An Operational Measure of Riskiness

Dean P. Foster
University of Pennsylvania

Sergiu Hart

An Operational Measure of Riskiness

Abstract

We propose a measure of riskiness of “gambles” (risky assets) that is objective: it depends only on the gamble and not on the decision maker. The measure is based on identifying for every gamble the critical wealth level below which it becomes “risky” to accept the gamble.

Disciplines

Statistical Theory | Statistics and Probability

An Operational Measure of Riskiness

Dean P. Foster

University of Pennsylvania

Sergiu Hart

Hebrew University of Jerusalem

We propose a measure of riskiness of "gambles" (risky assets) that is objective: it depends only on the gamble and not on the decision maker. The measure is based on identifying for every gamble the critical wealth level below which it becomes "risky" to accept the gamble.

I. Introduction

You are offered a gamble (a "risky asset") g in which it is equally likely that you gain \$120 or lose \$100. What is the risk in accepting g? Is there an *objective* way to measure the *riskiness* of g? "Objective" means that the measure should depend on the gamble itself and not on the decision maker; that is, only the outcomes and the probabilities (the "distribution") of the gamble should matter.

Such objective measures exist for the "return" of the gamble—its

Research was partially supported by the Institute for Advanced Studies at the Hebrew University of Jerusalem and by a grant of the Israel Science Foundation. We thank Robert Aumann for many illuminating discussions and Elchanan Ben-Porath, Eddie Dekel, Tom Ferguson, Martin Hellwig, Eugene Kandel, Paul Klemperer, Michael Lachmann, Haim Levy, Andreu Mas-Colell, Igal Milchtaich, Benny Moldovanu, Stephen Morris, John Nash, Ignacio Palacios-Huerta, Ron Peretz, Motty Perry, Doron Puder, Phil Reny, Frank Riedel, Arthur Robson, Ariel Rubinstein, Uzi Segal, Roberto Serrano, Oscar Volij, Zvi Wiener, Shlomo Yitzhaki, Peyton Young, Shmuel Zamir, Yechezkel Zilber, the anonymous referees, and the editor for useful comments and suggestions. A presentation is available at http://www.ma.huji.ac.il/hart/abs/risk-p.html.

[*Journal of Political Economy,* 2009, vol. 117, no. 5]

expectation, here $\mathbf{E}[g] = \$10$—and for the "spread" of the gamble—its standard deviation, here $\boldsymbol{\sigma}[g] = \$110$. While the standard deviation is at times used also as a measure of riskiness, it is well known that it is not a very good measure in general. One important drawback is that it is not monotonic: a better gamble h, that is, a gamble with higher gains and lower losses, may well have a higher standard deviation and thus be wrongly viewed as having a higher riskiness.[1]

We propose here a *measure of riskiness* for gambles that, like the expectation and the standard deviation, is objective and is measured in the same units as the outcomes; moreover, it is monotonic and has a simple "operational" interpretation.

Let us return to our gamble g. The risk in accepting g clearly depends on how much wealth you have. If all you have is \$100 or less, then it is extremely risky to accept g: you risk going bankrupt (assume there is no "Chapter 11," etc.). But if your wealth is, say, \$1 million, then accepting g is not risky at all (and recall that the expectation of g is positive). While one might expect a smooth transition between these two situations, we will show that there is in fact a well-defined *critical wealth level* that separates between two very different "regimes": one in which it is "risky" to accept the gamble and the other in which it is not.[2]

What does "risky" mean, and what is this critical level? For this purpose we consider a very simple model, in which a decision maker faces an unknown sequence of gambles. Each gamble is offered in turn and may be either accepted or rejected; or, in a slightly more general setup, any proportion of the gamble may be accepted.

We show that for every gamble g there exists a unique critical wealth level $\mathbf{R}(g)$ such that accepting gambles g when the current wealth is below the corresponding $\mathbf{R}(g)$ leads to "bad" outcomes, such as decreasing wealth and even bankruptcy in the long run; in contrast, not accepting gambles g when the current wealth is below $\mathbf{R}(g)$ yields "good" outcomes: no-bankruptcy is guaranteed, and wealth can only increase in the long run.[3] In fact, almost any reasonable criterion—such as no-loss, an assured gain, or no-bankruptcy—will be shown to lead to exactly the same critical point $\mathbf{R}(g)$. We will call $\mathbf{R}(g)$ the *riskiness* of the gamble g since it provides a sharp distinction between the "risky" and the "nonrisky" decisions. The risky decisions are precisely those of accepting gambles g whose riskiness $\mathbf{R}(g)$ is too high, specifically, higher than the

[1] Take the gamble g above; increasing the gain from \$120 to \$150 and decreasing the loss from \$100 to \$90 makes the standard deviation increase from \$110 to \$120.

[2] We distinguish between the terms "risky" and "riskiness": the former is a property of decisions, the latter of gambles. Thus, accepting a gamble in a certain situation may be a *risky decision* (or not), whereas the *riskiness of a gamble* is a measure that, as we shall see, determines when the decision to accept the gamble is risky.

[3] All these occur with probability one (i.e., almost surely); see Secs. III and IV for the precise statements.

current wealth W (i.e., $\mathbf{R}(g) > W$): they lead to bad outcomes and possibly bankruptcy. In short, a "phase transition" occurs at $\mathbf{R}(g)$.

Moreover, the riskiness measure **R** that we obtain satisfies all our initial desiderata: it is objective (it depends only on the gamble), it is scale-invariant and thus measured in the same unit as the outcomes,[4] it is monotonic (increasing gains and/or decreasing losses lowers the riskiness), it has a simple operational interpretation, and, finally, it is given by a simple formula. We emphasize that our purpose is not to analyze the most general investment and bankruptcy models, but rather to use such simple operational setups as a sort of "thought experiment" in order to determine the riskiness of gambles.

In summary, what we show is that there is a clear and robust way to identify exactly when it becomes risky to accept a gamble, and then

> the *riskiness* of a gamble g is defined as the *critical wealth* below which accepting g becomes risky.

The starting point of our research was the "economic index of riskiness" recently developed by Aumann and Serrano (2008).[5] While attempting to provide an "operational" interpretation for it, we were led instead to the different measure of riskiness of the current paper. A detailed comparison of the two can be found in Section VI.A. Here we will only mention that the "index" compares gambles in terms of their riskiness, whereas our **R** is a "measure" that is defined separately for each gamble[6] and, moreover, has a clear interpretation, in monetary terms.[7]

The paper is organized as follows. The basic model of no-bankruptcy is presented in Section II, followed in Section III by the result that yields the measure of riskiness. Section IV extends the setup and shows the robustness of the riskiness measure; an illustrating example is provided at the end of the section. The properties of the riskiness measure are studied in Section V. Section VI discusses the literature and other pertinent issues, in particular, the work of Aumann and Serrano (2008) on the "economic index of riskiness" and of Rabin (2000) on "calibration." The proofs are relegated to the Appendix.

[4] That is, the unit ("currency") in which the outcomes are measured does not matter: rescaling all outcomes by a constant factor $\lambda > 0$ rescales the riskiness by the same λ. Most measures of riskiness satisfy this condition; see Secs. VI.D, VI.E.1, and VI.E.9.

[5] This index was used in the technical report of Palacios-Huerta, Serrano, and Volij (2004); see Aumann and Serrano (2008, 810n).

[6] This explains the use of the different terms "index" and "measure."

[7] Such an interpretation is problematic for the Aumann-Serrano index, which is determined only up to a positive multiple.

II. The Basic Model

This section and the next deal with the simple basic model; it is generalized in Section IV.

A. *Gambles*

A *gamble* g is a real-valued random variable[8] having some negative values—losses are possible—and positive expectation, that is, $\mathbf{P}[g<0]>0$ and $\mathbf{E}[g]>0$. For simplicity[9] we assume that each gamble g has finitely many values, say $x_1, x_2, \ldots, x_m$, with respective probabilities $p_1, p_2, \ldots, p_m$ (where $p_i>0$ and $\sum_{i=1}^{m} p_i = 1$). Let $\mathcal{G}$ denote the collection of all such gambles.

Some useful notation: $L(g) := -\min_i x_i > 0$ is the *maximal loss* of g; $M(g) := \max_i x_i > 0$ is the *maximal gain* of g; and $\|g\| := \max_i |x_i| = \max\{M(g), L(g)\}$ is the (ℓ_∞) *norm* of g. One way to view g is that one buys a "ticket" to g at a cost of $L(g)>0$; this ticket yields various prizes $L(g)+x_i$ with probability p_i each (and so there is a positive probability of getting no prize—when $x_i = -L(g)$).

B. *Gambles and Wealth*

Let the initial wealth be W_1. At every period $t = 1, 2, \ldots$, the decision maker, whose current wealth we denote W_t, is offered a gamble $g_t \in \mathcal{G}$ that he may either accept or reject. If he accepts g_t, then his wealth next period will be $W_{t+1} = W_t + g_t$;[10] and if he rejects g_t, then $W_{t+1} = W_t$. Exactly which gamble g_t is offered may well depend on the period t and the past history (of gambles, wealth levels, and decisions); thus, there are no restrictions on the stochastic dependence between the random variables g_t. Let G denote the *process* $(g_t)_{t=1,2,\ldots}$. We emphasize that there is no underlying probability distribution on the space of processes from which G is drawn; the setup is *non-Bayesian*, and the analysis is "worst-case." Thus, at time t the decision maker knows nothing about which future gambles he will face nor how his decisions will influence them.

To avoid technical issues, it is convenient to consider only *finitely generated* processes; such a process G is generated by a finite set of gambles $\mathcal{G}_0 = \{g^{(1)}, g^{(2)}, \ldots, g^{(m)}\} \subset \mathcal{G}$ such that the gamble g_t that is offered following any history is a nonnegative multiple of some gamble

[8] We take g to be a random variable for convenience; only the distribution of g will matter. $\mathbf{P}$ denotes "probability."

[9] A significant assumption here is that of "limited liability"; see Sec. VI.E.7.

[10] That is, the gamble g_t is realized, and with x denoting its outcome, $W_{t+1} = W_t + x$.

in $\mathcal{G}_0$; that is, it belongs to the finitely generated cone $\mathcal{G}_0 = \{\lambda g : \lambda \geq 0$ and $g \in \mathcal{G}_0\}$.[11]

C. *Critical Wealth and Simple Strategies*

As discussed in the introduction, we are looking for simple rules that distinguish between situations that are deemed "risky" and those that are not: the offered gamble is rejected in the former and accepted in the latter. Such rules—think of them as candidate riskiness measures—are given by a "threshold" that depends only on the distribution of the gamble and is scale-invariant. That is, there is a *critical-wealth function* Q that associates to each gamble $g \in \mathcal{G}$ a number $Q(g)$ in $[0, \infty]$, with $Q(\lambda g) = \lambda Q(g)$ for every $\lambda > 0$, and which is used as follows: a gamble g is rejected at wealth W if $W < Q(g)$, and is accepted if $W \geq Q(g)$. We will refer to the behavior induced by such a function Q as a *simple strategy* and denote it s_Q. Thus s_Q accepts g at wealth $Q(g)$ and at any higher wealth, and rejects g at all lower wealths: $Q(g)$ is the minimal wealth at which g is accepted. In the two extreme cases, $Q(g) = 0$ means that g is always accepted (i.e., at every wealth $W > 0$), whereas $Q(g) = \infty$ means that g is always rejected.[12]

D. *No-Bankruptcy*

Since risk has to do with losing money and, in the extreme, bankruptcy, we start by studying the simple objective of avoiding bankruptcy. Assume that the initial wealth is positive (i.e., $W_1 > 0$) and that borrowing is not allowed (so $W_t \geq 0$ for all t).[13] *Bankruptcy* occurs when the wealth becomes zero[14] or, more generally, when it converges to zero, that is, $\lim_{t\to\infty} W_t = 0$. The strategy s yields *no-bankruptcy* for the process G and the initial wealth W_1 if the probability of bankruptcy is zero, that is, $\mathbf{P}[\lim_{t\to\infty} W_t = 0] = 0$.[15] Finally, the strategy *s guarantees no-bankruptcy* if it yields no-bankruptcy for every process G and every initial wealth W_1. Thus, no matter what the initial wealth is and what the sequence of

[11] The term λg means that the values of g are rescaled by the factor λ, whereas the probabilities do not change (this is not to be confused with the "dilution" of Sec. V).

[12] See Sec. VI.E.6 and Sec. G in the Appendix for more general strategies.

[13] If borrowing is allowed up to some maximal credit limit C, then shift everything by C (see also Sec. VI.C).

[14] We emphasize that "bankruptcy" is to be taken in the simple, naive, and literal sense of losing all the wealth (rather than the legal and regulatory sense—e.g., Chapter 11—where losses may be limited and issues of agency, moral hazard, and risk shifting may arise).

[15] $\mathbf{P} \equiv \mathbf{P}_{W_1,G,s}$ is the probability distribution induced by the initial wealth W_1, the process G, and the strategy s.

gambles will be, the strategy s guarantees that the wealth will not go to zero (with probability one).

III. The Measure of Riskiness

The result in the basic setup is

THEOREM 1. For every gamble $g \in \mathcal{G}$ there exists a unique real number $\mathbf{R}(g) > 0$ such that a simple strategy $s \equiv s_Q$ with critical-wealth function Q guarantees no-bankruptcy if and only if $Q(g) \geq \mathbf{R}(g)$ for every gamble $g \in \mathcal{G}$. Moreover, $\mathbf{R}(g)$ is uniquely determined by the equation

$$\mathbf{E}\left[\log\left(1 + \frac{1}{\mathbf{R}(g)}g\right)\right] = 0. \tag{1}$$

The condition $Q(g) \geq \mathbf{R}(g)$ says that the minimal wealth level $Q(g)$ at which g is accepted must be $\mathbf{R}(g)$ or higher, and so g is for sure rejected at all wealth levels below $\mathbf{R}(g)$, that is, at all $W < \mathbf{R}(g)$. Therefore, we get

COROLLARY 1. A simple strategy s guarantees no-bankruptcy if and only if for every gamble $g \in \mathcal{G}$

$$s \text{ rejects } g \text{ at all } W < \mathbf{R}(g). \tag{2}$$

Thus $\mathbf{R}(g)$ is the minimal wealth level at which g may be accepted; as discussed in Section I, it is the *measure of riskiness of g*.

Simple strategies s satisfying (2) differ in terms of which gambles are accepted. The "minimal" strategy, with $Q(g) = \infty$ for all g, never accepts any gamble; the "maximal" one, with $Q(g) = \mathbf{R}(g)$ for all g, accepts g as soon as the wealth is at least as large as the riskiness of g; these two strategies, as well as any strategy in between, guarantee no-bankruptcy (see also proposition 6 in the Appendix, Sec. A). We emphasize that condition (2) does *not* say when to accept gambles, but merely when a simple strategy *must reject* them, to avoid bankruptcy. Therefore, $\mathbf{R}(g)$ may also be viewed as a sort of minimal "reserve" needed for g.

Some intuition for the formula (1) that determines $\mathbf{R}$ will be provided in the next section. To see how it is applied, consider gambles g in which gaining a and losing b are equally likely (with $0 < b < a$ so that $g \in \mathcal{G}$); it is immediate to verify that $\mathbf{E}[\log(1 + g/R)] = 0$ if and only if $(1 + a/R)(1 - b/R) = 1$, and so $\mathbf{R}(g) = ab/(a - b)$ by formula (1). In particular, for $a = 120$ and $b = 100$ we get $\mathbf{R}(g) = 600$, and for $a = 105$ and $b = 100$ we get $\mathbf{R}(g) = 2{,}100$.

The proof of theorem 1 is relegated to the Appendix, Section A; an illustrating example is provided in Section IV.

IV. Extension: The Shares Setup

We will now show that the distinction made in the previous section between the two "regimes" is robust and does not hinge on the extreme case of long-run bankruptcy. To do so we slightly extend our setup by allowing the decision maker to take *any proportion* of the offered gamble. This results in a sharper distinction—bankruptcy on one side and wealth growing to infinity on the other—which moreover becomes evident already after finitely many periods. The intuitive reason is that it is now possible to overcome short-term losses by taking appropriately small proportions of the offered gambles (which is not the case in the basic model of Sec. III, where all future gambles may turn out to be too risky relative to the wealth).

Formally, in this setup—which we call the *shares setup*—the decision maker can accept any nonnegative multiple of the offered gamble g_t (i.e., $\alpha_t g_t$ for some $\alpha_t \geq 0$) rather than just accept or reject g_t (which corresponds to $\alpha_t \in \{0, 1\}$). Think, for instance, of investments that can be made in arbitrary amounts (shares of equities). Let $Q : \mathcal{G} \to (0, \infty)$ be a critical-wealth function (we no longer allow $Q(g) = 0$ and $Q(g) = \infty$) with $Q(\lambda g) = \lambda Q(g)$ for all $\lambda > 0$. The corresponding *simple shares strategy* $s \equiv s_Q$ is as follows: at wealth $Q(g)$ one accepts g (i.e., $\alpha = 1$), and at any wealth W one accepts the proportion $\alpha = W/Q(g)$ of g; that is, the gamble αg that is taken is exactly the one for which $Q(\alpha g) = W$. The result is

THEOREM 2. Let $s \equiv s_Q$ be a simple shares strategy with critical-wealth function Q. Then:

(i) $\lim_{t\to\infty} W_t = \infty$ (almost surely (a.s.)) for every process G if and only if $Q(g) > \mathbf{R}(g)$ for every gamble $g \in \mathcal{G}$.

(ii) $\lim_{t\to\infty} W_t = 0$ (a.s.) for some process G if and only if $Q(g) < \mathbf{R}(g)$ for some gamble $g \in \mathcal{G}$.

Theorem 2 is proved in the Appendix, Section B (proposition 8 there provides a more precise result). Thus, our measure of riskiness $\mathbf{R}$ provides the threshold between two very different "regimes": bankruptcy (i.e., $W_t \to 0$ a.s., when the riskiness of the accepted gambles is higher than the wealth), and infinite wealth (i.e., $W_t \to \infty$ a.s., when the riskiness of the accepted gambles is lower than the wealth). As a consequence, one may replace the "no-bankruptcy" criterion with various other criteria, such as:

- *no-loss*: $\liminf_{t\to\infty} W_t \geq W_1$ (a.s.);
- *bounded loss*: $\liminf_{t\to\infty} W_t \geq W_1 - C$ (a.s.) for some $C < W_1$, or $\liminf_{t\to\infty} W_t \geq cW_1$ (a.s.) for some $c > 0$;
- *assured gain*: $\liminf_{t\to\infty} W_t \geq W_1 + C$ (a.s.) for some $C > 0$, or $\liminf_{t\to\infty} W_t \geq (1 + c)W_1$ (a.s.) for some $c > 0$;
- *infinite gain*: $\lim_{t\to\infty} W_t = \infty$ (a.s.).

Moreover, in the no-bankruptcy as well as any of the above conditions, one may replace "almost surely" (a.s.) by "with positive probability." For each one of these criteria, theorem 2 implies that the threshold is the same: it is given by the riskiness function $\mathbf{R}$. For example:

COROLLARY 2. A simple shares strategy s_Q guarantees no-loss if $Q(g) > \mathbf{R}(g)$ for all g, and only if $Q(g) \geq \mathbf{R}(g)$ for all g.

By way of illustration, take the gamble g of Section I in which it is equally likely to gain \$120 or lose \$100, and consider the situation in which one faces a sequence of gambles g_t that are independent draws from g. Let $q := Q(g)$ be the critical wealth that is used for g; then in each period one takes the proportion $\alpha_t = W_t/q$ of g_t. Therefore,

$$W_{t+1} = W_t + \alpha_t g_t = W_t + \left(\frac{W_t}{q}\right) g_t = W_t\left(1 + \frac{1}{q} g_t\right),$$

and so $W_{t+1} = W_1 \prod_{i=1}^{t} (1 + (1/q)g_i)$. Assume first that $Q(g) = \$200$; then $1 + (1/q)g_t$ equals either $1 + 120/200 = 1.6$ or $1 - 100/200 = 0.5$ with equal probabilities (these are the *relative gross returns* of g when the wealth is \$200; in *net* terms, a gain of 60 percent or a loss of 50 percent). In the long run, by the Law of Large Numbers, about half the time the wealth will be multiplied by a factor of 1.6 and about half the time by a factor of 0.5. So, on average, the wealth will be multiplied by a factor of $\gamma = \sqrt{1.6 \cdot 0.5} < 1$ per period, which implies that it will almost surely converge to zero:[16] bankruptcy! Now assume that we use $Q(g) = \$1{,}000$ instead; the relative gross returns become $1 + 120/1{,}000 = 1.12$ or $1 - 100/1{,}000 = 0.9$, which yield a factor of $\gamma = \sqrt{1.12 \cdot 0.9} > 1$ per period, and so the wealth will almost surely go to infinity rather than to zero. The critical point is at $Q(g) = \$600$, where the per-period factor becomes $\gamma = 1$; the riskiness of g is precisely $\mathbf{R}(g) = \$600$.[17]

Indeed, accepting g when the wealth is less than \$600 yields "risky" returns—returns of the kind that if repeated lead in the long run to bankruptcy; in contrast, accepting g only when the wealth is more than \$600 yields returns of the kind that guarantee no-bankruptcy and lead to increasing wealth in the long run. We point out that these conclusions do not depend on the independent and identically distributed (i.i.d.) sequence that we have used in the illustration above; any sequence of returns of the first kind leads to bankruptcy, and of the second kind, to infinite growth.

The criteria up to now were all formulated in terms of the limit as t goes to infinity. However, the distinction between the two situations can

[16] Indeed, W_{t+1} will be close to $W_1(1.6)^{t/2}(0.5)^{t/2} = W_1\gamma^t \to_{t\to\infty} 0$. Intuitively, to offset a loss of 50 percent, it needs to be followed by a gain of 100 percent (since the basis has changed); a 60 percent gain does not suffice.

[17] It is easy to see that the growth factor is larger than 1 if and only if the expectation of the log of the relative gross returns is larger than 0; this explains formula (1).

be seen already after relatively few periods: the distribution of wealth will be very different. In the example above, the probability that there is no loss after t periods (i.e., $\mathbf{P}[W_{t+1} \geq W_1]$) is, for $t = 100$, about 0.027 when one uses $Q(g) = \$200$ and about 0.64 when $Q(g) = \$1{,}000$; these probabilities become 10^{-9} and 0.87, respectively, for $t = 1{,}000$. In terms of the median wealth, after $t = 100$ periods, it is only 0.000014 times the original wealth when $Q(g) = \$200$, in contrast to 1.48 times when $Q(g) = \$1{,}000$ (for $t = 1{,}000$, these numbers are 10^{-48} and 53.7, respectively).[18]

V. Properties of the Measure of Riskiness

The riskiness measure enjoys many useful properties; they all follow from formula (1). A number of basic properties are collected in proposition 1 below, following which we discuss two issues of particular interest: stochastic dominance and continuity.

Some notation: Given $0 < \lambda < 1$ and the gamble g that takes the values $x_1, x_2, \dots, x_m$ with respective probabilities $p_1, p_2, \dots, p_m$, the λ-*dilution of* g, denoted $\lambda * g$, is the gamble that takes the same values $x_1, x_2, \dots, x_m$, but now with probabilities $\lambda p_1, \lambda p_2, \dots, \lambda p_m$, and takes the value 0 with probability $1 - \lambda$; that is, with probability λ the gamble g is performed, and with probability $1 - \lambda$ there is no gamble.

PROPOSITION 1. For all gambles $g, h \in \mathcal{G}$:[19]

- (i) *Distribution*: If g and h have the same distribution, then $\mathbf{R}(g) = \mathbf{R}(h)$.
- (ii) *Homogeneity*: $\mathbf{R}(\lambda g) = \lambda \mathbf{R}(g)$ for every $\lambda > 0$.
- (iii) *Maximal loss*: $\mathbf{R}(g) > L(g)$.
- (iv) *Subadditivity*: $\mathbf{R}(g + h) \leq \mathbf{R}(g) + \mathbf{R}(h)$.
- (v) *Convexity*: $\mathbf{R}(\lambda g + (1 - \lambda)h) \leq \lambda \mathbf{R}(g) + (1 - \lambda)\mathbf{R}(h)$ for every $0 < \lambda < 1$.
- (vi) *Dilution*: $\mathbf{R}(\lambda * g) = \mathbf{R}(g)$ for every $0 < \lambda \leq 1$.
- (vii) *Independent gambles*: If g and h are independent random variables, then $\min\{\mathbf{R}(g), \mathbf{R}(h)\} < \mathbf{R}(g + h) < \mathbf{R}(g) + \mathbf{R}(h)$.

Moreover, there is equality in (iv) and (v) if and only if g and h are proportional (i.e., $h = \lambda g$ for some $\lambda > 0$).

Thus, only the distribution of a gamble determines its riskiness; the riskiness is always larger than the maximal loss (which may be viewed as an "immediate one-shot risk"); the riskiness measure is positively homogeneous of degree one and subadditive, and thus convex; diluting

[18] Taking $Q(g) = \$500$ and $Q(g) = \$700$ (closer to $\mathbf{R}(g) = \$600$) yields after $t = 1{,}000$ periods a median wealth that is 0.018 and 7.66, respectively, times the original wealth.

[19] In (iv), (v), and (vii), g and h are random variables defined on the same probability space.

a gamble does not affect the riskiness;[20] and the riskiness of the sum of independent gambles lies between the minimum of the two riskinesses and their sum.

Proofs can be found in the Appendix, Section C; see proposition 9 there for sequences of i.i.d. gambles.

A. *Stochastic Dominance*

There are certain situations in which one gamble g is clearly "less risky" than another gamble h. One such case occurs when in every instance the value that g takes is larger than the value that h takes. Another occurs when some values of h are replaced in g by their expectation (this operation of going from g to h is called a "mean-preserving spread"). These two cases correspond to "first-order stochastic dominance" and "second-order stochastic dominance," respectively (see Hadar and Russell 1969; Hanoch and Levy 1969; Rothschild and Stiglitz 1970, 1971).

Formally, a gamble g *first-order stochastically dominates* a gamble h, which we write $g\,\mathrm{SD}_1\,h$, if there exists a pair of gambles g' and h' that are defined on the same probability space such that: g and g' have the same distribution; h and h' have the same distribution; and $g' \geq h'$ and $g' \neq h'$. Similarly, *g second-order stochastically dominates h*, which we write $g\,\mathrm{SD}_2\,h$, if there exist g' and h' as above, but now the condition "$g' \geq h'$" is replaced by "$g' \geq h''$ and h' is obtained from h'' by a finite sequence of mean-preserving spreads, or as the limit of such a sequence."

The importance of stochastic dominance lies in the fact that, for expected-utility decision makers (who have a utility function u on outcomes and evaluate each gamble g by $\mathbf{E}[u(g)]$),[21] we have the following: $g\,\mathrm{SD}_1\,h$ if and only if g is strictly preferred to h whenever the utility function u is strictly increasing; and $g\,\mathrm{SD}_2\,h$ if and only if g is strictly preferred to h whenever the utility function u is also strictly concave.

Our riskiness measure is monotonic with respect to stochastic dominance: a gamble that dominates another has a lower riskiness. In contrast, this desirable property is not satisfied by most existing measures of riskiness (see Sec. VI.D).

Proposition 2. If g first-order stochastically dominates h or if g second-order stochastically dominates h, then $\mathbf{R}(g) < \mathbf{R}(h)$.

Proposition 2 is proved in the Appendix, Section D.

[20] In our setup of sequences of gambles, dilution by a factor λ translates into "rescaling time" by a factor of $1/\lambda$ (e.g., $\lambda = 1/2$ corresponds to being offered a gamble on average once every two periods). Such a rescaling does not affect the long-run outcome, which explains why the riskiness does not change.

[21] Or, if the wealth W is taken into account, by $\mathbf{E}[u(W+g)]$.

B. Continuity

The natural notion of convergence for gambles is convergence in distribution; after all, only the distribution of the gamble determines the riskiness; see proposition 1(i). Roughly speaking, gambles are close in distribution if they take similar values with similar probabilities. Formally, a sequence of gambles $(g_n)_{n=1,2,\ldots} \subset \mathcal{G}$ *converges in distribution* to a gamble $g \in \mathcal{G}$, denoted $g_n \xrightarrow{\mathcal{D}} g$, if $\mathbf{E}[\phi(g_n)] \to \mathbf{E}[\phi(g)]$ for every bounded and uniformly continuous real function ϕ (see Billingsley 1968). We get the following result:

PROPOSITION 3. Let $(g_n)_{n=1,2,\ldots} \subset \mathcal{G}$ be a sequence of gambles with uniformly bounded values; that is, there exists a finite K such that $|g_n| \leq K$ for all n. If $g_n \xrightarrow{\mathcal{D}} g \in \mathcal{G}$ and $L(g_n) \to L(g)$ as $n \to \infty$, then $\mathbf{R}(g_n) \to \mathbf{R}(g)$ as $n \to \infty$.

Proposition 3 is proved in the Appendix, Section E, as a corollary of a slightly more general continuity result (proposition 10).

To see that the condition $L(g_n) \to L(g)$ is indispensable, let g_n take the values 2, -1, and -3 with probabilities $(1/2)(1 - 1/n)$, $(1/2)(1 - 1/n)$, and $1/n$, respectively, and let g take the values 2 and -1 with probabilities 1/2 and 1/2. Then $g_n \xrightarrow{\mathcal{D}} g$ but $L(g_n) = 3 \neq 1 = L(g)$, and $\mathbf{R}(g_n) \to 3 \neq 2 = \mathbf{R}(g)$.

Though at first sight the discontinuity in the above example may seem disconcerting, it is nevertheless natural, and our setup helps to clarify it.[22] Even if the maximal loss $L(g_n)$ has an arbitrarily small—but *positive*—probability, it still affects the riskiness. After all, this maximal loss will eventually occur, and to avoid bankruptcy the wealth must be sufficiently large to overcome it. The fact that the probability is small implies only that it may take a long time to occur. But occur it will!

Interestingly, a similar point has been recently made by Taleb (2005): highly improbable events that carry a significant impact (called "black swans") should not be ignored. One may make money for a very long time, but if one ignores the very low probability possibilities, then one will eventually lose everything.

VI. Discussion and Literature

This section is devoted to several pertinent issues and connections to the existing literature. We start with the recently developed "index of riskiness" of Aumann and Serrano (2008), continue with matters concerning utility, risk aversion, wealth, and the "calibration" of Rabin (2000), discuss other measures of riskiness, and conclude with a number of general comments.

[22] Other measures of riskiness, such as that of Aumann and Serrano (2008), are continuous even when $L(g_n)$ does not converge to $L(g)$.

A. Aumann and Serrano's Index of Riskiness

Aumann and Serrano (2008) have recently developed the *economic index of riskiness*, which associates to every gamble $g \in \mathcal{G}$ a unique number $R^{\mathrm{AS}}(g) > 0$ as follows.[23] Consider the decision maker with constant (Arrow-Pratt) absolute risk aversion coefficient α (his utility function is thus[24] $u(x) = -\exp(-\alpha x)$) who is indifferent between accepting and rejecting g; put $R^{\mathrm{AS}}(g) = 1/\alpha$. The following equation thus defines $R^{\mathrm{AS}}(g)$ uniquely:

$$\mathbf{E}\left[\exp\left(-\frac{1}{R^{\mathrm{AS}}(g)}g\right)\right] = 1. \tag{3}$$

Aumann and Serrano's approach is based on a duality axiom, which essentially asserts that less risk-averse decision makers accept "riskier" gambles.[25] Together with positive homogeneity of degree one, this leads to the above index R^{AS}.

Comparing this to our approach, we note the following distinctions:

(i) R^{AS} is an *index* of riskiness, based on comparing the gambles in terms of their riskiness. Our $\mathbf{R}$ is a *measure* of riskiness, defined for each gamble separately (see Sec. VI.E.2).

(ii) R^{AS} is based on risk-averse expected-utility decision makers. Our approach completely dispenses with utility functions and risk aversion, and just compares two situations: bankruptcy versus no-bankruptcy, or, even better (Sec. IV), bankruptcy versus infinite growth, or loss versus no-loss, and so forth.

(iii) R^{AS} is based on the critical level of risk aversion, whereas our $\mathbf{R}$ is based on the critical level of wealth. Moreover, the comparison between decision makers in Aumann and Serrano (2008)—being "more" or "less" risk averse—must hold at all wealth levels. We thus have an interesting "duality": R^{AS} looks for the critical risk aversion coefficient regardless of wealth, whereas $\mathbf{R}$ looks for the critical wealth regardless of risk aversion.

[23] This index was used in the technical report of Palacios-Huerta et al. (2004); see Aumann and Serrano (2008, 810n).

[24] This is the class of CARA utility functions; $\exp(x)$ stands for e^x.

[25] For an alternative approach that is based on a simple "riskiness order," see Hart (2008).

(iv) Our approach yields a measure $\mathbf{R}$ whose unit and normalization are well defined, whereas Aumann and Serrano are free to choose any positive multiple of R^{AS}. Moreover, the number $\mathbf{R}(g)$ has a clear operational interpretation, which, at this point, has not yet been obtained for $R^{AS}(g)$. In fact, our work originally started as an attempt to provide such an interpretation for $R^{AS}(g)$, but it led to a different measure of riskiness.

The two approaches thus appear quite different in many respects, both conceptually and practically. Nevertheless, they share many properties (compare Sec. V above with Sec. V in Aumann and Serrano 2008).[26] Moreover, they turn out to yield similar values in many examples. To see why, rewrite (3) as $\mathbf{E}[1 - \exp(-g/R^{AS}(g))] = 0$, and compare it to our equation (1), $\mathbf{E}[\log(1 + g/\mathbf{R}(g))] = 0$. Now the two relevant functions, $\log(1 + x)$ and $1 - \exp(-x)$, are close for small x: their Taylor series around $x = 0$ are

$$\log(1 + x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \cdots$$

and

$$1 - \exp(-x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 - \tfrac{1}{24}x^4 + \cdots.$$

The two series differ only from their third-order terms on; this suggests that when $g/R(g)$ is small—that is, when the riskiness is large relative to the gamble—the two approaches should yield similar answers.

To see this formally, it is convenient to keep the gambles bounded, from above and from below, and let their riskiness go to infinity (recall that both $\mathbf{R}$ and R^{AS} are homogeneous of degree one); as we will see below, this is equivalent to letting their expectation go to zero. The notation $a_n \sim b_n$ means that $a_n/b_n \to 1$ as $n \to \infty$.

PROPOSITION 4. Let $(g_n)_{n=1,2,\ldots} \subset \mathcal{G}$ be a sequence of gambles such that there exist $K < \infty$ and $\kappa > 0$ with $|g_n| \le K$ and $\mathbf{E}[|g_n|] \ge \kappa$ for all n. Then the following three conditions are equivalent:

(i) $\mathbf{E}[g_n] \to 0$ as $n \to \infty$.
(ii) $\mathbf{R}(g_n) \to \infty$ as $n \to \infty$.
(iii) $R^{AS}(g_n) \to \infty$ as $n \to \infty$.

Moreover, in this case $\mathbf{R}(g_n) \sim R^{AS}(g_n)$ as $n \to \infty$.

Thus, when the expectation goes to zero, both measures go to infinity; and if one of them goes to infinity, then the other does so too—and, moreover, they become approximately equal. Proposition 4 is proved in the Appendix, Section F. We note here another general relation that

[26] The only differences concern continuity and independent gambles.

has been obtained in Aumann and Serrano (2008):[27] for every $g \in \mathcal{G}$,

$$-L(g) < R^{\mathrm{AS}}(g) - \mathbf{R}(g) < M(g). \tag{4}$$

At this point one may wonder which riskiness measure or index could be said to be "better." Our view is that there is no definite answer. Each one of them captures certain aspects of riskiness; after all, a whole distribution is summarized into one number (a "statistic"). Further research should help clarify the differences and tell us when it is appropriate to use them.[28]

B. Utility and Risk Aversion

Consider an expected-utility decision maker with utility function u, where $u(x)$ is the utility of wealth x. The utility function u generates a strategy $s \equiv s^u$ as follows: accept the gamble g when the wealth is W if and only if by doing so the expected utility will not go down, that is,

$$\text{accept } g \text{ at } W \text{ if and only if } \mathbf{E}[u(W+g)] \geq u(W); \tag{5}$$

equivalently, the expected utility from accepting g at W is no less than the utility from rejecting g at W.

A special case is the *logarithmic utility* $u(x) = \log x$ (also known as the "Bernoulli utility"). The riskiness measure $\mathbf{R}$ turns out to be characterized by the following property. For every gamble g, the logarithmic utility decision maker is indifferent between accepting and rejecting g when his wealth W equals exactly $\mathbf{R}(g)$, and he strictly prefers to reject g at all $W < \mathbf{R}(g)$ and to accept g at all $W > \mathbf{R}(g)$; this follows from (1) and lemma 1 in the Appendix (Sec. A) since

$$\mathbf{E}[\log(1 + g/\mathbf{R}(g))] = \mathbf{E}[\log(\mathbf{R}(g) + g)] - \log(\mathbf{R}(g)).$$

Therefore, the condition (2) of rejecting a gamble when its riskiness is higher than the current wealth, that is, when $W < \mathbf{R}(g)$, can be restated as follows: *reject any gamble that the logarithmic utility rejects.*

The logarithmic utility is characterized by a *constant relative risk aversion* coefficient of 1 (i.e., $\gamma_u(x) := -xu''(x)/u'(x) = 1$ for all $x > 0$). More generally, consider the class *CRRA* of utility functions that have a constant relative risk aversion coefficient, that is, $\gamma_u(x) = \gamma > 0$ for all $x > 0$; the corresponding utility functions are $u_\gamma(x) = x^{1-\gamma}/(1-\gamma)$ for $\gamma \neq 1$ and $u_1(x) = \log x$ for $\gamma = 1$. It can be checked that these are exactly the utility functions for which the resulting strategy s^u turns out to be a

[27] Aumann and Serrano (2008) show that a decision maker with log utility accepts g at all $W > R^{\mathrm{AS}}(g) + L(g)$ and rejects g at all $W < R^{\mathrm{AS}}(g) - M(g)$, and so (see Sec. VI.B below) $\mathbf{R}(g)$ must lie between these two bounds.

[28] For a similar point, which one is "better"—the mean or the median? (For an illuminating discussion on "multiple solutions," see Aumann 1985, esp. Sec. 4.)

simple strategy (i.e., with a critical-wealth function that is homogeneous; use for instance corollary 3 and lemma 4 in Sec. X.A of Aumann and Serrano 2008). Since a higher risk aversion means that more gambles are rejected, our main result (see corollary 1) yields the following: *no-bankruptcy is guaranteed for a CRRA utility u_γ if and only if the relative risk aversion coefficient γ satisfies* $\gamma \geq 1$.[29]

Given a general utility function u (which is not necessarily CRRA, and therefore the resulting strategy s^u is not necessarily simple), assume for simplicity that the relative risk aversion coefficient at 0 is well defined; that is, the limit $\gamma_u(0) := \lim_{x\to 0^+} \gamma_u(x)$ exists. Then proposition 11 in the Appendix (Sec. G) yields the following result: $\gamma_u(0) > 1$ guarantees no-bankruptcy, and guaranteed no-bankruptcy implies that $\gamma_u(0) \geq 1$.[30] It is interesting how the conclusion of a relative risk aversion coefficient of at least 1 has been obtained from the simple and basic requirement of no-bankruptcy or any of the alternative criteria in Section IV.[31]

C. Wealth and Calibration

We come now to the issue of what is meant by "wealth." Our basic setup assumes that the decision maker wants to avoid bankruptcy (i.e., $W_t \to 0$). This can be easily modified to accommodate any other minimal level of wealth $\underline{W}$ that must be guaranteed: just add $\underline{W}$ throughout. Thus, rejecting g at W when $W < \underline{W} + \mathbf{R}(g)$ guarantees that $W_t \geq \underline{W}$ for all t and $\mathbf{P}[\lim_{t\to\infty} W_t = \underline{W}] = 0$ (this follows from proposition 6 in Sec. A of the Appendix).

If, say, $\underline{W}$ is the wealth that is needed and earmarked for purposes such as living expenses, housing, consumption, and so on, then $\mathbf{R}(g)$ should be viewed as the least "reserve wealth" that is required to cover the possible losses without going bankrupt, or, more generally, without going below the minimal wealth level $\underline{W}$. That is, $\mathbf{R}(g)$ is not the total wealth needed, but only the additional amount above $\underline{W}$. Therefore, if that part of the wealth that is designated for no purpose other than taking gambles—call it "gambling wealth" or "risky investment wealth"—is below $\mathbf{R}(g)$, then g must be rejected.

This brings us to the "calibration" of Rabin (2000). Take a risk-averse

[29] It is interesting how *absolute* risk aversion and CARA utilities have come out of the Aumann and Serrano (2008) approach, and *relative* risk aversion and CRRA utilities out of ours—in each case, as a *result* and not an assumption.

[30] The knife-edge case of $\gamma_u(0) = 1$ can go either way: consider $u^1(x) = \log x$ and

$$u^2(x) = \exp(-\sqrt{-\log x})$$

for small x.

[31] Many—though not all—empirical studies indicate relative risk aversion coefficients larger than 1 (see, e.g., Palacios-Huerta and Serrano 2006). Perhaps (and take this *cum grano salis*) agents with a coefficient less than 1 may already be bankrupt and thus not part of the studies.

expected-utility decision maker and consider, for example, the following two gambles: the gamble g in which he gains \$105 or loses \$100 with equal probabilities, and the gamble h in which he gains \$5.5 million or loses \$10,000 with equal probabilities. Rabin proves that: if (i) g is rejected at all wealth levels $W <$ \$300,000, then (ii) h must be rejected at wealth $W =$ \$290,000.

If one were to interpret the wealth W as gambling wealth, then our result suggests that the premise (i) that one rejects g at all $W <$ \$300,000 is not plausible, since $\mathbf{R}(g)$ is only \$2,100. If, on the other hand, wealth were to be interpreted as total wealth, then, as we saw above, (i) is consistent with wanting to preserve a minimal wealth level $\underline{W}$ of at least \$297,900 = \$300,000 − \$2,100. If that is the case, then a wealth of \$290,000 is below the minimal level $\underline{W}$, and so it makes sense to reject h there.

Thus, if wealth in the Rabin setup is gambling wealth, then the assumption (i) is not reasonable and so it does not matter whether the conclusion (ii) is reasonable or not.[32] And if it is total wealth, then both (i) and (ii) are reasonable, because such behavior is consistent with wanting to keep a certain minimal wealth level $\underline{W} \geq$ \$297,000. In either case, our setup suggests that there is nothing "implausible" here, as Rabin argues there is (and which leads him to cast doubts on the usefulness and appropriateness of expected utility theory[33]).[34]

D. *Other Measures of Riskiness*

Risk is a central issue, and various measures of riskiness have been proposed (see the survey of Machina and Rothschild 2008 and Sec. 7 in Aumann and Serrano 2008). We have already discussed in Section VI.A the recent index of Aumann and Serrano (2008), which is the closest to ours.

Most of the riskiness measures in the literature (and in practice) turn out to be nonmonotonic with respect to first-order stochastic dominance, which, as has been repeatedly pointed out by various authors, is a very reasonable—if not necessary—requirement. Indeed, if gains increase and losses decrease, how can the riskiness not decrease? Nevertheless, riskiness measures, particularly those based on the variance or other measures of "dispersion" of the gamble (and also "Value-at-

[32] Palacios-Huerta and Serrano (2006) argue that (i) is unreasonable from an *empirical* point of view (their paper led to the theoretical work of Aumann and Serrano 2008).

[33] Safra and Segal (2008) show that similar issues arise in many non–expected utility models as well. Rubinstein (2001) makes the point that expected utility need not be applied to final wealth, and there may be inconsistencies between the preferences at different wealth levels.

[34] Of course, this applies provided that there is no "friction," such as hidden costs (e.g., in collecting the prizes) or "cheating" in the realization of the gambles.

Risk" [VaR];[35] see Sec. VI.E.1), do not satisfy this monotonicity condition.

Artzner et al. (1999) have proposed the notion of a "coherent measure of risk," which is characterized by four axioms: "translation invariance" (T), "subadditivity" (S), "positive homogeneity" (PH), and "monotonicity" (M). Our measure **R** satisfies the last three axioms: (PH) and (S) are the same as (iii) and (iv) in proposition 1, and (M), which is weak monotonicity with respect to first-order stochastic dominance, follows from proposition 2. However, **R** does not satisfy (T), which requires that $R(g + c) = R(g) - c$ for every constant c (assuming no discounting; see Sec. VI.E.7); that is, adding the same number c to all outcomes of a gamble decreases the riskiness by exactly c. To see why this requirement is not appropriate in our setup, take for example the gamble g of Section I in which one gains 120 or loses 100 with equal probabilities; its riskiness is $\mathbf{R}(g) = 600$. Now add $c = 100$ to all payoffs; the new gamble $g + 100$ has no losses, and so its riskiness should be 0, not $500 = 600 - 100$.[36] See also Section VI.E.1 below.

E. General Comments

1. *Universal and objective measure.* Our approach looks for a "universal" and "objective" measure of riskiness. First, it abstracts away from the goals and the preference order of specific decision makers (and so, a fortiori, from utility functions, risk aversion, and so on). The only property that is assumed is that no-bankruptcy is preferred to bankruptcy; or, in the shares setup, that infinite growth is preferred to bankruptcy or no-loss to loss.[37] Second, we make no assumptions on the sequence of gambles the decision maker will face. And third, our measure does not depend on any ad hoc parameters that need to be specified (as is the case, e.g., with the measure Value-at-Risk, which depends on a "confidence level" $\alpha \in (0, 1)$).

Of course, if additional specifications are available—such as how the sequence of gambles is generated—then a different measure may result. The measure that we propose here may be viewed as an idealized benchmark.

2. *Single gamble.* While our model allows arbitrary sequences of gambles, the analysis can be carried out separately for any single gamble g (together with its multiples); see the example in the shares setup of

[35] Increasing one of the possible gains leaves VaR unchanged.

[36] Formally, $g + 100$ is not a gamble; so take instead, say, $g + 99.99$, where one gains 219.99 or loses 0.01; its riskiness can hardly be 500.01. The index of Aumann and Serrano (2008) likewise satisfies (S), (PH), and (M), but not (T).

[37] In particular, the fact that gambles with positive expectation are sometimes rejected—i.e., "risk aversion"—is a *consequence* of our model, not an assumption.

Section IV and proposition 7 in the Appendix, Section A. The riskiness $\mathbf{R}(g)$ of a gamble g is thus determined by considering g *only*; no comparisons with other gambles are needed.

3. *Returns.* One may restate our model in terms of returns: accepting a gamble g at wealth W yields *relative gross returns* $X = (W + g)/W = 1 + g/W$. We will say that X has *B-returns* if $\mathbf{E}[\log X] < 0$, and that it has *G-returns* if $\mathbf{E}[\log X] > 0$ (B stands for "Bad" or "Bankruptcy," and G for "Good" or "Growth"):[38] a sequence of i.i.d. B-returns leads to bankruptcy, and of G-returns to infinite wealth (a.s.). Now, accepting g at W yields B-returns if and only if $W < \mathbf{R}(g)$, and G-returns if and only if $W > \mathbf{R}(g)$ (see lemma 1 in the Appendix, Sec. A), and so $\mathbf{R}(g)$ is the *critical wealth level below which the returns become B-returns.*

4. *Acceptance.* As pointed out in Section III, our approach tells us when we must reject gambles—namely, when their riskiness exceeds the available wealth—but it does not say when to accept gambles. Any strategy satisfying condition (2) guarantees no-bankruptcy (see proposition 6 in the Appendix, Sec. A). Therefore, additional criteria are needed to decide when to accept a gamble. For example, use a utility function and decide according to condition (5) in Section VI.B; or see point 5 below.

5. *Maximal growth rate.* In the shares setup, one may choose that proportion of the gamble that *maximizes* the expected growth rate (rather than just guarantees that it is at least 1, as the riskiness measure does). This yields a number $K \equiv K(g)$, where $\mathbf{E}[\log(1 + g/K)]$ is maximal over $K > 0$; equivalently (taking the derivative), $K(g)$ is the unique positive solution of the equation $\mathbf{E}[g/(1 + g/K(g))] = 0$; for example, when g takes the values 105 and -100 with equal probabilities, $K(g) = 4{,}200$ and $\mathbf{R}(g) = 2{,}100$.[39] There is an extensive literature on the maximal growth rate; see, for example, Kelly (1956), Samuelson (1979), Cover and Thomas (1991, chap. 6), and Algoet (1992). While the log function appears there too, our approach is different. We do not ask who will win and get more than everyone else (see, e.g., Blume and Easley 1992), but rather who will not go bankrupt and will get good returns. It is like the difference between "optimizing" and "satisficing."

6. *Nonhomogeneous strategies.* A simple strategy is based on a riskiness-like function and is thus homogeneous of degree one. This raises the question of what happens in the case of general nonhomogeneous strategies, where the critical-wealth function $Q : \mathcal{G} \to [0, \infty]$ may be arbitrary.

[38] The returns in the knife-edge case $\mathbf{E}[\log X] = 0$ may be called *C-returns* (C for "Critical" or "Constant").

[39] For 1/2-1/2 gambles g it is easy to prove that $K(g) = 2\mathbf{R}(g)$; of course, $K(g) > \mathbf{R}(g)$ holds for every gamble $g \in \mathcal{G}$ (see lemma 1 and fig. A1b in the Appendix: $K(g)$ is the point at which ψ is maximal). It may be checked that $K(g)$ is that wealth level at which a CRRA utility with $\gamma = 2$ is indifferent between accepting and rejecting g.

In the basic no-bankruptcy setup of Section III, for instance, condition (2) that $Q(g) \geq \mathbf{R}(g)$ for all g is sufficient to guarantee no-bankruptcy, whether Q is homogeneous or not (see proposition 6 in the Appendix, Sec. A). However, this condition is no longer necessary: a nonhomogeneous Q allows one to behave differently depending on whether the wealth is large or small. It turns out that no-bankruptcy is guaranteed if and only if, roughly speaking, condition (2) holds when the wealth is small—provided that immediate ruin is always avoided and so the wealth remains always positive (i.e., $Q(g) > L(g)$ for all g). See Section G in the Appendix.

7. *Limited liability.* Our approach yields infinite riskiness when the losses are unbounded (since $\mathbf{R}(g) > L(g)$; see also the discussion in Sec. V.B). This may explain the need to bound the losses, that is, have limited liability. It is interesting that, historically, the introduction of limited-liability contracts did in fact induce many people to invest who would otherwise have been hesitant to do so.

8. *Risk-free asset and discounting.* We have assumed no discounting over time and a risk-free rate of return $r_f = 1$. Allowing for discounting and an r_f different from 1 can, however, be easily accommodated, either directly or by interpreting future outcomes as being expressed in present-value terms.

9. *Axiomatic approach.* It would be useful to characterize the riskiness measure $\mathbf{R}$ by a number of reasonable axioms; this may also help clarify the differences between $\mathbf{R}$ and R^{AS}. See Foster and Hart (2008).

10. *Riskiness instead of standard deviation and VaR.* As pointed out in Sections V.A and VI.D, commonly used measures of risk—such as the standard deviation $\boldsymbol{\sigma}$ and VaR—may be problematic. We propose the use of $\mathbf{R}$ instead.

Indeed, $\mathbf{R}$ shares many good properties with $\boldsymbol{\sigma}$ (see proposition 1); but it has the added advantage of being monotonic with respect to stochastic dominance (see proposition 2). For instance, one could use $\mathbf{R}$ to determine "efficient portfolios" (Markowitz 1952, 1959; Sharpe 1964): rather than maximize the expected return for a fixed standard deviation, maximize the expected return for a fixed riskiness. Moreover, one may try to use $\mathbf{E}[g]/\mathbf{R}(g)$ in place of the Sharpe (1966) ratio $\mathbf{E}[g]/\boldsymbol{\sigma}[g]$.

The measures VaR are used for determining bank reserves. Since our measure $\mathbf{R}$ may be viewed as the minimum "reserve" needed to guarantee no-bankruptcy, it is a natural candidate to apply in this setup.

All this of course requires additional study.

Appendix

Proofs

The proofs are collected in this Appendix, together with a number of additional results.

A. *Proof of Theorem 1*

We prove here the main result, theorem 1, together with a number of auxiliary results (in particular lemma 1) and extensions (propositions 6 and 7). We start by showing that $\mathbf{R}(g)$ is well defined by equation (1).

LEMMA 1. For every $g \in \mathcal{G}$ there exists a unique number $R > 0$ such that $\mathbf{E}[\log(1 + g/R)] = 0$. Moreover: $R > L \equiv L(g)$ (the maximal loss of g); $\mathbf{E}[\log(1 + g/r)] < 0$ if and only if $L < r < R$; and $\mathbf{E}[\log(1 + g/r)] > 0$ if and only if $r > R$.

Proof. Let

$$\phi(\lambda) := \mathbf{E}[\log(1 + \lambda g)] = \sum_{i=1}^{m} p_i \log(1 + \lambda x_i)$$

for $0 \le \lambda < 1/L$. It is straightforward to verify that

$$\begin{aligned}
\phi(0) &= 0;\\
\lim_{\lambda \to (1/L)^-} \phi(\lambda) &= -\infty;\\
\phi'(\lambda) &= \sum_i \frac{p_i x_i}{1 + \lambda x_i};\\
\phi'(0) &= \sum_i p_i x_i = \mathbf{E}[g] > 0;\\
\phi''(\lambda) &= -\sum_i \frac{p_i x_i^2}{(1 + \lambda x_i)^2} < 0
\end{aligned}$$

for every $\lambda \in [0, 1/L)$. Therefore, the function ϕ is a strictly concave function that starts at $\phi(0) = 0$ with a positive slope ($\phi'(0) > 0$) and goes to $-\infty$ as λ increases to $1/L$. Hence (see fig. A1*a*) there exists a unique $0 < \lambda^* < 1/L$ such that $\phi(\lambda^*) = 0$, and moreover $\phi(\lambda) > 0$ for $0 < \lambda < \lambda^*$ and $\phi(\lambda) < 0$ for $\lambda^* < \lambda < 1/L$. Now let $R = 1/\lambda^*$. QED

Note that the function $\psi(r) := \mathbf{E}[\log(1 + g/r)]$ is *not* monotonic in r since g has negative values (see fig. A1*b*).

From lemma 1 it follows that $\mathbf{R}$ is positively homogeneous of degree one:

LEMMA 2. $\mathbf{R}(\lambda g) = \lambda \mathbf{R}(g)$ for every $g \in \mathcal{G}$ and $\lambda > 0$.

Proof. $0 = \mathbf{E}[\log(1 + g/\mathbf{R}(g))] = \mathbf{E}[\log(1 + (\lambda g)/(\lambda \mathbf{R}(g)))]$, and so $\lambda \mathbf{R}(g) = \mathbf{R}(\lambda g)$ since equation (1) determines $\mathbf{R}$ uniquely. QED

We recall a result on martingales:

PROPOSITION 5. Let $(X_t)_{t=1,2,\ldots}$ be a martingale defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and adapted to the increasing sequence of σ-fields $(\mathcal{F}_t)_{t=1,2,\ldots}$. Assume that (X_t) has bounded increments; that is, there exists a finite K such that

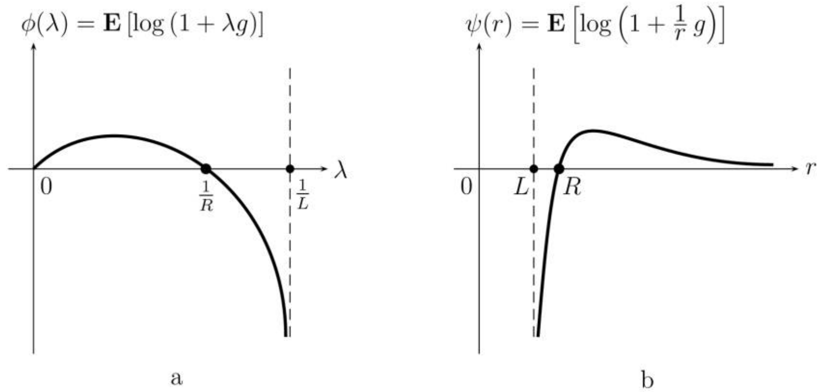

FIG. A1.—The functions $\phi(\lambda)$ and $\psi(r)$ (see lemma 1)

$|X_{t+1} - X_t| \le K$ for all $t \ge 1$. Then for almost every $\omega \in \Omega$ either: (i) $\lim_{t\to\infty} X_t(\omega)$ exists and is finite; or (ii) $\liminf_{t\to\infty} X_t(\omega) = -\infty$ and $\limsup_{t\to\infty} X_t(\omega) = +\infty$. Moreover, define the random variable $A_\infty := \sum_{t=1}^{\infty} \mathbf{E}[(X_{t+1} - X_t)^2 | \mathcal{F}_t] \in [0, \infty]$; then (i) holds for almost every $\omega \in \Omega$ with $A_\infty(\omega) < \infty$, and (ii) holds for almost every $\omega \in \Omega$ with $A_\infty(\omega) = \infty$.

Proof. Follows from proposition VII-3-9 in Neveu (1975). QED

Thus, almost surely either the sequence of values of the martingale converges or it oscillates infinitely often between arbitrarily large and arbitrarily small values. The term A_∞ may be interpreted as the "total one-step conditional variance."

Theorem 1 will follow from the next two propositions, which provide slightly stronger results.

PROPOSITION 6. If a strategy s satisfies condition (2), then s guarantees no-bankruptcy.

We emphasize that this applies to *any* strategy, not only to simple strategies; the function Q may be nonhomogeneous, or there may not be a critical-wealth function at all.

Proof of proposition 6. Consider a process G generated by a finite set $\mathcal{G}_0 \subset \mathcal{G}$. When g_t is accepted at W_t, we have $W_t \ge \mathbf{R}(g_t) > L(g_t)$, and so $W_{t+1} \ge W_t - L(g_t) > 0$; by induction, it follows that $W_t > 0$ for every t. Put

$$Y_t := \log W_{t+1} - \log W_t, \tag{A1}$$

and let d_t be the decision at time t; the history before d_t is $f_{t-1} := (W_1, g_1, d_1; \ldots; W_{t-1}, g_{t-1}, d_{t-1}; W_t, g_t)$. We have $\mathbf{E}[Y_t | f_{t-1}] \ge 0$; indeed, $Y_t = 0$ when g_t is rejected, and $Y_t = \log(W_t + g_t) - \log W_t = \log(1 + g_t/W_t)$ when it is accepted, and then $\mathbf{E}[Y_t | f_{t-1}] = \mathbf{E}[\log(1 + g_t/W_t) | f_{t-1}] \ge 0$ by (2) and lemma 1.

If g_t is accepted, then $W_t \ge \mathbf{R}(g_t)$, which implies that $1 + g_t/W_t \le 1 +$

$M(g_t)/\mathbf{R}(g_t) < \infty$ and $1 + g_t/W_t \geq 1 - L(g_t)/\mathbf{R}(g_t) > 0$. Therefore,

$$Y_t = \log\left(1 + \frac{g_t}{W_t}\right) \leq \sup_{g\in \mathrm{cone}\,\mathcal{G}_0} \log\left(1 + \frac{M(g)}{\mathbf{R}(g)}\right)$$
$$= \max_{g\in\mathcal{G}_0} \log\left(1 + \frac{M(g)}{\mathbf{R}(g)}\right) < \infty$$

and, similarly, $Y_t \geq \min_{g\in\mathcal{G}_0} \log(1 - L(g)/\mathbf{R}(g)) > -\infty$ (since $\mathcal{G}_0$ is finite and the functions M, L, and $\mathbf{R}$ are homogeneous of degree one); the random variables Y_t are thus uniformly bounded.

Put

$$X_T := \sum_{t=1}^{T} (Y_t - \mathbf{E}[Y_t | f_{t-1}]); \tag{A2}$$

then $(X_T)_{T=1,2,\ldots}$ is a martingale with bounded increments. Recalling that $\mathbf{E}[Y_t|f_{t-1}] \geq 0$, we have

$$X_T \leq \sum_{t=1}^{T} Y_t = \sum_{t=1}^{T} (\log W_{t+1} - \log W_t) = \log W_{T+1} - \log W_1.$$

Now bankruptcy means $\log W_T \to -\infty$, and so $X_T \to -\infty$; but the event $\{X_T \to -\infty\}$ has probability zero by proposition 5 (it is disjoint from both (i) and (ii) there), and so bankruptcy occurs with probability zero. QED

Proposition 7. Let s_Q be a simple strategy with $Q(\tilde{g}) < \mathbf{R}(\tilde{g})$ for some $\tilde{g} \in \mathcal{G}$. Then there exists a process $G = (g_t)$ such that $\lim_{t\to\infty} W_t = 0$ (a.s.); moreover, all the g_t are multiples of $\tilde{g}$.

Thus there is bankruptcy with probability one, not just with positive probability.

Proof of proposition 7. Let $q := Q(\tilde{g})$; we have $q > L(g)$ (otherwise there is immediate bankruptcy starting with $W_1 = q$ and accepting $g_1 = \tilde{g}$; indeed, once the wealth becomes zero, it remains so forever by the no-borrowing condition $W_t \geq 0$, since no gambles may be accepted). Therefore, $L(\tilde{g}) < q < \mathbf{R}(\tilde{g})$, and so $\mu := \mathbf{E}[\log(1 + \tilde{g}/q)] < 0$ by lemma 1. Let $(\tilde{g}_t)_{t=1,2,\ldots}$ be a sequence of i.i.d. gambles with each one having the same distribution as $\tilde{g}$, and take $g_t = \lambda_t \tilde{g}_t$ with $\lambda_t = W_t/q$. Now $Q(g_t) = (W_t/q)Q(\tilde{g}) = W_t$, and so g_t is accepted at W_t. Therefore, $Y_t = \log(1 + g_t/W_t) = \log(1 + \tilde{g}_t/Q)$ is an i.i.d. sequence, and so, as $T\to\infty$,

$$\frac{1}{T}(\log W_{T+1} - \log W_1) = \frac{1}{T}\sum_{t=1}^{T} Y_t \to \mathbf{E}\left[\log\left(1 + \frac{1}{q}\tilde{g}\right)\right] = \mu < 0$$

(a.s.), by the Strong Law of Large Numbers. Therefore, $\log W_T \to -\infty$, that is, $W_T \to 0$ (a.s.). QED

B. *Proof of Theorem 2*

The result in the shares setup will follow from the following proposition, which gives a more precise result.

Proposition 8. Let s_Q be a simple shares strategy, and let G be a process generated by a finite $\mathcal{G}_0$. Then:

(>) If $Q(g) > \mathbf{R}(g)$ for every $g \in \mathcal{G}_0$, then $\lim_{t\to\infty} W_t = \infty$ a.s.
(≥) If $Q(g) \geq \mathbf{R}(g)$ for every $g \in \mathcal{G}_0$, then $\limsup_{t\to\infty} W_t = \infty$ a.s.
(=) If $Q(g) = \mathbf{R}(g)$ for every $g \in \mathcal{G}_0$, then $\limsup_{t\to\infty} W_t = \infty$ and $\liminf_{t\to\infty} W_t = 0$ a.s.
(≤) If $Q(g) \leq \mathbf{R}(g)$ for every $g \in \mathcal{G}_0$, then $\liminf_{t\to\infty} W_t = 0$ a.s.
(<) If $Q(g) < \mathbf{R}(g)$ for every $g \in \mathcal{G}_0$, then $\lim_{t\to\infty} W_t = 0$ a.s.

Proof. Define Y_t and X_T as in the proof of proposition 6 above, by (A1) and (A2), respectively. Since the gamble taken at time t is $\alpha_t g_t$, where $\alpha_t = W_t/Q(g_t)$, we have

$$Y_t = \log\left(1 + \frac{\alpha_t}{W_t} g_t\right) = \log\left(1 + \frac{1}{Q(g_t)} g_t\right).$$

Next,

$$\lim \frac{1}{T} X_T = 0 \tag{A3}$$

and

$$\limsup X_T = \infty \quad \text{and} \quad \liminf X_T = -\infty \tag{A4}$$

a.s. as $T \to \infty$. Indeed, the random variables Y_t are uniformly bounded (since each g has finitely many values and $\mathcal{G}_0$ is finite), and so (A3) follows from the Strong Law of Large Numbers for Dependent Random Variables (see Loève 1978, vol. 2, theorem 32.1.E). As for (A4), it follows from proposition 5 applied to the martingale X_T, since for every history f_{t-1},

$$\mathbf{E}[(X_t - X_{t-1})^2 | f_{t-1}] \geq \min_{g \in \mathcal{G}_0} \mathbf{Var}\left[\log\left(1 + \frac{1}{Q(g)} g\right)\right] =: \delta > 0$$

(we have used the homogeneity of Q and the finiteness of $\mathcal{G}_0$; **Var** denotes variance), and so $A_\infty = \infty$.

We can now complete the proof in the five cases.

(>) The assumption that $Q(g) > \mathbf{R}(g)$ for every $g \in \mathcal{G}_0$ implies by lemma 1 that

$$\mathbf{E}[Y_t | f_{t-1}] \geq \min_{g \in \mathcal{G}_0} \mathbf{E}\left[\log\left(1 + \frac{1}{Q(g)} g\right)\right] =: \delta' > 0,$$

and so, as $T \to \infty$ (a.s.),

$$\begin{aligned} \liminf \frac{1}{T}(\log W_{T+1} - \log W_1) &= \liminf \frac{1}{T} \sum_{t=1}^{T} Y_t \\ &\geq \lim \frac{1}{T} X_T + \delta' = \delta' > 0 \end{aligned}$$

(recall (A3)); therefore, $\lim W_T = \infty$.

(<) Similar to the proof of (>), using

$$\mathbf{E}[Y_t|f_{t-1}] \le \max_{g\in\mathcal{G}_0} \mathbf{E}\left[\log\left(1+\frac{g}{Q(g)}\right)\right] =: \delta'' < 0.$$

(≥) Here we have $\mathbf{E}[Y_t|f_{t-1}] \ge 0$, and so

$$\limsup\,(\log W_{T+1} - \log W_1) = \limsup \sum_{t=1}^{T} Y_t \ge \limsup X_T = \infty$$

by (A4).

(≤) Similar to the proof of (≥), using now $\mathbf{E}[Y_t|f_{t-1}] \le 0$.

(=) Combine (≥) and (≤). QED

Proof of theorem 2. Follows from proposition 8: (>) and (≤) yield (i) (if $Q(\tilde{g}) \le \mathbf{R}(\tilde{g})$, then take G to be an i.i.d. sequence $(\tilde{g}_t)$ with all $\tilde{g}_t$ having the same distribution as $\tilde{g}$), and similarly (≥) and (<) yield (ii). QED

C. *Proof of Proposition 1*

We prove here the basic properties of the riskiness measure, followed by an additional result on sequences of i.i.d. gambles.

Proof of proposition 1. (i), (ii), and (iii) are immediate from (1) and lemmas 1 and 2.

(iv) Let $r := \mathbf{R}(g)$ and $r' := \mathbf{R}(h)$, and put $\lambda := r/(r+r') \in (0, 1)$. Since $(g+h)/(r+r') = \lambda(g/r) + (1-\lambda)(h/r')$, the concavity of the log function gives

$$\mathbf{E}\left[\log\left(1+\frac{g+h}{r+r'}\right)\right] \ge \lambda\mathbf{E}\left[\log\left(1+\frac{g}{r}\right)\right] + (1-\lambda)\mathbf{E}\left[\log\left(1+\frac{h}{r'}\right)\right] = 0,$$

and so $r + r' \le \mathbf{R}(g+h)$ by lemma 1.

(v) follows from (ii) and (iv).

(vi) Put $h := \lambda * g$; then

$$\mathbf{E}\left[\log\left(1+\frac{h}{\mathbf{R}(g)}\right)\right] = \lambda\mathbf{E}\left[\log\left(1+\frac{g}{\mathbf{R}(g)}\right)\right] + (1-\lambda)\log(1+0) = 0,$$

and so $\mathbf{R}(h) = \mathbf{R}(g)$.

(vii) The second inequality is (iv) (it is strict since only constant random variables can be both independent and equal [or proportional], and gambles in $\mathcal{G}$ are never constant). To prove the first inequality, recall the concave function $\phi(\lambda) := \mathbf{E}[\log(1+\lambda g)]$ of the proof of lemma 1 (see fig. A1*a*): it decreases at its second root $\lambda = 1/\mathbf{R}(g)$, and so $\phi'(\lambda) = \mathbf{E}[g/(1+\lambda g)] < 0$ for $\lambda = 1/\mathbf{R}(g)$, and thus for all $\lambda \ge 1/\mathbf{R}(g)$.

Without loss of generality assume that $\mathbf{R}(g) \le \mathbf{R}(h)$. Put $\rho := 1/\mathbf{R}(g)$; then $\mathbf{E}[\log(1+\rho g)] = 0 \ge \mathbf{E}[\log(1+\rho h)]$ and, as we have seen above,

$$\mathbf{E}\left[\frac{g}{1+\rho g}\right] < 0 \quad\text{and}\quad \mathbf{E}\left[\frac{h}{1+\rho h}\right] < 0 \tag{A5}$$

(since $\rho = 1/\mathbf{R}(g)$ and $\rho \geq 1/\mathbf{R}(h)$). Now

$$\mathbf{E}[\log(1+\rho(g+h))] = \mathbf{E}[\log(1+\rho g)] + \mathbf{E}[\log(1+\rho h)] + \mathbf{E}\left[\log\left(1 - \frac{\rho^2 gh}{(1+\rho g)(1+\rho h)}\right)\right].$$

The first term vanishes, the second is ≤ 0, and for the third we get

$$\mathbf{E}\left[\log\left(1 - \frac{\rho^2 gh}{(1+\rho g)(1+\rho h)}\right)\right] \leq \mathbf{E}\left[-\frac{\rho^2 gh}{(1+\rho g)(1+\rho h)}\right] = -\rho^2 \mathbf{E}\left[\frac{g}{1+\rho g}\right]\mathbf{E}\left[\frac{h}{1+\rho h}\right] < 0$$

(we have used $\log(1-x) \leq -x$, the independence of g and h, and (A5)). Altogether $\mathbf{E}[\log(1+\rho(g+h))] < 0$, and so $1/\rho < \mathbf{R}(g+h)$ (by lemma 1), proving our claim (recall that $1/\rho = \mathbf{R}(g) = \min\{\mathbf{R}(g), \mathbf{R}(h)\}$). QED

Let $g_1, g_2, \ldots, g_n, \ldots$ be a sequence of i.i.d. gambles; then (vii) implies that $\mathbf{R}(g_1) < \mathbf{R}(g_1 + g_2 + \cdots + g_n) < n\mathbf{R}(g_1)$. In fact, we can get a better estimate.

Proposition 9. Let $(g_n)_{n=1}^{\infty} \subset \mathcal{G}$ be a sequence of i.i.d. gambles. Then

$$\max\{\mathbf{R}(g_1),\ nL(g_1)\} < \mathbf{R}(g_1 + g_2 + \cdots + g_n) < \mathbf{R}(g_1) + nL(g_1) + M(g_1).$$

Moreover, $\lim_{t\to\infty}\mathbf{R}(\bar{g}_n) = L(\bar{g}_n) = L(g_1)$, where $\bar{g}_n := (g_1 + g_2 + \cdots + g_n)/n$.

Proof. Let $h_n := g_1 + g_2 + \cdots + g_n$. The left-hand-side inequality follows from proposition 1 (vii) and (ii); for the right-hand-side inequality, use (4), $R^{AS}(h_n) = R^{AS}(g_1)$ (see Aumann and Serrano 2008, Sec. V.H), and again (4):

$$\mathbf{R}(h_n) < R^{AS}(h_n) + L(h_n) = R^{AS}(g_1) + nL(g_1) < \mathbf{R}(g_1) + M(g_1) + nL(g_1).$$

The "moreover" statement follows from the homogeneity of $\mathbf{R}$. QED

For small n, if $\mathbf{R}(g_1)$ is large relative to g_1, then $\mathbf{R}(g_1 + g_2 + \cdots + g_n)$ is close to $\mathbf{R}(g_1)$ (compare Sec. V.H in Aumann and Serrano 2008). For large n, the average gamble $\bar{g}_n$ converges to the positive constant $\mathbf{E}[g_1]$ by the Law of Large Numbers, and so its riskiness decreases; however, as the maximal loss stays constant ($L(\bar{g}_n) = L(g_1)$), the riskiness of $\bar{g}_n$ converges to it (compare Sec. V.B).

D. *Proof of Proposition 2*

We prove here that $\mathbf{R}$ is monotonic with respect to stochastic dominance.

Proof of proposition 2. Let $r_0 := \mathbf{R}(g)$ and $u_0(x) := \log(1 + x/r_0)$. If $g\,\mathrm{SD}_1\,h$ or $g\,\mathrm{SD}_2\,h$, then $\mathbf{E}[u_0(g)] > \mathbf{E}[u_0(h)]$ since u_0 is strictly monotonic and strictly concave. But $\mathbf{E}[u_0(g)] = 0$ by (1), and so $\mathbf{E}[\log(1 + h/r_0)] < 0$, which implies that $\mathbf{R}(g) = r_0 < \mathbf{R}(h)$ by lemma 1. QED

E. *Proof of Proposition 3*

We will prove a slightly more precise continuity result that implies proposition 3.

Proposition 10. Let $(g_n)_{n=1,2,\dots} \subset \mathcal{G}$ be a sequence of gambles satisfying $\sup_{n\geq 1} M(g_n) < \infty$. If $g_n \xrightarrow{\mathcal{D}} g \in \mathcal{G}$ and $L(g_n) \to L_0$ as $n \to \infty$, then $\mathbf{R}(g_n) \to \max\{\mathbf{R}(g), L_0\}$ as $n \to \infty$.

Thus $\mathbf{R}(g_n) \to \mathbf{R}(g)$ except when the limit L_0 of the maximal losses $L(g_n)$ exceeds $\mathbf{R}(g)$, in which case $\mathbf{R}(g_n) \to L_0$.

Proof. Denote $R_0 := \max\{\mathbf{R}(g), L_0\}$. Now $g_n \xrightarrow{\mathcal{D}} g$ implies $\liminf_n L(g_n) \geq L(g)$ since $\liminf_n \mathbf{P}[g_n < -L(g) + \varepsilon] \geq \mathbf{P}[g < -L(g) + \varepsilon] > 0$ for every $\varepsilon > 0$ (see Billingsley 1968, theorem 2.1(iv)), and thus $L_0 \geq L(g)$. Let r be a limit point of the sequence $\mathbf{R}(g_n)$, possibly $\pm\infty$; without loss of generality, assume that $\mathbf{R}(g_n) \to r$. Since $R(g_n) > L(g_n) \to L_0$, it follows that

$$r \geq L_0, \tag{A6}$$

and so either r is finite or $r = \infty$.

We will now show that $r \geq \mathbf{R}(g)$. Indeed, when r is finite (if $r = \infty$, there is nothing to prove here), let $0 < \varepsilon < 1$ and $q := (1+\varepsilon)^2 r$; then for all large enough n we have

$$\mathbf{R}(g_n) < q \tag{A7}$$

and

$$L(g_n) < (1+\varepsilon)L_0 \leq (1+\varepsilon)r = \frac{1}{1+\varepsilon}q \tag{A8}$$

(the second inequality by (A6)). Hence $\mathbf{E}[\log(1 + g_n/q)] > 0$ (by lemma 1 and (A7)), and $\log(1 + g_n/q)$ is uniformly bounded: from above by $\log(1 + \sup_n M(g_n)/q)$, and from below by $\log(\varepsilon/(1+\varepsilon))$ since $g_n/q \geq -L(g_n)/q > -1/(1+\varepsilon)$ by (A8). Therefore, $\mathbf{E}[\log(1 + g/q)] = \lim_n \mathbf{E}[\log(1 + g_n/q)] \geq 0$ (since $g_n \xrightarrow{\mathcal{D}} g$), which implies that $q = (1+\varepsilon)^2 r \geq \mathbf{R}(g)$ (again by lemma 1). Now $\varepsilon > 0$ was arbitrary, and so we got

$$r \geq \mathbf{R}(g). \tag{A9}$$

Now (A6) and (A9) imply that $r \geq R_0$. If $r > R_0$, then take $0 < \varepsilon < 1$ small enough so that $q := (1+\varepsilon)^2 R_0 < r$. For all large enough n we then have

$$q < \mathbf{R}(g_n) \tag{A10}$$

and

$$L(g_n) < (1+\varepsilon)L_0 \leq (1+\varepsilon)R_0 = \frac{1}{1+\varepsilon}q. \tag{A11}$$

Hence $\mathbf{E}[\log(1 + g_n/q)] < 0$ (by lemma 1 and (A10)), and $\log(1 + g_n/q)$ is again uniformly bounded (the lower bound by (A11)). Therefore, $\mathbf{E}[\log(1 + g/q)] = \lim_n \mathbf{E}[\log(1 + g_n/q)] \leq 0$ (since $g_n \xrightarrow{\mathcal{D}} g$), contradicting $q = (1+\varepsilon)^2 R_0 \geq (1+\varepsilon)^2 \mathbf{R}(g) > \mathbf{R}(g)$ (by lemma 1 and $\mathbf{R}(g) > L(g) > 0$).

Therefore, $r = R_0$ for every limit point of $\mathbf{R}(g_n)$, or $\mathbf{R}(g_n) \to R_0$. QED

Proof of proposition 3. If $L_0 = L(g)$, then $\max\{\mathbf{R}(g), L_0\} = \mathbf{R}(g)$; apply proposition 10. QED

To see why the values need to be uniformly bounded from above (i.e., $\sup_{n\geq 1} M(g_n) < \infty$), let g_n take the values $-3/4$, 3, and $2^{n-1} - 1$ with probabilities

$(3/4)(1-1/n)$, $(1/4)(1-1/n)$, and $1/n$, respectively, and let g take the values $-3/4$ and 3 with probabilities 3/4 and 1/4. Then $g_n \xrightarrow{\mathcal{D}} g$ and $L(g_n) = L(g) = -3/4$, but $M(g_n) \to \infty$ and $\mathbf{R}(g_n) = 1 \neq 5.72 = \mathbf{R}(g)$.

F. *Proof of Proposition 4*

We prove the result connecting the measures of riskiness.

Proof of proposition 4. Statement (4) yields the equivalence of (ii) and (iii) and the "moreover" statement.

(ii) implies (i): Let $r_n := \mathbf{R}(g_n) \to \infty$. Using $\log(1+x) = x - x^2/2 + o(x^2)$ as $x \to 0$ for each value of g_n/r_n (all these values are uniformly bounded) and then taking expectation yields

$$0 = \mathbf{E}[\log(1 + g_n/r_n)] = \mathbf{E}[g_n]/r_n - \mathbf{E}[g_n^2]/(2r_n^2) + o(1/r_n^2).$$

Multiplying by r_n^2 gives $r_n\mathbf{E}[g_n] - \mathbf{E}[g_n^2]/2 \to 0$, and thus $\mathbf{E}[g_n] \to 0$ (since $r_n \to \infty$ and the $\mathbf{E}[g_n^2]$ are bounded), that is, (i).

(i) implies (ii): Assume that $\mathbf{E}[g_n] \to 0$. For every $0 < \delta < 1$, let $q_n = (1 - \delta)\mathbf{E}[g_n^2]/(2\mathbf{E}[g_n])$; then $q_n \to \infty$ and

$$\begin{aligned} q_n\mathbf{E}\left[\log\left(1 + \frac{1}{q_n}g_n\right)\right] &= \mathbf{E}[g_n] - \frac{\mathbf{E}[g_n^2]}{2q_n} + o\left(\frac{1}{q_n}\right) \\ &= -\frac{\delta}{1-\delta}\mathbf{E}[g_n] + o\left(\frac{1}{q_n}\right). \end{aligned}$$

Therefore, for all large enough n we have $\mathbf{E}[\log(1 + g_n/q_n)] < 0$, and thus $\mathbf{R}(g_n) > q_n \to \infty$. QED

Remark. One may define another measure on gambles: $R^0(g) = \mathbf{E}[g^2]/(2\mathbf{E}[g])$ for every $g \in \mathcal{G}$. It is easy to see that $R^0(g) \to \infty$ if and only if (i)–(iii) hold, and then $R^0(g) \sim \mathbf{R}(g_n) \sim R^{AS}(g_n)$ as $n \to \infty$. However, R^0 does not satisfy monotonicity.[40]

G. *Nonhomogeneous Strategies*

As discussed in Section VI.E.6, we take the basic setup of Section III and consider strategies s_Q with arbitrary critical-wealth functions $Q: \mathcal{G} \to [0, \infty]$ that are not necessarily homogeneous of degree one. To avoid inessential technical issues, we make a mild regularity assumption: a strategy s_Q is called *regular* if the limit $Q_1(g) := \lim_{\lambda \to 0^+} Q(\lambda g)/\lambda$ exists for every $g \in \mathcal{G}$ (see remark 2 below for general strategies).[41] The result is

Proposition 11. Let $s \equiv s_Q$ be a regular strategy with $Q(g) > L(g)$ for all $g \in \mathcal{G}$. Then s guarantees no-bankruptcy if $Q_1(g) > \mathbf{R}(g)$ for every $g \in \mathcal{G}$, and only if $Q_1(g) \geq \mathbf{R}(g)$ for every $g \in \mathcal{G}$.

[40] Let g take the values 500 and -100 with equal probabilities, and h take the values 300 and -100 with equal probabilities; then $g\, \mathrm{SD}_1\, h$, but $R^0(g) = 325 > 250 = R^0(h)$.

[41] Note that Q_1 is by definition positively homogeneous of degree one.

Thus, for nonhomogeneous strategies, one needs to consider only "small" gambles (i.e., λg with $\lambda \to 0$); but, again, $\mathbf{R}(g)$ provides the critical threshold.

Proof of proposition 11. We start by showing that $Q_1(g) > \mathbf{R}(g)$ for every $g \in \mathcal{G}$ implies that for every finite set of gambles $\mathcal{G}_0 \subset \mathcal{G}$ there exists $\varepsilon > 0$ such that

$$Q(g) \geq \min\{\mathbf{R}(g),\ \varepsilon\} \tag{A12}$$

for every $g \in$ cone $\mathcal{G}_0$. Indeed, otherwise we have sequences $\varepsilon_n \to 0^+$ and $g_n \in$ cone $\mathcal{G}_0$ with $Q(g_n) < \min\{\mathbf{R}(g_n),\ \varepsilon_n\}$ for every n. Since $\mathcal{G}_0$ is finite, without loss of generality we can take all g_n to be multiples of the same $g_0 \in \mathcal{G}_0$, say $g_n = \lambda_n g_0$. Now $\lambda_n \to 0$ since $\varepsilon_n > Q(g_n) > L(g_n) = \lambda_n L(g_0) > 0$ (the second inequality since $Q(g) > L(g)$ for every g); also $Q(\lambda_n g_0)/\lambda_n < \mathbf{R}(\lambda_n g_0)/\lambda_n = \mathbf{R}(g_0)$ (since $\mathbf{R}$ is homogeneous of degree one by lemma 2), and so $Q_1(g_0) \leq \mathbf{R}(g_0)$, contradicting our assumption.

Assume $Q_1(g) > \mathbf{R}(g)$ for all g. Given a process G generated by a finite set $\mathcal{G}_0 \subset \mathcal{G}$, fix $\varepsilon > 0$ that satisfies (A12). Let $Z_t := W_{t+1}/W_t$ and define $Z'_t := Z_t$ if $W_t < \varepsilon$ and $Z'_t := 1$ otherwise. Now $W_T \to 0$ implies that $W'_T := \prod_{t=1}^{T} Z'_t \to 0$ (indeed, let T_0 be such that $W_T < \varepsilon$ for all $T \geq T_0$; then $W'_T = [W'_{T_0}/W_{T_0}]W_T$ for all $T \geq T_0$ and so $W'_T \to 0$ too). We proceed as in the proof of the first part of theorem 1, but with $Y_t := \log Z'_t$, to obtain $\mathbf{P}[W'_T \to 0] = 0$, and thus $\mathbf{P}[W_T \to 0] = 0$.

Conversely, assume that there is $\tilde{g} \in \mathcal{G}$ with $Q_1(\tilde{g}) < \mathbf{R}(\tilde{g})$. Let q be such that $Q_1(\tilde{g}) < q < \mathbf{R}(\tilde{g})$; then there exists $\delta > 0$ such that for all $\lambda < \delta$ we have $Q(\lambda\tilde{g}) < \lambda q$, and thus $\lambda\tilde{g}$ is accepted at λq. Equivalently, $(W/q)\tilde{g}$ is accepted at W for all $W < \delta q$. We now proceed as in the proof of proposition 7. Let $\tilde{g}_t$ be an i.i.d. sequence with $\tilde{g}_t$ having the same distribution as $\tilde{g}$ for every t; let $G = (g_t)$ be the process with $g_t = (W_t/q)\tilde{g}_t$ for every t; put $U_T := \sum_{t=1}^{T} Y_t = \sum_{t=1}^{T} \log(1 + \tilde{g}_t/q)$. Then $U_T/T \to \mu := \mathbf{E}[\log(1 + \tilde{g}/q)] < 0$, and so

$$U_T \to -\infty \tag{A13}$$

a.s. as $T \to \infty$.

This does not yield bankruptcy, however, since the wealth W_T may go above δq, where we have no control over the decisions, and then $\log(W_{T+1}/W_T)$ need no longer equal Y_T. What we will thus show is that the probability of that happening is strictly less than one, and so bankruptcy indeed occurs with positive probability.

First, we claim that there exists $K > 0$ large enough such that

$$\mathbf{P}[U_T \leq K \text{ for all } T] > 0. \tag{A14}$$

Indeed, the Y_t are i.i.d., with $\mathbf{E}[Y_t] = \mu < 0$ and $a \leq Y_t \leq b$ for $a = \log(1 - L(\tilde{g})/q)$ and $b = \log(1 + M(\tilde{g})/q)$; applying the "large deviations" inequality of Hoeffding (1963, theorem 2) yields

$$\begin{aligned}\mathbf{P}[U_T > K] &= \mathbf{P}[U_T - \mu T > K + |\mu|T] \\ &\leq \exp\left(-\frac{2(K + |\mu|T)^2}{T(b-a)^2}\right) < \exp(-cT - dK)\end{aligned}$$

for appropriate constants $c,\ d > 0$ (specifically, $c = 2\mu^2/(b-a)^2$ and $d =$

$4|\mu|/(b-a)^2)$. Therefore,

$$\mathbf{P}[U_T > K \text{ for some } T] < \sum_{T=1}^{\infty} \exp(-cT - dK) = \frac{\exp(-c-dK)}{1-\exp(-c)},$$

which can be made < 1 for an appropriately large K; this proves (A14).

Start with $W_1 < \delta q \exp(-K)$. We claim that if $U_T \le K$ for all T, then g_T is accepted for all T. Indeed, assume by induction that $g_1, g_2, \dots, g_{T-1}$ have been accepted; then $W_T = W_1 \exp(U_{T-1}) \le W_1 \exp(K) < \delta q$, and so $g_T = (W_T/q)\tilde{g}_T$ is also accepted (at W_T). But if g_T is accepted for all T, then $W_T = W_1 \exp(U_{T-1})$ for all T; since $U_T \to -\infty$ a.s. (see (A13)), it follows that $W_T \to 0$ a.s. on the event $\{U_T \le K$ for all $T\}$. Therefore, $\mathbf{P}[W_T \to 0] \ge \mathbf{P}[U_T \le K$ for all $T] > 0$ (see (A14)), and so the process G leads to bankruptcy with positive probability. QED

Remark 1. In the proof we have shown that $Q_1(g) > \mathbf{R}(g)$ for all g implies that for every finite set of gambles $\mathcal{G}_0 \subset \mathcal{G}$ there exists $\varepsilon > 0$ such that $Q(g) \ge \min\{\mathbf{R}(g), \varepsilon\}$ for every $g \in$ cone $\mathcal{G}_0$, or[42]

$$s \text{ rejects } g \text{ at all } W < \mathbf{R}(g) \text{ with } W < \varepsilon. \tag{A15}$$

Compare (2): the addition here is "$W < \varepsilon$." Condition (A15) means that the policy of rejecting gambles whose riskiness exceeds the wealth (i.e., $W < \mathbf{R}(g)$) applies only at small wealths (i.e., $W < \varepsilon$); see Section VI.E.6.

Remark 2. Slight modifications of the above proof show that for a general strategy s that need not be regular or have a critical-wealth function (but does reject g when $W \le L(g)$), a sufficient condition for guaranteeing no-bankruptcy is that for every $g \in \mathcal{G}$, if $W < \mathbf{R}(g)$ then s rejects λg at λW for *all* small enough λ (i.e., there is $\delta > 0$ such that this holds for all $\lambda < \delta$); a necessary condition is that for every $g \in \mathcal{G}$, if $W < \mathbf{R}(g)$ then s rejects λg at λW for *arbitrarily* small λ (i.e., for every $\delta > 0$ there is $\lambda < \delta$ where this holds). If we let $Q_s(g) := \inf\{W > 0 : s \text{ accepts } g \text{ at } W\}$, then $\liminf_{\lambda \to 0^+} Q_s(\lambda g)/\lambda > \mathbf{R}(g)$ for all g is a sufficient condition; and when s is a threshold strategy (i.e., s accepts g at all $W > Q_s(g)$), then $\limsup_{\lambda \to 0^+} Q_s(\lambda g)/\lambda \ge \mathbf{R}(g)$ for all g is a necessary condition.

References

Algoet, Paul. 1992. "Universal Schemes for Prediction, Gambling and Portfolio Selection." *Ann. Probability* 20:901–41.

Artzner, Philippe, Freddy Delbaen, Jean-Marc Eber, and David Heath. 1999. "Coherent Measures of Risk." *Math. Finance* 9:203–28.

Aumann, Robert J. 1985. "What Is Game Theory Trying to Accomplish?" In *Frontiers of Economics*, edited by K. J. Arrow and S. Honkapohja, 28–76. Oxford: Blackwell.

Aumann, Robert J., and Roberto Serrano. 2008. "An Economic Index of Riskiness." *J.P.E.* 116:810–36.

Billingsley, Patrick. 1968. *Convergence of Probability Measures.* New York: Wiley.

Blume, Lawrence, and David Easley. 1992. "Evolution and Market Behavior." *J. Econ. Theory* 58:9–40.

[42] Moreover, it is easy to see that this condition implies that $Q_1(g) \ge \mathbf{R}(g)$ for all g.

Cover, Tom M., and Joy A. Thomas. 1991. *Elements of Information Theory*. New York: Wiley.
Foster, Dean P., and Sergiu Hart. 2008. "A Reserve-Based Axiomatization of the Measure of Riskiness." Manuscript, Center Study Rationality, Hebrew Univ. Jerusalem.
Hadar, Josef, and William Russell. 1969. "Rules for Ordering Uncertain Prospects." *A.E.R.* 59:25–34.
Hanoch, Giora, and Haim Levy. 1969. "The Efficiency Analysis of Choices Involving Risk." *Rev. Econ. Studies* 36:335–46.
Hart, Sergiu. 2008. "A Simple Riskiness Order Leading to the Aumann-Serrano Index of Riskiness." Manuscript, Center Study Rationality, Hebrew Univ. Jerusalem.
Hoeffding, Wassily. 1963. "Probability Inequalities for Sums of Bounded Random Variables." *J. American Statis. Assoc.* 58:13–30.
Kelly, John L., Jr. 1956. "A New Interpretation of Information Rate." *IEEE Trans. Information Theory* 2:185–89.
Loève, Michael. 1978. *Probability Theory*. 4th ed. Berlin: Springer.
Machina, Mark J., and Michael Rothschild. 2008. "Risk." In *The New Palgrave Dictionary of Economics*, 2nd ed., edited by S. N. Durlauf and L. E. Blume. New York: Palgrave Macmillan.
Markowitz, Harry. 1952. "Portfolio Selection." *J. Finance* 7:77–91.
———. 1959. *Portfolio Selection: Efficient Diversification of Investment*. New Haven, CT: Yale Univ. Press.
Neveu, Jacques. 1975. *Discrete-Parameter Martingales*. Amsterdam: North-Holland.
Palacios-Huerta, Ignacio, and Roberto Serrano. 2006. "Rejecting Small Gambles under Expected Utility." *Econ. Letters* 91:250–59.
Palacios-Huerta, Ignacio, Roberto Serrano, and Oscar Volij. 2004. "Rejecting Small Gambles under Expected Utility." Manuscript, Dept. Econ., Brown Univ.
Rabin, Matthew. 2000. "Risk Aversion and Expected-Utility Theory: A Calibration Theorem." *Econometrica* 68:1281–92.
Rothschild, Michael, and Joseph Stiglitz. 1970. "Increasing Risk: I. A Definition." *J. Econ. Theory* 2:225–43.
———. 1971. "Increasing Risk: II. Its Economic Consequences." *J. Econ. Theory* 3:66–84.
Rubinstein, Ariel. 2001. "Comments on the Risk and Time Preferences in Economics." Manuscript, School Econ., Tel Aviv Univ.; Dept. Econ., Princeton Univ.
Safra, Zvi, and Uzi Segal. 2008. "Calibration Results for Non–Expected Utility Theories." *Econometrica* 76:1143–66.
Samuelson, Paul. 1979. "Why We Should Not Make the Mean Log of Wealth Big Though Years to Act Are Long." *J. Banking and Finance* 3:305–7.
Sharpe, William F. 1964. "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk." *J. Finance* 19:425–42.
———. 1966. "Mutual Fund Performance." *J. Bus.* 39:119–38.
Taleb, Nassim N. 2005. *Fooled by Randomness: The Hidden Role of Chance in Life and in the Markets*. New York: Random House.