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AN OCCUPANCY PROBLEM

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ABSTRACT. Under a certain restriction, n numbered balls are distributed into m urns. A match occurs if the ball numbered i lies in the urn with the number i. The number of matches and its asymptotic probability distribution are found.

1. Introduction. Let the n numbered balls be distributed into m numbered equal urns under the restriction (R); if the ball numbered i lies in the urn numbered j, then no ball with a number higher than i is permitted to lie in an urn with a smaller number than j, $1 \leqq i \leqq n$, $1 \leqq j \leqq m$.

It is clear that the number of distributions $A(n, m)$ of n balls into the m urns under the restriction (R) is equal to the number of increasing words formed of n letters out of m distrinct letters and thus (see [3, p. 10]), $A(n, m) = \binom{n+m-1}{n}$. We say a match has occurred in the jth urn if the ball numbered j lies in the urn with number j, $1 \leqq j \leqq \min(n, m)$. Let $W(n, m, r)$ be the number of distributions in which there are exactly r matches $0 \leqq r \leqq \min(n, m)$. We will prove the following theorems.

THEOREM 1.
(a) *For* $n \neq m$, $W(n, m, r) = W(n-1, m, r) + W(n, m-1, r)$,
(b) *For* $n = m$, $W(n, n, r) = W(n-1, n, r-1) + W(n, n-1, r)$,
(c) *For* $r \neq 0$, $W(n, m, r) = W(m, n, r)$

It is found that the value of $W(n, m, r)$ which satisfies Theorem 1 is given by Theorem 2,

THEOREM 2.

$$W(n, m, r) = \begin{cases} \binom{n+m-1}{m-r} \dfrac{n-m+2r}{n+r} & m \leqq n, \\ \binom{n+m-1}{n-r} \dfrac{m-n+2r}{m+r} & m > n. \end{cases}$$

As an application in probability theory we obtain the following result.

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THEOREM 3. *Let* $W = W(n, m, r)/\binom{n+m-1}{n}$ *and* $Z = W/\sqrt{n}$, *then* $\lim P(Z > z) = e^{-\alpha z - z^2}$, *where the limit on n and m are such that* $\alpha = \lim_{n,m\to\infty}(n - m)/\sqrt{n}$.

Similar problems were solved in [1] and [2] leading to theorems corresponding to theorems 1, 2, 3.

2. Proof of Theorem 1. (a) If the nth ball is placed into the mth urn then we have to distribute the n-1 remaining balls into the m urns under the rstriction (R) to give r matches, the number of these possibilities equals $W(n - 1, m, r)$.

If on the other hand the nth ball is not in the mth urn, then the n balls are to be distributed under (R), into the $m - 1$ urns to get r matches, the number of these possibilities equals $W(n, m - 1, r)$, hence.

$$W(n, m, t) = W(n - 1, m, r) + W(n, m - 1, r). \tag{2.1}$$

(b) The first summand is as in the last summand in part (a), the second summand has a different shape, since if the $(n - 1)$-*st* ball is in the nth urn this gives a match; therefore the $n - 1$ remaining balls have to be distributed into the n urns (including the nth urn) under (R) to get $r - 1$ matches.

(c) We prove, $W(n, m, r) = W(m, n, r)$, $r \neq 0$, by double induction on m, n. Assume that i) $W(n - 1, m, r) = W(m, n - 1, r)$ and ii) $W(n, m - 1, r) = W(m - 1, n, r)$.

Adding corresponding summands in (i) and (ii) and by (a) of theorem 1, we get

$$W(n, m, r) = W(m, n, r), \qquad r \neq 0 \tag{2.2}$$

The following tables give the number of cases in which there are exactly r matches for certain r.

Urns

$r=1$	1	2	3	4
1	1	1	1	1
2	1	2	3	4
3	1	3	5	9
4	1	4	9	14

$r=2$	1	2	3	4
	0	0	0	0
	0	1	1	1
	0	1	4	5
	0	1	5	14

$r=3$	1	2	3	4
	0	0	0	0
	0	0	0	0
	0	0	1	1
	0	0	1	6

3. Proof of Theorem 2. By double induction on $m, n, m \leqq n$, assume,

$$W(n - 1, n, r) = \binom{n + m - 2}{m - r} \frac{n - m + 2r - 1}{n + r - 1},$$

$$W(n, m - 1, r) = \binom{n + m - 2}{m - r - 1} \frac{n - m + 2r + 1}{n + r}.$$

By a simple calculation and (a) of Theorem 1, we get

(3.1) $$W(n-1,m,r)+W(n,m-1,r)=\binom{n+m-1}{n-r}\frac{n-m+2r}{n+r}=W(n,m,r).$$

For $n < m$ the result is obtained by interchanging n by m and using (c) of Theorem 1. The case $n = m$ can be proved similarly.

4. Proof of Theorem 3. In what follows θ denotes on unspecified bounded function which may be different from one line to another, such that $0 < \theta < 1$, over the range of argument specified. Using $\text{Ln}(1 - Z) = Z + (1/2)Z^2 + \theta(Z)\, Z^3$, then

(4.1) $$\begin{aligned} P(W > k) &= \binom{n+m-1}{m-k}\Big/\binom{n+m-1}{n} \\ &= \left(\frac{m}{n}\right)^{k-1}\frac{\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\cdots\left(1-\frac{k-1}{m}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{k-1}{n}\right)}, \qquad m \leqq n. \end{aligned}$$

Hence,

(4.2) $$\begin{aligned} \ln P(W > k) &= \ln\left(\frac{m}{n}\right)^{k-1} + \sum_{i=1}^{k-1}\ln\left(1-\frac{i}{m}\right) - \sum_{j=1}^{k-1}\ln\left(1+\frac{j}{n}\right) \\ &= \ln\left(\frac{m}{n}\right)^{k-1} - \left(\frac{1}{m}+\frac{1}{n}\right)\frac{k(k+1)}{2} \\ &\quad - \frac{1}{2}\left(\frac{1}{m^2}-\frac{1}{n^2}\right)\frac{k(k+1)(2k+1)}{6} \\ &\quad - \theta(k)\frac{1}{3}\left(\frac{1}{m^3}+\frac{1}{n^3}\right)\frac{k^2(k+1)^2}{4}. \end{aligned}$$

Replace k by $\sqrt{n}\,, z$ and m by $n - \alpha\sqrt{n}$ such that $\alpha \to 0$ as $n \to m$, and taking the limit as $n, m \to \infty$ all terms in $P(W > k)$ tend to 0 except the first and the second term, hence

(4.3) $$\begin{aligned} \lim_{n\to\infty} P(Z > z) &\simeq \lim_{n\to\infty}\left(1-\frac{\alpha}{n}\right)^{\sqrt{n}z} e^{-\left(\frac{2n-\alpha\sqrt{n}}{n(n-\alpha\sqrt{n})}\right)\frac{\sqrt{n}\,z(\sqrt{n}z-1)}{2}} \\ &\simeq e^{(-z^2+\alpha z)}\,. \end{aligned}$$

The case $n < m$ gives the same result where $\alpha \to 0$ as $m \to n$.

References

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