AN OCTONIONIC CONSTRUCTION OF E_8 AND THE LIE ALGEBRA MAGIC SQUARE

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ABSTRACT. We give a new construction of the Lie algebra of type E_8, in terms of 3×3 matrices, such that the Lie bracket has a natural description as the matrix commutator. This leads to a new interpretation of the Freudenthal–Tits magic square of Lie algebras, acting on themselves by commutation.

Dedicated to the memory of Jacques Tits

1. INTRODUCTION

The Chevalley construction of E_8, as complex Lie algebra and complex Lie group, generalizes easily to finite and algebraically closed fields [2]. However, over the real numbers there are three different forms of E_8, known as the split, semi-split and compact real forms. The Chevalley construction naturally produces the split form, written as 248×248 real matrices. By a straightforward change of basis, the other real forms can be written as 248×248 complex matrices, but writing them as 248×248 real matrices is more of a challenge. One approach to this problem is to use the Freudenthal–Tits 'magic square'.

According to Vinberg's interpretation [9] of the Freudenthal–Tits magic square, the compact real form of the Lie algebra of type E_8 can be identified with an algebra of traceless anti-Hermitian 3×3 matrices over $\mathbb{K} \otimes \mathbb{L}$, plus the derivations of $\mathbb{K}$ and $\mathbb{L}$, where $\mathbb{K} \cong \mathbb{L} \cong \mathbb{O}$ is the division algebra of real octonions. If $\mathbb{K} \cong \mathbb{O}$, but $\mathbb{L} \cong \mathbb{O}'$ is the algebra of spilt real octonions, then the semi-split real form of E_8 is obtained instead, and if $\mathbb{K} \cong \mathbb{L} \cong \mathbb{O}'$, the split real form is obtained by the same process. However, the definition of the Lie bracket given by Vinberg is not straightforward, and involves some quite complicated and unintuitive formulae. In one sense, the complications arise because the subalgebra of type $D_4 + D_4$ is divided up between the diagonal matrices and the derivations.

Another interpretation, due to Barton and Sudbery [1], unifies the subalgebra of type $D_4 + D_4$ in what they call the triality construction, but throws away the matrix interpretation. In this paper we combine the best of the two approaches into a triality-invariant, 3×3 matrix construction of E_8. The key to our constructions is a simple octonionic notation for both the split and compact real forms of the Lie algebra of type D_4, which enables us to put these Lie algebras on the diagonal of 3×3 matrices, leaving the off-diagonal matrices to be anti-Hermitian as in the Vinberg interpretation. Then it turns out, somewhat surprisingly, that the Lie bracket can be obtained as a natural extension of the usual matrix commutator.

The same ideas give rise to a simple construction of the magic square of 2×2 matrix algebras (see [1]), which we discuss first. In this case, the real Lie algebras which appear are all of the form $\mathfrak{so}(p, q)$, and the corresponding spin representations can be used to show that we actually construct associative algebras whose Lie algebras are those which appear in the magic square. This approach is fully discussed by Dray, Huerta and Kincaid [4]. Since these associative algebras are just algebras of matrices, with linear operators as matrix entries, the Jacobi identity follows automatically.

The 2×2 matrix algebras over $\mathbb{O}$ and $\mathbb{O}'$ are isomorphic to the real Lie algebras $\mathfrak{so}(9)$ and $\mathfrak{so}(5, 4)$ respectively. Extending to 3×3 matrices is then accomplished by introducing an explicit triality automorphism, which itself comes automatically from the exceptional Jordan algebra, or Albert algebra, in order to identify three copies of $\mathfrak{so}(8)$ or $\mathfrak{so}(4, 4)$. This identification is in effect a quotient of the vector space of matrices, which means that the matrices themselves do not form an algebra (either associative or Lie). However, the quotient space does act on the Albert algebra, which implies that it forms a Lie algebra of type $\mathfrak{f}_4$ under commutation. Finally we derive the adjoint representation, in order to facilitate extending to $\mathfrak{e}_8$.

The most interesting case, of course, is the algebra of 3×3 matrices over $\mathbb{O} \otimes \mathbb{O}$ (resp. $\mathbb{O} \otimes \mathbb{O}'$, $\mathbb{O}' \otimes \mathbb{O}'$), which is expected to give compact (resp. semisplit, split) $\mathfrak{e}_8$. The algebra we construct below is certainly generated by two copies of $\mathfrak{f}_4$, and it is fairly easy to see that the underlying vector space has dimension 248. Since the algebra acts on this vector space, the Jacobi identity holds automatically. The final piece of the argument, therefore, is to show that the 248-dimensional space is closed under taking commutators.

In this paper, all algebras are real unless otherwise specified, including octonion algebras and Lie algebras, and Jordan algebras. Our constructions work equally well for both compact and split versions of the octonions (and quaternions, and complex numbers), so give explicit versions of all the real forms of $\mathfrak{e}_8$.

In a subsequent paper we show how to restrict the construction of $\mathfrak{e}_8$ to give constructions of the entire magic square, first in the adjoint representations. This is not as straightforward as in the case of 2×2 matrices, where the spin representations permit direct construction of each entry in the magic square. In particular, the constructions of $\mathfrak{e}_6$ and $\mathfrak{e}_7$ exhibit interesting features, and permit construction not only of the adjoint representations, but also of the minimal representations.

The paper is organised as follows. In Section 2 we recall necessary background material on octonions and $\mathfrak{so}(8)$, including triality. In Section 3 we analyse the 2×2 octonion matrix algebra, and the related Lie algebra $\mathfrak{so}(9)$. This material is then extended in two independent ways. First, in Section 4, we extend to the tensor product of two octonion algebras, in order to construct $\mathfrak{so}(16)$. Second, in Section 5, we extend to 3×3 matrices, in order to construct the Albert algebra and the related Lie algebra $\mathfrak{f}_4$. Finally, in Section 6, we make both extensions at the same time, to construct $\mathfrak{e}_8$.

2. Octonions and $\mathfrak{so}(8)$

2.1. **Octonions and split octonions.** The (real) octonions are an 8-dimensional division algebra $\mathbb{O}$, with (real) basis 1, i, j, k, ℓ, $i\ell$, $j\ell$, $k\ell$, with the property that all the 7 imaginary basis units square to -1, any two associate but anti-commute, $ij = -ji = k$, and any three whose product is not ± 1 anti-associate, in the sense

that, for example, $(ij)\ell = -i(j\ell)$. From this the entire multiplication table can be computed. There is an involution $x \mapsto \overline{x}$ which fixes 1 and negates all the other basic units, and a (squared) norm $N(x) = x\overline{x}$. There is also a 7-fold symmetry which cycles $(i, j, \ell, k, j\ell, -k\ell, i\ell)$, and a 3-fold symmetry $(i, j, k)(i\ell, j\ell, k\ell)$. Moreover, if i, j, k are fixed, then any of $i\ell, j\ell, k\ell$ can be used in place of ℓ.

The split octonions $\mathbb{O}'$ are similar, except that four of the imaginary units square to $+1$ instead of -1. For example, we may take a basis conisting of $1, I, J, K, L, IL, JL, KL$, and decree that L, IL, JL, KL square to $+1$. We lose the 7-fold symmetry, of course, but keep the 3-fold symmetry. It is also still true that any of IL, JL, or KL can be used in place of L.

In what follows, we shall notate two copies of the octonions by lower-case and upper-case letters, and usually (but not always) assume that i, j, k, I, J, K square to -1, and explicitly state $\ell^2 = 1$ or $\ell^2 = -1$, and $L^2 = 1$ or $L^2 = -1$, in all cases where it actually matters. In general, we prove results only for the division algebra $\mathbb{O}$, and note that the same proof works for $\mathbb{O}'$, with a few signs changed.

Many results do not depend on the actual basis, and will therefore be stated for more general p, q, r, s, which will usually denote arbitrary mutually orthogonal pure imaginary octonions. However, in most cases there is no loss of generality in assuming that p, q, r, s belong to the standard basis. Similarly, a, b, c will often denote arbitrary octonions, not assumed imaginary, and not assumed to have norm ± 1.

2.2. **The $\mathfrak{so}(8)$ Lie algebra.** The (compact, real) $\mathfrak{so}(8)$ Lie algebra may be generated by the left actions of the octonions on themselves. As a vector space this algebra is then spanned by the 7 single left multiplications $L_i, \ldots L_{k\ell}$ and the 21 'nested' multiplications $L_i \circ L_j$ (meaning L_j followed by L_i), $\ldots$, $L_{j\ell} \circ L_{k\ell}$. In the associative algebra generated by these operations, $L_p \circ L_q = -L_q \circ L_p$, and $L_p \circ L_p = L_{p^2}$, and we easily deduce the following rules, where p, q, r, s are (arbitrary) orthogonal imaginary octonions:

$$
\begin{aligned}
L_p \circ (L_p \circ L_q) &= -(L_p \circ L_q) \circ L_p \\
&= p^2 L_q \\
(L_r \circ L_p) \circ (L_p \circ L_q) &= -(L_p \circ L_q) \circ (L_r \circ L_p) \\
&= p^2 (L_r \circ L_q) \\
L_r \circ (L_p \circ L_q) &= (L_p \circ L_q) \circ L_r \\
(L_r \circ L_s) \circ (L_p \circ L_q) &= (L_p \circ L_q) \circ (L_r \circ L_s)
\end{aligned}
$$

(1)

Where it does not cause confusion, we shall write fg for $f \circ g$ and $[f, g]$ for $fg - gf$. Hence in the Lie algebra we have

$$
\begin{aligned}
[L_p, L_q] &= 2L_p L_q \\
[L_p, L_p L_q] &= 2p^2 L_q \\
[L_r L_p, L_p L_q] &= 2p^2 L_r L_q \\
[L_r, L_p L_q] &= 0 \\
[L_p L_q, L_r L_s] &= 0.
\end{aligned}
$$

(2)

In the compact case, where $p^2 = -1$ for all imaginary elements of norm 1, the map taking L_p to $1 \wedge p$, and $L_p L_q$ to $p \wedge q$, for p, q imaginary elements of the standard basis, shows the isomorphism with the standard construction of (the compact real form of) $\mathfrak{so}(8)$. Here the Lie bracket is given by

$$
[p \wedge q, p \wedge r] = 2q \wedge r,
$$

where we now also allow the variables p, q, r to be real. If we use instead the split octonions, we obtain a different real form, namely the split real form $\mathfrak{so}(4,4)$.

2.3. Right actions.
For any $a \in \mathbb{O}$, define left, right, and double actions of a on $\mathbb{O}$ by

$$
\begin{array}{rcl}
L_a & : & x \mapsto ax \\
R_a & : & x \mapsto xa \\
\end{array}
$$
(3)
$$
B_a := -L_a - R_a \quad : \quad x \mapsto -ax - xa
$$

As is well-known, the Lie algebra $\mathfrak{so}(8)$ is generated by any one of the left, right, or double actions of the imaginary part of $\mathbb{O}$ on $\mathbb{O}$. To see this explicitly, we may restrict to the standard basis of $\mathbb{O}$, and then by symmetry it suffices to verify that

(4)
$$
L_\ell \;\; = \;\; (-R_\ell + R_i R_{i\ell} + R_j R_{j\ell} + R_k R_{k\ell})/2
$$

This is an easy exercise. (Here we use the natural convention for functions, that $R_i R_j$ means R_j followed by R_i, rather than the natural convention for right multiplications, that $R_i R_j$ means R_i followed by R_j.) To obtain the full set of such equations, apply the 7-cycle $(i, j, \ell, k, j\ell, -k\ell, i\ell)$ to the indices. For convenience, we record the following useful identities.

(5)
$$
\begin{array}{rcl}
L_i L_{i\ell} & = & (-R_\ell + R_i R_{i\ell} - R_j R_{j\ell} - R_k R_{k\ell})/2 \\
L_i & = & (-R_i + R_{i\ell} R_\ell + R_k R_j + R_{j\ell} R_{k\ell})/2
\end{array}
$$

Triality cycles L_p to R_p to B_p, while duality swaps $-L_p$ with R_p, and therefore negates B_p. Hence we obtain corresponding formulae expressing each of the operators of type L, R or B in terms of either of the other two types.

Analogous formulae hold in the split octonions, sometimes with different signs. The rule for signs is that whenever ℓ appears twice in a single term, the corresponding term is negated in the split octonions relative to the compact octonions. For example, in the split octonions

(6)
$$
L_I \;\; = \;\; (-R_I - R_{IL} R_L + R_K R_J - R_{JL} R_{KL})/2
$$

3. The Lie algebra $\mathfrak{so}(9) = \mathfrak{su}(2, \mathbb{O})$

3.1. The spin representation.
We are now ready to start looking at the $n = 2$ magic square. We start the investigation with the $\mathfrak{so}(9)$ case, that is the top-right or bottom-left corner, as that's analogous to $\mathfrak{f}_4$ in the $n = 3$ square. This algebra is already well-understood as an algebra of 2×2 octonion matrices (see Manogue and Schray [8]), and what we give here is just a new notation, and a slightly different interpretation.

Our approach is to start with the off-diagonal matrices, and use them to construct the rest of the algebra. We simply take them to be anti-Hermitian matrices in the usual sense, spanned by

(7)
$$
X_1 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad X_p := \begin{pmatrix} 0 & p \\ p & 0 \end{pmatrix}
$$

where p can be any pure imaginary octonion. We let these matrices act on 2-component columns (of octonions) in the obvious way. Since these actions, regarded as maps on real 16-space, are linear, they generate an associative algebra. It follows that under commutation they generate a Lie algebra.

On the other hand, the octonions are not associative, so the commutator of two actions, each expressed as a 2×2 octonionic matrix, *cannot* in general be expressed

as the action of the commutator of the matrices. This forces us to incorporate into the matrices a notation for a commutator of left actions of octonions. Let us write $p \circ q = [L_p, L_q]/2$ for this kind of 'product' of p and q. With this definition, we obtain the following diagonal matrices, considered as left actions on octonionic (column) 2-vectors, where p and q are orthogonal imaginary octonions:

$$D_p = D_{1,p} = [X_1, X_p]/2 = \begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix}$$

$$(8) \qquad D_{p,q} = -D_{q,p} = [X_p, X_q]/2 = \begin{pmatrix} p \circ q & 0 \\ 0 & p \circ q \end{pmatrix}$$

It is now clear that the matrices D_p and $D_{p,q}$, acting on 2-component column vectors, satisfy the same commutation relations as the operators L_p and $L_p L_q$, acting on octonions. In particular they are closed under commutation. Therefore the matrices D_p and $D_{p,q}$ give a representation of $\mathfrak{so}(8)$. The first octonion coordinate gives the half-spin representation defined by left octonion multiplication. The second octonion coordinate is equivalent to the other half-spin representation, which is defined here by left-multiplication by the octonion conjugate, but which can equivalently be defined by right-multiplication by the octonions themselves.

3.2. **The Lie algebra.** We claim that this $\mathfrak{so}(8)$ together with X_1 and the X_p, for p any imaginary basic unit, generate a copy of $\mathfrak{so}(9)$. In order to prove this fact, it is necessary first to check closure under commutation. Since the commutators of two Xs or two Ds have already been considered, all that remains is to check the commutators of the actions of the X_a with the actions of D_p and $D_{p,q}$. This is almost a triviality: we compute the following

$$
\begin{array}{rcl}
[X_1, D_p] &=& -2X_p \\
[X_p, D_p] &=& -2p^2 X_1 \\
[X_q, D_p] &=& 0 \\
[X_1, D_{p,q}] &=& 0 \\
[X_p, D_{p,q}] &=& 2p^2 X_q \\
[X_r, D_{p,q}] &=& 0
\end{array}
$$

(9)

Notice that the first three rules can be expressed in a single equation as

$$(10) \qquad [D_p, X_a] = X_{ap} + X_{pa}$$

which will be useful in later calculations. Moreover, the last three of the six rules follow immediately from the first three together with the Jacobi identity for matrix actions, thus:

$$
\begin{array}{rcl}
[X_1, D_{p,q}] &=& [X_1, [X_p, X_q]]/2 \\
&=& [X_p, [X_1, X_q]]/2 - [X_q, [X_1, X_p]]/2 \\
&=& [X_p, D_q]/2 - [X_q, D_p]/2 \\
&=& 0 \\
[X_p, D_{p,q}] &=& [X_p, [D_p, D_q]]/2 \\
&=& -[D_p, [D_q, X_p]]/2 - [D_q, [X_p, D_p]]/2 \\
&=& p^2 [D_q, X_1] \\
&=& 2p^2 X_q \\
2[X_r, D_{p,q}] &=& [X_r, [D_p, D_q]] \\
&=& -[D_p, [D_q, X_r]] - [D_q, [X_r, D_p]] \\
(11) &=& 0
\end{array}
$$

3.3. **The adjoint representation.** By this stage we know we have a Lie algebra of dimension 36, containing $\mathfrak{so}(8)$. It is easy to see that it is simple, and that it is a copy of $\mathfrak{so}(9)$. More explicitly, an isomorphism with the standard copy of $\mathfrak{so}(9)$ in its adjoint representation can be obtained as follows. If we label the 9 coordinates for $\mathfrak{so}(9)$ as $1', 1, i, j, k, \ell, i\ell, j\ell, k\ell$ then the isomorphism with the usual Lie algebra, namely the skew square of the natural representation, is given by

(12)
$$\begin{aligned} X_a &\mapsto 1' \wedge a \\ D_p &\mapsto 1 \wedge p \\ D_{p,q} &\mapsto p \wedge q \end{aligned}$$

where a, p, q are elements of the standard basis of $\mathbb{O}$, and p, q are imaginary. Here the Lie bracket is given by

$$[a \wedge b, a \wedge c] = 2[b \wedge c]$$

and other products are zero.

Strictly speaking, the adjoint representaton ad is defined by

$$\mathrm{ad}\, x : y \mapsto [x, y],$$

for every x, y in the algebra. Thus $\mathrm{ad}\, x$ is a linear map on the underlying vector space of the algebra. This somewhat pedantic distinction between the algebra itself and its adjoint representation becomes helpful later on when we come to consider the $\mathfrak{f}_4$ algebra and its representation on the Albert algebra, and even more so when we come to build $\mathfrak{e}_8$.

The representation of the algebra on 2-component octonion columns is of course the spin representation of $\mathfrak{so}(9)$, of real dimension 16.

3.4. **The natural representation.** We show how the natural representation of $\mathfrak{so}(9)$ can also be constructed using 2×2 octonion matrices. This will be required in the construction of $\mathfrak{f}_4$ below, in its action on the Albert algebra.

Let $\mathcal{U}$ be the space of 2×2 trace 0 Hermitian matrices, spanned by

(13) $U_{1'} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $U_a = \begin{pmatrix} 0 & a \\ \overline{a} & 0 \end{pmatrix}$ for $a = 1, i, j, k, \ell, i\ell, j\ell, k\ell$.

We define the action of the X_b on $\mathcal{U}$ via

(14) $X_b \quad : \quad U_a \mapsto X_b U_a - U_a X_b$

and check that X_b maps $U_{1'}$ to $-2U_b$, and U_b to $2U_{1'}$, while sending all other basis elements U_a to 0. Since this is (up to an overall scalar factor) the natural action of the X_b on a 9-dimensional orthogonal space, it extends to a representation of the whole of $\mathfrak{so}(9)$, provided we define the actions of the commutators to be the commutators of the actions.

In the case $D_p = [X_1, X_p]/2$ there is enough associativity to ensure that the action of D_p is the same as the obvious action of the matrix $\begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix}$. In the case $D_{p,q}$ we compute the action as $U_p \mapsto U_q \mapsto -U_p$, and every other U_a maps to 0. Now let us compare this with the action of a hypothetical 'matrix' $\begin{pmatrix} p \circ q & 0 \\ 0 & p \circ q \end{pmatrix}$. To obtain the correct action on $\mathcal{U}$, it is necessary and sufficient to interpret the commutators as

(15) $[p \circ q, a] \quad = \quad p \circ q \circ a - a \circ p \circ q$

since this evaluates to 0 except when $a = p$ or $a = q$, when it evaluates to $2q$ and $-2p$ respectively. (If $a = 1$ we interpret $p \circ q \circ a = a \circ p \circ q = p \circ q$.)

3.5. Matrix commutators. When working with matrices over an associative algebra, the Lie bracket automatically agrees with the commutator of matrices. Over octonions, one cannot expect this. However, it is remarkable how close we can get to this aim in this case. The commutators of the Xs were used to define the Ds. The commutators of the Ds have already been shown to follow the same rules as the Ls. These rules explain how to interpret commutators of 'nested' octonions.

It remains to consider the commutators of the Xs with the Ds. These contain off-diagonal entries which are either of the form $ap + pa$ (in the case of $[X_a, D_p]$), or of the form $a(p \circ q) - (p \circ q)a$ (in the case of $[X_a, D_{p,q}]$). The former gives the correct commutator as a matrix if the products are just multiplied out as octonions. The latter also gives the correct commutator provided the given entry is evaluated as $[a, p \circ q]$ using the same rules as for the diagonal entries. However, it does not make sense to try to evaluate the individual terms $a(p \circ q)$ or $(p \circ q)a$ separately.

This interpretation now gives an *abstract* construction of the Lie algebra, without reference to its action on column vectors. It consists of anti-Hermitian matrices, in the extended sense which includes 'nested' diagonal matrices, and the Lie bracket is just the matrix commutator, extended by a more-or-less obvious interpretation of commutators involving nested octonions. Since this result is by no means obvious, it is perhaps worth stating formally.

Theorem 1. *Let $\mathcal{L}$ be the real 36-space spanned by the 2×2 octonionic matrices*

$$X_a = \begin{pmatrix} 0 & a \\ -\overline{a} & 0 \end{pmatrix}, D_p = \begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix}, D_{p,q} = \begin{pmatrix} p \circ q & 0 \\ 0 & p \circ q \end{pmatrix}$$

where p, q are any distinct imaginary basic units, and a is any basic octonionic unit. Define a commutator on $\mathcal{L}$ as the matrix commutator, subject to evaluating all entries as left-multiplications. This is equivalent to the following special rules:

(1) *Diagonal entries are evaluated as normal except that $pq - qp$ is evaluated as $2p \circ q$, so that $[X_p, X_q] = 2D_{p,q}$.*
(2) *Off-diagonal entries $ab + ba$ are evaluated as normal octonions, so are 0 on the basis unless $a = 1$ or $b = 1$ or $a = b$.*
(3) *Off-diagonal entries $[p \circ q, r]$ are evaluated as $p \circ q \circ r - r \circ p \circ q$, so are 0 on the basis unless $r = p$ or $r = q$.*

Then $\mathcal{L}$ with this commutator map is isomorphic to the Lie algebra $\mathfrak{so}(9)$, if the octonion division algebra is used, or to $\mathfrak{so}(5,4)$, if the split octonions are used.

4. THE LIE ALGEBRA $\mathfrak{so}(16) = \mathfrak{su}(2, \mathbb{O} \otimes \mathbb{O})$

4.1. The spin representation. The same idea enables us to construct $\mathfrak{so}(16)$ by taking matrices defined over the tensor product of two copies of the octonions. The hard work, such as it was, has already been done. Let $\mathbb{K}$ and $\mathbb{L}$ be two copies of the octonions. This time, we let the algebra act on two-component vectors with entries in $\mathbb{K} \otimes \mathbb{L}$. This vector space has real dimension 128, and affords the spin representation of $\mathfrak{so}(16)$. (See [4].)

The algebra may be generated by two copies of $\mathfrak{so}(9)$, that is $\mathfrak{su}(2, \mathbb{K})$ and $\mathfrak{su}(2, \mathbb{L})$ as constructed in Section 3. These each act in the obvious way on one component of the tensor products, and intersect in a copy of $\mathfrak{so}(2)$ spanned by X_1. The

diagonal subalgebras $\mathfrak{so}(\mathbb{K}) \cong \mathfrak{so}(8)$ and $\mathfrak{so}(\mathbb{L}) \cong \mathfrak{so}(8)$ commute with each other, by construction.

We already have 15 dimensions of off-diagonal matrices, and there are 49 more of type

$$(16) \qquad X_{pP} := [D_p, X_P]/2 \quad = \quad \begin{pmatrix} 0 & pP \\ -pP & 0 \end{pmatrix},$$

making 64 in total. This makes up the full $28 + 28 + 64 = 120$ dimensions of $\mathfrak{so}(16)$. It is straightforward to show that this 120-space is closed under commutation, and is a compact simple Lie algebra containing $\mathfrak{so}(8) + \mathfrak{so}(8)$, so is $\mathfrak{so}(16)$. However, we shall give a more explicit isomorphism with the standard copy of $\mathfrak{so}(16)$ in Section 4.2 below.

4.2. **The Lie bracket.** It is easy to see that the commutators of the Ds and Xs are given by the same formulae as before, with the extra subscript on the Xs being carried through. More explicitly, we have

$$(17) \qquad \begin{array}{rcl} [D_p, X_{aA}] & = & X_{apA} + X_{paA} \\ [D_{p,q}, X_{qA}] & = & 2q^2 X_{pA} \\ [D_{p,q}, X_{aA}] & = & 0 \text{ for } a \perp p, q \end{array}$$

and the corresponding equations with the roles of the two alphabets interchanged.

It remains therefore to consider the commutators of two Xs. (As usual matrices should be understood to act on 2-component columns.) Assume that a, b are orthogonal basic units in $\mathbb{K}$ and A, B are orthogonal basic units in $\mathbb{L}$. Then

$$[X_{aA}, X_{bB}] \quad = \quad \begin{pmatrix} 0 & aA \\ -\overline{a}\overline{A} & 0 \end{pmatrix} \begin{pmatrix} 0 & bB \\ -\overline{b}\overline{B} & 0 \end{pmatrix} - \begin{pmatrix} 0 & bB \\ -\overline{b}\overline{B} & 0 \end{pmatrix} \begin{pmatrix} 0 & aA \\ -\overline{a}\overline{A} & 0 \end{pmatrix}$$

which evaluates to

$$\begin{pmatrix} -(a \circ \overline{b})(A \circ \overline{B}) + (b \circ \overline{a})(B \circ \overline{A}) & 0 \\ 0 & -(\overline{a} \circ b)(\overline{A} \circ B) + (\overline{b} \circ a)(\overline{B} \circ A) \end{pmatrix}$$

We may assume that a, b, A, B belong to the standard basis, and that either $a \neq b$ or $A \neq B$. The arguments in the two cases are the same, so without loss we may assume $A \neq B$. First disjoin cases for a and b. In the case $a = b$, so that $a \circ \overline{b} = b \circ \overline{a} = \pm 1$, the top-left entry reduces to $a\overline{a}(B \circ \overline{A} - A \circ \overline{B})$, while if $a \neq b$ it reduces to $(b \circ \overline{a})(B \circ \overline{A} + A \circ \overline{B})$. Now disjoin cases for A and B. In the case $A = 1$ and B is imaginary, the former reduces to $2a\overline{a}B$ and the latter to 0. Similarly, if both A and B are imaginary, then the former reduces to $2a\overline{a}A \circ B$, and the latter to 0 again. We find that the commutators are zero except for

$$(18) \qquad \begin{array}{rcl} [X_{aA}, X_{aB}] & = & 2a\overline{a}D_{A,B} \\ [X_{aA}, X_{bA}] & = & 2A\overline{A}D_{a,b} \end{array}$$

To see the isomorphism with $\mathfrak{so}(16)$ a little more explicitly, we may imagine 16 coordinates labelled $1, i, \ldots, k\ell, 1', I, \ldots, KL$ and identify

$$(19) \qquad \begin{array}{rcl} D_i & = & 1 \wedge i \\ D_{i,j} & = & i \wedge j \\ D_I & = & 1' \wedge I \\ D_{I,J} & = & I \wedge J \\ X_{aA} & = & a \wedge A \end{array}$$

Then it is easy to check that all the non-zero commutator relations are of the form

$$[x \wedge y, x \wedge z] = 2y \wedge z.$$

4.3. **Matrix commutators.** Again, the above computations are all done with actions on 2-component column vectors. However, just as in Section 3 above, we observe *post hoc* that matrix commutators, computed with the same rules as before, give the same answers. In other words, we may define an abstract Lie algebra of type $\mathfrak{so}(16)$, spanned by these matrices, and with the Lie bracket defined by the matrix commutator, suitably interpreted.

Indeed, the only new commutators we need to consider are the $[X_{aA}, X_{bB}]$. But the computation above works exactly the same whether we work with actions, or with abstract matrices (i.e. multiplying out $a \circ b$ and $A \circ B$ as octonion products wherever they arise). The reason for this is that the only rules we are using are commutation rules between octonion actions, and these are the same as the corresponding rules for octonion multiplications: essentially just that $L_p \circ L_q = -L_q \circ L_p$ corresponds to $pq = -qp$. Hence no new rules of matrix multiplication are required.

For completeness we write out this result as a formal theorem.

Theorem 2. *Let $\mathcal{L}$ be the real 120-space spanned by the following 2×2 matrices over $\mathbb{K} \otimes \mathbb{L}$, where $\mathbb{K}$ and $\mathbb{L}$ are octonion algebras:*

$$X_{aA} = \begin{pmatrix} 0 & aA \\ -\overline{a}\,\overline{A} & 0 \end{pmatrix}, \quad D_p = \begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix}, \quad D_{p,q} = \begin{pmatrix} p \circ q & 0 \\ 0 & p \circ q \end{pmatrix}$$

(20)
$$D_P = \begin{pmatrix} P & 0 \\ 0 & -P \end{pmatrix}, \quad D_{P,Q} = \begin{pmatrix} P \circ Q & 0 \\ 0 & P \circ Q \end{pmatrix}$$

where p, q are any distinct imaginary basic units of $\mathbb{K}$, and a is any basic unit, and similarly for $P, Q, A \in \mathbb{L}$. Define a commutator on $\mathcal{L}$ as the matrix commutator, subject to evaluating all entries as left-multiplications. Then $\mathcal{L}$ with this commutator map is isomorphic to the Lie algebra $\mathfrak{so}(16)$, if both $\mathbb{K}$ and $\mathbb{L}$ are division algebras, or to $\mathfrak{so}(8,8)$ if both are split octonion algebras, or to $\mathfrak{so}(12,4)$, if they are one of each type.

5. The Lie algebra $\mathfrak{f}_4 = \mathfrak{su}(3, \mathbb{O})$

5.1. **Introduction.** The Lie algebra $\mathfrak{f}_4$ may be defined abstractly as the algebra of derivations on the exceptional Jordan algebra, or Albert algebra, that is an algebra of 3×3 Hermitian matrices over $\mathbb{O}$. However, this does not lead easily to a nice description of $\mathfrak{f}_4$, as there is no obvious canonical parametrisation of the derivations. In effect, each derivation has many names.

A construction of the Lie *group* F_4, acting as automorphisms of the Albert algebra, is described by Dray and Manogue [3], in terms of 3×3 anti-Hermitian octonion matrices. Indeed, generators for this group are already given by Jacobson in 1960 [6]. Naively, the space of anti-Hermitian matrices has dimension 45, and there are significant technical difficulties in turning this into a workable description of 52-dimensional F_4. In particular, the elements of the group generated by the actions of the diagonal matrices do not have an obvious canonical notation. In a sense, it is straightforward to adapt these constructions to construct the corresponding action of the Lie algebra $\mathfrak{f}_4$ on the Albert algebra, in effect by differentiating the group generators. But the problem of non-canonical names for automorphisms and derivations is one that remains to be solved at some point.

In order to prepare the ground for our eventual construction of $\mathfrak{e}_8$, we shall take a concrete 'first-principles' approach, and build $\mathfrak{f}_4$ explicitly from three copies of $\mathfrak{so}(9)$, as constructed in Section 3. We shall not use the Albert algebra structure directly, so we shall not prove directly that the elements of our Lie algebra are derivations. Instead we shall prove that our Lie algebra is simple, and compact, and has dimension 52, so it follows from the classification that it is the compact real form of $\mathfrak{f}_4$.

5.2. **The action of $\mathfrak{so}(9)$ on the Albert algebra.** We construct $\mathfrak{f}_4$ from the union of three copies of $\mathfrak{so}(9)$, each acting on two of the three coordinates. Thus we first define the three types of off-diagonal anti-Hermitian matrices as

$$(21) \quad X_a = \begin{pmatrix} 0 & a & 0 \\ -\overline{a} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y_a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -\overline{a} & 0 \end{pmatrix}, \quad Z_a = \begin{pmatrix} 0 & 0 & -\overline{a} \\ 0 & 0 & 0 \\ a & 0 & 0 \end{pmatrix}.$$

The action of such a matrix A on an element H of the Albert algebra is given by

$$(22) \qquad\qquad A \;:\; H \mapsto AH + H\overline{A}^\top = AH - HA$$

where $\overline{A}^\top$ denotes the octonion-conjugate transpose matrix, and $\overline{A}^\top = -A$ and $\overline{H}^\top = H$.

Writing

$$(23) \qquad\qquad H \;=\; \begin{pmatrix} \rho & \gamma & \overline{\beta} \\ \overline{\gamma} & \sigma & \alpha \\ \beta & \overline{\alpha} & \tau \end{pmatrix}$$

we compute

$$(24) \qquad [X_a, H] \;=\; \begin{pmatrix} a\overline{\gamma} + \gamma\overline{a} & a(\sigma - \rho) & a\alpha \\ \overline{a}(\sigma - \rho) & -\overline{a}\gamma - \overline{\gamma}a & -\overline{a}(\overline{\beta}) \\ \overline{\alpha}(\overline{a}) & -\beta a & 0 \end{pmatrix}$$

Hence we see that the X_a generate a copy of $\mathfrak{so}(9)$, acting on the left on $\begin{pmatrix} \overline{\beta} \\ \alpha \end{pmatrix}$ as the spin representation. and on the right on $\begin{pmatrix} \beta & \overline{\alpha} \end{pmatrix}$ as the dual copy of the spin representation, and on both sides on $\begin{pmatrix} (\rho - \sigma)/2 & \gamma \\ \overline{\gamma} & (\sigma - \rho)/2 \end{pmatrix}$ as the natural representation. This leaves two copies of the trivial representation, spanned by τ and $\begin{pmatrix} (\rho + \sigma)/2 & 0 \\ 0 & (\rho + \sigma)/2 \end{pmatrix}$.

Thus we obtain the following diagonal matrices as commutators, which can be computed directly with the matrices, just as in Section 3.5.

$$D_p = [X_1, X_p]/2 \;=\; \begin{pmatrix} p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$E_p = [Y_1, Y_p]/2 \;=\; \begin{pmatrix} 0 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & -p \end{pmatrix}$$

$$F_p = [Z_1, Z_p]/2 \;=\; \begin{pmatrix} -p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p \end{pmatrix}$$

$$D_{p,q} = [X_p, X_q]/2 = \begin{pmatrix} p \circ q & 0 & 0 \\ 0 & p \circ q & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$E_{p,q} = [Y_p, Y_q]/2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & p \circ q & 0 \\ 0 & 0 & p \circ q \end{pmatrix}$$

$$(25) \qquad F_{p,q} = [Z_p, Z_q]/2 = \begin{pmatrix} p \circ q & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p \circ q \end{pmatrix}$$

We now have three copies of $\mathfrak{so}(9)$, generated by

(1) the D_p, $D_{p,q}$ and X_a;
(2) the E_p, $E_{p,q}$ and Y_a; and
(3) the F_p, $F_{p,q}$ and Z_a.

All commutators within one of the three copies of $\mathfrak{so}(9)$ can be computed by matrix operations as in Section 3.5 above. Before we compute the rest of the commutators, we investigate the intersection of the three copies of $\mathfrak{so}(9)$.

5.3. **Triality.** The trace 0 part of the Albert algebra is spanned by the elements

$$U_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad U_a = \begin{pmatrix} 0 & a & 0 \\ \overline{a} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$V_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad V_a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & \overline{a} & 0 \end{pmatrix}$$

$$(26) \qquad W_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad W_a = \begin{pmatrix} 0 & 0 & \overline{a} \\ 0 & 0 & 0 \\ a & 0 & 0 \end{pmatrix}$$

We compute the action of the diagonal matrices D_p, E_p and F_p on the Albert algebra as follows:

$$
\begin{aligned}
D_p &: & U_a &\mapsto U_{pa} + U_{ap} \\
E_p &: & U_a &\mapsto -U_{ap} \\
(27) \qquad F_p &: & U_a &\mapsto -U_{pa}
\end{aligned}
$$

as well as $U_0 \mapsto 0$, and corresponding actions on V_a and W_a.

It then follows that these actions of D_p, E_p and F_p on U_a are given by the negatives of the actions of B_p, R_p and L_p on the subscript a, in the three cases. We have already described the triality relations which hold between these three types of operators on the octonions, and it follows that the same relations hold between the actions of the Ds, Es and Fs. Specifically, this induces the following equivalence between matrices:

$$
\begin{aligned}
E_\ell &\equiv (-D_\ell - D_{i,i\ell} - D_{j,j\ell} - D_{k,k\ell})/2 \\
(28) \qquad F_\ell &\equiv (-D_\ell + D_{i,i\ell} + D_{j,j\ell} + D_{k,k\ell})/2
\end{aligned}
$$

These formulae appear for example in Wangberg's PhD thesis [10]. From them one easily deduces the equivalent formulae for the remaining elements:

$$
\begin{aligned}
E_{i,i\ell} &\equiv (D_\ell + D_{i,i\ell} - D_{j,j\ell} - D_{k,k\ell})/2 \\
(29) \qquad F_{i,i\ell} &\equiv (-D_\ell + D_{i,i\ell} - D_{j,j\ell} - D_{k,k\ell})/2
\end{aligned}
$$

These relations imply that in their actions on the Albert algebra, any two of the three copies of $\mathfrak{so}(9)$ intersect in the same $\mathfrak{so}(8)$ subalgebra. Hence the dimension of their sum is $28 + 8 + 8 + 8 = 52$. We must now prove that this sum is closed under commutation, in order to complete the proof that it is a Lie algebra. (Of course, in a sense this is all well-known. The point, however, is to find a proof which generalises nicely to $\mathfrak{e}_8$, where the results are not already known.)

5.4. **An abstract $\mathfrak{f}_4$ Lie algebra.** In terms of actions, the algebra is spanned by X_a, Y_a, Z_a, and the D_p and $D_{p,q}$. The commutator relations between Xs and Ds have all been computed above. Similar relations hold between the Ys and Es, and between the Zs and the Fs. Using the computed linear equations between the Ds, Es and Fs, we can deduce the commutator relations between the Ds and both the Ys and Zs. It is worth noting the particular cases

(30)
$$\begin{aligned} [D_p, Y_b] &= -Y_{pb} \\ [D_p, Z_c] &= -Z_{cp} \end{aligned}$$

This leaves only the commutators between Xs, Ys and Zs to compute. The computations are non-trivial so we sketch them here. The following technical lemma will be useful.

Lemma 1. *If $y \neq b$ and $z \neq c$ are basic octonion units, then $\overline{b}(yc) = -\overline{y}(bc)$ and $(bz)\overline{c} = -(bc)\overline{z}$.*

Proof. We prove the first equation, the second being essentially the same. First, if $y = 1$, so $b \neq 1$, then
$$\overline{b}(yc) = \overline{b}c = -bc = -\overline{y}(bc).$$

Similarly, if $b = 1$. Now we can assume $b \neq 1 \neq y$, and either b, c, y associate, in which case
$$\overline{b}(yc) = (\overline{b}y)c = -(\overline{y}b)c = -\overline{y}(bc),$$

or they anti-associate, in which case only the signs of the middle two expressions change. $\qquad\square$

Lemma 2. *In the action on the Albert algebra, the relation*
$$[Y_b, Z_c] = -X_{\overline{bc}}$$

holds for any octonions b, c.

Proof. We first compute the action of X_a directly, with special cases

(31)
$$\begin{aligned} U_0 &\mapsto -2U_a \\ U_a &\mapsto 2U_0 \\ V_0 &\mapsto W_a \\ W_0 &\mapsto V_a \end{aligned}$$

and generic cases

(32)
$$\begin{aligned} U_x &\mapsto 0 \\ V_y &\mapsto W_{\overline{ay}} \\ W_z &\mapsto -V_{\overline{za}} \end{aligned}$$

where a, b, x, y, z all denote basic octonion units, with $x \neq a$.

The actions of Y_b and Z_c are given by the same formulae with the letters U, V, W cycled in the obvious way. Hence we can compute the action of $[Y_b, Z_c] = Y_b Z_c - Z_c Y_b$. The special cases are straightforward:

$$
\begin{aligned}
U_0 &\mapsto 2U_{\overline{bc}} \\
V_0 &\mapsto -W_{\overline{bc}} \\
W_0 &\mapsto -V_{\overline{bc}} \\
U_{\overline{bc}} &\mapsto 2V_0 + 2W_0 = -2U_0
\end{aligned}
$$

(33)

but the generic cases require some extra work:

$$
\begin{aligned}
U_x &\mapsto 0 \\
V_y &\mapsto W_{\overline{b}(yc)} - 2\delta_{yb}W_c \\
W_z &\mapsto -V_{(bz)\overline{c}} + 2\delta_{zc}V_b
\end{aligned}
$$

(34)

To see this, observe first that in the case $y = b$ the image of V_y simplifies to $-W_c$, and in the case $z = c$ the image of W_z simplifies to V_b. The remaining cases are dealt with by Lemma 1. $\qquad\square$

5.5. **Matrix commutators.** We have already seen that the Lie bracket in $\mathfrak{so}(9)$ agrees with the matrix commutator, subject to some quite intuitive rules for extending the octonion multiplication to incorporate terms of the form $p \circ q$. In addition, it is easy to check that the commutator rule $[X_a, Y_b] = -Z_{\overline{ab}}$ also corresponds exactly to the natural matrix commutator. It follows that the Lie bracket can be completely calculated using matrices, provided that diagonal matrices are always expressed in the appropriate form for the calculation being performed.

It is interesting, however, to explore to what extent mixed commutators, such as $[D_i, E_j]$ or $[D_i, Y_j]$, can be computed directly, without first converting the Ds, Es or Fs into the appropriate form. Let us consider first the commutators between one diagonal and one off-diagonal element. If we compute

$$
\begin{aligned}
[E_\ell, X_i] &= [-D_\ell - D_{i,i\ell} - D_{j,j\ell} - D_{k,k\ell}, X_i]/2 \\
&= -X_{i\ell}
\end{aligned}
$$

(35)

then we get exactly what we would expect by naively computing the commutator of the matrices:

$$
(36) \qquad
\left[
\begin{pmatrix} 0 & 0 & 0 \\ 0 & \ell & 0 \\ 0 & 0 & -\ell \end{pmatrix},
\begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\right]
=
\begin{pmatrix} 0 & -i\ell & 0 \\ \ell i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

By symmetry, this shows that most of the commutators of Xs, Ys or Zs with 'single-index' Ds, Es or Fs can be computed as matrix commutators. The remaining cases $[E_p, X_1]$ and $[F_p, X_1]$ are easy, since there is sufficient associativity in these cases to identify the action of the commutator matrix with the commutator of the actions.

Next we investigate the commutators of Ys and Zs with 'double-index' Ds. We first calculate with the actions on the Albert algebra, using the Jacobi identity, which holds automatically for actions.

$$
\begin{aligned}
2[Y_a, D_{p,q}] &= [Y_a, [D_p, D_q]] \\
&= -[D_p, [D_q, Y_a]] - [D_q, [Y_a, D_p]] \\
&= [D_p, Y_{jqa}] - [D_q, Y_{pa}] \\
&= -Y_{p(qa)} + Y_{q(pa)} \\
&= -2Y_{p(qa)}
\end{aligned}
$$

(37)

Now if we try computing matrix commutators we obtain

$$[Y_a, D_{p,q}]_M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -\overline{a} & 0 \end{pmatrix} \begin{pmatrix} p \circ q & 0 & 0 \\ 0 & p \circ q & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$- \begin{pmatrix} p \circ q & 0 & 0 \\ 0 & p \circ q & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -\overline{a} & 0 \end{pmatrix}$$

$$(38) \qquad = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -(p \circ q)a \\ 0 & -\overline{a}(p \circ q) & 0 \end{pmatrix}$$

Clearly, if these two methods of computation are to give the same answer, then we require

$$(39) \qquad \begin{aligned} (p \circ q)a &:= p(qa) \\ a(p \circ q) &:= (ap)q \end{aligned}$$

which is again a natural convention. That is, $p \circ q$ is interpreted as a composition of operators, acting on left or right as appropriate, and with the innermost operator acting first. Then the above matrix reduces to

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -p(qa) \\ 0 & -(\overline{a}p)q & 0 \end{pmatrix} = Y_{-p(qa)}$$

$$(40) \qquad\qquad\qquad\qquad\qquad = Y_{q(pa)}$$

as required.

Similarly, for $[Z_a, D_{p,q}]$, again calculating in the Lie algebra we have constructed, and using the Jacobi identity, we have

$$\begin{aligned} 2[Z_a, D_{p,q}] &= [Z_a, [D_p, D_q]] \\ &= -[D_p, [D_q, Z_a]] - [D_q, [Z_a, D_p]] \\ &= [D_p, Z_{aq}] - [D_q, Z_{ap}] \\ &= -Z_{(aq)p} + Z_{(ap)q} \\ (41) \qquad\qquad &= 2Z_{(ap)q} \end{aligned}$$

Then we compare with the naive computation of commutators of matrices, using the same interpretations of $(p \circ q)a$ and $a(p \circ q)$ as above. We have

$$[Z_a, D_{p,q}]_M = \begin{pmatrix} 0 & 0 & -\overline{a} \\ 0 & 0 & 0 \\ a & 0 & 0 \end{pmatrix} \begin{pmatrix} p \circ q & 0 & 0 \\ 0 & p \circ q & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$- \begin{pmatrix} p \circ q & 0 & 0 \\ 0 & p \circ q & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -\overline{a} \\ 0 & 0 & 0 \\ a & 0 & 0 \end{pmatrix}$$

$$(42) \qquad = \begin{pmatrix} 0 & 0 & (p \circ q)\overline{a} \\ 0 & 0 & 0 \\ a(p \circ q) & 0 & 0 \end{pmatrix}$$

and again we get the correct answer.

We turn now to the computation of commutators of two diagonal elements. In the Lie algebra, we have

$$[E_\ell, D_i] = -[D_\ell + D_{i,i\ell} + D_{j,j\ell} + D_{k,k\ell}, D_i]/2$$

$$\begin{aligned}
&= & D_{i,\ell} - D_{i\ell} \\
(43) \qquad &= & D_{i,\ell} + E_{i,\ell} - F_{i,\ell}
\end{aligned}$$

The same computation with matrices gives

$$\begin{aligned}
[E_\ell, D_i]_M &= & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2i \circ \ell & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
(44) \qquad &= & D_{i,\ell} + E_{i,\ell} - F_{i,\ell}
\end{aligned}$$

By symmetry, therefore, the commutators of single-index Ds, Es and Fs can all be calculated correctly by naive matrix commutator calculations.

Unfortunately, the same is not true for double-index elements. The easiest way to translate these elements between D, E and F is probably to use the equations

$$(45) \qquad\qquad E_{p,q} - E_{pq} = F_{p,q} + F_{pq}$$

and their images under rotations of D, E, F.

To summarise the results of this section, we have shown how to compute all commutators between elements of the basis consisting of D_p, $D_{p,q}$, X_a, Y_a, Z_a, in such a way that the matrix commutator, suitably interpreted, agrees with the commutator of the action on the Jordan algebra. We have derived certain triality relations between the bases of type D, E and F for $\mathfrak{so}(8)$, which describe relations betwen different linear combinations of basis vectors which act in the same way on the Albert algebra. These relations are used in an essential way in the computation of matrix commutators, and we thereby obtain an abstract definition of the Lie algebra of type $\mathfrak{f}_4$, with respect to any one of three triality-related bases.

More formally:

Theorem 3. *Let $\mathcal{L}$ be the real 52-space spanned by the 3×3 octonionic matrices X_a, Y_a, Z_a together with either D_p, $D_{p,q}$, or E_p, $E_{p,q}$, or F_p, $F_{p,q}$ defined above, subject to the relations given above which identify the three 28-spaces spanned by the Ds, the Es and the Fs. Define a commutator on $\mathcal{L}$ as the matrix commutator, subject to the following special rules:*

(1) *Commutators of two diagonal matrices must be computed in one of the three languages D, E or F, and not mixed;*

(2) *Commutators within each of the three copies of $\mathfrak{so}(9)$ are computed as in Theorem 1.*

(3) *Commutators of any X with any Y or Z, or of any Y with any Z, are computed with the usual octonion multiplication.*

(4) *Commutators of any D with any Y or Z (and similarly for E with X or Z, and for F with X or Y) may be computed by using the rules $(p \circ q)a = p(qa)$ and $a(p \circ q) = (ap)q$.*

Then $\mathcal{L}$ with this commutator map is isomorphic to the compact real form of the Lie algebra $\mathfrak{f}_4$, if the octonion division algebra is used, or to the form $\mathfrak{f}_{4(4)}$, if the split octonions are used.

6. The Lie algebra $\mathfrak{e}_8 = \mathfrak{su}(3, \mathbb{O} \otimes \mathbb{O})$

6.1. Introduction. The minimal representation of $\mathfrak{e}_8$ is the adjoint representation of the Lie algebra on itself. A concrete approach is to build the algebra out of three copies of $\mathfrak{so}(16)$, each acting on an explicitly defined 248-dimensional vector space. These actions then generate an associative subalgebra of the full 248×248 matrix

algebra. We show that any two of these copies of $\mathfrak{so}(16)$ intersect in $\mathfrak{so}(8) + \mathfrak{so}(8)$, and that they therefore span a space of dimension 248 inside the matrix algebra. Finally we verify explicitly that the commutators of these 248 linear maps also lie in this 248-space. Thus the space is closed under commutation, and since the Jacobi identity holds automatically, we identify the algebra as $\mathfrak{e}_8$.

6.2. **A** 248**-dimensional space.** We begin by defining the 248-dimensional vector space. Let $\mathbb{K}$ and $\mathbb{L}$ be two copies of the octonions, with standard basis elements $1, i, j, k, \ell, i\ell, j\ell, k\ell \in \mathbb{K}$ and $1, I, J, K, L, IL, JL, KL \in \mathbb{L}$, such that $ij = k$, $IJ = K$, and i, j, ℓ generate $\mathbb{K}$ and I, J, L generate $\mathbb{L}$. For basis elements a of $\mathbb{K}$ and A of $\mathbb{L}$, define

$$
X_{aA} = \begin{pmatrix} 0 & a \otimes A & 0 \\ -\overline{a} \otimes \overline{A} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

$$
Y_{aA} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \otimes A \\ 0 & -\overline{a} \otimes \overline{A} & 0 \end{pmatrix}
$$

(46)
$$
Z_{aA} = \begin{pmatrix} 0 & 0 & -\overline{a} \otimes \overline{A} \\ 0 & 0 & 0 \\ a \otimes A & 0 & 0 \end{pmatrix}
$$

Under matrix commutation, as defined in Section 4, the matrices X_{aA} generate a copy of $\mathfrak{so}(16)$, containing the matrices of type D_p, $D_{p,q}$, D_P and $D_{P,Q}$. These together with the X_{aA} form a 120-dimensional vector space, on which $\mathfrak{so}(16)$ acts in its adjoint representation. The Y_{aA} and Z_{aA} form a 128-dimensional space, on which $\mathfrak{so}(16)$ acts in its spin representation. Let $\mathcal{V}$ to be the 248-dimensional vector space spanned by the matrices just mentioned.

Similarly we may define spaces $\mathcal{V}'$ and $\mathcal{V}''$ on which the other two copies of $\mathfrak{so}(16)$ act. Now restricting the subscripts to lie in $\mathbb{K}$, we must recover the construction of $\mathfrak{f}_4$ from three copies of $\mathfrak{so}(9)$ intersecting in $\mathfrak{so}(8)$. Therefore we must have the same relations as before expressing the E and F elements as linear combinations of the D type elements. The same applies, of course, to the restriction to $\mathbb{L}$. These relations are enough to identify the three vector spaces $\mathcal{V}$, $\mathcal{V}'$ and $\mathcal{V}''$, and hence they allow us to construct actions of all three copies of $\mathfrak{so}(16)$ on the same 248-space.

6.3. **The space acts on itself.** As a vector space, $\mathcal{V}$ is the sum of three copies of $\mathfrak{so}(16)$, any two of which intersect in $\mathfrak{so}(8) \oplus \mathfrak{so}(8)$. The next step is to show that every one of our generators for $\mathfrak{so}(16)$ acts on this 248-space, as a combination of its action on both the adjoint and spin representations. The argument is much the same as for $\mathfrak{so}(9)$ in $\mathfrak{f}_4$ The basic idea is to use the action of 3×3 anti-Hermitian matrices on each other via

(47)
$$
A \quad : \quad X \mapsto AX - XA.
$$

But we need to verify that this really is the same as the action of $\mathfrak{so}(16)$ on the sum of the adjoint and spin representations as described in Section 4. Breaking up the 3×3 matrices in blocks of sizes $2 + 1$ we can write the diagonal parts of both A and X so that they have 0 in the third place. Then, writing A' and X' for the top-left 2×2 blocks of A and X, we have

(48)
$$
\begin{pmatrix} A' & 0 \\ 0 & 0 \end{pmatrix} \quad : \quad \begin{pmatrix} X' & v \\ -v^{\dagger} & 0 \end{pmatrix} \mapsto \begin{pmatrix} A'X' - X'A' & A'v \\ v^{\dagger}A' & 0 \end{pmatrix}
$$

Now we see the adjoint action given by matrix commutation in the 2×2 block, and the spinor action $A' : v \mapsto A'v$ exactly as described in Section 4. The bottom row $v^\dagger A' = -v^\dagger A'^\dagger = -(A'v)^\dagger$ just repeats the spinor action. It follows that our generators for the three copies of $\mathfrak{so}(16)$ generate a subalgebra of the associative algebra of 248×248 matrices. Therefore under commutation they generate a Lie algebra.

6.4. **Closure.** It suffices now to show that all the commutators are already in the 248-space spanned by the generators. Within $\mathfrak{so}(16)$ we already know that the commutator of the actions on $\mathcal{V}$ is given by the matrix commutator as described in Section 4. In particular, all such commutators lie in the 248-space, which consists of matrices modulo the relations given above.

The only case not already dealt with inside one of the three copies of $\mathfrak{so}(16)$ is the matrix commutator

$$(49) \qquad [X_{aA}, Y_{bB}]_M \quad = \quad -Z_{\overline{abAB}},$$

which we must now prove is equal to the commutator of the actions of X_{aA} and Y_{bB}.

Lemma 3. *In the action on $\mathcal{V}$, we have $[X_{aA}, Y_{bB}] = -Z_{\overline{abAB}}$ for all basic units $a, b \in \mathbb{K}$ and $A, B \in \mathbb{L}$.*

Proof. First consider the case $bB = 1$, that is $b = 1$ and $B = 1$. If we compute directly the action of $[X_{aA}, Y_1]$ on any vector in the 248-space, there are no issues with non-associativity, and therefore the action is the same as the matrix commutator $[X_{aA}, Y_1]_M = -Z_{\overline{aA}}$. The same applies to the commutators $[X_A, Y_b]$.

Now the Jacobi identity, which holds automatically for linear actions on a vector space, allows us to extend this to the general case. We deduce first that for $a \neq 1$, $b \neq 1$ and $B = 1$

$$
\begin{aligned}
[X_{aA}, Y_b] \quad &= \quad [X_{aA}, [Y_1, D_b]] \\
&= \quad -[Y_1, [D_b, X_{aA}]] - [D_b, [X_{aA}, Y_1]] \\
(50) \qquad &= \quad -[Y_1, X_{abA}] - [Y_1, X_{baA}] + [D_b, Z_{\overline{aA}}]
\end{aligned}
$$

which in the case $b \neq a$ evaluates to $-Z_{\overline{ab}\overline{A}} = -Z_{\overline{abA}}$ and in the case $b = a$ evaluates to

$$-2Z_{\overline{abA}} - Z_{\overline{ab}\overline{A}} = -Z_{\overline{abA}}.$$

Similarly if $B \neq 1$ we have

$$
\begin{aligned}
[X_{aA}, Y_{bB}] \quad &= \quad [X_{aA}, [Y_b, D_B]] \\
&= \quad -[Y_b, [D_B, X_{aA}]] - [D_B, [X_{aA}, Y_b]] \\
(51) \qquad &= \quad -[Y_b, X_{aAB}] - [Y_b, X_{aBA}] - [D_B, Z_{\overline{abA}}]
\end{aligned}
$$

which we again evaluate separately in the cases $B = A$ and $B \neq A$. This completes the proof. $\square$

6.5. **The main result.** Pulling together all the various strands, we have the following main result.

Theorem 4. *Let $\mathbb{K}$ and $\mathbb{L}$ be two copies of the octonions (either split or compact, or one of each), with typical elements $a, p, q \in \mathbb{K}$ and $A, P, Q \in \mathbb{L}$, such that a, p, q are orthogonal, and p, q are imaginary, and similarly A, P, Q are orthogonal and P, Q are imaginary.*

Define diagonal matrices $D_p, D_P, D_{p,q}, D_{P,Q}$ as

$$D_p = \begin{pmatrix} p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad D_{p,q} = \begin{pmatrix} p \circ q & 0 & 0 \\ 0 & p \circ q & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and similarly for $\mathbb{L}$. Define similarly matrices E_p and $E_{p,q}$ as the images of D_p and $D_{p,q}$ under rotating the three coordinates down and to the right, and F_p and $F_{p,q}$ as images under the inverse rotation.

Define off-diagonal matrices with entries in $\mathbb{K} \otimes \mathbb{L}$ as

$$(52) \qquad X_{aA} \quad = \quad \begin{pmatrix} 0 & a \otimes A & 0 \\ -\overline{a} \otimes \overline{A} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and similarly Y_{aA} and Z_{aA} by rotating respectively down-and-right and up-and-left.

Define a real vector space $\mathcal{V}$ spanned by all the matrices just defined, by imposing the linear relations given in Section 5.3, that is, if $\mathbb{K}$ is compact, the relations

$$(53) \qquad \begin{array}{rcl} E_\ell & \equiv & -(D_\ell + D_{i,i\ell} + D_{j,j\ell} + D_{k,k\ell})/2 \\ F_\ell & \equiv & (-D_\ell + D_{i,i\ell} + D_{j,j\ell} + D_{k,k\ell})/2 \end{array}$$

together with the images under the 7-cycle $(i, j, \ell, k, j\ell, -k\ell, i\ell)$, and mutatis mutandis *for $\mathbb{L}$ and for split octonions. Then $\mathcal{V}$ has dimension 248.*

Define an antisymmetric product on $\mathcal{V}$ via

$$[V, W] = VW - WV,$$

where commutators of diagonal matrices are computed in each copy of $\mathfrak{so}(8)$ in any one of the three languages D, E, F (but not mixed!), and octonion operations are extended to nested octonions via the rules

$$(54) \qquad \begin{array}{rcl} [p \circ q, a] & = & p \circ q \circ a - a \circ p \circ q \quad (= 0 \text{ unless } a = p \text{ or } a = q) \\ (p \circ q)a & = & p(qa) \\ a(p \circ q) & = & (ap)q. \end{array}$$

Then this product is well-defined, i.e. does not depend on the representing matrices V, W for elements of $\mathcal{V}$. Moreover, this product satisfies the Jacobi identity, and makes $\mathcal{V}$ into a copy of the compact real form of the Lie algebra $\mathfrak{e}_8$ (if both $\mathbb{K}$ and $\mathbb{L}$ are division algebras), or the split real form (if both $\mathbb{K}$ and $\mathbb{L}$ are split octonions), or the other real form $\mathfrak{e}_{8(-24)}$ (if one of each).

References

[1] C. H. Barton and A. Sudbery, Magic squares and matrix models of Lie algebras, *Advances in Math.* **180** (2003), 596–647.

[2] C. Chevalley, Sur certains groupes simples, *Tohoku Math. J.* **7** (1955), 14–66.

[3] Tevian Dray and Corinne A. Manogue, Octonionic Cayley spinors and E_6, *Comment. Math. Univ. Carolin.* **51** (2010), 193–207.

[4] Tevian Dray, John Huerta and Joshua Kincaid, The 2×2 Lie group magic square, *Lett. Math. Phys.* **104** (2014), 1445–1468.

[5] N. Jacobson, Some groups of transformations defined by Jordan algebras. I. *J. reine angew. Math.* **201** (1959), 178–195.

[6] N. Jacobson, Some groups of transformations defined by Jordan algebras. II. Groups of type F_4. *J. reine angew. Math.* **204** (1960), 74–98.

[7] N. Jacobson, Some groups of transformations defined by Jordan algebras. III. Groups of type E_{6I}. *J. reine angew. Math.* **207** (1961), 61–85

[8] Corinne A. Manogue and Jörg Schray, Finite Lorentz transformations, automorphisms, and division algebras, *J. Math. Phys.* **34** (1993), 3746–3767.

[9] E. B. Vinberg, Construction of the exceptional simple Lie algebras (in Russian), *Trudy Sem. Vekt. Tenz. Anal.* **13** (1966), 7–9. (Translated into English, *Amer. Math. Soc. Transl.* **213** (2005), 241–242.)

[10] A. Wangberg, The structure of E_6, PhD thesis, Oregon State University, 2007.

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