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AN OBSTRUCTION TO THE EXISTENCE OF CONSTANT SCALAR CURVATURE KÄHLER METRICS

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Abstract

We prove that polarised manifolds that admit a constant scalar curvature Kähler (cscK) metric satisfy a condition we call slope semistability. That is, we define the slope μ for a projective manifold and for each of its subschemes, and show that if X is cscK then $\mu(Z) \leq \mu(X)$ for all subschemes Z.

This gives many examples of manifolds with Kähler classes which do not admit cscK metrics, such as del Pezzo surfaces and projective bundles. If $\mathbb{P}(E) \to B$ is a projective bundle which admits a cscK metric in a rational Kähler class with sufficiently small fibres, then E is a slope semistable bundle (and B is a slope semistable polarised manifold). The same is true for *all* rational Kähler classes if the base B is a curve.

We also show that the slope inequality holds automatically for smooth curves, canonically polarised and Calabi-Yau manifolds, and manifolds with $c_1(X) < 0$ and L close to the canonical polarisation.

1. Introduction

An important problem in Kähler geometry is that of finding a constant scalar curvature Kähler (cscK) metric in a given Kähler class on a complex manifold X. For a curve this is provided by the uniformisation theorem. For general X the class $[\omega] \in H^2(X, \mathbb{R})$ admits a Kähler-Einstein metric (which is therefore cscK) when $c_1(X) = 0$ [**Y1**], or when $c_1(X) < 0$ and $[\omega] = -\lambda[c_1(X)]$ [**Au**, **Y1**].

The first known obstructions to the existence of cscK metrics came from the holomorphic automorphism group. The most famous is the Calabi-Futaki invariant of the Kähler class. This is a character on the Lie algebra $\mathfrak{aut}(X)$ of the holomorphic automorphism group which must vanish if the class admits a cscK metric [**Fut**].

Tian defined a finer obstruction called K-stability, arising from certain degenerations (or *test configurations*) of X [**Ti2**, **Ti3**]. Moreover it is conjectured that K-polystability is a necessary and sufficient condition for the existence of cscK metrics; see Conjecture 2.8. One direction

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of this conjecture is now almost proved: it is known that cscK implies K-semistability [**Do5**]. Thus test configurations can provide obstructions to cscK metrics. In particular those arising from a $\mathbb{C}^\times$-action recover the Calabi-Futaki obstruction, and these "product configurations" are currently the only test configurations that have been systematically studied.

In this paper we consider test configurations canonically associated to subschemes of X, yielding a new obstruction to the existence of cscK metrics. These configurations are more complicated than product configurations; in particular the central fibre is non-normal. The motivation is an analogy with stability for vector bundles; just as subsheaves can destabilise a sheaf or bundle we show how subschemes can destabilise X. In Section 3 we define, by analogy with vector bundles, a notion of slope (semi)stability of a manifold and rational Kähler class. We prove in Section 4 that this gives an obstruction to K-semistability, and hence to cscK metrics.

Thus, K-semistability implies slope semistability. A partial converse is given in Theorem 6.1 of [**RT**]; in particular the two are shown to be equivalent for curves. In trying to form moduli of varieties in algebraic geometry using Geometric Invariant Theory, other stability conditions arise, for instance Chow stability. Different notions of slope are given for these in [**RT**], and stability is shown to imply the relevant slope stability (Proposition 4.33 and Theorem 7.2 of [**RT**]).

In Section 5.2 we show slope stability for the canonical class when $c_1(X) < 0$, and for arbitrary classes when $c_1(X) = 0$ (as expected from the existence of their Kähler-Einstein metrics). We also show slope stability for classes close to the canonical class when $c_1(X) < 0$, and compare to some similar analytical results of [**We**]. Slope stability of smooth curves is proved in 5.3, which by Corollary 6.7 of [**RT**] implies K-stability. As far as we know this is the only direct, non-analytic proof of K-stability of smooth curves.

We apply the slope formula to study unstable projective bundles in Section 5.4, providing a converse to the results of Hong [**Ho**]. When the base is a curve, the Narasimhan-Seshadri theorem gives a cscK metric on the projectivisation of any polystable bundle (of arbitrary rank) in any Kähler class, and we are able to give an almost complete converse (there is a small discrepancy for bundles which are strictly semistable but not polystable until the results of [**Do2**, **Do5**, **Mab**] are improved to give K-polystability).

Other examples include unstable blow ups in Section 5.5 and unstable rational manifolds in Section 5.6. In particular we give examples of Kähler classes on surfaces with trivial automorphism group which do not admit cscK metrics (5.32).

One might hope that in the continuity method to find a cscK metric, the multiplier ideal sheaf along which the C^0-estimates required for closedness fail [**Na**] defines a subscheme which slope destabilises the variety. In particular, if one could show this for canonically polarised manifolds then Theorem 4.2 combined with Nadel's results would solve the Kähler-Einstein problem for Fano manifolds.

Notation and Terminology. In this paper (X, L) will be a smooth complex manifold of dimension n with a polarisation L (*i.e.* an ample line bundle on X). Furthermore Z will denote an arbitrary subscheme of X defined by an ideal sheaf $\mathscr{I}_Z$. When Z is smooth $\nu_Z = (\mathscr{I}_Z/\mathscr{I}_Z^2)^*$ will denote its normal bundle.

The blow up along Z is denoted by $\pi\colon \widehat{X} \to X$, with exceptional divisor E. Note that $\pi_*\mathcal{O}(-jE) = \mathscr{I}_Z^j$ for $j \gg 0$. For convenience we often suppress pullback maps and use the same letter to denote a divisor and the associated line bundle. For example, on $\widehat{X}$ we denote $(\pi^*L \otimes \mathcal{O}(-E))^{\otimes k}$ by $L^{\otimes k}(-kE)$. The intersection product of divisors $D_1, \ldots, D_n$ on X is denoted by $\int_X c_1(D_1). \ldots c_1(D_n)$, and this is abbreviated to $D_1.D_2 \ldots D_n$ in sections 5.1 and 5.2.

A $\mathbb{Q}$-divisor is a formal sum of divisors with rational coefficients; some multiple is therefore a divisor with an associated line bundle. A $\mathbb{Q}$-divisor is said to be ample if it can be written as a formal sum of ample divisors with positive rational coefficients. We recall that a nef line bundle (or divisor) is one whose intersection with every curve in X is nonnegative, and this extends to $\mathbb{Q}$-line bundles. By the Kleiman criterion [**Kl**] these divisors are precisely those in the closure of the ample cone. In notation like $H^0(L^{\otimes k})$ we always tacitly restrict to those k for which $L^{\otimes k}$ is an honest line bundle.

Any finite-dimensional vector space V with a $\mathbb{C}^\times$-action splits into one-dimensional weight spaces $V = \bigoplus_i V_i$, where $t \in \mathbb{C}^\times$ acts on V_i by t^{w_i}. The integers w_i are the *weights* of the action, and $w(V) = \sum_i w_i$ is the *total weight* of the action; i.e., the weight of the induced action on the top exterior power $\Lambda^{\max} V$.

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2. Definition of K-stability

Tian [**Ti2**, **Ti3**] introduced a notion of K-stability using differential geometry. Donaldson [**Do3**] gave an algebro-geometric definition that allows arbitrarily singular central fibre and which we use here. The relation between the two is studied in [**PT**].

There is a strong formal link between K-stability and stability notions in Geometric Invariant Theory; in particular the test configurations defined below are what one gets by applying a one parameter subgroup of projective linear transformations to the Kodaira embedding of $(X, L^{\otimes k})$, and what we call the Donaldson-Futaki invariant is the GIT weight of the induced action on a certain line. We will not attempt to describe this further but instead refer the interested reader to [**Do3**, **RT**].

Definition 2.1. Suppose that (X, L) is a polarised variety with Hilbert polynomial $\mathcal{P}(k) := \chi(L^{\otimes k})$. A **test configuration with general fibre** (X, L) consists of

1) A flat projective family of $\mathbb{Q}$-polarised schemes $(\mathscr{X}, \mathcal{L}) \to \mathbb{C}$.

2) An action of $\mathbb{C}^\times$ on $(\mathscr{X}, \mathcal{L})$ covering the usual action of $\mathbb{C}^\times$ on $\mathbb{C}$, such that the fibre $(\mathscr{X}_t, \mathcal{L}|_{\mathscr{X}_t)})$ is isomorphic to (X, L) for one, and so all, $t \in \mathbb{C}\backslash\{0\}$.

The flatness condition is that the fibres $(X_t, \mathcal{L}_t)$ all have the same Hilbert polynomial $\mathcal{P}(k)$ ([**Ha**] Theorem III.9.9). We call a test configuration a **product configuration** if $\mathscr{X} \cong X \times \mathbb{C}$, and a **trivial configuration** if in addition $\mathbb{C}^\times$ acts only on the second factor. Since $0 \in \mathbb{C}$ is fixed, we get an induced action on the central fibre $(\mathscr{X}_0, \mathcal{L}|_{\mathscr{X}_0})$ and hence on $H^0(\mathscr{X}_0, \mathcal{L}^{\otimes k}|_{\mathscr{X}_0})$ for all k.

Definition 2.2. Suppose $(\mathscr{X}, \mathcal{L})$ is a test configuration with general fibre (X, L). Let $w(k)$ be the weight of the induced action on $H^0(\mathscr{X}_0, \mathcal{L}|_{\mathscr{X}_0}^{\otimes k})$, which by the equivariant Riemann-Roch formula is a polynomial of degree $n+1$ for $k \gg 0$, so there is an expansion

$$\frac{w(k)}{k\mathcal{P}(k)} = f_0 + f_1 k^{-1} + O(k^{-2}).$$

We define the **Donaldson-Futaki invariant** of a test configuration to be $F_1 = -f_1$ (so this has the opposite sign to the definition in [**Do3**]).

Writing $\mathcal{P}(k) = a_0 k^n + a_1 k^{n-1} + O(k^{n-2})$ and $w(k) = b_0 k^{n+1} + b_1 k^n + O(k^{n-1})$, then

$$F_1 = \frac{b_0 a_1 - b_1 a_0}{a_0^2}. \tag{2.3}$$

Definition 2.4.

- We say that (X, L) is **algebraically K-stable** (resp. **algebraically K-semistable**) if for all non-trivial test configurations with general fibre (X, L) we have $F_1 > 0$ (resp. $F_1 \geq 0$).

- We say that (X, L) is **algebraically K-polystable** if it is K-semistable, and any test configuration with general fibre (X, L) and $F_1 = 0$ is a product configuration. That is, the only instability arises from $\mathbb{C}^\times$-actions on (X, L).

Remarks 2.5.

- The property of being K-(semi/poly)stable is preserved under replacing L by $L^{\otimes r}$, and so makes sense when L or $\mathcal{L}$ is an ample $\mathbb{Q}$-line bundle. The definition of a test configuration given here differs from that in Definition 3.6 of [**RT**], but is the same after twisting $\mathcal{L}$ by some power.
- When the central fibre $(\mathscr{X}_0, \mathcal{L}|_{\mathscr{X}_0})$ is smooth, F_1 is, up to a constant, the usual Calabi-Futaki invariant, with respect to the class $c_1(L)$, of the vector field induced by the S^1-action [**Do3**].
- The Donaldson-Futaki invariant can be interpreted in terms of the Mumford weight function in Geometric Invariant Theory [**Do3**] (see also Theorem 3.9 of [**RT**]).

It is possible to strengthen the definition of K-stability. We define an **analytic test configuration** with general fibre (X, L) exactly the same way as we defined a test configuration, but allow $\mathcal{L}$ to be an ample $\mathbb{R}$-divisor. (By an ample $\mathbb{R}$-divisor we mean a formal sum $\mathcal{L} = \sum_{i=1}^m \alpha_i D_i$ with each D_i an ample divisor and α_i a positive real; a $\mathbb{C}^\times$-action on $\mathcal{L}$ is a choice of $\mathbb{C}^\times$-action on each line bundle $\mathcal{O}(D_i)$).

For any test configuration the Donaldson-Futaki invariant can be calculated using the equivariant Riemann-Roch theorem in terms of the equivariant first Chern class of $\mathcal{L}$ with its $\mathbb{C}^\times$-action. The resulting expression makes sense even if $\mathcal{L}$ is an ample $\mathbb{R}$-divisor, and we take this to be the definition of F_1 in this case.

Definition 2.6. We say that (X, L) is **analytically K-stable** (resp. **analytically K-semistable**) if for all non-trivial analytic test configurations $(\mathscr{X}, \mathcal{L})$ with general fibre (X, L) we have $F_1 > 0$ (resp. $F_1 \geq 0$). It is **analytically K-polystable** if it is analytically K-semistable and any analytic test configuration with $F_1 = 0$ is a product configuration.

Remark 2.7. As analytic K-semistability is equivalent to algebraic K-semistability, we will drop the qualifier when dealing with K-semistability.

2.1. Relationship to constant scalar curvature Kähler metrics.

The precise conjecture relating K-stability to the existence of cscK metrics is the following [**Y2**, **Ti3**, **Do3**]:

Conjecture 2.8 (Yau-Tian-Donaldson)**.** Let (X, L) be a polarised manifold. Then there exists a constant scalar curvature Kähler metric in the class of $c_1(L)$ if and only if (X, L) is K-polystable.

One direction of this conjecture, that existence of a cscK metric implies stability, has almost been proved: in [**Do5**] it is shown that a cscK metric implies K-semistability. Before that paper one had a slightly weaker result by using balanced metrics: if $\mathfrak{aut}(X) = 0$ then the existence of a cscK metric implies that the Kodaira embedding of $(X, L^{\otimes r})$ can be "balanced" for $r \gg 0$ [**Do2**]. This implies it is asymptotically Chow stable [**Zh**, **P**, **Wa**], which in turn implies that (X, L) is K-semistable (see for example Theorem 3.9 of [**RT**]). Mabuchi [**Mab**] extended this proof to manifolds with non-discrete automorphism group satisfying a certain stability condition.

Another path to stability is through the K-energy, also called the Mabuchi functional. The existence of a cscK metric implies the K-energy map is bounded from below [**Do4**, **CT**] (resp. proper in the Kähler-Einstein Fano case when $\mathfrak{aut}(X) = 0$ [**Ti3**]); in turn this implies K-semistability [**PT**] (resp. K-stability).

We remark that a recent example in [**ACGT**] suggests that algebraic K-stability may not be enough to guarantee the existence of a cscK metric, and that the stronger analytic definition of K-stability may be required. (The authors wish to thank V. Apostolov and D. Calderbank for discussions on this point). Moreover it may be that we have to allow non-projective central fibres (see Section 4.4).

It is expected that the deep results mentioned above proving stability are not optimal, and that K-polystability can be proved. However, the fact that a cscK metric implies K-semistability is enough to give a new obstruction in terms of the subschemes of X which we now describe.

3. Definition of slope stability

Fix a polarised manifold (X, L) and write the Hilbert polynomial as

$$\mathcal{P}(k) = \chi(L^{\otimes k}) = a_0 k^n + a_1 k^{n-1} + O(k^{n-2}).$$

Definition 3.1. The **slope** of (X, L) is

$$\mu(X) = \mu(X, L) = \frac{a_1}{a_0}.$$

By the Riemann-Roch theorem,

$$a_0 = \frac{1}{n!}\int_X c_1(L)^n, \quad \text{and} \quad a_1 = -\frac{1}{2(n-1)!}\int_X c_1(K_X).c_1(L)^{n-1},$$

so

$$\mu(X) = -\frac{n\int_X c_1(K_X).c_1(L)^{n-1}}{2\int_X c_1(L)^n}.$$

For a subscheme Z of X let $\widehat{X}$ be the blow up of X along Z, with exceptional divisor E.

Definition 3.2. The **Seshadri constant** of Z is

$$\begin{aligned}\epsilon(Z) &= \epsilon(Z, X, L)\\ &= \sup\{c : L^{\otimes k} \otimes \mathscr{I}_Z^{ck} \text{ is globally generated for } k \gg 0\}\\ &= \sup\{c : L(-cE) \text{ is ample on } \widehat{X}\}\\ &= \max\{c : L(-cE) \text{ is nef on } \widehat{X}\}.\end{aligned}$$

We say the global sections of $L \otimes \mathscr{I}_Z$ **saturate** $\mathscr{I}_Z$ if they generate the line bundle $L(-E)$ on $\widehat{X}$. This is weaker than (i.e., is implied by) $L \otimes \mathscr{I}_Z$ being globally generated (see [**RT**] Section 2).

For fixed $x \in \mathbb{Q}$ define $a_i(x)$ by

$$\chi(L^{\otimes k}(-xkE)) = a_0(x)k^n + a_1(x)k^{n-1} + O(k^{n-2}) \quad k \gg 0, xk \in \mathbb{N}. \tag{3.3}$$

Since $\chi(L^{\otimes k}(-rE))$ is a polynomial in k and r of total degree at most n, $a_i(x)$ is a polynomial in x of degree at most $n-i$, and can be extended to all of $\mathbb{R}$. We have

$$a_0(x) = \frac{1}{n!}\int_{\widehat{X}} c_1(L(-xE))^n, \tag{3.4}$$

and, when Z is a codimension p submanifold, by the Riemann-Roch formula on $\widehat{X}$,

$$a_1(x) = -\frac{1}{2(n-1)!}\int_{\widehat{X}} c_1(K_{\widehat{X}}).c_1(L(-xE))^{n-1}, \tag{3.5}$$

where $K_{\widehat{X}} = K_X((p-1)E)$ is the canonical divisor of $\widehat{X}$.

The $a_i(x)$ can also be defined in terms of the ideal sheaf of Z. Fix j_0 such that $\pi_*(-jE) = \mathscr{I}_Z^j$ for all $j \geq j_0$ (when Z is smooth we can take $j_0 = 0$). Then for $xk \in \mathbb{N}$, $x < \epsilon(Z)$ and $k \gg 0$ (in particular $kx \geq j_0$),

$$h^0(L^{\otimes k} \otimes \mathscr{I}_Z^{xk}) = \chi(L^{\otimes k} \otimes \mathscr{I}_Z^{xk}) = a_0(x)k^n + a_1(x)k^{n-1} + O(k^{n-2}). \tag{3.6}$$

Thus $a_0(0) = a_0$. When X and Z are smooth, taking $j_0 = 0$ shows that we also have $a_1(0) = a_1$. More generally this holds when X is normal ([**RT**] Remarks 4.21).

Definition 3.7. The **slope of Z with respect to c** is

$$\mu_c(\mathscr{I}_Z) = \mu_c(\mathscr{I}_Z, L) = \frac{\int_0^c \left(a_1(x) + \frac{a_0'(x)}{2}\right)dx}{\int_0^c a_0(x)dx}.$$

Definition 3.8.

- We say that (X, L) is **slope semistable with respect to** Z if $\mu_c(\mathscr{I}_Z) \leq \mu(X)$ for all $c \in (0, \epsilon(Z)]$.
- We say (X, L) is **slope stable with respect to** Z if $\mu_c(\mathscr{I}_Z, L) < \mu(X)$ for every $c \in (0, \epsilon(Z))$, *and* for $c = \epsilon(Z)$ if $\epsilon(Z)$ is rational and the global sections $L^{\otimes k} \otimes \mathscr{I}_Z^{\epsilon(Z)k}$ saturate $\mathscr{I}_Z^{\epsilon(Z)k}$ for $k \gg 0$.

- We say (X, L) is **slope polystable with respect to** Z if it is slope semistable, and if (Z, c) is any pair such that $\mu_c(\mathscr{I}_Z) = \mu(X)$, then $c = \epsilon(Z) \in \mathbb{Q}$ and, on the deformation to the normal cone (Section 4.1) of Z, $\mathcal{L}_c = L(-cP)$ is pulled back from a product test configuration $(X \times \mathbb{C}, L)$.
- Finally (X, L) is said to be **slope (semi/poly)stable** if it is so with respect to all subschemes Z.

Remark 3.9. The definition above of slope semistability agrees with that in [**RT**]. However, the definitions given here of slope (poly)stability are slightly stronger as we require the relevant condition to hold even for irrational c. Thus what we have defined as slope (poly)stability might more properly be referred to as *analytic* slope (poly)stability, and is the notion relevant to the analytic K-stability of Definition 2.6.

An example of a slope polystable variety is provided by $\mathbb{P}^n$ (whose Fubini-Study metric is cscK). When $c = \epsilon(p)$, $\mu_c(\mathscr{I}_p) = \mu(\mathbb{P}^n)$ and the deformation to the normal cone (Section 4.1) of a point $p \in \mathbb{P}^n$ collapses to $\mathbb{P}^n \times \mathbb{C}$, with a non-trivial $\mathbb{C}^\times$-action with Donaldson-Futaki invariant 0. Generalisations of this example are provided by the projective bundles of (5.14).

Remarks 3.10.

- We say that Z **destabilises** (resp. **strictly destabilises**) if (X, L) is not slope stable (resp. slope semistable) with respect to Z.
- Slope (semi/poly)stability is preserved under twisting L, since $\epsilon(Z, L^{\otimes r}) = r\epsilon(Z, L)$, $\mu(X, L) = r\mu(X, L^{\otimes r})$, and $\mu_c(\mathscr{I}_Z, L) = r\mu_{rc}(\mathscr{I}_Z, L^{\otimes r})$.
- If $0 < x < \epsilon(Z)$ then from the fact that $L(-xE)$ is ample,

$$
\begin{aligned}
a_0(x) &= \frac{1}{n!}\int_{\widehat{X}} c_1(L(-xE))^n > 0, \\
a_0'(x) &= -\frac{1}{n!}\int_{\widehat{X}} c_1(L(-xE))^{n-1}.E < 0.
\end{aligned}
\tag{3.11}
$$

  In particular, for $0 < c \leq \epsilon(Z)$, $\int_0^c a_0(x)dx > 0$ so $\mu_c(\mathscr{I}_Z)$ is finite.
- $\lim_{c\to 0} \mu_c(\mathscr{I}_Z) = \frac{a_1(0)+a_0'(0)/2}{a_0} < \frac{a_1(0)}{a_0}$ by (3.11). For X normal this is $\frac{a_1}{a_0} = \mu(X)$ (by Remarks 4.21 of [**RT**]), so (Z, c) does not destabilise for small $c > 0$. Therefore, on defining $\mu(\mathscr{I}_Z) := \max_{0\leq x\leq c} \mu_c(\mathscr{I}_Z)$, slope semistability is equivalent to $\mu(\mathscr{I}_Z) \leq \mu(X)$. This is how it was presented in the Abstract.

Remarks 3.12.

- In the slope inequality we may assume without loss of generality that Z is not a thickening of any other subscheme. For if $Z = mZ'$,

$m \geq 1$, then $\epsilon(Z) = \frac{1}{m}\epsilon(Z')$ and, as $a_0'(x) < 0$ (3.11),

$$\mu_{c/m}(\mathscr{I}_{Z'}^m) = \mu_c(\mathscr{I}_{Z'}) + (m-1)\frac{\int_0^c a_0'(x)dx}{2\int_0^c a_0(x)dx} < \mu_c(\mathscr{I}_{Z'}).$$

- If Z strictly destabilises then so does one connected component of Z, and smooth points do not destabilise a smooth X ([**RT**] Theorem 4.29). Thus, for the strict inequality, we may assume without loss of generality that Z is connected.

Definition 3.13. Let $\tilde{a}_i(x)$ be defined by

$$\chi(L^{\otimes k}\otimes\mathcal{O}_{xkZ}) = \chi(L^{\otimes k}/(L^{\otimes k}\otimes\mathscr{I}_Z^{xk})) = \tilde{a}_0(x)k^n + \tilde{a}_1(x)k^{n-1} + O(k^{n-2}),$$

so $\tilde{a}_i(x) = a_i - a_i(x)$. The **quotient slope** of Z with respect to c is (in slightly misleading notation)

$$(3.14) \qquad \begin{aligned} \mu_c(\mathcal{O}_Z) &= \mu_c(\mathcal{O}_Z, L) \\ &= \frac{\int_0^c \left(\tilde{a}_1(x) + \frac{\tilde{a}_0'(x)}{2}\right)dx}{\int_0^c \tilde{a}_0(x)dx} \\ &= \frac{\int_0^c \left(a_1(x) + \frac{a_0'(x)}{2}\right)dx - ca_1}{\int_0^c a_0(x)dx - ca_0}, \end{aligned}$$

which is finite for $0 < c \leq \epsilon(Z)$. Notice that

$$\mu_c(\mathscr{I}_Z) < \mu(X) \iff \mu(X) < \mu_c(\mathcal{O}_Z) \iff \mu_c(\mathscr{I}_Z) < \mu_c(\mathcal{O}_Z),$$

due to the implications

$$\frac{A}{B} < \frac{C}{D} \iff \frac{C}{D} < \frac{C-A}{D-B} \iff \frac{A}{B} < \frac{C-A}{D-B}$$

for $0 < B < D$, on setting $B = \int_0^c a_0(x)dx$ and $D = ca_0$ (so $D - B = \int_0^c \tilde{a}_0(x)dx > 0$).

So slope stability can be phrased in terms of the quotient slope $\mu_c(\mathcal{O}_Z)$.

Proposition 3.15. *For fixed $x \in \mathbb{Q}_{>0}$, define $\alpha_i(x)$ by*

$$(3.16) \qquad \chi(L^{\otimes k}\otimes\mathscr{I}_Z^{xk}/\mathscr{I}_Z^{xk+1}) = \alpha_1(x)k^{n-1} + \alpha_2(x)k^{n-2} + O(k^{n-3})$$

for $k \gg 0, xk \in \mathbb{N}$. (So if Z is smooth with normal bundle ν_Z then

$$\chi(L^{\otimes k}|_Z\otimes S^{xk}\nu_Z^*) = \alpha_1(x)k^{n-1} + \alpha_2(x)k^{n-2} + O(k^{n-3}),\ k \gg 0,\ xk \in \mathbb{N},$$

where $S^r(\cdot)$ denotes the r-th symmetric product.)

Then

$$\mu_c(\mathcal{O}_Z) = \frac{\int_0^c(c-x)\alpha_2(x)dx + \frac{c}{2}\alpha_1(0)}{\int_0^c(c-x)\alpha_1(x)dx}.$$

Proof. Fix $x > 0$ and let $\bar{x} = x + 1/k$. Clearly

$$\chi(L^{\otimes k} \otimes \mathscr{I}_Z^{xk}/\mathscr{I}_Z^{xk+1}) = \chi(L^{\otimes k} \otimes \mathscr{I}_Z^{xk}) - \chi(L^{\otimes k} \otimes \mathscr{I}_Z^{xk+1}).$$

By (3.6) and the Taylor expansion of $a_i(x)$ this equals, for $k \gg 0$,

$$\begin{aligned}&[a_0(x) - a_0(\bar{x})]k^n + [a_1(x) - a_1(\bar{x})]k^{n-1} + \cdots \\ &\qquad = -a_0'(x)k^{n-1} - \frac{a_0''(x)}{2}k^{n-2} - a_1'(x)k^{n-2} + O(k^{n-3}).\end{aligned}$$

(Note that this holds when $n = 1$ for then $a_0''(x) = a_1'(x) = 0$.) Hence

$$\alpha_1(x) = -a_0'(x) \quad \text{and} \quad \alpha_2(x) = -a_1'(x) - \frac{a_0''(x)}{2}. \tag{3.17}$$

Thus the denominator of the quotient slope of Z is

$$\int_0^c \tilde{a}_0(x)dx = \int_0^c \int_0^x \alpha_1(y)dydx = \int_0^c (c-x)\alpha_1(x)dx.$$

The calculation for the numerator is similar. q.e.d.

4. Slope stability as a necessary condition for K-stability

4.1. Deformation to the normal cone. Fix a subscheme Z of (X, L). Let $\mathscr{X}$ be the deformation to the normal cone of Z, so $\mathscr{X}$ is the blow up of $X \times \mathbb{C}$ along $Z \times \{0\}$, and denote the exceptional divisor by P. The central fibre $\mathscr{X}_0$ is isomorphic to the blow up $\widehat{X}$ glued to P along E (see Figure 1). When Z is a submanifold, $E = \mathbb{P}(\nu_Z)$ and P is isomorphic to the projective completion of the normal bundle of Z, i.e., $P = \mathbb{P}(\nu_Z \oplus \underline{\mathbb{C}})$, with a copy $Z' := \mathbb{P}(\underline{\mathbb{C}})$ of Z as its zero section.

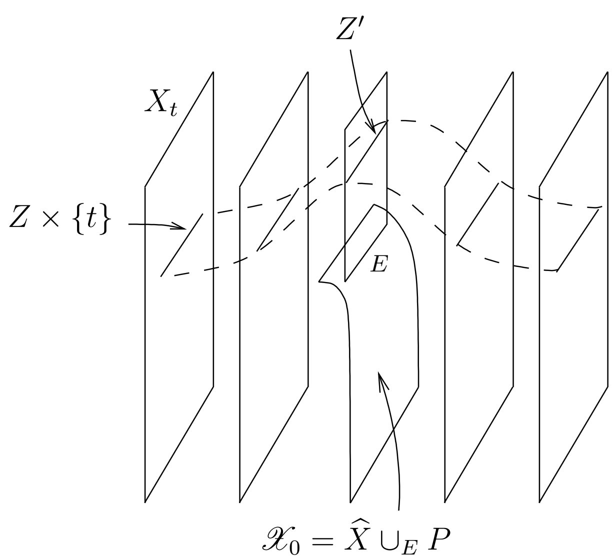

Figure 1. The deformation to the normal cone of Z.

Consider the product action on $(X\times\mathbb{C}, L)$ (where as usual we suppress the pullback map on L), which acts trivially on (X, L) but scales $\mathbb{C}$ with weight 1. This fixes $Z\times\{0\}$ and so induces an action on $\mathscr{X}$ and on P. The induced action on the central fibre $\mathscr{X}_0 = \widehat{X}\cup_E P$ is trivial on $\widehat{X}$, and $\lambda\in\mathbb{C}^\times$ acts on $P = \mathbb{P}(\nu_Z\oplus\underline{\mathbb{C}})$ as $\operatorname{diag}(1,\lambda)$.

We define a $\mathbb{Q}$-line bundle on $\mathscr{X}$ by $\mathcal{L}_c = L(-cP)$ for $c\in\mathbb{Q}$.

Lemma 4.1. *For rational $c\in(0,\epsilon(Z))$, the line bundle $\mathcal{L}_c$ is ample.*

Proof. Let $p\colon \mathscr{X}\to X$ be the composition of the projections, and let $c = r/q$. Choose q and r sufficiently large so that $L^{\otimes q}$ and $L^{\otimes q}(-rE)$ are globally generated. Then away from $Z' = \mathbb{P}(\underline{\mathbb{C}})\subset P$, the line bundle $\mathcal{L}_c^{\otimes q} = L^{\otimes q}(-rP)$ is generated by $p^*H^0(L^{\otimes q}\otimes\mathscr{I}_Z^r)$, while on Z', it is generated by $t^r p^*H^0(L^{\otimes q})$. That is, $\mathcal{L}_c^{\otimes q}$ is globally generated for all $0<c<\epsilon(Z)$ and so nef for $c\in[0,\epsilon(Z)]$. Since $\mathcal{L}_c$ is ample for c sufficiently small, the fact that the ample cone is convex and the interior of the nef cone [**Kl**] implies that $\mathcal{L}_c$ is ample for rational $0<c<\epsilon(Z)$. q.e.d.

4.2. Slope stability as an obstruction to K-stability. Slope stability with respect to Z is precisely K-stability restricted to test configurations arising from the degeneration to the normal cone of Z. In ([**RT**] Theorem 4.18) it is shown that K-semistability implies slope semistability. Moreover, using the algebraic definitions of slope (polystability) in [**RT**] it is also shown that K-(poly)stability implies slope (poly)stability. Here we give a proof of the part of this result which is sufficient for our examples and applications to cscK metrics.

Theorem 4.2. *Suppose (X, L) is K-semistable. Then it is slope semistable with respect to any smooth subscheme Z.*

Proof. We need to show that $\mu_c(\mathcal{O}_Z)\geq\mu(X)$ for all $0<c\leq\epsilon(Z)$. By continuity of μ with respect to c it is sufficient to consider rational $c<\epsilon(Z)$. So $\mathcal{L}_c$ is ample (4.1) and hence so is $\mathcal{L}_c|_{\mathscr{X}_0}$.

By the definition of the blow up in $\mathscr{I}_{Z\times\{0\}}\subset\mathcal{O}_{X\times\mathbb{C}}$ (*i.e.* $\mathscr{I}_Z+(t)\subset\mathbb{C}[t]\otimes\mathcal{O}_X$, where t is the coordinate on $\mathbb{C}$), for $k\gg 0$ and $ck\in\mathbb{N}$,

$$
\begin{aligned}
(4.3)\quad H^0(\mathscr{X},\mathcal{L}_c^{\otimes k}) &= H^0(\mathscr{X},(L(-cE))^{\otimes k})\\
&= H^0(X\times\mathbb{C}, L^{\otimes k}\otimes\mathscr{I}_{Z\times\{0\}}^{ck})\\
&= H^0(X\times\mathbb{C}, L^{\otimes k}\otimes(\mathscr{I}_Z+(t))^{ck})\\
&= \bigoplus_{i=1}^{ck} t^{ck-i}H^0(X, L^{\otimes k}\otimes\mathscr{I}_Z^i)\ \oplus\ t^{ck}\mathbb{C}[t]H^0(L^{\otimes k}).
\end{aligned}
$$

Similarly, for k sufficiently large and $j\geq 1$,

$$
0 = H^j(\mathscr{X},\mathcal{L}_c^{\otimes k}) = \bigoplus_{i=1}^{ck} t^{ck-i}H^j(X, L^{\otimes k}\otimes\mathscr{I}_Z^i)\oplus t^{ck}\mathbb{C}[t]H^j(L^{\otimes k}),
$$

so that

$$H^j(L^{\otimes k}\otimes \mathscr{I}_Z^i)=0 \quad \text{for } j\ge 1,\ k\gg 0,\ ck\in\mathbb{N},\ i=0,\dots,ck. \tag{4.4}$$

Thus, for instance, $H^0(L^{\otimes k}\otimes \mathscr{I}_Z^i)/H^0(L^{\otimes k}\otimes \mathscr{I}_Z^{i+1}) = H^0\big(L^{\otimes k}\otimes \big(\mathscr{I}_Z^i/\mathscr{I}_Z^{i+1}\big)\big) = H^0(L^{\otimes k}|_Z\otimes S^i\nu_Z^*)$. So from (4.3) we get the splitting, for $k\gg 0$,

$$\begin{aligned} &H^0(\mathcal{L}_c^{\otimes k}|_{\mathscr{X}_0}) \\ &= H^0(\mathscr{X},\mathcal{L}_c^{\otimes k})/tH^0(\mathscr{X},\mathcal{L}_c^{\otimes k}) \\ &= H^0(X,L^{\otimes k}\otimes \mathscr{I}_Z^{ck})\oplus \bigoplus_{i=0}^{ck-1} t^{ck-i}H^0(L^{\otimes k}|_Z\otimes S^i\nu_Z^*), \end{aligned} \tag{4.5}$$

of the functions on $\mathscr{X}_0$ into those on $\widehat{X}$ and the polynomials on the ν_Z-fibres of P. In particular $h^0(\mathcal{L}_c^{\otimes k}|_{\mathscr{X}_0})$ equals

$$\begin{aligned} &h^0(L^{\otimes k}\otimes \mathscr{I}_Z^{ck})+\sum_{i=0}^{ck-1}\Big(h^0(L^{\otimes k}\otimes \mathscr{I}_Z^{i})-h^0(L^{\otimes k}\otimes \mathscr{I}_Z^{i+1})\Big) \\ &= h^0(L^{\otimes k})=\mathcal{P}(k). \end{aligned}$$

This proves flatness, so $(\mathscr{X},\mathcal{L}_c)$ is a test configuration with general fibre (X,L).

Now $\mathbb{C}^\times$ acts trivially on (X,L) and so also on ν_Z^* and $L|_Z$, but with weight -1 on t, so (4.5) is also the weight space decomposition of $H^0(\mathcal{L}_c^{\otimes k}|_{\mathscr{X}_0})$ into the pieces of weight $-(ck-i)$. Thus the total weight of the action on $H^0(\mathcal{L}_c^{\otimes k}|_{\mathscr{X}_0})$ is

$$\begin{aligned} w(k) &= -\sum_{i=0}^{ck-1}(ck-i)h^0(L^{\otimes k}|_Z\otimes S^i\nu_Z^*) \\ &= -\sum_{i=0}^{ck-1}(ck-i)\chi(L^{\otimes k}|_Z\otimes S^i\nu_Z^*) \\ &= -\sum_{i=0}^{ck-1}(ck-i)\left(\alpha_1(i/k)k^{n-1}+\alpha_2(i/k)k^{n-2}+O(k^{n-3})\right), \end{aligned}$$

using the fact that $H^j(L^{\otimes k}\otimes S^i\nu_Z^*)=0$ for $j>0,\ k\gg 0,\ ck\in\mathbb{N},\ i=0,\dots,ck-1$ (by (4.4) and $S^i\nu_Z^*=\mathscr{I}_Z^i/\mathscr{I}_Z^{i+1}$). Here the α_i are as in (3.16) and (3.17). The k^{n+1} and k^n terms of $w(k)$ can be calculated using the trapezium rule (Lemma 4.7), giving $w(k)=b_0k^{n+1}+b_1k^n+$

$O(k^{n-1})$, where

$$b_0 = -\int_0^c (c-x)\alpha_1(x)dx = \int_0^c a_0(x)dx - ca_0, \tag{4.6}$$

$$b_1 = -\int_0^c \left((c-x)\alpha_2(x) + \frac{\alpha_1(0)}{2}\right) dx$$

$$= \int_0^c \left(a_1(x) + \frac{a_0'(x)}{2}\right) dx - ca_1,$$

where each line follows from integration by parts and (3.17).

As (X, L) is assumed to be K-semistable, the Donaldson-Futaki invariant F_1 of the test configuration $(\mathscr{X}, \mathcal{L}_c)$ is nonnegative so

$$0 \le F_1 = \frac{1}{a_0^2}(b_0a_1 - b_1a_0) = \frac{-b_0}{a_0}\left(\frac{b_1}{b_0} - \frac{a_1}{a_0}\right) = \frac{-b_0}{a_0}\left(\mu_c(\mathcal{O}_Z) - \mu(X)\right),$$

where the last equality uses (3.13) and (4.6). Using (3.17, 3.10), $\alpha_1(x) = -a_0'(x)$ is positive for $0 < x < \epsilon(Z)$. By equation (4.6) this shows that $b_0 < 0$, and hence $\mu_c(\mathcal{O}_Z) < \mu(X)$ as required. q.e.d.

Lemma 4.7. *Let $f(x)$ be a polynomial. Then*

$$\sum_{i=0}^{ck-1}(ck-i)f(i/k) = \int_0^c \left(k^2(c-x)f(x) + \frac{k}{2}f(0)\right) dx + O(k^0).$$

Proof. If $f(x) = \alpha$ is constant then both sides equal $\frac{\alpha}{2}ck(ck+1)$. So by linearity we may assume $f(x) = x^m,\ m \ge 1$. Using $\sum_{i=0}^k i^m = \frac{1}{m+1}k^{m+1} + \frac{1}{2}k^m + O(k^{m-1})$ we get

$$\sum_{i=0}^{ck-1}(ck-i)f(i/k) = k^{-m}\sum_{i=0}^{ck}(ck-i)i^m$$

$$= \int_0^c k^2(c-x)x^m dx + O(k^0),$$

as required. q.e.d.

Although we will not use it, we indicate how this result extends to K-stability.

Theorem 4.8. *Suppose (X, L) is analytically K-stable. Then it is slope stable with respect to any smooth subscheme.*

Proof. Suppose that $\mu_c(\mathcal{O}_Z) = \mu(X)$ for some $0 < c < \epsilon(Z)$ (with c possibly irrational). By convexity of the ample cone and (4.1) $\mathcal{L}_c$ is ample and thus the degeneration to the normal cone $(\mathscr{X}, \mathcal{L}_c)$ is an analytic test configuration. For rational d close to c we have from the previous proof a test configuration $(\mathscr{X}, \mathcal{L}_d)$ with Futaki invariant

$$F_1(d) = \frac{-b_0}{a_0}\left(\mu_d(\mathcal{O}_Z) - \mu(X)\right).$$

Thus the Futaki invariant of $(\mathscr{X}, \mathcal{L}_c)$ is

$$F_1 = \lim_{d\to c} F_1(d) = \lim_{d\to c} \frac{-b_0}{a_0}\left(\mu_d(\mathcal{O}_Z) - \mu(X)\right) = 0$$

since $\lim_{d\to c}\mu_d(\mathcal{O}_Z) - \mu(X) = \mu_c(\mathcal{O}_Z) - \mu(X) = 0$. Thus (X, L) is not analytically K-stable.

Now if $c = \epsilon(X)$ is rational and $\mu_c(\mathcal{O}_Z) = \mu(X)$ then it is shown in [**RT**] Theorem 4.18 that (X, L) is not algebraically K-stable, and thus is not analytically K-stable either. q.e.d.

4.3. Toric test configurations. For toric varieties we can relate Donaldson's weight computation [**Do3**] to ours by an application of Fubini's theorem; *i.e.* a change of order of integration. Let $X_P = (X, L)$ be toric, defined by an integral polytope $P \subset \mathbb{R}^n$ such that $kP \cap \mathbb{Z}^n \cong H^0(X, L^{\otimes k})$. Let $f\colon P \to \mathbb{R}$ be a strictly positive, rational, concave and piecewise linear function. Then the polytope

$$Q = \{(p,t) \in P \times \mathbb{R} : 0 \le t \le f(p)\}$$

defines a toric variety with a $\mathbb{Q}$-polarisation $\mathcal{L}$, a $\mathbb{C}^\times$-action and an equivariant flat map to $\mathbb{P}^1$. Removing the fibre over $\{\infty\} \in \mathbb{P}^1$ gives a test configuration $(\mathscr{X}, \mathcal{L})$ with general fibre (X, L) and $\mathbb{C}^\times$ acting on the section $(s, i) \in kQ \cap \mathbb{Z}^{n+1} \cong H^0(\mathscr{X}, \mathcal{L}^{\otimes k})$ with weight $-i$.

Let $\#(kQ)$ denote the number of lattice points in kQ. When f is *integral*, Donaldson [**Do3**] shows that the weight of this degeneration is $w_k = \#(kP) - \#(kQ) = b_0k^{n+1} + b_1k^n + O(k^{n-1})$ where

$$b_0 = -\int_P f\,d\mu = -\operatorname{vol}(Q), \qquad b_1 = -\frac{1}{2}\int_{\partial P} f\,d\sigma. \tag{4.9}$$

Here $d\mu$ is the standard measure on $\mathbb{R}^n$ and $d\sigma$ is defined by requiring that on any face of P given by a primitive integral conormal vector $h : \mathbb{R}^n \to \mathbb{R}$, we have $d\sigma \wedge dh = \pm d\mu$.

Any toric subvariety of X is defined by a face of P. Such a face is an intersection of codimension 1 faces. Pick primitive integral conormal vectors $\{f_i\}_{i=0}^m$ to the faces, with their signs chosen so that $f_i \ge 0$ on P. Then the ideal of the subvariety is generated by the monomials

$$\left\{p \in P \cap \mathbb{Z}^n \colon f_i(p) \ge 1 \text{ for some } i\right\} = \left\{p \in P \cap \mathbb{Z}^n \colon \sum_{i=1}^m f_i(p) \ge 1\right\},$$

since $f_i(p) \ge 0$ for all $p \in P$. Therefore, more generally, the ideal of any integrally closed toric *subscheme* Z (with multiplicities m_i in the direction of f_i) is generated by the monomials

$$\left\{p \in P \cap \mathbb{Z}^n \colon \sum_{i=1}^m \frac{f_i(p)}{m_i} \ge 1\right\}. \tag{4.10}$$

(We have lost nothing by passing to the integral closure of $\mathscr{I}_Z$; this corresponds to taking the normalisation of the deformation to the normal cone of Z, and in testing K- or slope stability one need only consider normal test configurations (by Proposition 5.1 of [**RT**]) since their Futaki invariants are smaller and so less stable.)

The deformation to the normal cone $(\mathscr{X}, \mathcal{L}_c)$ of this Z corresponds to taking the positive, rational, concave, piecewise linear function $f = \min\left(c, \sum_{i=1}^m \frac{f_i}{m_i}\right)$ in Donaldson's construction (see Figure 2, which should of course be compared to Figure 1). (This f is ≥ 0 but not everywhere > 0, so to get the right geometry we must add a positive constant to it. Since the resulting Donaldson-Futaki invariant is independent of the constant, we calculate without it.) Thus $f\colon P \to [0, c]$ and

$$f^{-1}[x, c] = P_x := \left\{ p \in P\colon \sum_{i=1}^m \frac{f_i(p)}{m_i} \geq x \right\}, \tag{4.11}$$

which from (4.10) is seen to have integral points in $\frac{1}{k}\mathbb{Z}^n$ which form a basis for $H^0(X, L^{\otimes k} \otimes \mathscr{I}_Z^{xk})$. So comparing coefficients in $h^0(X, L^{\otimes k} \otimes \mathscr{I}_Z^{xk}) = \#(kP_x) = \mathrm{vol}(P_x)k^n + \frac{1}{2}\mathrm{vol}(\partial P_x)k^{n-1} + O(k^{n-2})$ yields

$$\mathrm{vol}(P_x) = a_0(x), \qquad \mathrm{vol}(\partial P_x) = 2a_1(x). \tag{4.12}$$

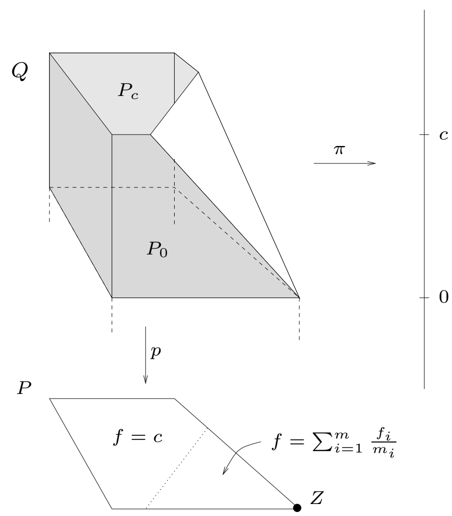

Figure 2. Toric representation $Q = \mathrm{graph}(f)$ of the deformation to the normal cone of $Z \subset X_P$.

To relate Donaldson's weight formula (4.9) to ours (4.6) we change the order of integration with respect to the two projections $Q \xrightarrow{p} P$ and $Q \xrightarrow{\pi} \mathbb{R}$. That is, using (4.12),

$$\int_P f = \int_P p_* 1 = \int_Q 1 = \int_0^c \pi_* 1 = \int_0^c \mathrm{vol}(P_x)dx = \int_0^c a_0(x)dx. \tag{4.13}$$

Similarly we can compute the volume of $\partial Q \backslash (P_0 \cup P_c)$ (the "sides" of Q) as $\int_0^c (\pi|_{\partial Q})_* 1 = \int_0^c \mathrm{vol}(\partial P_x)dx = 2\int_0^c a_1(x)dx$ using (4.12). Now $\mathrm{vol}(P_0) = a_0$ and $\mathrm{vol}(P_c) = a_0(c)$, so $\mathrm{vol}(\partial Q) = 2\int_0^c a_1(x)dx + a_0 + a_0(c)$. But this can be computed differently as the volume of the shaded area in Figure 2, plus the volumes of the top and bottom. If f is integral (which is always assumed in [**Do3**]) then the top and bottom are (piecewise) integrally affine isomorphic by the projection p; equivalently they have the same number of integral points and so the same volume a_0. If f is rational there are less points on the top face, so the result has larger Futaki invariant, *i.e.* it is more stable. Alternatively we can multiply f by an integer N to make it integral; this corresponds to taking the Nth power of the $\mathbb{C}^\times$-action and normalising the resulting test configuration. Again, by Proposition 5.1 of [**RT**], this has Futaki invariant more stable than (N times) the old Futaki invariant. So either way we may as well assume, like Donaldson, that f is integral.

The area of the shaded region is computed by $\int_{\partial P} f$, so we have found that

$$2\int_0^c a_1(x)dx + a_0 + a_0(c) = \int_{\partial P} f + 2a_0,$$

and so

$$\text{(4.14)}\quad \frac{1}{2}\int_{\partial P} f = \int_0^c a_1(x)dx + \frac{1}{2}(a_0(c) - a_0) = \int_0^c \left(a_1(x) + \frac{a_0'(x)}{2}\right)dx.$$

(4.13) and (4.14) differ from $-b_0$ and $-b_1$ in (4.6) by ca_0 and ca_1 respectively, which cancel in the Futaki invariant (2.3) (or can be removed by adding c to f). So we recover Donaldson's formulae in this case.

4.4. Extension to Kähler manifolds. The definition of K-(poly/semi)stability given in (2.4) cannot be defined when the Kähler class is not rational, but slope (poly/semi)stability can. The same issue arises for bundles; GIT cannot construct moduli for bundles over non-projective manifolds, but the slope criterion for stability generalises to all Kähler manifolds and Uhlenbeck-Yau proved that it is equivalent to the existence of a HYM connection in this generality.

To define slope stability we must define the slope of an analytic subspace Z of a Kähler manifold (X, ω). We work on the blow up $\pi\colon \widehat{X} \to X$ of X in Z, with exceptional divisor E. By the singular Hirzebruch-Riemann-Roch formula for analytic spaces [**Ful**] we can define a Todd *homology* class of $\widehat{X}$, and then define the polynomials $a_i(x)$ by the formula

$$\int_{\mathrm{Td}\,(\widehat{X})} \exp(k\pi^*\omega - xke) = a_0(x)k^n + a_1(x)k^{n-1} + O(k^{n-2}).$$

Here e denotes any differential form Poincaré dual to E, and when X is projective with $\omega = c_1(L)$ this gives the same definition as (3.3). In particular $a_0(x) = \frac{1}{n!}\int_{\widehat{X}}(\pi^*\omega - xe)^n$, while we can write $a_1(x)$ in terms of any resolution of singularities $p\colon \overline{X} \to \widehat{X}$:

$$a_1(x) = \frac{1}{2(n-1)!}\int_{\overline{X}}((p\circ\pi)^*\omega - xp^*e)^{n-1}c_1(\overline{X}).$$

Take any $c > 0$ such that $\omega - ce$ has nonnegative volume on any analytic subvariety of $\widehat{X}$ (if $\widehat{X}$ is smooth then this is the condition that $\omega - ce$ be in the closure of the Kähler cone of $\widehat{X}$ [**DP**]). Then define the slope of $\mathscr{I}_Z$, with respect to ω and c, as before:

$$\mu_c(\mathscr{I}_Z) := \frac{\int_0^c \left(a_1(x) + \frac{a_0'(x)}{2}\right)dx}{\int_0^c a_0(x)dx}.$$

We say that X is slope semistable if $\mu_c(\mathscr{I}_Z) \le \mu(X)$ for all proper $Z \subset X$ and c such that $\omega - ce$ has nonnegative volume on any analytic subvariety of $\widehat{X}$. For slope stability we require that $\mu_c(\mathscr{I}_Z) < \mu(X)$ for all c such that $\omega - ce$ is the pullback of a Kähler form on a Kähler variety. We define X to be slope polystable if it is slope semistable and $\mu_c(\mathscr{I}_Z,\omega) = \mu(X)$ implies that $\omega - ce$ on the deformation to the normal cone of Z is pulled back from a map to the product $X \times \mathbb{C}$.

Since the $\mathbb{C}^\times$-action on the degeneration to the normal cone is trivial on the central fibre except on the component P, one can use the localisation formula on P to calculate the Calabi-Futaki invariant in terms of the resulting vector field. This gives an alternate, but fundamentally equivalent, definition of slope for an analytic Z in a Kähler manifold. Then one would expect that the usual argument that the derivative of the Mabuchi functional is the Futaki invariant of the central fibre (defined in terms of the vector field) should show that if X is not slope polystable then the Mabuchi functional is not proper, and so the class $[\omega]$ does not admit a cscK metric.

5. Examples

5.1. Slope of divisors and curves.

Theorem 5.1. *Let (X, L) be a polarised manifold of dimension $n \ge 2$ and suppose that Z is a smooth curve in X of genus g with normal bundle ν_Z. Then*

$$\mu_c(\mathcal{O}_Z) = \frac{n^2(n^2-1)(L.Z) - cn(n+1)[(n-2)c_1(\nu_Z) + 2(g-1)]}{2nc[(n+1)(L.Z) - cc_1(\nu_Z)]}.$$

Proof. The Riemann-Roch theorem for curves yields

$$\begin{aligned}\chi(L^{\otimes k}|_Z \otimes S^{xk}\nu_Z^*) &= \operatorname{rank} S^{xk}\nu_Z \cdot (kL.Z - \frac{xkc_1(\nu_Z)}{n-1} + 1 - g) \\ &= \alpha_1(x)k^{n-1} + \alpha_2(x)k^{n-2} + O(k^{n-3}),\end{aligned}$$

since $\frac{c_1(S^i\nu_Z^*)}{\operatorname{rank} S^i\nu_Z} = -i\frac{c_1(\nu_Z)}{n-1}$. Now $\operatorname{rank} S^{xk}\nu_Z = \binom{xk+n-2}{n-2}$ equals

$$\frac{1}{(n-2)!}\left(x^{n-2}k^{n-2} + \frac{(n-2)(n-1)}{2}x^{n-3}k^{n-3} + O(k^{n-4})\right).$$

(This makes sense even if $n = 2$ as in that case the k^{n-3} term vanishes.) Thus

$$\begin{aligned}\alpha_1(x) &= \frac{x^{n-2}}{(n-2)!}\left(L.Z - \frac{xc_1(\nu_Z)}{n-1}\right),\\ \alpha_2(x) &= \frac{x^{n-3}}{(n-2)!}\left(\frac{(n-2)(n-1)}{2}\left(L.Z - \frac{xc_1(\nu_Z)}{n-1}\right) + x(1-g)\right).\end{aligned}$$

Integration and rearranging (3.15) gives the formula for $\mu_c(\mathcal{O}_Z)$. q.e.d.

Theorem 5.2. *Suppose that Z is a divisor in (X, L). Then*

$$\mu_c(\mathcal{O}_Z) = \frac{n\left(L^{n-1}.Z - \sum_{j=1}^{n-1}\binom{n-1}{j}\frac{(-c)^j}{j+1}L^{n-1-j}.Z^j.(K_X(Z))\right)}{2\sum_{j=1}^{n}\binom{n}{j}\frac{(-c)^j}{j+1}L^{n-j}.Z^j}.$$

Proof. As Z is a divisor, $\widehat{X} = X$ so (3.4, 3.5)

$$\tilde{a}_0(x) = a_0 - a_0(x) = \frac{1}{n!}(L^n - (L-xZ)^n) = -\frac{1}{n!}\sum_{j=1}^{n}\binom{n}{j}L^{n-j}.(-xZ)^j,$$

and

$$\begin{aligned}\tilde{a}_1(x) + \frac{\tilde{a}_0'(x)}{2} &= \frac{1}{2(n-1)!}\left(-K_X.L^{n-1} + (K_X(Z)).(L-xZ)^{n-1}\right)\\ &= \frac{1}{2(n-1)!}\sum_{j=1}^{n-1}\binom{n-1}{j}L^{n-1-j}.(-xZ)^j.(K_X(Z))\\ &\quad + \frac{1}{2(n-1)!}L^{n-1}.Z.\end{aligned}$$

Integrating these expressions gives the required formula. q.e.d.

The formulae (5.1) and (5.2) agree for curves in surfaces; the result simplifies to the following.

Corollary 5.3. *Let Z be a smooth curve in a smooth polarised surface (X, L). Then*

$$\begin{aligned}\mu(X) &= -\frac{K_X.L}{L^2},\\ \mu_c(\mathcal{O}_Z) &= \frac{3[2L.Z - c(K_X.Z + Z^2)]}{2c(3L.Z - cZ^2)}.\end{aligned}$$

If Z is a smooth rational curve then

$$\mu_c(\mathcal{O}_Z) = \frac{3(L.Z + c)}{c(3L.Z - cZ^2)}.$$

We use these formulae in Section 5.6 to give examples of unstable rational surfaces.

5.2. Manifolds with nonpositive first Chern class. The existence of Kähler-Einstein metrics when $c_1(X) \leq 0$ gives K-semistability in these cases by the results of Donaldson [**Do5**]. We give a direct proof that smooth subschemes $Z \subset X$ do not destabilise; a more general proof for arbitrary Z (and extended to varieties X with canonical singularities) is given in Theorem 8.4 of [**RT**].

Theorem 5.4. *A polarised manifold (X, L) is slope stable with respect to smooth subschemes if either*

1) *K_X is numerically trivial, or*
2) *K_X is ample and L is a multiple of K_X.*

Proof. In both cases, $K_X \sim \alpha L$ is numerically equivalent to a nonnegative multiple $\alpha \geq 0$ of the polarisation. So $\mu(X) = a_1/a_0 = -nK_X.L^{n-1}/2L^n = -n\alpha/2$. If $Z \subset X$ is a codimension p submanifold, the canonical divisor of the blow up is $K_{\widehat{X}} = K_X((p-1)E)$. Letting $L_x := L(-xE)$,

$$\begin{aligned} -\mu(X)a_0(x) + a_1(x) &= \frac{\alpha}{2(n-1)!}L_x^n - \frac{1}{2(n-1)!}K_{\widehat{X}}.L_x^{n-1} \\ &= -\frac{1}{2(n-1)!}(\alpha x + p - 1)L_x^{n-1}.E \ \leq \ 0, \end{aligned}$$

since L_x is nef for $x \in (0, \epsilon(Z))$. As $a_0'(x) < 0$ (3.11), integration gives

$$-\mu(X)\int_0^c a_0(x)dx + \int_0^c a_1(x) + \frac{a_0'(x)}{2}dx < 0 \quad \text{for } c \in (0, \epsilon(Z)].$$

Rearranging this gives slope stability, $\mu_c(\mathscr{I}_Z) < \mu(X)$. q.e.d.

With more work these results can be extended to show that when K_X is nef and $K_X^n > 0$, then X is slope stable for L sufficiently close to K_X. More precisely, using additive notation ($aL + bK := L^{\otimes a} \otimes K^{\otimes b}$) for line bundles,

Theorem 5.5. *Fix a polarised manifold (X, L) with K_X nef and $K_X^n > 0$. Then (X, L) is slope stable with respect to smooth subschemes if*

1) *$2\mu(X, L)L + nK_X$ is nef, or*
2) *$-2\mu(X, L)L - nK_X$ is nef.*

Moreover, for any divisor G there is a $\delta_0 > 0$ such that if $0 \leq \delta < \delta_0$ and $L = K_X(\delta G)$ is ample then (X, L) is slope stable with respect to smooth subschemes.

Remark 5.6. Suppose K_X is ample. Then there exists an open set around $-[c_1(X)]$ of classes which admit cscK metrics [**LeBS**], so one has slope-semistability for these classes.

It is shown in [**We**] that if

$$-2\mu(X,L)L-(n-1)K_X$$

is ample then the Mabuchi functional associated to the class $c_1(L)$ is bounded from below, confirming the second result of Theorem 5.5.

Proof. Fix a Z and suppose $0<x<c\le\epsilon(Z)$. Let

$$f(x)=2n!(n-1)![a_0a_1(x)-a_1a_0(x)].$$

We will show that $\int_0^c f(x)dx\le 0$ for all smooth subschemes Z and all $0<c\le\epsilon(Z)$, which implies $\mu_c(\mathscr{I}_Z)<\mu(X,L)$ since $a_0'(x)<0$. For the third part we will show this holds as long as $\delta<\delta_0$ where δ_0 will be chosen independently of Z and c.

For $x\in(0,\epsilon(Z))$, $L_x:=L(-xE)$ is nef, so as $K_{\widehat{X}}-K_X=(p-1)E$ is effective,

$$\begin{aligned} f(x) &= -(L^n)L_x^{n-1}.K_{\widehat{X}}+(K_X.L^{n-1})L_x^n \\ &\le -(L^n)L_x^{n-1}.K_X+(K_X.L^{n-1})L_x^n \\ &= L_x^{n-1}.(B-x(K_X.L^{n-1})E), \end{aligned}$$

where

$$\begin{aligned} B &:= (K_X.L^{n-1})L-(L^n)K_X \\ &= (K_X.L^{n-1})(L-K_X)-((L-K_X).L^{n-1})K_X \\ &= \delta(K_X.L^{n-1})G-\delta(G.L^{n-1})K_X. \end{aligned}$$

Notice that $B.L^{n-1}=0$. Now, if $-B=\frac{L^n}{n}(2\mu(X,L)L+nK_X)$ is nef then, as L_x is nef, $f(x)\le 0$, which proves (1). When $n=1$ (so X is a smooth curve), B is numerically trivial so $f(x)\le 0$ and we are done. So we suppose that $n\ge 2$.

As $B=O(\delta)$ we certainly have $f(x)\le 0$ for δ sufficiently small for any fixed value of x. However, since such a choice of δ is not uniform in x, we integrate:

$$\int_0^c f(x)dx=I_1-(K_X.L^{n-1})I_2, \tag{5.7}$$

where $I_1=\int_0^c L_x^{n-1}.Bdx$ and $I_2=\int_0^c xL_x^{n-1}.Edx$. Then

$$\begin{aligned} L_x^{n-1}.B &= L^{n-1}.B+(L_x-L).\sum_{j=0}^{n-2}L^j.L_x^{n-2-j}.B \\ &= -x\sum_{j=0}^{n-2}L^j.L_x^{n-2-j}.E.B, \qquad \text{as } L^{n-1}.B=0. \end{aligned}$$

We claim that for any a and b,

$$\sum_{j=0}^{n-2}\int_0^c xa^j(a-xb)^{n-2-j}dx = \frac{c^2}{n}\sum_{j=0}^{n-2}(j+1)a^j(a-cb)^{n-2-j},$$

which can be shown by comparing the coefficient of c^{n-j} on both sides for $j=0,\dots,n-2$ and using the identity

$$\sum_{j=0}^{i}(j+1)\binom{n-2-j}{i-j} = \frac{n}{n-i}\sum_{j=0}^{i}\binom{n-2-j}{i-j} \quad \text{for } i=0,\dots,n-2.$$

Hence

$$I_1 = \int_0^c L_x^{n-1}.Bdx \le c^2\sum_{j=0}^{n-2}L^j.L_c^{n-2-j}.E.B. \tag{5.8}$$

Similarly, as

$$\int_0^c x(a-xb)^{n-1}dx = \frac{c^2}{n(n+1)}\sum_{j=0}^{n-1}(n-j)a^j(a-cb)^{n-1-j},$$

$$I_2 = \int_0^c xL_x^{n-1}.Edx \ge \frac{c^2}{n(n+1)}\sum_{j=0}^{n-2}L^{j+1}.L_c^{n-2-j}.E. \tag{5.9}$$

Putting (5.7), (5.8), (5.9) together

$$\begin{aligned}\int_0^c f(x)dx &\le I_1-(K_X.L^{n-1})I_2\\ &\le -c^2\left(B+\frac{(K_X.L^{n-1})}{n(n+1)}L\right).\sum_{j=0}^{n-2}L^j.L_c^{n-2-j}.E.\end{aligned}$$

Recall that L and L_c are nef classes. So it is now sufficient to prove that $B+\frac{(K_X.L^{n-1})}{n(n+1)}L$ is also nef. But

$$B+\frac{(K_X.L^{n-1})}{n(n+1)}L = \frac{L^n}{n}\left(-2\mu(X,L)L-nK_X-\frac{2\mu(X,L)}{n(n+1)}L\right).$$

As $\mu(X,L)\le 0$, this is nef when $-2\mu(X,L)L-nK_X$ is, proving (2).

To prove the third part we must show that $\int_0^c f(x)\le 0$ uniformly with respect to δ. Notice that the statement of the theorem is unchanged if we scale G by some positive number. So without loss of generality we suppose that $K_X(G)$ is ample. Now

$$\begin{aligned}B+\frac{(K_X.L^{n-1})}{n(n+1)}L &= \frac{(K_X.L^{n-1})}{2n(n+1)}(L+(2n(n+1)+1)\delta G)\\ &\quad+\left(\frac{(K_X.L^{n-1})}{2n(n+1)}-\delta(G.L^{n-1})\right)K_X.\end{aligned}$$

For positive δ sufficiently small the line bundle

$$L + (2n(n+1)+1)\delta G = K_X + (2n(n+1)+2)\delta G$$

is ample, for it lies on the line between K_X (which is nef) and $K_X(G)$ (which is ample). Moreover

$$\frac{(K_X.L^{n-1})}{2n(n+1)} - \delta(G.L^{n-1}) = \frac{K_X^n}{2n(n+1)} + O(\delta)$$

is positive for δ sufficiently small, since $K_X^n > 0$. Hence $B + \frac{(K_X.L^{n-1})}{n(n+1)}L$ is nef for δ sufficiently small, and the proof is complete. q.e.d.

5.3. Slope stability of smooth curves. Since smooth curves always have cscK metrics, they should be stable. We give a direct proof that they are slope (poly)stable:

Theorem 5.10. *Any smooth polarised curve (Σ, L) of genus g is slope stable if $g \geq 1$ and strictly slope polystable if $g = 0$.*

Proof. Any nonempty subscheme Z is a divisor of degree $d > 0$, so

$$\chi(L^{\otimes k} \otimes \mathscr{I}_Z^{xk}) = k \deg L - xdk + 1 - g$$

which shows that $\tilde{a}_0(x) = xd$ and $\tilde{a}_1(x) = 0$. Thus $\mu_c(\mathcal{O}_Z) = \frac{cd}{c^2 d} = \frac{1}{c} > 0 \geq \frac{1-g}{\deg L} = \mu(X)$ for $g \geq 1$, proving slope stability.

For $g = 0$, c may take values up to and *including* $\epsilon(Z) = \deg L/d$, since $L^{\otimes d} \otimes \mathscr{I}_Z^{\deg L} = \mathcal{O}_{\mathbb{P}^1}(d \deg L - d \deg L) = \mathcal{O}_{\mathbb{P}^1}$ is globally generated. Thus $\mu_c(\mathcal{O}_Z) \geq \frac{d}{\deg L} \geq \frac{1}{\deg L} = \mu(X)$ with equality (strict semistability) only for $d = 1$, *i.e.* Z a single point, and $c = \epsilon(Z)$. Since the deformation to the normal cone of a single point on $\mathbb{P}^1$ blows down to $\mathbb{P}^1 \times \mathbb{C}$ (with a nontrivial $\mathbb{C}^\times$-action) from which the relevant line bundle $\mathcal{L}_\epsilon$ pulls back, we find $\mathbb{P}^1$ is in fact slope polystable. q.e.d.

Remark 5.11. In Corollary 6.7 of [**RT**] it is shown that, for smooth curves, slope (semi/poly)stability is equivalent to K-(semi/poly)stability. Thus smooth curves are algebraically K-stable for $g \geq 1$ and algebraically K-polystable if $g = 0$.

5.4. Projective bundles. Fix a polarised manifold $(B, \mathcal{O}_B(1))$ of dimension b, and let E be a vector bundle on B with $r + 1 := \operatorname{rank} E \geq 2$. We show that the stability of $\mathbb{P}(E)$ is related to slope stability of the bundle E (as defined in 5.4.1) and slope stability of the base B. Let $n = \dim \mathbb{P}(E) = b + r$ and

$$L_m = \mathcal{O}_{\mathbb{P}(E)}(1) \otimes \mathcal{O}_B(m),$$

which is ample for m sufficiently large.

Theorem 5.12. *If $(\mathbb{P}(E), L_m)$ is slope semistable for all $m \gg 0$ then E is a slope semistable vector bundle and $(B, \mathcal{O}_B(1))$ is a slope semistable manifold. Moreover, there is an m_0 which depends only on E and $(B, \mathcal{O}_B(1))$ such that if $(\mathbb{P}(E), L_m)$ is slope semistable for some $m \geq m_0$ then E is a slope semistable vector bundle.*

Thus if E is a strictly slope unstable bundle or if $(B, \mathcal{O}_B(1))$ is a strictly slope unstable manifold, then $\mathbb{P}(E)$ does not admit a cscK metric in $[c_1(L_m)]$ for $m \gg 0$.

For bundles (of any rank) over curves, we get stronger results.

Theorem 5.13. *Suppose B is a smooth curve of genus $g \geq 1$ and that L_m is ample. If $(\mathbb{P}(E), L_m)$ is slope (semi/poly)stable then E is slope (semi/poly)stable.*

If E is polystable then $\mathbb{P}(E)$ has a cscK metric in every Kähler class. Conversely if E is strictly unstable then $\mathbb{P}(E)$ does not admit a cscK metric in any rational Kähler class. Finally, if E is not polystable then $\mathbb{P}(E)$ is not algebraically K-polystable.

The proofs appear after a calculation of the relevant slopes and Seshadri constants. It is well known that if E is polystable and B is a curve then $\mathbb{P}(E)$ admits a cscK metric in every Kähler class [**BdB**]. So Theorem 5.13 gives an almost complete converse. If, as expected, a cscK metric implies algebraic K-polystability then Theorem 5.13 would be a full converse. Moreover it would imply that slope polystability is equivalent to algebraic K-polystability for projective bundles over curves of genus $g \geq 1$.

There is also a partial converse to Theorem 5.12. Suppose that E is slope stable, and B is a manifold with $\mathfrak{aut}(B) = 0$ and a cscK metric in $c_1(\mathcal{O}_B(1))$. Then there exists a cscK metric on $\mathbb{P}(E)$ in $c_1(L_m)$ for m sufficiently large [**Ho**].

In the rank $E = 2$, $\dim B = 1$ case, it is known that if a ruled surface $\mathbb{P}(E)$ has a cscK metric in any class then E is a polystable bundle. This is proved by [**BdB**] in the scalar-flat case, by [**LeB**] in the case that $g \geq 2$, $-\int_X c_1(K_X).c_1(L) < 0$, and [**AT**] in general.

The stability of ruled surfaces has also been studied by Morrison [**Mo**]. If E is unstable then $\mathbb{P}(E)$ is Chow unstable with respect to what he calls "good" polarisations (in particular $(\mathbb{P}(E), L_m^{\otimes k})$ is Chow unstable for $k \gg 0$). By [**Do2**] this implies that $\mathbb{P}(E)$ does not have a cscK metric in any class. Morrison also shows that if E is stable then for suitable m, $(\mathbb{P}(E), L_m)$ is Chow stable, and he conjectures that this holds for $(\mathbb{P}(E), L_m^{\otimes k})$ with $k \gg 0$. Since there exists a cscK metric in $c_1(L_m)$, this conjecture follows from [**Do2**] when $g \geq 2$ and E is simple.

Remark 5.14. Suppose that $E \to B$ has a subbundle F. Let $\mathscr{X}$ be the degeneration to the normal cone of $\mathbb{P}(F) \subset (\mathbb{P}(E), L_m)$ with $c = \epsilon(\mathbb{P}(F)) = 1$. Then $\mathcal{L}_c$ is only semi-ample but not ample, and contracts a component of the central fibre $\mathscr{X}_0$ (test configurations with semi-ample polarisation are studied in Proposition 5.1 of [**RT**]). This contraction is a test configuration which is the projectivisation of the degeneration of bundles taking the extension

$$0 \to F \to E \to G \to 0, \quad \text{defined by} \ \ e \in \mathrm{Ext}^1(G, F),$$

to the direct sum $F \oplus G$ (via the family of extensions $\lambda e,\ \lambda \in \mathbb{C}$). If $e = 0$ (*i.e.* $E = F \oplus G$ to begin with) then we get a product degeneration. We show below that if F and E have the same slope then the Futaki invariant is 0 (on curves, and to the two top orders in m for general B). So we recover the usual notion of polystability for bundles.

5.4.1. Slope stability of vector bundles. For brevity write $\mu(B) = \mu(B, \mathcal{O}_B(1))$. For any coherent sheaf E on B it is convenient to define

$$\mu_E = \frac{\deg E}{a_0^B (b-1)! \operatorname{rank} E} + \mu(B),$$

where $\chi(\mathcal{O}_B(k)) = a_0^B k^b + a_1^B k^{b-1} + O(k^{b-2})$. Note that this differs from the usual definition of slope for a sheaf. However, for any coherent subsheaf F,

$$\mu_E - \mu_F = \frac{1}{a_0^B (b-1)!} \left(\frac{\deg E}{\operatorname{rank} E} - \frac{\deg F}{\operatorname{rank} F} \right).$$

Thus E is a slope stable (resp. semistable) vector bundle if and only if $\mu_F < \mu_E$ (resp. $\mu_F \leq \mu_E$) for all coherent subsheaves $F < E$. And E is polystable if and only if it is a direct sum $E = \oplus F_i$ of slope stable sheaves, with $\mu_{F_i} = \mu_E$ for all i.

Lemma 5.15. *Let E and F be torsion free coherent sheaves on B. Then*

1) $\chi(E \otimes \mathcal{O}_B(m)) = a_0^B \operatorname{rank} E (m^b + \mu_E m^{b-1}) + O(m^{b-2})$, *where the* $O(m^{b-2})$ *is understood to be zero when* $b = \dim B = 1$,
2) $\mu_{S^k E^*} = (1 + k)\mu(B) - k\mu_E$,
3) $\mu_{E \otimes F} = \mu_E + \mu_F - \mu(B)$,
4) *if* $F < E$ *and* E/F *is also torsion free then*

$$(\operatorname{rank} E)\mu_E = (\operatorname{rank} F)\mu_F + (\operatorname{rank}(E/F))\mu_{E/F}.$$

Proof. From the definition of μ_E and the Riemann-Roch theorem,

$$\begin{aligned} \chi(E \otimes \mathcal{O}_B(m)) &= \int_B \mathrm{ch}(\mathcal{O}_B(m))\mathrm{ch}(E)\mathrm{Td}_B \\ &= \int_B e^{mc_1(\mathcal{O}_B(1))}(\operatorname{rank} E + c_1(E) + \cdots)\mathrm{Td}_B \\ &= a_0^B \operatorname{rank} E (m^b + \mu_E m^{b-1}) + O(m^{b-2}). \end{aligned}$$

Now as E is torsion free, we can calculate the degrees of E and S^kE^* by restricting to the set where E is locally free, since its complement has codimension ≥ 2. We compute $\mu_{S^kE^*}$ to be

$$\begin{aligned}\frac{\deg S^kE^*}{a_0^B(b-1)!\operatorname{rank} S^kE}+\mu(B) &= -k\frac{\deg E}{a_0^B(b-1)!\operatorname{rank} E}+\mu(B)\\ &= (1+k)\mu(B)-k\mu_E,\end{aligned}$$

where the second equality follows from the splitting principle. Also,

$$\mu_{E\otimes F}=\frac{\operatorname{rank} F\deg E+\operatorname{rank} E\deg F}{a_0^B(b-1)!\operatorname{rank} E\operatorname{rank} F}+\mu(B)=\mu_E+\mu_F-\mu(B).$$

Finally, if $F<E$ then comparing the m^{b-1} terms in $\chi(E\otimes\mathcal{O}_B(m))=\chi(F\otimes\mathcal{O}_B(m))+\chi((E/F)\otimes\mathcal{O}_B(m))$ gives $(\operatorname{rank} E)\mu_E=(\operatorname{rank} F)\mu_F+(\operatorname{rank}(E/F))\mu_{E/F}$. q.e.d.

5.4.2. Seshadri constants of projective subbundles. For the rest of this section let $Z=\mathbb{P}(F)$, so the Seshadri constant $\epsilon(\mathbb{P}(F),L_m)$ is defined as in (3.2).

Lemma 5.16. *For E be a vector bundle over a curve B, $\deg_B E^*=\deg_{\mathbb{P}(E)}\mathcal{O}_{\mathbb{P}(E)}(1)$.*

Proof. Let ω denote $c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$ on $\mathbb{P}(E)$. The general Grothendieck formula $\sum_{i=0}^{r+1}\omega^{r+1-i}c_i(E)=0$ reduces over a curve to $-c_1(E)\omega^r=\omega^{r+1}$, whose left hand side is $-\deg_B E$. q.e.d.

Proposition 5.17. *There is an m_0 (depending only on E and the pair $(B,\mathcal{O}_B(1))$ such that for any $m\geq m_0$ and any* saturated *subsheaf F of the bundle E* (i.e. *E/F is torsion free*) *with $\mu_F\geq\mu_E$ we have $\epsilon=\epsilon(\mathbb{P}(F),L_m)=1$.*

Suppose that B is a curve, $F<E$ is saturated, E/F is semistable and $\mu_F\geq\mu_E$. Then for any m such that L_m is ample, $\epsilon(\mathbb{P}(F),L_m)=1$ and the global sections of $L_m^{\otimes k}\otimes\mathscr{I}^k_{\mathbb{P}(F)}$ generate $\mathscr{I}^k_{\mathbb{P}(F)}$ for $k\gg 0$.

Proof. Since $\mathbb{P}(F_p)\subset\mathbb{P}(E_p)$ is a linear subspace for any $p\in B$, and $L_m|_{\mathbb{P}(E_p)}=\mathcal{O}_{\mathbb{P}(E_p)}(1)$, it follows that $\epsilon\leq 1$. To show ϵ is at least 1 it is sufficient to show that $L_m\otimes\mathscr{I}_{\mathbb{P}(F)}$ is generated by global sections.

Let $G=E/F$. As the set of quotients G of E with $\mu_G\leq\mu_E$ is bounded ([**HL**] Lemma 1.7.9) there is an m_0 (depending only on E and $(B,\mathcal{O}_B(1))$) such that for all $m\geq m_0$, $G^*(m)$ is globally generated and has no higher cohomology and L_m is ample.

Working on $\mathbb{P}(E)$, $\mathcal{O}_{\mathbb{P}(E)}(-1)$ is a subbundle of (the pullback of) E giving a canonical element $u\in\operatorname{Hom}(\mathcal{O}_{\mathbb{P}(E)}(-1),G)$ obtained by composition with the projection from E to G. Thinking of u as a section of $G\otimes\mathcal{O}_{\mathbb{P}(E)}(1)$, its zero set is precisely $\mathbb{P}(F)$.

Now turn to $\mathbb{P}=\mathbb{P}(H^0(L_m)^*)=\mathbb{P}(H^0(E^*(m))^*)$ and let $m\geq m_0$. The exact sequence $0\to H^0(G^*(m))\to H^0(E^*(m))\to H^0(F^*(m))\to 0$

yields a canonical section v of $H^0(G^*(m))^* \otimes \mathcal{O}_{\mathbb{P}}(1)$ whose zero set is $\mathbb{P}(H^0(F^*(m))^*)$.

Since $G^*(m)$ is globally generated, $G(-m)$ injects into $H^0(G^*(m))^*$. Tensoring with L_m shows that $G \otimes \mathcal{O}_{\mathbb{P}(E)}(1)$ injects into $H^0(G^*(m))^* \otimes L_m$, and u maps to v. Thus $\mathbb{P}(F)$ is the intersection of $\mathbb{P}(E)$ with the subspace $\mathbb{P}(H^0(F^*(m))^*)$ of $\mathbb{P}$. Hence $L_m \otimes \mathscr{I}_{\mathbb{P}(F)}$ is generated by global sections, so $\epsilon = 1$ as claimed.

Now suppose that B is a curve, m is chosen so that L_m is ample, $\mu_F \geq \mu_E$, F is destabilising and saturated, and $G = E/F$ is semistable. Since F is saturated, G is torsion free, so both are locally free since B is a curve. Then $\mu_E \geq \mu_G$, so

$$\deg(G^* \otimes \mathcal{O}_B(m)) \geq \deg(E^* \otimes \mathcal{O}_B(m)) = \deg L_m > 0,$$

by Lemma 5.16. Thus $G^* \otimes \mathcal{O}_B(m)$ is a semistable bundle of positive degree on a curve B, so it is ample, *i.e.* $\mathcal{O}_{\mathbb{P}(G)}(1) \otimes \mathcal{O}_B(m)$ is ample ([**La**] 6.4.11). So for $k \gg 0$, $\mathcal{O}_{\mathbb{P}(G)}(k) \otimes \mathcal{O}_B(km)$ is globally generated and thus so is its pushdown $S^k G^* \otimes \mathcal{O}_B(km)$.

Now $\pi_*(L_m^{\otimes k} \otimes \mathscr{I}^k_{\mathbb{P}(F)}) = S^k G^* \otimes \mathcal{O}_B(mk) \ \big(< S^k E^* \otimes \mathcal{O}_B(mk)\big)$ generates $\mathscr{I}^k_{\mathbb{P}(F)}$ on each fibre, so the global sections of $L_m^{\otimes k} \otimes \mathscr{I}^k_{\mathbb{P}(F)}$ generate $\mathscr{I}^k_{\mathbb{P}(F)}$ as claimed. From the definition of the Seshadri constant we again get that $\epsilon = 1$. q.e.d.

5.4.3. Slope of projective bundles. In calculating the quantities $\mu(\mathbb{P}(E), L_m)$ and $\mu(\mathcal{O}_{\mathbb{P}(F)}, L_m)$ it is convenient to make the change of variables

$$\widetilde{m} = m + \frac{1}{b}(\mu(B) - \mu_E). \tag{5.18}$$

(The reader may prefer to assume that $\deg E = 0$, in which case $\widetilde{m} = m$.) We write $\mu(\mathbb{P}^r) := \mu(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)) = r(r+1)/2$.

Lemma 5.19. *Let* $\chi(\mathbb{P}(E), L_m^{\otimes k}) = a_0 k^n + a_1 k^{n-1} + O(k^{n-2})$. *Then* a_0 *and* a_1 *are polynomials in* m. *In fact if* $\widetilde{m}$ *is defined as in* (5.18) *then*

$$\begin{aligned} a_0 &= \frac{a_0^B}{r!}\widetilde{m}^b + O(\widetilde{m}^{b-2}), \qquad \textit{and} \\ a_1 &= \frac{a_0^B}{r!}\Big(\mu(\mathbb{P}^r)\widetilde{m}^b + \mu(B)\widetilde{m}^{b-1}\Big) + O(\widetilde{m}^{b-2}), \end{aligned}$$

where if $\dim B = 1$ *we interpret* $O(\widetilde{m}^{b-2})$ *as being zero. Moreover the* $O(\widetilde{m}^{b-2})$ *terms depend only on* $(B, \mathcal{O}_B(1))$ *and the Chern classes of* E.

Proof. Let $\pi\colon \mathbb{P}(E) \to B$ be the projection. As L_m is relatively ample, for $k \gg 0$,

$$\begin{aligned}
\chi(L_m^{\otimes k}) &= \chi(\pi_*(L_m^{\otimes k}))\\
&= \chi(S^k E^* \otimes \mathcal{O}_B(mk))\\
&= a_0^B \operatorname{rank} S^k E \cdot \left(m^b k^b + \mu_{S^k E^*} m^{b-1} k^{b-1}\right) + O(m^{b-2})\\
&= a_0^B \operatorname{rank} S^k E \cdot \left(m^b k^b + [(1+k)\mu(B) - k\mu_E] m^{b-1} k^{b-1}\right)\\
&\quad + O(m^{b-2}),
\end{aligned}$$

where in the last line we have used (5.15 (2)) and the $O(m^{b-2})$ term is zero if $b = 1$. Now the rank term is

$$\operatorname{rank} S^k E = \binom{r+k}{k} = \frac{1}{r!}\left[k^r + \mu(\mathbb{P}^r)k^{r-1} + O(k^{r-2})\right].$$

Expanding and taking the k^n and k^{n-1} terms gives

$$\begin{aligned}
a_0 &= \frac{a_0^B}{r!}\left(m^b + [\mu(B) - \mu_E] m^{b-1}\right) + O(m^{b-2}),\\
a_1 &= \frac{a_0^B}{r!}\left(\mu(\mathbb{P}^r) m^b + \mu(\mathbb{P}^r)[\mu(B) - \mu_E] m^{b-1} + \mu(B) m^{b-1}\right)\\
&\quad + O(m^{b-2}),
\end{aligned}$$

and the change of variables from m to $\widetilde{m}$ gives the expressions in the statement of the lemma. As the Chern character of $S^k E$ depends only on k and the Chern classes of E ([**Ha**] Appendix A3) we see that the $O(m^{b-2})$ terms (and hence the $O(\widetilde{m}^{b-2})$ terms) depend only on $(B, \mathcal{O}_B(1))$ and the Chern classes of E. q.e.d.

Lemma 5.20. *Let F be a saturated coherent subsheaf of E. Define $\alpha_i(x)$ for $\mathbb{P}(F) \subset \mathbb{P}(E)$ as in Proposition* 3.15*, and let $\widetilde{m}$ be defined as in* (5.18)*. Then,*

$$\begin{aligned}
&\int_0^1 (1-x)\alpha_1(x)dx\\
&= \frac{a_0^B(s+1)}{(r+1)!}\left[\widetilde{m}^b + \frac{1}{r+2}(\mu_E - \mu_F)\widetilde{m}^{b-1}\right] + O(\widetilde{m}^{b-2}),\\
&\int_0^1 \left((1-x)\alpha_2(x) + \frac{\alpha_1(0)}{2}\right) dx\\
&= \frac{a_0^B(s+1)}{2(r+1)!}\left(2\mu(\mathbb{P}^r)\widetilde{m}^b + [2\mu(B) + (r+1)(\mu_E - \mu_F)]\,\widetilde{m}^{b-1}\right)\\
&\quad + O(\widetilde{m}^{b-2}),
\end{aligned}$$

where $O(\widetilde{m}^{b-2})$ is understood to be zero if $\dim B = 1$*. Both expressions are polynomials in $\widetilde{m}$, and the $O(\widetilde{m}^{b-2})$ terms depend only on $(B, \mathcal{O}_B(1))$ and the Chern classes of E and F.*

Proof. Let F have rank $s+1$ and $G=E/F$ have rank $t+1$, so

$$s+t+2=r+1. \tag{5.21}$$

Since F is saturated, G is torsion free, as is F since it sits inside a locally free sheaf E. Thus E, F and G are locally free on an open set U whose complement has codimension at least 2, on which their first Chern classes (and those of their symmetric powers) can be calculated. Since G is locally free on U, $F\hookrightarrow E$ has constant rank so $\mathbb{P}(F|_U)$ sits inside $\mathbb{P}(E|_U)$ as a smooth submanifold with normal bundle

$$\nu=\nu_{\mathbb{P}(F|_U)}=\pi^*G\otimes\mathcal{O}_{\mathbb{P}(F)}(1),$$

where, by abuse of notation, we let $\pi=\pi|_{\mathbb{P}(F)}$. As the complement of $V=\mathbb{P}(E|_U)$ also has codimension at least 2, we can calculate $\alpha_1(x)$ and $\alpha_2(x)$ on V. Then for $0<x<1$,

$$\pi_*(L_m^{\otimes k}\otimes S^{xk}\nu^*)=S^{xk}G^*\otimes S^{(1-x)k}F^*\otimes\mathcal{O}_B(mk).$$

Furthermore, for $0<x\ll 1$ the higher cohomology of both $L_m^{\otimes k}\otimes\mathscr{I}_{\mathbb{P}(F)}^{xk}$ and $L_m^{\otimes k}\otimes\mathscr{I}_{\mathbb{P}(F)}^{xk+1}$ vanish for $k\gg 0$ and hence the same is true for $L_m^{\otimes k}\otimes\mathscr{I}_{\mathbb{P}(F)}^{xk}/\mathscr{I}_{\mathbb{P}(F)}^{xk+1}$, so

$$\chi(S^{xk}G^*\otimes S^{(1-x)k}F^*\otimes\mathcal{O}_B(mk))=\alpha_1(x)k^{n-1}+\alpha_2(x)k^{n-2}+O(k^{n-3}). \tag{5.22}$$

Now let $R=\operatorname{rank}S^{(1-x)k}F\cdot\operatorname{rank}S^{xk}G$, which equals

$$\begin{aligned}&\frac{1}{s!t!}\left[(1-x)^sk^s+\frac{s(s+1)}{2}(1-x)^{s-1}k^{s-1}+\cdots\right]\\ &\qquad\cdot\left[x^tk^t+\frac{t(t+1)}{2}x^{t-1}k^{t-1}+\cdots\right]\\ &=\frac{1}{s!t!}\left((1-x)^sx^tk^{r-1}+\delta(x)k^{r-2}+O(k^{r-3})\right),\end{aligned}$$

where $2\delta(x)=s(s+1)(1-x)^{s-1}x^t+t(t+1)(1-x)^sx^{t-1}$. Notice that this holds even if s or t are zero, for then the k^{s-1} or k^{t-1} terms vanish. Much calculation with Lemma 5.15 computes

$$\begin{aligned}&\chi(S^{xk}G^*\otimes S^{(1-x)k}F^*\otimes\mathcal{O}_B(mk))\\ &=a_0^BR\big(m^bk^b+[(1+k)\mu(B)-kx\mu_G-k(1-x)\mu_F]m^{b-1}k^{b-1}\big)\\ &\quad+O(m^{b-2})\\ &=a_0^BR((m^b+\mu(B)m^{b-1})k^b+\mu(B)m^{b-1}k^{b-1})\\ &\quad-a_0^BR(x\mu_G+(1-x)\mu_F)m^{b-1}k^b+O(m^{b-2}).\end{aligned}$$

Now $m^b + \mu(B)m^{b-1} = \widetilde{m}^b + \mu_E \widetilde{m}^{b-1} + O(\widetilde{m}^{b-2})$, and

$$\begin{aligned} \gamma(x) &:= \mu_E - (x\mu_G + (1-x)\mu_F) \\ &= \mu_E - \frac{x}{t+1}((r+1)\mu_E - (s+1)\mu_F) - (1-x)\mu_F \\ &= (\mu_E - \mu_F)\left(1 - \frac{x(r+1)}{t+1}\right), \end{aligned}$$

where the last line uses (5.21). Thus $\chi(S^{xk}G^* \otimes S^{(1-x)k}F^* \otimes \mathcal{O}_B(mk))$ is

$$a_0^B R(\widetilde{m}^b k^b + \gamma(x)\widetilde{m}^{b-1}k^b + \mu(B)\widetilde{m}^{b-1}k^{b-1}) + O(\widetilde{m}^{b-2}).$$

Now $\alpha_1(x)$ and $\alpha_2(x)$ (5.22) are polynomials in x and extend uniquely from $0 < x \ll 1$ to all of $\mathbb{R}$, and the above shows that

$$\begin{aligned} \alpha_1(x) &= \frac{a_0^B}{s!t!}(1-x)^s x^t(\widetilde{m}^b + \gamma(x)\widetilde{m}^{b-1}) + O(\widetilde{m}^{b-2}), \quad \text{and} \\ \alpha_2(x) &= \frac{a_0^B}{s!t!}\delta(x)(\widetilde{m}^b + \gamma(x)\widetilde{m}^{b-1}) + \frac{a_0^B}{s!t!}\mu(B)(1-x)^s x^t \widetilde{m}^{b-1} \\ &\quad + O(\widetilde{m}^{b-2}). \end{aligned}$$

To calculate the required integrals of the $\alpha_i(x)$ one has to consider four cases, depending on whether s or t vanish. In all four cases, repeated applications of the identity $\int_0^1 (1-x)^s x^t dx = \frac{s!t!}{(s+t+1)!}$ give the formula in the statement of the Lemma. These expressions depend only on $(B, \mathcal{O}_B(1))$ and the Chern characters of the symmetric powers of E and F, and thus only on $(B, \mathcal{O}_B(1))$ and the Chern classes of E and F by ([**Ha**] Appendix A3). q.e.d.

Proposition 5.23. *Let F be a saturated coherent subsheaf of E and suppose that $\epsilon(\mathbb{P}(F), L_m) = 1$. Then $\mu_1(\mathcal{O}_{\mathbb{P}(F)}, L_m) - \mu(\mathbb{P}(E), L_m)$ equals*

$$C\left((\mu_E - \mu_F)\left[(r+1)\widetilde{m}^{2b-1} - \mu(B)\widetilde{m}^{2b-2}\right] + O(\widetilde{m}^{2b-3})\right),$$

where $C = C(\widetilde{m})$ is positive. Here the $O(\widetilde{m}^{2b-3})$ term is understood to be zero if B is a curve. Moreover the $O(\widetilde{m}^{2b-3})$ terms depend only on $(B, \mathcal{O}_B(1))$ and the Chern classes of F and E.

Proof. Using the expressions in Lemmas 5.19 and 5.20 gives

$$\begin{aligned} &a_0 \int_0^1 \left((1-x)\alpha_2(x) + \frac{\alpha_1(0)}{2}\right)dx - a_1 \int_0^1 (1-x)\alpha_1(x)dx \\ &= \frac{(a_0^B)^2(s+1)}{(r+2)!r!}(\mu_E - \mu_F)[(r+1)\widetilde{m}^{2b-1} - \mu(B)\widetilde{m}^{2b-2}] + O(\widetilde{m}^{2b-3}). \end{aligned}$$

Thus $\mu_1(\mathcal{O}_{\mathbb{P}(F)}, L_m) - \mu(\mathbb{P}(E), L_m)$ equals

$$C(\widetilde{m})\left((\mu_E - \mu_F)[(r+1)\widetilde{m}^{2b-1} - \mu(B)\widetilde{m}^{2b-2}] + O(\widetilde{m}^{2b-3})\right),$$

where

$$C(\widetilde{m}) = \frac{(a_0^B)^2(s+1)}{(r+2)!r!a_0\int_0^1(1-x)\alpha_1(x)dx},$$

which is positive. As the $O(\widetilde{m}^{b-2})$ terms of a_0 and a_1, as well as $\alpha_1(x)$ and $\alpha_2(x)$, depend only on $(B,\mathcal{O}_B(1))$ and the Chern classes of F and E so does the $O(\widetilde{m}^{2b-3})$ term above. q.e.d.

Proof of Theorem 5.13. If E is not slope stable (resp. strictly unstable) then there is a maximally destabilising subsheaf of E^* which is saturated and so locally free. Call its dual G and let F be the kernel of $E \to G \to 0$. Then $F < E$ is saturated and locally free, G is semistable, and $\mu_F \geq \mu_E$ (resp. $\mu_F > \mu_E$). Therefore by Proposition 5.17 $\epsilon(\mathbb{P}(F), L_m) = 1$ and the global sections of $L_m^k \otimes \mathscr{I}_{\mathbb{P}(F)}^k$ saturate $\mathscr{I}_{\mathbb{P}(F)}^k$ for $k \gg 0$.

As $\deg L_m = a_0^B(r+1)\widetilde{m} > 0$ (Lemma 5.16) we have $\widetilde{m} > 0$. And as $g \geq 1$, $\mu(B) \leq 0$, so $(r+1)\widetilde{m} - \mu(B) > 0$. From Proposition 5.23,

$$\mu_1(\mathcal{O}_{\mathbb{P}(F)}, L_m) - \mu(\mathbb{P}(E), L_m) = C(\mu_E - \mu_F)[(r+1)\widetilde{m} - \mu(B)],$$

where $C > 0$. Thus if E is not slope (semi)stable then $\mathbb{P}(E)$ is not slope (semi)stable.

Finally suppose that E is not polystable. Then there is a subbundle F with either $\mu_F > \mu_E$, which we have already dealt with, or $\mu_F = \mu_E$ and F *is not a direct summand*. The degeneration to the normal cone of $\mathbb{P}(F)$ with $c = \epsilon = 1$ gives a test configuration with zero Futaki invariant whose central fibre is $\mathbb{P}(F \oplus E/F)$ (5.14). This cannot be a product configuration since the central fibre is not isomorphic to $\mathbb{P}(E)$. So $(\mathbb{P}(E), L_m)$ is not slope polystable. q.e.d.

We could similarly now prove the first part of Theorem 5.12, but to prove all of it we first calculate the slope and Seshadri constant of $\mathbb{P}(E|_{B'})$.

Lemma 5.24. *Let C and D be torsion free sheaves on B and suppose that $\mu_C = \mu(B)$. Then*

$$\chi(C \otimes D \otimes \mathcal{O}_B(mk)) = \operatorname{rank} C \cdot \chi(D \otimes \mathcal{O}_B(mk)) + O(m^{b-2}).$$

Proof. By Lemma 5.15 (3) the hypotheses imply that $\mu_{C\otimes D} = \mu_D$. Now apply Lemma 5.15 (1) twice. q.e.d.

Proposition 5.25. *Let B' be a subscheme of B. Then for $m \gg 0$, $\epsilon(\mathbb{P}(E|_{B'}), L_m) \geq m\epsilon(B', \mathcal{O}_B(1)) + O(m^0)$, and*

$$\begin{aligned}&\mu_{cm}(\mathscr{I}_{\mathbb{P}(E|_{B'})}, L_m) - \mu(\mathbb{P}(E), L_m)\\ &= \frac{1}{m}[\mu_c(\mathscr{I}_{B'}, \mathcal{O}_B(1)) - \mu(B)] + O(m^{-2}).\end{aligned}$$

Proof. Pick an integer u so that $E^*(u)$ is globally generated; then so is $S^k(E^*(u)) = S^kE^* \otimes \mathcal{O}_B(ku)$ for all k. We first show that

$$\epsilon(\mathbb{P}(E|_{B'}), L_m) \geq (m-u)\epsilon(B', \mathcal{O}_B(1)) = m\epsilon(B') + O(m^0).$$

By the definition of the Seshadri constant, if $c < (m-u)\epsilon(B', \mathcal{O}_B(1))$ then $\mathcal{O}_B((m-u)k) \otimes \mathscr{I}_{B'}^{ck}$ is globally generated for k sufficiently large. Hence for $k \gg 0$ the sheaf

$$\begin{aligned} \pi_*(L_m^{\otimes k} \otimes \mathscr{I}_{\mathbb{P}(E|_{B'})}^{ck}) &= \mathcal{O}_B(mk) \otimes S^kE^* \otimes \mathscr{I}_{B'}^{ck} \\ &= \mathcal{O}_B(mk - uk) \otimes S^k(E^*(u)) \otimes \mathscr{I}_{B'}^{ck} \end{aligned}$$

is also globally generated, and thus so is $L_m^{\otimes k} \otimes \mathscr{I}_{\mathbb{P}(E|_{B'})}^{ck}$, because $L_m^{\otimes k}$ is globally generated along the fibres. This implies that $c \leq \epsilon(\mathbb{P}(E|_{B'}), L_m)$, so $(m-u)\epsilon(B', \mathcal{O}_B(1)) \leq \epsilon(\mathbb{P}(E|_{B'}), L_m)$.

Now we calculate the slope of $\mathbb{P}(E|_{B'})$. Since we are interested in $m \gg 0$, we may twist E by some power of $\mathcal{O}_B(1)$ to assume, without loss of generality, that $\mu_E = \mu(B)$ (*i.e.* $\deg E = 0$). This power may not be integral, but that does not affect the purely numerical argument below; we just have to allow rational m. Let

$$\begin{aligned} \chi_{\mathbb{P}(E)}(L_m^{\otimes k} \otimes \mathscr{I}_{\mathbb{P}(E|_{B'})}^{xk}) &= a_0(x)k^n + a_1(x)k^{n-1} + O(k^{n-2}), \quad \text{and} \\ \chi_B(\mathcal{O}_B(k) \otimes \mathscr{I}_{B'}^{xk}) &= b_0(x)k^n + b_1(x)k^{n-1} + O(k^{n-2}). \end{aligned}$$

Fix $x < \epsilon(\mathbb{P}(E|_{B'}), L_m)$ and suppose $k \gg 0$. Then

$$\pi_*(L_m^{\otimes k} \otimes \mathscr{I}_{\mathbb{P}(E|_{B'})}^{xk}) = \mathscr{I}_{B'}^{xk} \otimes \mathcal{O}_B(mk) \otimes S^kE^*,$$

with the higher pushdowns zero. From Lemma 5.15 (2), we have that $\mu_{S^kE} = \mu(B)$ for all k, so Lemma 5.24 yields

$$\begin{aligned} \chi(L_m^{\otimes k} \otimes \mathscr{I}_{\mathbb{P}(E|_{B'})}^{xk}) &= \chi(\mathscr{I}_{B'}^{xk} \otimes \mathcal{O}_B(mk) \otimes S^kE^*) \\ &= \operatorname{rank} S^kE \cdot \chi(\mathscr{I}_{B'}^{xk} \otimes \mathcal{O}_B(mk)) + O(m^{b-2}) \\ &= \operatorname{rank} S^kE \cdot (b_0(x/m)m^bk^b + b_1(x/m)m^{b-1}k^{b-1}) \\ &\quad + O(m^{b-2}). \end{aligned}$$

Now

$$\operatorname{rank} S^kE = \binom{r+k}{r} = \frac{1}{r!}\left(k^r + \mu(\mathbb{P}^r)k^{r-1} + \cdots\right),$$

where $\mu(\mathbb{P}^r) = \mu(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)) = r(r+1)/2$. Thus

$$\begin{aligned} a_0(x) &= \frac{1}{r!}b_0(x/m)m^b + O(m^{b-2}), \qquad \text{and} \\ a_1(x) &= \frac{1}{r!}\left[\mu(\mathbb{P}^r)b_0(x/m)m^b + b_1(x/m)m^{b-1}\right] + O(m^{b-2}). \end{aligned}$$

Hence

$$\begin{aligned}
&\mu_{mc}(\mathscr{I}_{\mathbb{P}(E|_{B'})}, L_m) \\
&= \frac{\int_0^{mc}\left(a_1(x) + \frac{a_0(x)'}{2}\right)dx}{\int_0^{mc} a_0(x)dx} \\
&= \frac{\int_0^{c}\left(a_1(mx) + \frac{a_0'(mx)}{2}\right)dx}{\int_0^{c} a_0(mx)dx} \\
&= \frac{\int_0^{c}\left(\mu(\mathbb{P}^r)m^b b_0(x) + m^{b-1}(b_1(x) + \frac{b_0(x)'}{2})\right)dx + O(m^{b-2})}{\int_0^{c} m^b b_0(x)dx + O(m^{b-2})} \\
&= \mu(\mathbb{P}^r) + \frac{1}{m}\mu_c(\mathscr{I}_{B'}, \mathcal{O}_B(1)) + O(m^{-2}).
\end{aligned}$$

On the other hand, by assuming $\deg E = 0$ we have $\widetilde{m} = m$ so from Lemma (5.19),

$$\mu(\mathbb{P}(E), L_m) = \frac{a_1}{a_0} = \mu(\mathbb{P}^r) + \frac{1}{m}\mu(B) + O(m^{-2}).$$

Thus

$$\mu_{mc}(\mathscr{I}_{\mathbb{P}(E|_{B'})}, L_m) - \mu(\mathbb{P}(E), L_m) = \frac{1}{m}[\mu(\mathscr{I}_{B'}, \mathcal{O}_B(1)) - \mu(B)] + O(m^{-2}),$$

as required. q.e.d.

Proof of Theorem 5.12. By Proposition 5.17 there is an m_0 such that for all $m \geq m_0$, $\epsilon(\mathbb{P}(F), L_m) = 1$ for all saturated coherent subsheaves $F < E$ with $\mu_F \geq \mu_E$.

As the family of destabilising subsheaves of F of E is bounded, the set $\{c_i(F) \in H^{2i}(B)\colon F < E, \mu_F \geq \mu_E, 0 \leq i \leq n\}$ is finite. Thus we can bound the $O(\widetilde{m}^{2b-3})$ terms in Proposition 5.23 independently of F. Furthermore there is a $\delta > 0$ (again independent of F) such that $\mu_F > \mu_E$ implies $\mu_F \geq \mu_E + \delta$. Hence for all saturated coherent subsheaves $F < E$ with $\mu_F \geq \mu_E$ and $m \geq m_0$,

$$\begin{aligned}
(5.26)\qquad &\mu_1(\mathcal{O}_{\mathbb{P}(F)}, L_m) - \mu(\mathbb{P}(E), L_m) \\
&= C\left((r+1)(\mu_E - \mu_F)\widetilde{m}^{2b-1} + O(\widetilde{m}^{2b-2})\right) \\
&\leq -C(r+1)\big(\delta m^{2b-1} + O(m^{2b-2})\big),
\end{aligned}$$

where $C = C(\widetilde{m}) > 0$ is independent of F.

Now suppose that E is not slope semistable. Then there exists a coherent $F < E$ with $\mu_F > \mu_E$. Replace F by its saturation (*i.e.* the kernel of $E \to (E/F)/\text{torsion}$), which has slope $\geq \mu_F > \mu_E$. Making m_0 larger if necessary we have that $\mu_1(\mathcal{O}_{\mathbb{P}(F)}, L_m) < \mu(\mathbb{P}(E), L_m)$ for $m \geq m_0$ by (5.26), so $(\mathbb{P}(E), L_m)$ is not slope semistable.

Similarly, if $(B, \mathcal{O}_B(1))$ is not slope semistable then there is a B' and c with $c < \epsilon(B)$ and $\mu_c(\mathscr{I}_B) > \mu(B)$. Therefore, for $m \gg 0$, $c + O(m^0)/m < \epsilon(B)$, so by Proposition 5.25, $mc < \epsilon(\mathbb{P}(E|_{B'}), L_m)$ and $\mu_{cm}(\mathscr{I}_{\mathbb{P}(E|_{B'})}, L_m) > \mu(\mathbb{P}(E), L_m)$. Thus $\mathbb{P}(E|_{B'})$ strictly destabilises $(\mathbb{P}(E), L_m)$ for $m \gg 0$. q.e.d.

5.5. Unstable blow ups. Fix $Z \subset X$. If we form the blow up $\pi\colon \widehat{X} \to X$ of X along Z, with exceptional divisor E, then since for $k \gg 0$

$$H^0_X(L^{\otimes k} \otimes \mathscr{I}_Z^{xk}) \cong H^0_{\widehat{X}}(\pi^* L^{\otimes k} \otimes \mathscr{I}_E^{xk}),$$

there is a strong link between $Z \subset X$ and $E \subset \widehat{X}$. Morally, Z destabilises (X, L) if and only if E destabilises $(\widehat{X}, \pi^* L)$, but the latter line bundle is only semi-ample. However, $L_d := \pi^* L(-dE)$ is ample for $0 < d < \epsilon(Z)$, and with respect to this polarisation the Seshadri constants are related by $\epsilon(E) = \epsilon(Z) - d$. For $k \gg 0$,

$$H^0_{\widehat{X}}(L_d^{\otimes k} \otimes \mathscr{I}_E^{xk}) \cong H^0_X(L^{\otimes k} \otimes \mathscr{I}_Z^{(x+d)k}),$$

so for $d < c \leq \epsilon(Z)$,

$$\mu_{c-d}(\mathscr{I}_E, L_d) = \frac{\int_d^c \left(a_1(x) + \frac{a_0'(x)}{2}\right) dx}{\int_d^c a_0(x) dx}.$$

As $d \to 0$, $\mu_{c-d}(\mathscr{I}_E) \to \mu_c(\mathscr{I}_Z)$ and $\mu(\widehat{X}) \to \mu(X)$ as expected.

This can be applied in the following way. Suppose that a singular point strictly destabilises a variety X, and that its blow up is smooth. More generally, fix an ideal sheaf $\mathscr{I}_Z \subset \mathcal{O}_X$ whose blow up is smooth (this exists by resolution of singularities) and suppose that $\mathscr{I}_Z$ strictly destabilises (X, L). Then for small d, $(\widehat{X}, L_d)$ is also strictly unstable, and so has no cscK metric. This gives an easy way of producing smooth polarised varieties without cscK metrics.

5.6. Unstable rational manifolds.

Example 5.27. $\mathbb{P}^2$ blown up at 1 point. Any polarisation on $\pi\colon X \to \mathbb{P}^2$ is a multiple of $L = L_q = \mathcal{O}_{\mathbb{P}^2}(1) - qE$ for $q \in (0,1) \cap \mathbb{Q}$. We claim that the exceptional curve E destabilises (X, L) for any such q. Let $Z = E$. Then $L(-cZ) = \mathcal{O}_{\mathbb{P}^2}(1)\big(-(q+c)E\big)$ is nef for $c < 1 - q$, hence $\epsilon = \epsilon(Z, L) = 1 - q$. By (5.3)

$$\mu(X, L) = \frac{3-q}{1-q^2}, \quad \text{and}$$

$$\mu_\epsilon(\mathcal{O}_Z) = \frac{3}{(1-q)(2q+1)}.$$

Since

$$(3-q)(1-q)(2q+1) - 3(1-q^2) = 2q(1-q)^2 > 0,$$

we have $\mu_\epsilon(\mathcal{O}_Z) < \mu(X, L)$ for all $0 < q < 1$.

In fact this example is covered by Section 5.4, since X is a projective bundle $X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ (with E the projectivisation of the destabilising subbundle $\mathcal{O}_{\mathbb{P}^1}(1)$). It is significant that E destabilises X for *all* q, since *a priori* X is unstable for all polarisations because of the non-reductivity of $\mathrm{Aut}(X)$.

To find destabilising examples it is convenient to allow L to tend to a divisor that is not necessarily ample.

By analogy with (3.4, 3.5) we use the Riemann-Roch formula on $\widehat{X}$ to define the slope with respect to any divisor F by

$$\begin{aligned} a_0^F(x) &= \frac{1}{n!}\int_{\widehat{X}} c_1(F(-xE))^n, \\ a_1^F(x) &= -\frac{1}{2(n-1)!}\int_{\widehat{X}} c_1(K_{\widehat{X}}).c_1(F(-xE))^{n-1}, \end{aligned}$$

and (3.14)

$$\begin{aligned} \mu(X,F) &= -\frac{n\int_X c_1(K_X).c_1(F)^{n-1}}{2\int_X c_1(F)^n}, \\ \mu_c(\mathcal{O}_Z, F) &= \frac{\int_0^c \left(\tilde{a}_1^F(x) + \frac{\tilde{a}_0^{F\prime}(x)}{2}\right)dx}{\int_0^c \tilde{a}_0^F(x)dx}, \end{aligned}$$

where $\tilde{a}_i^F(x) = a_i^F(0) - a_i^F(x)$. Note that since F is not assumed to be ample, these could be infinite.

Proposition 5.28. *Let F be a nef divisor on X. Suppose that there is a $c > 0$ such that $F(-cE)$ is nef on $\widehat{X}$, and*

$$\int_X c_1(F)^n \int_0^c \left(\tilde{a}_1^F(x) + \frac{\tilde{a}_0^{F\prime}(x)}{2}\right)dx$$

is strictly less than

$$-\frac{n\int_X c_1(K_X).c_1(F)^{n-1}}{2}\int_0^c \tilde{a}_0^F(x)dx.$$

(*In particular this holds if $\int_X c_1(F)^n > 0$ and*

$$\mu_c(\mathcal{O}_Z, F) < \mu(X, F) < \infty.)$$

Then Z strictly destabilises (X, L) for L sufficiently close to F. More precisely: if G is an ample divisor and $L = F(\delta G)$, then there is a $\delta_0 > 0$ such that Z strictly destabilises (X, L) for all $0 < \delta < \delta_0$.

Proof. Since $L(-cE) = F(-cE + \delta G)$ is nef we have $\epsilon(Z, L) \geq c$. Notice that $\int_X c_1(L)^n \int_0^c \tilde{a}_0^L(x)dx[\mu(\mathcal{O}_Z, L) - \mu(X, L)]$ equals

$$\int_X c_1(L)^n \int_0^c \left(\tilde{a}_1^L(x) + \frac{\tilde{a}_0^{L\prime}(x)}{2}\right)dx + \frac{n\int_X c_1(K_X).c_1(L)^{n-1}}{2}\int_0^c \tilde{a}_0^L(x)dx.$$

As δ tends to zero this tends to

$$\int_X c_1(F)^n \int_0^c \left(\tilde{a}_1^F(x) + \frac{\tilde{a}_0^{F\prime}(x)}{2} \right) dx + \frac{n \int_X c_1(K_X).c_1(F)^{n-1}}{2} \int_0^c \tilde{a}_0^F(x) dx,$$

which is assumed to be strictly negative. Since L is ample, and $c \leq \epsilon(Z, L)$, $\int_X c_1(L)^n \int_0^c \tilde{a}_0^L(x)dx > 0$. Thus $\mu_c(\mathcal{O}_Z, L) < \mu(X, L)$ for δ sufficiently small. q.e.d.

Corollary 5.29. Unstable blow ups. *Suppose that (X, L) is destabilised by Z, and let Y be the blow up of X along a centre disjoint from Z. Then for polarisations making the exceptional set small, Y is destabilised by the proper transform of Z.*

Example 5.30. $\mathbb{P}^2$ blown up at m distinct points. Let X be $\mathbb{P}^2$ blown up at $m \geq 1$ distinct points, with exceptional divisors $\{E_i\}_{i=1}^m$. Then applying the above to Example 5.27 shows that X is slope unstable with respect to suitable polarisations: those of the form $\mathcal{O}_{\mathbb{P}^2}(1)\big(- \sum_{i=1}^m q_i E_i\big)$ with $0 < q_i \ll q_1 < 1$ for $i \geq 2$.

Remark 5.31. It is important to note that these polarisations are far from the anticanonical polarisation. For generic configurations of points K_X^* is ample if $m \leq 8$, and this polarisation *does* admit a cscK (in fact Kähler-Einstein) metric, unless $m = 1$ or $m = 2$ [**Ti1**].

Remark 5.32 (The folklore conjecture)**.** The case of $\mathbb{P}^2$ blown up at ≥ 4 points gives smooth polarised del Pezzo surfaces with $\mathfrak{aut}(X) = 0$ but no cscK metric in certain classes. This is in contrast to the case of the anticanonical polarisation for which Tian [**Ti1**] proved the "folklore conjecture", that smooth Fano surfaces have a Kähler-Einstein metric if and only if their holomorphic automorphism group is reductive, and disproves the conjecture for manifolds of dimension $n \geq 3$. There are also examples of ruled surfaces [**BdB**] which show the folklore conjecture for cscK metrics on surfaces does not hold.

Remark 5.33 (Unstable elliptic surface)**.** If X is $\mathbb{P}^2$ blown up at 9 points which are the intersection of two cubics, then X is slope unstable with respect to suitable polarisations. Thus we have a polarised elliptic surface ("half a $K3$ surface") which is not K-semistable and does not admit a cscK metric.

Example 5.34 (-2 curves)**.** We now give an example of a destabilising -2-curve on a del Pezzo surface. Blow $\mathbb{P}^2$ up at a point, and let X be its blow up at a point on the exceptional divisor. Thus X contains a -2 curve E_1 and an exceptional -1-curve E_2. Thus $E_1^2 = -2$, $E_2^2 = -1$ and $E_1.E_2 = 1$. Notice that $\mathcal{O}_{\mathbb{P}^2}(1)\big(- \frac{1}{2}E_1 - E_2\big)$ is nef (X

is toric, and the line bundle's degree on each invariant curve is nonnegative, so the toric Kleiman criterion applies). Also, $\mathcal{O}_{\mathbb{P}^2}(1)$ is nef, as is $\mathcal{O}_{\mathbb{P}^2}(1)\big(-E_1-E_2\big)$ since it is the pullback of a nef bundle on $\mathbb{P}^2$ blown up at one point. By convexity of the ample cone, $\mathcal{O}_{\mathbb{P}^2}(1)\big(-qE_1-rE_2\big)$ is ample for $1\geq r\geq q\geq r/2\geq 0$.

Set $L=\mathcal{O}_{\mathbb{P}^2}(1)\big(-\frac{1}{2}E_1-rE_2\big)$, which is ample for $1>r>1/2$, and let $Z=E_1$ be the exceptional -2-curve in X. From the above, $c_r=r-\frac{1}{2}\leq\epsilon(Z,L)$. Then $L.Z=1-r\to 0$ as $r\to 1$, so (5.3)

$$\mu(X,L)=\frac{\mathcal{O}_{\mathbb{P}^2}(3)\big(-E_1-2E_2\big).L}{L^2}=\frac{6-2r}{1+2r-2r^2}\to 4 \text{ as } r\to 1,$$

$$\mu_{c_r}(\mathcal{O}_Z,L)=\frac{3(L.Z+c_r)}{c_r(L.Z+2c_r)}\to 3 \text{ as } r\to 1.$$

Thus for r close to 1, $\mu_{c_r}(\mathcal{O}_Z)<\mu(X,L)$ so Z strictly destabilises (X,L). Further blowing up X in some points disjoint from the -2-curve and taking a polarisation in which the new exceptional divisors are small, this also gives examples with no automorphism group.

Example 5.35 ($\mathbb{P}^n$ blown up at points)**.** The exceptional divisor E strictly destabilises $\mathbb{P}^n$ blown up at a point with respect to all polarisations. This is an application of Theorem 5.1; we omit the gory details.

Therefore the blow up of $\mathbb{P}^n$ at $m\geq 1$ distinct points (or a point and some disjoint subvarieties) is unstable with respect to polarisations which make one component of the exceptional set large, and the other $m-1$ small.

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