An M-theory flop as a large N duality

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An M-theory Flop as a Large N Duality

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We show how a recently proposed large N duality in the context of type IIA strings with $\mathcal{N}=1$ supersymmetry in 4 dimensions can be derived from purely geometric considerations by embedding type IIA strings in M-theory. The phase structure of M-theory on G_2 holonomy manifolds and an S^3 flop are the key ingredients in this derivation.

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1. Introduction

String propagation in the presence of branes has been studied from many different angles, and has been the intersection point of many fruitful ideas in the context of dualities. A beautiful example of it is in the context of the AdS/CFT correspondence [1]. This idea is a refinement of the statement that if we consider strings in the presence of branes, in certain regime of parameters the system is better described by strings propagating on the gravitational back reaction of light string modes to the presence of branes. This is an example of duality in the sense that two different theories are continuously connected by a change in parameters. On the one end one has a geometry involving branes and on the other extreme the geometry has been deformed and the branes have disappeared.

A recent example of a large N duality in type IIA superstring, was proposed in [2] based on embedding the large N Chern-Simons/topological gravity duality of [3] in type IIA superstrings. The duality states that if we consider N D6 branes wrapped on an S^3 in the deformed conifold geometry[1] T^*S^3, then the same system is equivalent to a type IIA geometry where the D-branes have disappeared but where an S^2 has blown up so that the CY geometry is $O(-1)+O(-1)$ bundle over $\mathbf{P}^1$. In other words, the topology is that of the so called "small resolution" of the conifold, where the S^2 has finite size. Moreover there are N units of 2-form field strength flux through $\mathbf{P}^1$, and the complexified Kahler parameter t of $\mathbf{P}^1$ is related to the volume V of the S^3 and the string coupling constant g_s by

$$(e^t-1)^N = a\ exp(-V/g_s) \tag{1.1}$$

Note that for large V and when $Ng^2_{YM}=\frac{Ng_s}{V}\ll 1$, the wrapped D-brane description is good and the blown up description is bad as $t\rightarrow 0$, and when $t\gg 0$ where the blown up $\mathbf{P}^1$ description is good the $V\rightarrow -\infty$ and the original wrapped D-brane description is bad. In this sense we only really have at most one good description in each regime of parameters and the parameters being related by (1.1). This situation is similar to other cases one encounters in the context of large N string dualities. For example with N D3 branes in R^{10} if $Ng^2_{YM}=Ng_s\ll 1$ the D-brane description ignoring the gravitational backreaction is fine; when $Ng_s\gg 1$ the gravitationally deformed background description without the D-brane is the right description.

[1] Mathematicians use the terminology "quadric" for the deformed conifold and "quadric cone" for a singular quadric. The small resolution of conifold is called the small resolution (blow up) of the quadric cone.

The main aim of this paper is to embed this type IIA duality in M-theory. We find that the statement of this duality in the context of M-theory translates to a simple geometric duality. Turning it around, we can derive the type IIA string duality of [2] from its relation with M-theory.

The organization of this paper is as follows. In section 2 we review a perturbative string theory duality which is a good exercise for the M-theory duality of interest. In section 3 we discuss M-theory in a 7 dimensional background with G_2 holonomy and a simple geometric duality. In section 4 we reinterpret the M-theory duality in the context of type IIA strings and obtain the duality of [2]. In the appendix we discuss some aspects of the G_2 holonomy metric.

The derivation of the large N duality in this case reinforces the philosophy advocated in [3] and [2] that large N dualities in general correspond to transitions in geometry. It would be interesting to try to understand other large N dualities in the same spirit.

While we were completing this paper we received [4] which has some overlap with this paper.

2. A String theory duality

Consider Type IIA superstrings propagating on a non-compact CY background given by $O(-1)+O(-1)$ bundle over $\mathbf{P}^1$. A powerful worldsheet description of this sigma model is in terms of linear sigma model [5] where one considers a $(2,2)$ supersymmetric $U(1)$ gauge theory with four fields $(\Phi_1,\Phi_2,\Phi_3,\Phi_4)$ with charges (+1,+1,-1,-1), with an FI term given by r and a $U(1)$ θ angle. The low energy vacuum of this theory is characterized by

$$\mathcal{V}: \qquad |\Phi_1|^2+|\Phi_2|^2-|\Phi_3|^2-|\Phi_4|^2=r \tag{2.1}$$

The actual vacuum of this theory is given by the gauge inequivalent solutions to (2.1), i.e, one considers $\mathcal{V}/U(1)$. This can be naturally identified with $O(-1)+O(-1)$ bundle over $\mathbf{P}^1$. If we take $r>0$ the $\mathbf{P}^1$ is identified as the locus $\Phi_3=\Phi_4=0$, and the normal directions are identified with Φ_3 and Φ_4. This space is also called the "small resolution" of the conifold, where the S^2 has finite size.

An important aspect of this theory is its phase structure and in particular what happens as $r\to 0$. In fact the natural phase structure for this theory is parameterized by the *complex* parameter

$$t=r+i\theta.$$

It turns out that both positive and negative r make sense and in fact are smoothly connected if we vary the θ because the only singularity in the moduli space of this theory is at the origin $t = 0$. From the equation (2.1) one may naively have thought that there should be some singularity at $r = 0$ for all θ, but this is lifted by worldsheet instanton corrections. Of course there is a simple reason why this had to happen. The structure of $(2,2)$ supersymmetry leads to a naturally complex moduli space and it does not allow any real locus singularity in moduli space.

2.1. A perturbative string duality

From the above description we notice a symmetry: If we replace $t \to -t$ we obtain the same geometry with the role of the $(\Phi_1, \Phi_2) \leftrightarrow (\Phi_3, \Phi_4)$ exchanged. Geometrically this is called a flop. Even though in the geometric setup there is a discontinuity at $r = 0$, and one just considers either $r > 0$ or $r < 0$, the situation in string theory is different because of the θ angle. Both regions are smoothly connected. In particular if we start with $r \gg 0$ and do some computations as a function of t, then analytic continuation of these quantities by $t \to -t$ should yield the answers for the other side. This in fact was directly checked at the level of instanton computations on both sides [5][6]. In particular the worldsheet instantons at genus 0 (with three points fixed on S^2) have a partition function

$$\partial_t^3 F_0 = C + \frac{q}{1-q}$$

where $q = e^{-t}$ (the constant term C is somewhat ambiguous and is related to the classical triple intersection of the 4-cycle dual to $\mathbf{P^1}$). This makes sense as an instanton expansion when $r \gg 0$. However if we analytically continue this quantity to $t \to -t$ we obtain in terms of $\tilde{q} = 1/q$

$$-\partial_t^3 F_0 = 1 - C + \frac{\tilde{q}}{1-\tilde{q}}$$

which is the statement of the symmetry under $t \to -t$. The shift in the constant term is reflecting the geometric fact that under flop the classical triple intersection shifts by one.

Suppose however we consider a modified theory where we mod out the Φ_i's by some discrete group G which does not necessarily act symmetrically under the exchange $(\Phi_1, \Phi_2) \leftrightarrow (\Phi_3, \Phi_4)$. Let us call the resulting theory $Q_G[t]$, exhibiting explicitly the dependence of the theory Q on the choice of the group G as well as the (complexified) size of $\mathbf{P}^1$ given by t. Now we ask what happens when we consider $t \to -t$. In this case in general we do not come back to the same theory because of the asymmetric action of G. We obtain

$$Q_G[-t] = Q_{G'}[t] \tag{2.2}$$

where G' is related to G by conjugation with the element exchanging the pairs

$$U: \qquad (\Phi_1, \Phi_2) \leftrightarrow (\Phi_3, \Phi_4)$$

$$G' = UGU^{-1}$$

To better appreciate the content of this duality statement let us consider a simple example. Consider the case where G is generated by the element

$$(\Phi_1, \Phi_2, \Phi_3, \Phi_4) \to (\omega\Phi_1, \omega^{-1}\Phi_2, \Phi_3, \Phi_4)$$

where ω is an n-th root of unity (this can also be viewed as introducing an additional Z_n gauge group). Geometrically this corresponds to considering the following action on $O(-1) + O(-1)$ over $\mathbf{P}^1$:

$$(\zeta_1, \zeta_2, z) \to (\omega\zeta_1, \omega\zeta_2, \omega^{-2}z)$$

where z denotes the coordinate of $\mathbf{P}^1$ (say near the north pole) and ζ_i denote the coordinates of the bundle $O(-1)+O(-1)$ over it. Note in particular that modding by this group leads to two orbifold singularities, namely the north and south poles at the origin of the ζ_i. The group G' on the other hand is obtained by conjugating G with U and is generated by

$$(\zeta_1, \zeta_2, z) \to (\omega\zeta_1, \omega^{-1}\zeta_2, z)$$

This is a very different group action from G and in particular it leads to an A_{n-1} singularity over $\mathbf{P}^1$. Thus changing $t \to -t$ has led to a totally new but nevertheless "dual" theory. Note that this duality is a perturbative string duality (i.e. can be understood genus by genus in worldsheet expansion).

3. An M-theory duality

We will be interested in compactifications of M-theory on G_2 holonomy manifolds (see [7] for construction of some compact 7-manifolds with G_2 holonomy). This leads to $\mathcal{N} = 1$ supersymmetry in $d = 4$. In the compact case[2], the number of moduli of the Ricci-flat metric is given by the dimension of the third homology of the manifold, i.e. b_3. The point on the moduli space of Ricci-flat metrics can be characterized by the volume of some basis for 3-cycles. Physically we know that these should correspond to lowest components of chiral fields in $\mathcal{N} = 1$ supersymmetry multiplets. Thus one again expects a complexification of the volumes. In fact this happens because there is a 3-form gauge field C in M-theory and its vev about each 3-cycle leads to the complexification of the volume elements. Moreover the moduli space of M-theory compactifications are given by analytic expressions and thus singularities occur in *complex* codimension 1 and higher. Thus there are in particular no boundary walls in moduli space of M-theory.

Let us now come to the concrete example that we will be interested in. Consider the non-compact 7-manifold given by the spin bundle over S^3. This has the topology of $R^4 \times S^3$ and admits a G_2 holonomy metric. We will present the metric later in this section. The topology of the manifold can be viewed as

$$(u_1^2 + u_2^2 + u_3^2 + u_4^2) - (v_1^2 + v_2^2 + v_3^2 + v_4^2) = V$$

where u_i, v_i are real parameters. For $V > 0$ the S^3 is identified as the locus $v_i = 0$, and v_i correspond to the R^4 normal directions over S^3. Note that if we consider $V < 0$ then the role of the u's and v's have been interchanged and another S^3 blows up, corresponding to $u_i = 0$. We can also write this in complex notation by

$$(|z_1|^2 + |z_2|^2) - (|z_3|^2 + |z_4|^2) = V$$

where z_i are complex variables. It should be emphasized that the G_2 holonomy manifold does not admit a complex structure (it is odd dimensional) and so there is no intrinsic meaning to writing the equation in terms of complex coordinates, other than for simpler book keeping. We can also use a quaternionic notation and write it as

$$|q_1|^2 - |q_2|^2 = V.$$

[2] In the non-compact case there could be additional deformations which change the asymptotic behaviour of the metric at infinity.

We can view the quaternion as $q_1 = \sum_i u_i\sigma_i$ and $q_2 = \sum_i v_i\sigma_i$, where σ_i denotes the realization of quaternionic generators as 2×2 matrices. This way of writing it suggests that one can have an $SU(2)^{1,2}_{L,R}$ symmetries, which act on each quaternion, as left and right multiplication by $SU(2)_{L,R}$. This is the same, as the $Spin(4)$ action on the u's or on the v's.

A G_2 holonomy metric can be defined on this manifold [8][9]. It is given as

$$ds^2 = \alpha^2 dr^2 + \gamma^2(\tilde{w}^a)^2 + \beta^2(w^a - \frac{1}{2}\tilde{w}^a)^2 \tag{3.1}$$

with

$$\alpha^{-2} = 1 - \frac{a^3}{r^3}\ , \qquad \beta^2 = \frac{r^2}{9}(1-\frac{a^3}{r^3})\ , \qquad \gamma^2 = r^2/12 \tag{3.2}$$

where $\tilde{w}^a$ and w^a are the left invariant one forms on $\tilde{S}^3$ and S^3 respectively. The two S^3's are associated to each of the two quaternions at fixed norm, and the r variable fills one of the S^3's depending on the sign of V. In the form shown above $r \geq a$ and it fills the S^3 associated with the left invariant one forms w^a, while $\tilde{w}^a$ are associated with $\tilde{S}^3$ which is topologically non-trivial. Note that the volume of the $\tilde{S}^3$ is proportional to a^3. This G_2 holonomy metric has an $[SU(2)]^3$ isometry group

$$SU(2)^1_L \times SU(2)^2_L \times [SU(2)^1_R \times SU(2)^2_R]_D$$

where $SU(2)^i_{L,R}$ denotes the L,R multiplication of q_i by the $SU(2)$ group and D denotes the diagonal subgroup. This is almost obvious from the above presentation of the metric, where $SU(2)^i_L$ do not act on the left-invariant forms and so do not modify the metric. The left-invariant 1-forms transform in the adjoint representation of the corresponding $SU(2)_R$. The diagonal combination of $SU(2)^i_R$ leaves the metric invariant (the last term in the metric is what requires choosing the diagonal $SU(2)_R$ as the symmetry group). To fix notation let us associate $SU(2)^1$ with the $\tilde{S}^3$ sphere and $SU(2)^2$ with the contractible S^3. The space (3.1) is asymptotic to a cone whose base has $\tilde{S}^3 \times S^3$ topology

$$ds^2 \sim dr^2 + \frac{r^2}{9}[(\tilde{w}^a)^2 + (w^a)^2 - \tilde{w}^a w^a]$$

As we will note later, when we discuss the connection to type IIA string theory we expect a one parameter family of G_2 holonomy metric deformations which breaks the $SU(2)^i_L$ to $U(1)$, for either $i=1$ or $i=2$.

As noted above, in the context of M-theory propagating in this background the moduli space of the theory is parameterized in addition to the volume V of the S^3 by the flux of the C-field through it. Let us denote this complex combination by $V_M = V + iC$. Just as in the case of string propagation on $O(-1)+O(-1)$ bundle over $\mathbf{P}^1$, the phase structure of M-theory as a function of V_M is expected to have a singularity at most only at the origin and that turning on the C-field should smooth out the singularity where $V = 0$ in moduli space. This follows from the number of supersymmetries we are preserving in 4 dimensions and the fact that V_M is the lowest component of a chiral field. A similar situation (with twice as many supersymmetries) in the context of M-theory and type IIA strings on Calabi-Yau threefolds containing an S^3 was analyzed in [10] where it was shown that Euclidean M2 brane instantons modify the moduli space and in fact remove the singularity at $V = 0$. Here also we expect the same to be true, though we do not know how to rigorously argue this. At any rate we can argue, based on supersymmetry alone, that a singularity at most will happen at isolated points in moduli space, and for the theory at hand this means potentially only at $V_M = 0$ (in particular $V = 0$ and $C \neq 0$ is not a singular point). This observation implies that we can continuously go from regions where the real part of V_M is large and positive to regions where it is large and negative without encountering any singularity.[3] Moreover it is clear that $V_M \to -V_M$ is a flop, and otherwise gives rise to an equivalent "dual" M-theory background. Let us denote M-theory in the presence of this background by $Q[V_M]$. Then we have

$$Q[V_M] = Q[-V_M]$$

Notice that for positive V_M, S^3 is contractible in the full geometry and $\tilde{S}^3$ is topologically nontrivial while the opposite is true of negative V_M.

Parallel to our discussion of the string duality in the previous section, we can consider modding out by some group actions which are isometries of the G_2 holonomy manifold. As discussed in the appendix this leads to manifolds (possibly with singularities) which continue to have G_2 holonomy. In this way we obtain a statement of duality where

$$Q_G[-V_M] = Q_{G'}[V_M]$$

[3] For a discussion of topology change in the context of G_2 holonomy manifolds see the recent work [11].

where $G = UG'U^{-1}$ and U is the Z_2 outer automorphism exchanging the u's and the v's, and acts on the $SU(2)$'s as

$$U[SU(2)^{1,2}_{L,R}]U^{-1} = SU(2)^{2,1}_{L,R}$$

The special case we will be interested in is when G is generated by a Z_N subgroup of $SU(2)^2_L$. In complex coordinates we can view this transformation as

$$g: \qquad (z_1, z_2, z_3, z_4) \to (z_1, z_2, \omega z_3, \omega z_4)$$

Then G' is generated by

$$g': \qquad (z_1, z_2, z_3, z_4) \to (\omega z_1, \omega z_2, z_3, z_4)$$

and we have an M-theory duality

$$Q_G[-V_M] = Q_{G'}[V_M] \tag{3.3}$$

Let us consider $Q_G[V_M]$ when $V \gg 0$. In this case the element g acts with fixed point: The $\tilde{S}^3$ defined by $z_3 = z_4 = 0$. Moreover the singularity is of the type of A_{N-1} singularity as the normal direction is R^4/Z_N in the usual action of Z_N on R^4. As is well known, this singularity in M-theory gives rise to an $SU(N)$ gauge symmetry on the singular locus. In the present case, taking the number of supersymmetries into account, we have an $\mathcal{N} = 1$ supersymmetric $SU(N)$ Yang-Mills gauge theory living on $\tilde{S}^3$ times the Minkowski space. On the other hand when we consider $Q_{G'}[V_M]$ for $V \gg 0$, the g' corresponds to the Z_N action with no fixed points (the would be fixed point locus $z_1 = z_2 = 0$ is not on the manifold for $V \gg 0$).

3.1. Gauge theoretic interpretation of the duality

From (3.3) we see that M-theory in one background is continuously connected with another one. In particular $Q_G[V_M]$ when $V \gg 0$ contains as light excitations $\mathcal{N} = 1$ Yang-Mills sector in 4 dimensions with the Yang-Mills coupling given by

$$\frac{1}{g^2_{YM}} + i\theta = V_M$$

where g^2_{YM} should be viewed as the gauge coupling at the Planck scale. As we know the coupling will start to run. More precisely, the effective coupling constant depends on the scale we probe. Through the above relation we see that V_M itself should run and its value will depend on which scale we measure it at. In particular $V_M(\mu)$ should decrease logarithmically at infrared as

$$V_M(\mu) = V_M + \text{const.}\log\frac{\mu}{M_{pl}}$$

This running should be induced by quantum effects in the presence of the Z_N singularity in measuring the volume of S^3, which is at the singular locus. For small μ, we expect V_M to become small. In fact if we trust the above formula we seem to get a *negative* volume V_M. Even though this is not allowed in the usual gauge theory (negative $1/g_{YM}^2$ naively does not make sense), here we can make sense of negative V_M as a flop. In fact we are thus led to view the infrared behavior of the same theory at negative and large values of $V \ll 0$. However this theory for negative V is best viewed as the dual theory $Q_G[-V_M] = Q_{G'}[V_M]$, in terms of which there is no singularity in geometry and we obtain an $\mathcal{N} = 1$ theory in four dimensions with no sign of $SU(N)$ gauge symmetry. This is exactly what one expects for a confining gauge theory. Moreover we should see N vacua. This is also present here; the G' group corresponds to modding out the S^3 by a Z_N. So the volume of the final S^3 is smaller by a factor of $V \to V/N$. However we also have to decide about the choice of the theta angle. If we change the theta angle by $2\pi k$ on S^3, which in the original $Q_G[V_M]$ corresponds to not changing the theory at all, as we go to negative V_M, it does give rise to a change. Namely quotienting the S^3 by a Z_N gives rise to a fractional change in the C-flux of the quotient theory by $2\pi k/N$. Thus we obtain N choices for the phase of the theory in the infrared. These are the N vacua of $\mathcal{N} = 1$ supersymmetric $SU(N)$ Yang-Mills, and we have thus found a purely geometric interpretation of them. One can also identify the domain wall of the $\mathcal{N} = 1$ system with the $M5$ brane wrapped over S^3/Z_N.

4. Re-interpretation of the M-theory duality in Type IIA string

We now view the same geometry from the type IIA perspective. In order to do this we need to choose the "11-th" circle. There are many ways to do this. In order to connect this to the duality of [2] we identify the 11-th direction with the fibers of the $U(1)$ sitting in $SU(2)_L^2$, where the Z_N that we modded out in the previous section, is a subgroup of it. In other words $Z_N \subset U(1) \subset SU(2)_L^2$. We start with $Q_G[V_M]$ with $V \gg 0$. Kaluza Klein reducing along this circle produces an R^3 fibration over $\tilde{S}^3$ with a singularity at the origin. The singularity at the origin has the interpretation of a $D6$ brane, before modding out by Z_N, or N units of $D6$ branes after modding out. Thus we expect this to correspond to type IIA string theory on the conifold background T^*S^3 with N D6 branes wrapped around S^3. A more precise statement is the following. Suppose we start with the deformed conifold before we put any branes on it. In M-theory we add an extra circle of constant radius so that we have a seven dimensional geometry which is the deformed conifold times a circle. Now we add N D6 branes on $\tilde{S}^3$. At large distances from the D6 branes, the presence of the branes is signaled in IIA theory by the presence of a two form field strength on the surrounding S^2. When we lift this up to M-theory this means that the eleventh circle S^1 is non-trivially fibered over the S^2. In fact the total topology of this S^1 fibration over S^2 ends up being that of S^3/Z_N. (Notice that for $N = 1$ we just have S^3). For very large r we expect that the dilaton will be constant, so that the size of the S^1 fiber is constant, while the size of $S^2 \times \tilde{S}^3$ should grow. For small r, on the other hand, we enter the near horizon region of the six branes. The dilaton decreases as we approach the "core" of a six brane. In fact, the near horizon region of N six branes in flat space lifts up in M-theory to an A_{N-1} singularity and the M-theory circle is just one of the angles on the three sphere. So when we wrap this on $\tilde{S}^3$ we expect a geometry which is that of the region $r \sim a$ of (3.1). The asymptotics of (3.1) is not what we expect in the IIA situation since the radius of the M-theory circle continues to grow as $r \to \infty$. In fact the geometry (3.1) looks more like the infinite coupling limit of the IIA geometry, where we take the limit in such a way to keep V_M finite. In principle we expect to find a gravity solution in M-theory that describes more precisely the situation we expect in IIA theory for finite string coupling constant. That should be a deformation of the above G_2 holonomy metric where the $SU(2)_L^2$ symmetry of (3.1) is broken to $U(1)_L$ so that the circle can have constant asymptotic size as $r \to \infty$. This deformation should exist for both signs of V_M which means, in terms of (3.1), that we should also find a second deformation where $SU(2)_L^1$ is broken to $U(1)$. This would describe the situation after the transition where, in IIA theory, we have the small resolution of the conifold with N units of F_2 flux. Again, we expect a solution where the string coupling asymptotes to a constant. Assuming this deformation of the metric exists, it is natural to expect that under the flop $V_M \to -V_M$ we get from one kind of deformation to the other. In other words the considerations of section 3 should also apply to this deformed metric. With this assumption we now rederive the large N type IIA duality, including the identification of parameters on both sides in the geometric regime.

Let us denote the volume of the $\tilde{S}^3$ in the type IIA setup by V_A, and consider N D6 branes wrapped over it. Then from the map between M-theory parameters and type IIA parameters one deduces that

$$V_M = V_A / g_s \tag{4.1}$$

where g_s is the type IIA coupling constant. Now let us consider the limit where $V_M \ll 0$. In this case the theory is better described by another M-theory background with group modding out by G', and where the volume of the S^3 is $-V_M$. Again we use the same 11-th direction for the circle fibration, which means that we choose G' to be a subgroup of the corresponding $U(1)$. Now the fibration we get gives a geometry which has an S^2 and the M-theory S^3 over it is a quotient of Hopf fibration by Z_N. This means, in the type IIA terminology that we have N units of RR 2-form field strength through S^2. Moreover the volume of the (minimal) S^2 is given, changing the parameters from M-theory to type IIA, as

$$t = -V_M / N \tag{4.2}$$

This relation is trustable for large t where the supergravity description of M-theory would be adequate. Thus we see that we have a duality which in the type IIA description corresponds to N units of $D6$ branes wrapping the S^3 of volume V_A and on the other side an S^2 with N units of RR flux through it, with Kahler class t and with the relation (combining (4.1) and (4.2))

$$t = \frac{-V_A}{N g_s} \tag{4.3}$$

This agrees, in the limit of large t with the result obtained in [2]:

$$(e^t - 1)^N = a \; exp(-V_A / g_s) \tag{4.4}$$

The modified relation (4.4) includes the contribution of worldsheet instantons, which is neglected in the identification at the large volume limit given in (4.3). Note that in [2] the parameter t was identified as the lowest component of the gaugino chiral field $t = g_s Tr W^2 = S$. Note that in the limit of large V, which corresponds to small t if we include the instanton correction, we deduce a gaugino condensation exactly as one would expect for the $\mathcal{N} = 1$ Yang-Mills theory, namely in the form $S^N = exp(-1/g_{YM}^2)$ (where the g_{YM} is the Yang-Mills coupling constant at the string scale).

From the relation to M-theory it is natural that the worldsheet instantons know about gaugino condensation. This is similar to the worldsheet instantons for the flopped geometry in the $O(-1)+O(-1)$ over $\mathbf{P}^1$. Thus the worldsheet instantons of the $O(-1)+O(-1)$ theory over $\mathbf{P}^1$ with N units of flux, which in M-theory correspond to superpotential corrections due to Euclidean $M2$ brane instantons, know, by analytic continuation, about the Euclidean $M2$ brane instantons of the flopped geometry, which correspond to the usual gauge theory instantons.

Note that regardless of whether we find the new deformed G_2 holonomy solutions or not, we do not expect to find a geometry that truly decouples from the bulk and which can be interpreted as a decoupled field theory. There is a limit where we expect a decoupled field theory, the large V_M limit, and the region very close to the singularity, but it is not a limit where we expect a weakly coupled geometrical description. This corresponds as we saw above to $t \sim 0$. Though a weakly coupled *string* desciption is expected for large N. The field theory we have been considering, therefore, has more degrees of freedom than pure $\mathcal{N}=1$ Yang-Mills. In particular, it has the parameter V_M which is not a parameter in $\mathcal{N}=1$ SYM.

It is important to remark that when we talk about the resolved conifold of IIA theory with some units of flux, we are characterizing the space in terms of its topology, but we do not have a complex manifold. So in what sense is the consideration of topological strings as used in [2] relevant? The answer turns out to be that if we consider the BPS charge measured by the $M2$ brane, when we integrate over the 11-th circle, it leads to a BPS charge seen by the fundamental string, which corresponds to a symplectic form on the quotient geometry, which topologically is $R^4 \times S^2$. The symplectic structure induced from this reduction agrees with the symplectic structure of $O(-1)+O(-1)$ bundle over $\mathbf{P}^1$, as is shown in the appendix (the corresponding symplectic two form k is obtained by integrating the G2 invariant three form Ω_3 over the 11th circle, $k=\int_{S^1}\Omega_3$). This implies that we expect to obtain the same results in consideration of topological strings for this reduction of the 7-manifold.

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Appendix A. Aspects of G_2 Holonomy Metric

A G_2 holonomy manifold is a seven dimensional manifold whose holonomy group is the simple group G_2. It can be proven that in such manifolds there is a special harmonic three form Ω, $d\Omega = d * \Omega = 0$, which is such that it locally determines the reduction of the holonomy group from Spin(7) to G_2. More precisely, at each point the subgroup of GL(7) that leaves Ω invariant is G_2.

It was shown in [9][8] that the following metric (on $R^4 \times S^3$) has G_2 holonomy

$$ds^2 = \alpha^2 dr^2 + \gamma^2 (\tilde{w}^a)^2 + \beta^2 (w^a - \frac{1}{2}\tilde{w}^a)^2 \tag{A.1}$$

with

$$\alpha^{-2} = 1 - \frac{a^3}{r^3} \ , \qquad \beta^2 = \frac{r^2}{9}(1 - \frac{a^3}{r^3}) \ , \qquad \gamma^2 = r^2/12 \tag{A.2}$$

where $\tilde{w}^a$ and w^a are left invariant one forms on two three spheres, which we denotes as $\tilde{S}^3$ and S^3. We think of S^3 as the SU(2) group manifold. We can use the following formulas:

$$\begin{aligned}
g =& e^{i\psi/2\sigma^3} e^{i\theta/2\sigma^1} e^{i\phi/2\sigma^3} \\
\frac{i}{2} w_R^a \sigma^a =& dgg^{-1} \qquad\qquad \frac{i}{2} w_L^a \sigma^a = g^{-1}dg \\
(w_R^1 + i w_R^2) =& e^{-i\psi}(d\theta + i \sin\theta d\phi) \qquad\qquad w_R^3 = d\psi + \cos\theta d\phi \\
(w_L^1 + i w_L^2) =& e^{+i\phi}(d\theta - i \sin\theta d\psi) \qquad\qquad w_L^3 = d\phi + \cos\theta d\psi \\
dw_R^a =& -\frac{1}{2}\epsilon^{abc} w_R^b w_R^c \\
dw_L^a =& \frac{1}{2}\epsilon^{abc} w_L^b w_L^c
\end{aligned} \tag{A.3}$$

We can see from these definitions that the forms w_L^a are invariant under left multiplications of g, $g \to h_L g$ while they transform in the adjoint representation of $SU(2)$ under right multiplication $g \to g h_R$. We can write the metric of the unit three sphere as

$$ds^2 = \frac{1}{4}\sum_a (w_L^a)^2 = \frac{1}{4}\sum_a (w_R^a)^2 \tag{A.4}$$

We can easily check that the metric (A.1) has $SU(2)^3$ isometry. Two $SU(2)$s arise from left multiplication in each of the S^3s while the third $SU(2)$ arises from right multiplication on both three spheres by the same group element. This last fact we can check by noticing that the index a, which transforms in the adjoint of $SU(2)$ is contracted in an $SU(2)$ invariant fashion.

Finally we can write the explicit form of the three form

$$\begin{aligned}\Omega =&\frac{a^3}{12}\frac{1}{6}\epsilon_{abc}\tilde{w}^a\tilde{w}^b\tilde{w}^c - \frac{1}{18}(r^3-a^3)\epsilon_{abc}(\tilde{w}^a w^b w^c - \tilde{w}^a\tilde{w}^b w^c) + \frac{r^2}{3}dr\tilde{w}^a w^a \\ =&\frac{a^3}{12}\frac{1}{6}\epsilon_{abc}\tilde{w}^a\tilde{w}^b\tilde{w}^c + d(\frac{r^3-a^3}{9}\tilde{w}^a w^a)\end{aligned} \tag{A.5}$$

From this expression we see that the $SU(2)^3$ isometry group of the metric also leaves the three form invariant. In other words, these symmetries leave the G_2 structure invariant.

The metric (A.1) is asymptotic at large r to

$$ds^2 \sim dr^2 + r^2\frac{1}{9}[(\tilde{w}^a)^2 + (w^a)^2 - w^a\tilde{w}^a] \tag{A.6}$$

The manifold on the right hand side of (A.6) is a cone whose base is topologically $\tilde{S}^3 \times S^3$. This cone has a singularity at $r = 0$. This singularity is eliminated in (A.1) by giving a finite volume to one of the three spheres, $\tilde{S}^3$. We could similarly consider a situation where we give a finite volume to the other sphere S^3. In that case the manifold we obtain is just given by (A.1) with $w^a \leftrightarrow \tilde{w}^a$. These two manifolds are related by a flop. We see from the explicit expression of Ω that we can continuously go from one to the other, passing through a singular manifold at the point where both spheres have zero volume.

If we qoutient this manifold by any subgroup of the isometry group described above we will obtain again a G_2 manifold, since the isometry group leaves the three form invariant. We will consider two different quotients.

A.1. Singular quotient

In this quotient we mod out by $Z_N \subset U(1) \subset SU(2)^2_L$ which acts on the coordinates of S^3. After modding out the metric (A.1) by Z_N as above we get a singular space. We get an A_{N-1} singularity wrapped over $\tilde{S}^3$. If we KK reduce over the circle associated to this $U(1)$ and we go to type IIA theory the singularity looks like the singularity we have in the near horizon region of N sixbranes wrapped on $\tilde{S}^3$. The normal bundle of this S^3 in IIA theory is the same as what we have when we wrap branes on the S^3 of a deformed conifold. In more mathematical notation, it is $T^*\tilde{S}^3$. Notice that in this IIA description, there is a singularity at the position of the branes even for $\mathcal{N} = 1$[4]. As usual, in the near

[4] Mathematically the point is that when we quotient $R^4 = C^2$ by the circle (acting as complex scalars) the resulting space can be naturally identifed with R^3 topologically but not differentiably - the identification being singular at the origin. This singularity in type IIA is interpreted as a D6 brane.

horizon region of six branes in IIA theory, the dilaton is varying and it is approaching zero at the core of the sixbranes [12]. The string metric in the directions along the six brane is also shrinking as we approach the core, but it does so in such a way that $V_M = V_A/g_s$ is constant and equal to the volume of $\tilde{S}^3$ in M-theory. The above remarks about the near horizon region of a six brane apply close to the singularity at $r = a$, for the region $r - a \ll a$. For large N there is a large region where the IIA description is valid [12]. As we go futher away from the singularity the geometry becomes more and more strongly coupled and the 11 dimensional description becomes better. For large r the dilaton goes to infinity. In principle there should be another solution where the dilaton goes to a constant.

A.2. Non-singular quotient

If we choose $Z_N \subset U(1) \subset SU(2)^1$. This group acts as left multiplication on $\tilde{S}^3$. Since the volume of this three sphere is nowhere vanishing we conclude that the quotient is non-singular. We can further Kaluza Klein reduce the metric along this circle. This produces a non-singular IIA metric on a space which has the topology of the small resolution of the conifold and with N units of two form flux on S^2. If N is very large, then this IIA geometry is weakly coupled at the origin. When we start moving out in the radial direction the string coupling starts increasing and it becomes infinite asymptotically. In principle there should be another solution where the string coupling does not diverge. If we integrate the three form (A.5) on this S^1 we find the symplectic form on the small resolution of the conifold. It should be noted that, though we get the same two form as the Kahler form of the local CY, the IIA geometry with the flux is not Kahler.

Let us see this more explicitly. The circle in question is parametrized by $\tilde{\psi}$. In order to do the KK reduction it is convenient to define the new one forms $\hat{w}^a$ and the vector n^a as follows

$$\tilde{w}^a_L = \hat{w}^a + n^a d\tilde{\psi} \tag{A.7}$$

Notice that $n^a n^a = 1$. We have extracted the dependence on $\tilde{\psi}$ in order to KK reduce. This splitting preserves $SU(2)^1_R$ but breaks $SU(2)^1_L$ to $U(1)_L$.

The IIA dilaton, metric and one form RR potential are

$$\begin{aligned} e^{4\phi/3} =& \frac{1}{N^2}[\gamma^2 + \frac{\beta^2}{4}] \\ ds^2_{str} =& e^{2\phi/3}\left[dx_4^2 + \alpha^2 dr^2 - e^{4\phi/3}A_1^2 + \gamma^2(\hat{w}^a)^2 + \beta^2(w^a - \frac{1}{2}\hat{w}^a)^2\right] \\ A_1 =& N\left[\hat{w}^a n^a - \frac{\beta^2}{2(\gamma^2+\beta^2/4)}w^a n^a\right] \end{aligned} \tag{A.8}$$

where A_1 is the RR one form potential.

Similarly we can integrate the three form over S^1 and we obtain

$$J = \frac{a^3}{12}\frac{1}{2}\epsilon_{abc}n^a\hat{w}^b\hat{w}^c - (\frac{r^3}{18} - \frac{a^3}{18})\epsilon_{abc}(n^a w^b w^c - 2n^a w^b \hat{w}^c) - \frac{r^2}{3}drn^a w^a \tag{A.9}$$

which is closed since Ω was closed.

We can further simplify these expressions by doing a coordinate transformation that amounts to switching from the left invariant one forms to the right invariant one forms in $\tilde{S}^3$. This translates into the following replacements

$$\hat{w}^a \to \breve{w}^a_R \qquad\qquad n^a \to \breve{n}^a_R \qquad\qquad w^a \to w^a + \breve{w}^a_R \tag{A.10}$$

Where now $\breve{w}^a_R, \breve{n}^a_R$ are defined through

$$\tilde{w}^a_R|_{\tilde{\psi}=0} = \breve{w}^a_R + \breve{n}^a_R d\tilde{\psi} \ .$$

In these variables

$$\breve{n}^a_R = \delta^a_3, \qquad \breve{w}^1 = d\tilde{\theta} \ , \qquad \breve{w}^2 = \sin\tilde{\theta}d\tilde{\phi} \ , \qquad \breve{w}^3 = \cos\tilde{\theta}d\tilde{\phi} \ .$$

In these variables the two form (A.9) becomes

$$J = \frac{a^3}{12}\sin\tilde{\theta}d\tilde{\theta}d\tilde{\phi} - d(\frac{r^3 - a^3}{9}(w^3 + \cos\tilde{\theta}d\tilde{\phi})) \tag{A.11}$$

This agrees with the two form of the small resolution of the conifold which is

$$k = t\sin\tilde{\theta}d\tilde{\theta}d\tilde{\phi} - d(h(\rho)(w^3 + \cos\tilde{\theta}d\tilde{\phi})) \tag{A.12}$$

where $h(\rho)$ is some function of the radial coordinate. these two expressions coincide if we identify $h(\rho) = (r^3 - a^3)/9$ which just amounts to a reparametrization of the radial coordinate.

We can also write the IIA metric (A.8) and the RR 1-form potential in a more explicit form

$$\begin{aligned}
ds^2_{str} =& e^{2\phi/3}\Big[dx_4^2 + \alpha^2 dr^2 + \gamma^2(d\tilde{\theta}^2 + \sin^2\tilde{\theta}d\tilde{\phi}^2) + \\
& \beta^2[(w^1 + \frac{1}{2}d\tilde{\theta})^2 + (w^2 + \frac{1}{2}\sin\tilde{\theta}d\tilde{\phi})^2 + (w^3 + \cos\tilde{\theta}d\tilde{\phi})^2] - \\
& \frac{\beta^4}{4(\gamma^2 + \beta^2/4)}(w^3 + \cos\tilde{\theta}d\tilde{\phi})^2\Big] \\
A_1 =& N\left[\cos\tilde{\theta}d\tilde{\phi} - \frac{\beta^2}{2(\gamma^2 + \beta^2/4)}(w^3 + \cos\tilde{\theta}d\tilde{\phi})\right]
\end{aligned} \tag{A.13}$$

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