, see §3.1. In this case we obtain the local equation

$$\begin{aligned}\lambda(\varphi_j u-w)-\Delta(\varphi_j u-w)+\nabla(\varphi_j(p-M_j))&\\ =\varphi_j f+\Delta w-2\nabla\varphi_j\cdot\nabla u-(\Delta\varphi_j)u-\lambda w+(\nabla\varphi_j)(p-M_j)&\end{aligned} \tag{3.29}$$

with $p-M_j=R'(\nabla p)$ and $w=R((\nabla\varphi_j)\cdot u)$ as above.

First let $2\leqslant q\leqslant 6$. Concerning w, we use the estimates above and the inequality

$$\|w\|_{L^q(U_j)}\leqslant C_1\|w\|_{W^{1,2}(U_j)}\leqslant C_2\|u\|_{L^2(U_j)},$$

$C_i=C_i(q,\alpha,\beta,K)>0$, $i=1,2$. For $p-M_j$ we use the above estimate and the inequality

$$\|p-M_j\|_{L^q(U_j)}\leqslant C(\|f\|_{L^q(U_j)}+|\lambda|\;\|u\|_{L^2(U_j)}+\|\nabla u\|_{L^q(U_j)})$$

with $C=C(q,\alpha,\beta,K)>0$. Further, to the local resolvent equation (3.29) we apply the estimate (3.4) with λ replaced by $\lambda+\lambda_0'$, where $\lambda_0'\geqslant 0$ is sufficiently large such that $|\lambda+\lambda_0'|\geqslant\lambda_0$ for $|\lambda|\geqslant\delta$, and λ_0 is as in (3.4). Then we combine these estimates and are led to the local inequality

$$\begin{aligned}\|\lambda\varphi_j u\|^q_{L^q(U_j)}+\|\varphi_j u\|^q_{L^q(U_j)}+\|\varphi_j\nabla^2 u\|^q_{L^q(U_j)}+\|\varphi_j\nabla p\|^q_{L^q(U_j)}&\\ \leqslant C(\|f\|^q_{L^q(U_j)}+\|u\|^q_{L^q(U_j)}+\|\nabla u\|^q_{L^q(U_j)}+\|\lambda u\|^q_{L^2(U_j)})&\end{aligned} \tag{3.30}$$

with $C=C(q,\delta,\varepsilon,\alpha,\beta,K)>0$. Next we take the sum over $j=1,\dots,N$ in the same way as in (3.19). This leads to the inequality

$$\begin{aligned}|\lambda|\,\|u\|_{L^q(\Omega)}+\|u\|_{L^q(\Omega)}+\|\nabla^2u\|_{L^q(\Omega)}+\|\nabla p\|_{L^q(\Omega)}\\ \leqslant C(\|f\|_{L^q(\Omega)}+\|u\|_{L^q(\Omega)}+\|\nabla u\|_{L^q(\Omega)}+|\lambda|\,\|u\|_{L^2(\Omega)})\end{aligned} \tag{3.31}$$

with $C=C(q,\delta,\varepsilon,\alpha,\beta,K)>0$, $|\lambda|\geqslant\delta$ and $2\leqslant q\leqslant 6$. Applying (3.13) we remove the term $\|\nabla u\|_{L^q(\Omega)}$ in (3.31) by the absorption principle.

If $q>6$, estimate (3.31) holds in the same way with the term $|\lambda|\,\|u\|_{L^2(\Omega)}$ on the right-hand side replaced by $|\lambda|\,\|u\|_{L^6(\Omega)}$. Now use the elementary estimate

$$|\lambda|\,\|u\|_{L^6(\Omega)}\leqslant\gamma\left(\frac{1}{\varepsilon}\right)^{1/\gamma}|\lambda|\,\|u\|_{L^2(\Omega)}+(1-\gamma)\varepsilon^{1/(1-\gamma)}|\lambda|\,\|u\|_{L^q(\Omega)}$$

with $0<\gamma<1$ such that

$$\frac{1}{6}=\frac{\gamma}{2}+\frac{1-\gamma}{q},$$

with sufficiently small $\varepsilon>0$, and use the absorption principle. This proves (3.31) for all $q\geqslant 2$ without the term $\|\nabla u\|_{L^q(\Omega)}$. Moreover, due to (3.14), the term $\|u\|_{L^q(\Omega)}$ may be removed from the right-hand side of (3.31). Now we combine this improved inequality (3.31) with estimate (3.9) for $|\lambda|\geqslant\delta$, and we apply (3.25) with $q=2$. This proves the desired estimate (3.28) for $2\leqslant q<\infty$.

Next let $1<q<2$ and consider in Ω the (well-defined) equation $\lambda u-\Delta u+\nabla p=f$ with $f\in L^2_\sigma+L^q_\sigma$, where $u\in D(A_2)+D(A_q)$, $\nabla p=(I-\widetilde{P}_q)\Delta u$ and $\lambda\in\mathcal{S}_\varepsilon$, $|\lambda|\geqslant\delta$. Using $f=\lambda u-\widetilde{P}_q\Delta u$ and (3.28) with $q'>2$ we first obtain that

$$\begin{aligned}\|f\|_{L^2_\sigma+L^q_\sigma}&=\sup\left\{\frac{|\langle\lambda u-\widetilde{P}_q\Delta u,v\rangle|}{\|v\|_{L^2_\sigma\cap L^{q'}_\sigma}}:0\neq v\in L^2_\sigma\cap L^{q'}_\sigma\right\}\\ &=\sup\left\{\frac{|\langle u,\lambda v-\widetilde{P}_{q'}\Delta v\rangle|}{\|v\|_{L^2_\sigma\cap L^{q'}_\sigma}}:0\neq v\in L^2_\sigma\cap L^{q'}_\sigma\right\}\\ &=\sup\left\{\frac{|\langle u,g\rangle|}{\|(\lambda I-\widetilde{P}_{q'}\Delta)^{-1}g\|_{L^2_\sigma\cap L^{q'}_\sigma}}:0\neq g\in L^2_\sigma\cap L^{q'}_\sigma\right\}\\ &\geqslant\frac{|\lambda|}{C}\sup\left\{\frac{|\langle u,g\rangle|}{\|g\|_{L^2_\sigma\cap L^{q'}_\sigma}}:0\neq g\in L^2_\sigma\cap L^{q'}_\sigma\right\}\\ &=\frac{|\lambda|}{C}\|u\|^*_{L^2_\sigma\cap L^q_\sigma}\end{aligned} \tag{3.32}$$

with C as in (3.28); see (2.11) concerning $\|u\|^*_{L^2_\sigma\cap L^q_\sigma}$. Hence we also get $|\lambda|\,\|u\|_{L^2_\sigma+L^q_\sigma}\leqslant C\|f\|_{L^2_\sigma+L^q_\sigma}$ and even

$$|\lambda|\,\|u\|_{L^2_\sigma+L^q_\sigma}+\|u\|_{L^2_\sigma+L^q_\sigma}+\|A_qu\|_{L^2_\sigma+L^q_\sigma}\leqslant C\|f\|_{L^2_\sigma+L^q_\sigma},\quad\lambda\in\mathcal{S}_\varepsilon,\ |\lambda|\geqslant\delta. \tag{3.33}$$

From the equivalence of the norms $\|\cdot\|_{D(A_q)}$ and $\|\cdot\|_{W^{2,q}}$, cf. (3.26), and from (2.2) with $B_1=A_2$ and $B_2=A_q$, we conclude that

$$C_1\|u\|_{W^{2,2}+W^{2,q}} \leqslant \|u\|_{L^2_\sigma+L^q_\sigma}+\|A_q u\|_{L^2_\sigma+L^q_\sigma} \leqslant C_2\|u\|_{W^{2,2}+W^{2,q}},$$

where $C_i=C_i(q,\varepsilon,\alpha,\beta,K)$, $i=1,2$. Then (3.33) and the identity $\nabla p=f-\lambda u+\Delta u$ lead to the estimate

$$|\lambda|\,\|u\|_{L^2_\sigma+L^q_\sigma}+\|u\|_{W^{2,2}+W^{2,q}}+\|\nabla p\|_{L^2+L^q} \leqslant C\|f\|_{L^2_\sigma+L^q_\sigma} \tag{3.34}$$

with $C=C(q,\delta,\varepsilon,\alpha,\beta,K)>0$.

Since Ω is bounded, we easily conclude that $\tilde{A}_q u=-\widetilde{P}_q\Delta u=A_q u$ for $u\in D(\tilde{A}_q)=D(A_q)$, $1<q<\infty$. The only new result in this case is the validity of the estimate

$$|\lambda|\,\|u\|_{\tilde{L}^q_\sigma}+\|u\|_{\widetilde{W}^{2,q}}+\|\nabla p\|_{\tilde{L}^q} \leqslant C\|f\|_{\tilde{L}^q_\sigma},\quad u\in D(\tilde{A}_q), \tag{3.35}$$

with $C=C(q,\delta,\varepsilon,\alpha,\beta,K)>0$ when $|\lambda|\geqslant\delta>0$. Thus the proof of Theorem 2.3 is complete for bounded Ω.

Step 2. Ω *unbounded.* In principle we use the same arguments as in Step 2 of §3.2 with the bounded subdomains $\Omega_j\subset\Omega$, $j\in\mathbf{N}$, see (2.8).

Let $f\in\tilde{L}^q_\sigma(\Omega)$, $1<q<\infty$, and $\lambda\in\mathcal{S}_\varepsilon$, $0<\varepsilon<\frac{1}{2}\pi$. Our aim is to construct a unique solution $u\in\widetilde{D}^q(\Omega)$ of the equation

$$\lambda u-\widetilde{P}_q\Delta u=\lambda u-\Delta u+\nabla p=f,\quad \nabla p=(I-\widetilde{P}_q)\Delta u\quad \text{in } \Omega$$

satisfying estimate (2.12). For this purpose set $f_j=\widetilde{P}_q f|_{\Omega_j}$ and consider the solution $u_j\in\widetilde{D}^q(\Omega_j)$ of the equation

$$\lambda u_j+\tilde{A}_q u_j=\lambda u_j-\Delta u_j+\nabla p_j=f_j,\quad \nabla p_j=(I-\widetilde{P}_q)\Delta u_j\quad \text{in } \Omega_j.$$

From (3.35) we obtain the uniform estimate

$$|\lambda|\,\|u_j\|_{\tilde{L}^q_\sigma(\Omega_j)}+\|u_j\|_{\widetilde{W}^{2,q}(\Omega_j)}+\|\nabla p_j\|_{\tilde{L}^q(\Omega_j)} \leqslant C\|f\|_{\tilde{L}^q_\sigma(\Omega)} \tag{3.36}$$

with $|\lambda|\geqslant\delta>0$ and $C=C(q,\delta,\varepsilon,\alpha,\beta,K)>0$. The same weak convergence argument as in Step 2 of §3.2 yields, suppressing subsequences, weak limits

$$u=\operatorname*{w-lim}_{j\to\infty} u_j\ \text{ in } \tilde{L}^q_\sigma(\Omega)\quad \text{and}\quad \nabla p=\operatorname*{w-lim}_{j\to\infty}\nabla p_j\ \text{ in } \tilde{L}^q(\Omega)$$

satisfying $u\in\widetilde{D}^q(\Omega)$, $\lambda u-\Delta u+\nabla p=\lambda u-\widetilde{P}_q\Delta u=f$ in Ω and (2.12).

To prove the uniqueness of u we assume that there is some $v\in\widetilde{D}^q(\Omega)$ and $\lambda\in\mathcal{S}_\varepsilon$ satisfying $\lambda v-\widetilde{P}_q\Delta v=0$. Given $f'\in\tilde{L}^{q'}(\Omega)$ let $u\in\widetilde{D}^{q'}(\Omega)$ be a solution of $\lambda u-\widetilde{P}_{q'}\Delta u=\widetilde{P}_{q'}f'$. Then

$$0=\langle\lambda v-\widetilde{P}_q\Delta v,u\rangle=\langle v,(\lambda-\widetilde{P}_{q'}\Delta)u\rangle=\langle v,\widetilde{P}_{q'}f'\rangle=\langle v,f'\rangle$$

for all $f'\in\tilde{L}^{q'}(\Omega)$; hence, $v=0$. Thus we get that the equation $\lambda u+\tilde{A}_q u=f$, $\lambda\in\mathcal{S}_\varepsilon$, has a unique solution $u=(\lambda I+\tilde{A}_q)^{-1}f$ satisfying (2.12).

3.4. Maximal regularity in $\tilde{L}^q$ for the nonstationary Stokes system; Proof of Theorem 2.5

Step 1. *Ω bounded.* In principle we use the same arguments as in the previous proofs. Given $0<T<\infty$ and $1<s,q<\infty$ let

$$\|\cdot\|_{L^s(X(\Omega))}=\|\cdot\|_{L^s(0,T;X(\Omega))}=\left(\int_0^T\|\cdot\|_X^s\,dt\right)^{1/s},$$

where $X(\Omega)$ is a Banach space of functions in Ω; furthermore, we use the operators $\mathcal{J}_{q,s}$ and $\mathcal{J}'_{q,s}$, see §3.1, and define $\widetilde{\mathcal{J}}_{q,s}$ and $\widetilde{\mathcal{J}}'_{q,s}$ for $f\in L^s(0,T;\tilde{L}^q_\sigma)$ by

$$(\widetilde{\mathcal{J}}_{q,s}f)(t)=\int_0^t e^{-(t-\tau)\tilde{A}_q}f(\tau)\,d\tau \quad\text{and}\quad (\widetilde{\mathcal{J}}'_{q,s}f)(t)=\int_t^T e^{-(\tau-t)\tilde{A}_q}f(\tau)\,d\tau,$$

$0\leqslant t\leqslant T$. Since $\tilde{A}'_q=\tilde{A}_{q'}$, we obtain for all $f\in L^s(0,T;\tilde{L}^q_\sigma)$ and $g\in L^{s'}(0,T;\tilde{L}^{q'}_\sigma)$ that

$$\langle\widetilde{\mathcal{J}}_{q,s}f,g\rangle_T=\langle f,\widetilde{\mathcal{J}}'_{q',s'}g\rangle_T.$$

First consider the case $u_0=0$ and let $s=q$. Then $u=\widetilde{\mathcal{J}}_{q,q}f$ solves the evolution system $u_t+\tilde{A}_qu=f$, $u(0)=0$, and $u=\widetilde{\mathcal{J}}'_{q,q}f$ is the solution of the system $-u_t+\tilde{A}_qu=f$, $u(T)=0$. Our aim is to prove in both cases the estimate

$$\|u_t\|_{L^q(\tilde{L}^q_\sigma(\Omega))}+\|u\|_{L^q(\widetilde{W}^{2,q}(\Omega))}+\|\nabla p\|_{L^q(\tilde{L}^q(\Omega))}\leqslant C\|f\|_{L^q(\tilde{L}^q_\sigma(\Omega))} \tag{3.37}$$

with $\nabla p=(I-\widetilde{P}_q)\Delta u$ and $C=C(T,q,\alpha,\beta,K)>0$.

Observe that it is sufficient to prove (3.37) for the case $u=\widetilde{\mathcal{J}}_{q,q}f$ only. The other case follows using the transformation $\tilde{u}(t)=u(T-t)$, $\tilde{f}(t)=f(T-t)$. Further, it is sufficient to prove (3.37) when $2\leqslant q<\infty$. For, using $(\widetilde{\mathcal{J}}'_{q,q})'=\widetilde{\mathcal{J}}_{q',q'}$ and the duality principle in the same way as in (3.32), the case $1<q<2$ is reduced to the case $2<q'<\infty$. In this context we note that it is sufficient to prove instead of (3.37) the estimate $\|u_t\|_{L^q(\tilde{L}^q_\sigma(\Omega))}\leqslant C\|f\|_{L^q(\tilde{L}^q_\sigma(\Omega))}$. Actually, (3.37) follows using $\tilde{A}_qu=f-u_t$, the simple identity $u(t)=\int_0^t u_t(\tau)\,d\tau$ leading to the estimate $\|u\|_{L^q(\tilde{L}^q_\sigma(\Omega))}\leqslant C\|u_t\|_{L^q(\tilde{L}^q_\sigma(\Omega))}$, $C=C(T)>0$, and the equivalence relation (3.26).

Thus it remains to prove (3.37) with $2\leqslant q<\infty$, where $u=\widetilde{\mathcal{J}}_{q,q}f$ solves

$$u_t+\tilde{A}_qu=u_t-\Delta u+\nabla p=f\in L^q(0,T;\tilde{L}^q_\sigma),\quad u(0)=0,$$

and $\nabla p=(I-\widetilde{P}_q)\Delta u$. Using the well-known estimate (3.11) for bounded domains we know that $u=\widetilde{\mathcal{J}}_{q,q}f$ satisfies (3.37) with $C=C(T,q,\Omega)>0$. Thus it remains to prove that C in (3.37) can be chosen depending only on T, q, α, β and K.

To prove this result consider the local equation

$$(\varphi_j u-w)_t-\Delta(\varphi_j u-w)+\nabla(\varphi_j(p-M_j))$$
$$=\varphi_j f-w_t+\Delta w-2\nabla\varphi_j\cdot\nabla u-(\Delta\varphi_j)u+(\nabla\varphi_j)(p-M_j)$$

in U_j, where $w=R((\nabla\varphi_j)\cdot u)\in L^q(0,T;W_0^{2,q}(U_j))$ solves the equations $\operatorname{div} w=(\nabla\varphi_j)\cdot u$ and $\operatorname{div} w_t=(\nabla\varphi_j)\cdot u_t$ for a.a. $t\in(0,T)$. Here U_j and φ_j, $1\leqslant j\leqslant N$, have the same meaning as in the previous proofs, and $M_j=M_j(p)$ is a constant depending on t defined by $p-M_j=R'(\nabla p)\in L^q(0,T;L_0^q(U_j))$.

First let $2\leqslant q\leqslant 6$. Then from (3.6) and (3.7) using $\nabla p=f-u_t+\Delta u$ we obtain the estimates

$$\begin{aligned}\|w_t\|_{L^q(L^q(U_j))}&\leqslant C\|u_t\|_{L^q(L^2(U_j))},\\ \|\nabla^2 w\|_{L^q(L^q(U_j))}&\leqslant C(\|u\|_{L^q(L^q(U_j))}+\|\nabla u\|_{L^q(L^q(U_j))}),\\ \|p-M_j\|_{L^q(L^q(U_j))}&\leqslant C(\|f\|_{L^q(L^q(U_j))}+\|u_t\|_{L^q(L^2(U_j))}+\|\nabla u\|_{L^q(L^q(U_j))}),\end{aligned}\tag{3.38}$$

with $C=C(q,\alpha,\beta,K)>0$. Applying the local estimate (3.5) and using (3.38) we are led to the inequality

$$\begin{aligned}&\|\varphi_j u_t\|^q_{L^q(L^q(U_j))}+\|\varphi_j u\|^q_{L^q(L^q(U_j))}+\|\varphi_j\nabla^2 u\|^q_{L^q(L^q(U_j))}+\|\varphi_j\nabla p\|^q_{L^q(L^q(U_j))}\\ &\quad\leqslant C(\|f\|^q_{L^q(L^q(U_j))}+\|u\|^q_{L^q(L^q(U_j))}+\|\nabla u\|^q_{L^q(L^q(U_j))}+\|u_t\|^q_{L^q(L^2(U_j))})\end{aligned}\tag{3.39}$$

with $C=C(T,q,\alpha,\beta,K)>0$. Next we argue in principle in the same way as in Step 1 of §3.3: Take the sum over $j=1,\dots,N$, remove the term $\|\nabla u\|_{L^q(L^q(\Omega))}$ with the absorption argument using (3.13), then apply the estimate (3.11) to $\|u_t\|_{L^q(L^2(\Omega))}$ with $C=C(q,T)>0$. If $q>6$, we have to replace the term $\|u_t\|_{L^q(L^2(\Omega))}$ by the term $\|u_t\|_{L^q(L^6(\Omega))}$, and use the interpolation inequality

$$\|u_t\|_{L^q(L^6(\Omega))}\leqslant\gamma\left(\frac{1}{\varepsilon}\right)^{1/\gamma}\|u_t\|_{L^q(L^2(\Omega))}+(1-\gamma)\varepsilon^{1/(1-\gamma)}\|u_t\|_{L^q(L^q(\Omega))}$$

with sufficiently small $\varepsilon>0$. This leads to the inequality

$$\begin{aligned}&\|u_t\|_{L^q(L^2_\sigma(\Omega)\cap L^q_\sigma(\Omega))}+\|u\|_{L^q(W^{2,2}(\Omega)\cap W^{2,q}(\Omega))}+\|\nabla p\|_{L^q(L^2(\Omega)\cap L^q(\Omega))}\\ &\qquad\leqslant C\|f\|_{L^q(L^2_\sigma(\Omega)\cap L^q_\sigma(\Omega))}\end{aligned}$$

for all $2\leqslant q<\infty$ with $C=C(T,q,\alpha,\beta,K)>0$, and completes the proof of (3.37) for $1<q<\infty$. In particular, this proves inequality (2.18) for the bounded domain Ω when $u_0=0$. To prove (2.18) with $u_0\in D(\tilde A_q)$ we solve the system $\tilde u_t+\tilde A_q\tilde u=\tilde f$, $\tilde u(0)=0$, with $\tilde f=f-\tilde A_q u_0$. Then $u(t)=\tilde u(t)+u_0$ yields the desired solution with $u_0\in D(\tilde A_q)$. This proves Theorem 2.5 for bounded Ω.

Step 2. *Ω unbounded.* Using the same arguments as in Step 2 of §3.3, let $f\in L^q(0,T;\tilde{L}^q_\sigma(\Omega))$, $1<q<\infty$, and consider the solution $u_j\in L^q(0,T;D(\tilde{A}_q))$ of the system

$$u_{j,t}+\tilde{A}_q u_j=f_j,\quad u_j(0)=0,$$

with $f_j=\widetilde{P}_q f|_{\Omega_j}$, $j\in\mathbf{N}$, following Step 1. Then (3.37) applied to the domains Ω_j yields the uniform estimate

$$\|u_{j,t}\|_{L^q(\tilde{L}^q_\sigma(\Omega_j))}+\|u_j\|_{L^q(\widetilde{W}^{2,q}(\Omega_j))}+\|\nabla p_j\|_{L^q(\tilde{L}^q(\Omega_j))}\leqslant C\|f\|_{L^q(\tilde{L}^q_\sigma(\Omega))} \tag{3.40}$$

with $\nabla p_j=(I-\widetilde{P}_q)\Delta u_j$ and $C=C(T,q,\alpha,\beta,K)>0$. Suppressing subsequences we obtain by the weak convergence argument the weak limits

$$u=\operatorname*{w-lim}_{j\to\infty}u_j\in L^q(0,T;\tilde{L}^q_\sigma(\Omega))\quad\text{and}\quad\nabla p=\operatorname*{w-lim}_{j\to\infty}\nabla p_j\in L^q(0,T;\tilde{L}^q(\Omega))$$

satisfying $u\in L^q(0,T;\widetilde{D}^q(\Omega))$, $u_t+\tilde{A}_q u=u_t-\Delta u+\nabla p=f$, $u(0)=0$, and the estimate

$$\|u_t\|_{L^q(\tilde{L}^q_\sigma(\Omega))}+\|u\|_{L^q(\widetilde{W}^{2,q}(\Omega))}+\|\nabla p\|_{L^q(\tilde{L}^q(\Omega))}\leqslant C\|f\|_{L^q(\tilde{L}^q_\sigma(\Omega))}, \tag{3.41}$$

with C as in (3.40), which is equivalent to inequality (2.18).

The uniqueness of u follows in the same way as in Step 2 of §3.3, and the case $u(0)=u_0\in D(\tilde{A}_q)$ is treated as above in Step 1. The other properties in Theorem 2.5 are obvious. This completes the proof.

3.5. Suitable weak solutions, strong energy inequality and Leray's structure result for general domains; Proof of Theorem 2.7

To construct a suitable weak solution u for the general uniform C^2-domain Ω of type α,β,K, we use approximate solutions u_k and the key estimate (2.18) in the formulation (2.21) with the exponent $q=\frac{5}{4}$; the reason for this exponent is the structure of the nonlinear term. Except for this estimate, all the other approximation arguments are well known in principle; here we follow the construction in [32, Chapter V]. However, it is easier first to consider a bounded domain Ω and then to treat the subdomains Ω_j with $j\to\infty$ as in the previous proofs. Furthermore, we may assume without loss of generality that $0<T<\infty$, and consequently that $T'=T$ in (2.25); if $T=\infty$ we consider a sequence $0<T_1<T_2<\ldots$ with $\lim_{j\to\infty}T_j=\infty$, and continue the construction of u step by step.

Moreover, we may assume that $u_0=0$ in the following proof. The case $u_0\neq 0$ can be reduced to this case in two steps: If $u_0\in D(\tilde{A}_q)$, we replace $u(t)$ by $\hat{u}(t)=u(t)-e^{-A_2 t}u_0$ in the linear part of the equation (2.23). Hence $\hat{u}(0)=0$, and the argument for $u_0=0$ yields

(2.25) with $\varepsilon=0$ and u replaced by $\hat{u}$. Since $u_0 \in D(\tilde{A}_q)$, we conclude that (2.25) holds for u with $\varepsilon=0$. If $u_0 \in L^2_\sigma$ only, we choose any $0<\varepsilon<T$, use that $e^{-A_2 t}u_0 = e^{-A_2(t-\varepsilon)}u_{0,\varepsilon}$ with $u_{0,\varepsilon} = e^{-A_2\varepsilon}u_0 \in D(A_2) \subset D(\tilde{A}_q)$, $q=\frac{5}{4}$, and conclude from the validity of (2.25) for $\hat{u}$ and $\varepsilon=0$, that (2.25) holds for u in the restricted interval (ε, T'). This information is sufficient to prove (2.26) and (2.27).

Thus we may assume that $u_0=0$ and $0<T'=T<\infty$, and we prove (2.25) with $\varepsilon=0$. Further let $f \in L^q(0,T;L^2(\Omega))$ and $q=\frac{5}{4}$.

Step 1. *Ω bounded.* Following [32, V.3.3], we use the Yosida operators

$$J_k = (I + k^{-1}A_2^{1/2})^{-1}, \quad k \in \mathbf{N},$$

and find solutions $u=u_k$ of the approximate Navier–Stokes system

$$u_t - \Delta u + (J_k u)\cdot\nabla u + \nabla p = f, \quad \operatorname{div} u = 0,\ u|_{\partial\Omega} = 0,\ u(0) = 0, \tag{3.42}$$

on $(0,T)$. Further, we recall the estimates

$$\begin{aligned} \tfrac{1}{2}\|u_k\|^2_{L^\infty(L^2_\sigma(\Omega))} + \|\nabla u_k\|^2_{L^2(L^2(\Omega))} &\leqslant C_0 \|f\|^2_{L^1(L^2(\Omega))}, \quad C_0>0, \\ \|u_k\|_{L^\gamma(L^\delta(\Omega))} &\leqslant C\|f\|_{L^1(L^2(\Omega))}, \end{aligned} \tag{3.43}$$

where $\delta \geqslant 2, \gamma \geqslant 2$, $2/\gamma + 3/\delta = 3/2$, $C=C(\gamma,\delta)>0$, and

$$\|J_k u_k \cdot \nabla u_k\|_{L^\gamma(L^\delta(\Omega))} \leqslant C\|f\|^2_{L^1(L^2(\Omega))},$$

where $1<\gamma,\delta<2$, $2/\gamma+3/\delta=4$, $C=C(\gamma,\delta)>0$; see [32, V.2.2, (2.2.5), and V.1.2] concerning these properties.

Moreover, due to (3.37),

$$\begin{aligned} &\|u_{k,t}\|_{L^q(L^q(\Omega))} + \|u_k\|_{L^q(W^{2,q}(\Omega))} + \|\nabla p_k\|_{L^q(L^q(\Omega))} \\ &\quad \leqslant C(\|f\|_{L^q(L^2(\Omega))} + \|f\|^2_{L^1(L^2(\Omega))}), \quad q=\tfrac{5}{4},\ C=C(T,\alpha,\beta,K)>0. \end{aligned} \tag{3.44}$$

Using these uniform boundedness properties we conclude letting $k\to\infty$ (and suppressing subsequences) that there exists a weak solution u of the system (2.23) with the following weak ("$\rightharpoonup$") and strong ("$\to$") convergence properties, respectively:

$$\begin{aligned} u_k &\rightharpoonup u && \text{in } L^2(0,T;W_0^{1,2}(\Omega)), \\ u_k &\to u && \text{in } L^2(0,T;L^2(\Omega)) \quad (\text{since } \Omega \text{ is bounded}), \\ \nabla u_k &\rightharpoonup \nabla u_k && \text{in } L^2(0,T;L^2(\Omega)), \\ u_k(t) &\to u(t) && \text{in } L^2_\sigma(\Omega) \text{ for a.a. } t\in[0,T), \end{aligned}$$

and $(u_{k,t}, u_k, \nabla u_k, \nabla^2 u_k, \nabla p_k) \rightharpoonup (u_t, u, \nabla u, \nabla^2 u, \nabla p)$ in $L^q(0,T;L^q(\Omega))$, where $q=\frac{5}{4}$. Moreover, Poincaré's inequality shows that

$$\|p_k - M_k\|_{L^q(L^r(\Omega))} \leqslant C\|\nabla p_k\|_{L^q(L^q(\Omega))}, \tag{3.45}$$

where $q=\frac{5}{4}$, $r=\frac{15}{7}$, $M_k=M_k(p_k)=(1/|\Omega|)\int_\Omega p_k\,dx$ and $C=C(T,\Omega)>0$.

Hence we conclude that the estimates (3.43) and (3.44) also hold with u_k and ∇p_k replaced by u and ∇p, and that

$$p_k - M_k \rightharpoonup \hat{p} \quad \text{in } L^q(0,T;L^r(\Omega))$$

for some $\hat{p}\in L^q(0,T;L^r(\Omega))$ satisfying $\nabla\hat{p}=\nabla p$. Choosing $M=M(t)$ such that $\hat{p}=p-M$, (3.45) holds with p_k-M_k and ∇p_k replaced by $p-M$ and ∇p.

Let $\phi\in C_0^\infty(\mathbf{R}^3)$. An elementary calculation yields for all $0\leqslant s\leqslant t\leqslant T$ the equality

$$\begin{aligned}\frac{1}{2}\|\phi u_k(t)\|_{L^2}^2 &+ \int_s^t \|\phi\nabla u_k\|_{L^2}^2\,d\tau \\ &= \frac{1}{2}\|\phi u_k(s)\|_{L^2}^2 + \int_s^t \langle \phi f, \phi u_k\rangle\,d\tau - \frac{1}{2}\int_s^t \langle \nabla |u_k|^2, \nabla\phi^2\rangle\,d\tau \\ &\quad + \int_s^t \langle \tfrac{1}{2}|u_k|^2, (J_k u_k)\cdot\nabla\phi^2\rangle\,d\tau + \int_s^t \langle p_k, u_k\cdot\nabla\phi^2\rangle\,d\tau.\end{aligned} \tag{3.46}$$

By the convergence properties above and writing the most problematic term in (3.46) in the form $\langle p_k, u_k\cdot\nabla\phi^2\rangle = \langle p_k - M_k, u_k\cdot\nabla\phi^2\rangle$, we may let k converge to infinity in each term, using Lebesgue's dominated convergence theorem. Because of the weak convergence property concerning ∇u_k, equality (3.46) yields (2.26) for a.a. $s\in[0,T)$ and all $t\in[s,T)$. Finally the strong energy inequality (2.27) is a consequence of (2.26) with $\phi\equiv 1$ on Ω. Recall that the restriction concerning ε in (2.25) is needed only for technical reasons if $0\neq u_0\in L^2_\sigma\backslash D(\tilde{A}_q)$.

Step 2. *Ω unbounded.* Consider the bounded subdomains $\Omega_j\subseteq\Omega$, $j\in\mathbf{N}$, as in (2.8), and let u_j be a weak solution in Ω_j according to Step 1 with associated pressure term ∇p_j, satisfying

$$u_{j,t} - \Delta u_j + u_j\cdot\nabla u_j + \nabla p_j = f_j, \quad \operatorname{div} u_j = 0,\ u_j(0)=0,\ u_j|_{\partial\Omega_j}=0, \tag{3.47}$$

where $f_j=f|_{\Omega_j}$. Applying the diagonal principle in the same way as in [32, Chapter V, (3.3.17)], we find a subsequence $\{\tilde{u}_j\}_{j=1}^\infty$ of the sequence $\{u_j\}_{j=1}^\infty$ and a weak solution u with pressure term ∇p of the system (2.23) with the following convergence properties as $j\to\infty$ (assuming for simplicity $\tilde{u}_j=u_j$):

(1) u_j converges to u weakly in $L^2(0,T;W^{1,2}(\Omega_{j_0}))$ and strongly in $L^2(0,T;L^2(\Omega_{j_0}))$ for each fixed j_0;

(2) ∇u_j converges to ∇u weakly in $L^2(0,T;L^2(\Omega_{j_0}))$;

(3) $u_j(t)$ converges to $u(t)$ strongly in $L^2(\Omega_{j_0})$ for a.a. $t\in[0,T)$.

Furthermore, uniformly in $j\in\mathbf{N}$,

$$\begin{aligned} \tfrac{1}{2}\|u_j\|^2_{L^\infty(L^2_\sigma(\Omega_j))}+\|\nabla u_j\|^2_{L^2(L^2(\Omega_j))} &\leqslant C_0\|f\|^2_{L^1(L^2(\Omega))}, \quad C_0>0,\\ \|u_j\|_{L^\gamma(L^\delta(\Omega_j))} &\leqslant C\|f\|_{L^1(L^2(\Omega))}, \end{aligned} \tag{3.48}$$

where $\gamma\geqslant 2$, $\delta\geqslant 2$, $2/\gamma+3/\delta=3/2$, $C=C(\gamma,\delta)>0$, and

$$\|u_j\cdot\nabla u_j\|_{L^\gamma(L^\delta(\Omega_j))}\leqslant C\|f\|^2_{L^1(L^2(\Omega))},$$

where $1<\gamma,\delta<2$, $2/\gamma+3/\delta=4$, $C=C(\gamma,\delta)>0$.

Using the maximal regularity estimate (2.18) in the form (2.21) combined with the last estimate we are led to the inequality

$$\begin{aligned} \|u_{j,t}\|_{L^q(L^2(\Omega_j)+L^q(\Omega_j))}+\|u_j\|_{L^q(W^{2,2}(\Omega_j)+W^{2,q}(\Omega_j))}+\|\nabla p_j\|_{L^q(L^2(\Omega_j)+L^q(\Omega_j))}&\\ \leqslant C(\|f\|_{L^q(L^2(\Omega))}+\|f\|^2_{L^1(L^2(\Omega))})& \end{aligned} \tag{3.49}$$

with $q=\frac{5}{4}$ and $C=C(T,\alpha,\beta,K)>0$ not depending on $j\in\mathbf{N}$. Thus we may conclude without loss of generality, see the previous proofs, that

$$(u_{j,t},u_j,\nabla u_j,\nabla^2 u_j,\nabla p_j)\rightharpoonup(u_t,u,\nabla u,\nabla^2 u,\nabla p)\quad\text{in } L^q(0,T,L^2(\Omega)+L^q(\Omega))$$

as $j\to\infty$, and that (3.49) holds with u_j and Ω_j replaced by u and Ω. This proves (2.25) for $u_0=0$.

To prove the local energy inequality (2.26) choose j_0 such that $\Omega\cap\operatorname{supp}\phi\subseteq\Omega_{j_0}$, use (2.26) from Step 1 for Ω_j and u_j, $j\geqslant j_0$, and let $j\to\infty$ using the convergence properties above. This proves (2.26) for u and Ω.

To prove (2.27) we choose a sequence $\phi_j\in C_0^\infty(\mathbf{R}^3)$, $j\in\mathbf{N}$, satisfying $0\leqslant\phi_j\leqslant 1$ and $|\nabla\phi_j^2|\leqslant C_0$ with some constant C_0, and with $\lim_{j\to\infty}\phi_j(x)=1$ and $\lim_{j\to\infty}\nabla\phi_j^2(x)=0$ for all $x\in\mathbf{R}^3$. Setting $\phi=\phi_j$ in (2.26) we obtain the desired inequality (2.27) by letting $j\to\infty$.

Now the proof of Theorem 2.7 is complete.

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